Constructing Morphisms for Arithmetic Subsequences of Fibonacci

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Abstract. From a general theorem by Dekking it follows that an arithmetic subsequence of any morphic sequence is morphic again. The construction of such a morphism is not directly obvious. In this note we demonstrate the explicit construction of a morphism generating an arbitrary arithmetic subsequence of the infinite fixed point of the Fibonacci morphism.

Keywords: morphic sequences · automata · Fibonacci

1 Introduction

This paper originates in discussions with Hans Zantema about the complexity of morphic and automatic sequences, see also [17,18]. One of the questions that arose was what the complexity is of the sequence of even (or odd) numbered entries of the well-known Fibonacci morphic sequence; the notion of complexity we use here is that of *morphic complexity*, see the next section for details. Because of a theorem of Dekking (see [1] 7.91), arithmetic subsequences of a morphic sequence are morphic again. The construction of a corresponding morphism is part of Dekking's proof. However, Dekking's theorem is more general and goes beyond arithmetic subsequences. This generality has two consequences. Firstly, it is not so easy to obtain the morphism explicitly in a concrete example involving arithmetic subsequences. Secondly, more efficient constructions can be found when restricting to special cases. In particular, arithmetic subsequences might be generated with less complex morphisms.

Our goal in this paper is to address these two issues. The proof of the main result in this paper (Theorem 1) gives an easy, explicit construction generating any arithmetic subsequence of Fibonacci by a morphism. Along the way, we measure and bound the morphic complexity of these arithmetic subsequences by properties of the construction; the paper [4] by Dekking only gives a general bound on the size of the alphabet.

Although some experimentation shows that the same construction can be applied much more generally (but not universally) to morphic sequences, we restrict to the Fibonacci case in this paper.

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2 Preliminaries and Main Result

Denote the Fibonacci numbers by F_n, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

A sequence over an alphabet Σ is called a *pure morphic sequence* if it can be obtained as an infinite fixed point of a morphism (or substitution) $g : \Sigma \to \Sigma^*$. Our key example is the morphic Fibonacci sequence on $\Sigma = \{0, 1\}$, defined by

$$\sigma = \mathcal{F}^\infty(0) \quad \text{where} \quad \begin{cases} \mathcal{F}(0) = 01, & \text{and} \\ \mathcal{F}(1) = 0. \end{cases}$$

An initial segment of σ is

$$01001010010010100101001001010010010100101001001010.$$

The segment above shows σ from $\sigma[0]$ up to $\sigma[49]$, where by $\sigma[j]$ we denote the j-th entry of σ, for $j \geq 0$.

If Σ and $\tilde{\Sigma}$ are alphabets, a *coding* is a map $\tau : \Sigma \to \tilde{\Sigma}$. A coding does not have to be injective. A sequence $\tilde{\rho} \in \tilde{\Sigma}^\infty$ is a *morphic sequence* if there exists a pure morphic sequence ρ on an alphabet Σ and a coding $\tau : \Sigma \to \tilde{\Sigma}$ such that $\tilde{\rho}[j] = \tau(\rho[j])$ for all $j \geq 0$. By the complexity of the morphism g, we mean the sum of the word lengths of the images of the letters $\sum_{a \in \Sigma} |g(a)|$. The *morphic complexity* of a morphic sequence is the minimal complexity of a morphism producing this sequence (with or without a coding). Note that there are various alternative notions of complexity for infinite sequences. In this paper, complexity of an infinite sequence always means morphic complexity. Every non-trivial morphic sequence has complexity at least 3. The morphic Fibonacci sequence σ is essentially the only sequence with complexity 3.

Our main result in this paper is the following for the subsequences formed by a given residue class k mod m of the indices.

Theorem 1. *Let σ be the morphic Fibonacci sequence. For any modulus $m > 1$ and any k with $0 \leq k < m$, the arithmetic subsequence $(\sigma[n \cdot m + k])_{n \geq 0}$ is morphic with morphic complexity at most $(m + 1)F_{2m+2}$.*

A sharper but less explicit upper bound for the complexity is $(m + 1)F_{z(m)+2}$, where $z(m)$ is the smallest index j for which F_j is divisible by m; see the discussions later on in this paper, where also a good approximation for the complexity of our morphism is given.

3 Fibonacci Numbers

The proof of Theorem 1 is *constructive* and only uses a few elementary properties of the sequence of Fibonacci numbers. The following result is well-known, see for example [15]; we also include a proof here.

Lemma 1. *For every integer $m > 1$ the sequence $(F_n \bmod m)_{n\geq 0}$ is purely periodic; in particular, there exists $j \geq 2$ such that $m|F_n$ if and only if $j|n$.*

Proof. Extend the definition of F_n to negative n. By this definition, for all $n \in \mathbb{Z}$

$$F_{n+1} \equiv F_n + F_{n-1} \bmod m.$$

The sequence $(F_n \bmod m)_{n\in\mathbb{Z}}$ takes on at most m different values. Therefore, there must exist $u < v$ such that

$$F_u \equiv F_v \bmod m, \qquad F_{u+1} \equiv F_{v+1} \bmod m.$$

The recursive definition now implies for all $n \in \mathbb{Z}$,

$$F_{u+n} \equiv F_{v+n} \bmod m.$$

Hence the sequence $(F_n \bmod m)_{n\geq 0}$ is periodic with period $v - u$, and $F_{v-u} \equiv 0 \bmod m$.

Let $j \geq 2$ be the first strictly positive index with $F_j \equiv 0 \bmod m$. Define $a \equiv F_{j+1} \bmod m$. Then

$$F_j \equiv 0 \equiv aF_0 \bmod m, \qquad F_{j+1} \equiv a \equiv aF_1 \bmod m,$$

and the recursion gives $F_{j+n} \equiv aF_n \bmod m$ for all n. It follows that $F_n \equiv 0 \bmod m$ if and only if $j|n$, since a is coprime to m: any common factor would divide F_{j+1} and F_j, and by the recursive relation also F_{j-1}, and eventually $F_1 = 1$.

The (minimal) period $\pi(m)$ of $(F_n \bmod m)_{n\geq 0}$ is called the Pisano period. Since a pair of consecutive values in $(F_n \bmod m)_{n\geq 0}$ determines the sequence, it is easy to see that $\pi(m)$ is at most the maximum of the distance between equal pairs. This gives $\pi(m) \leq m^2$. Sharper results are available in the literature: it is known that $\pi(m) \leq 6m$, which is tight; see [6,15].

Clearly, $F_{\pi(m)} \equiv 0 \bmod m$, but there could be a smaller Fibonacci number divisible by m. Let $z(m)$ be the smallest index j for which $F_j \equiv 0 \bmod m$. Then $z(m)$ divides $\pi(m)$. The Fibonacci numbers are known to show quite remarkable behaviour (see for instance [7]) and the divisibility properties are no exception; see Fig. 1 for a visualization of $\pi(m)$ and $z(m)$.

The following result will be useful for bounding the complexity of our morphisms.

Lemma 2 (Sallé, [13]). *Let $m \geq 2$. Then $z(m) \leq 2m$.*

This is the sharpest possible linear upper bound that works for all m. It is sharp if and only if $m = 6 \cdot 5^k$ for some $k \geq 0$, see [9]. However, the bound is quite weak for most values of m, as can already be seen from Fig. 1. Sharper bounds or exact values for different types of composite numbers m can be found in [9] and [10]. Tables with values of $z(m)$ up to $m = 10^5$ were published by the Fibonacci Association in 1965 ([16] and [8]), but they can of course easily be reproduced with the help of modern computers.

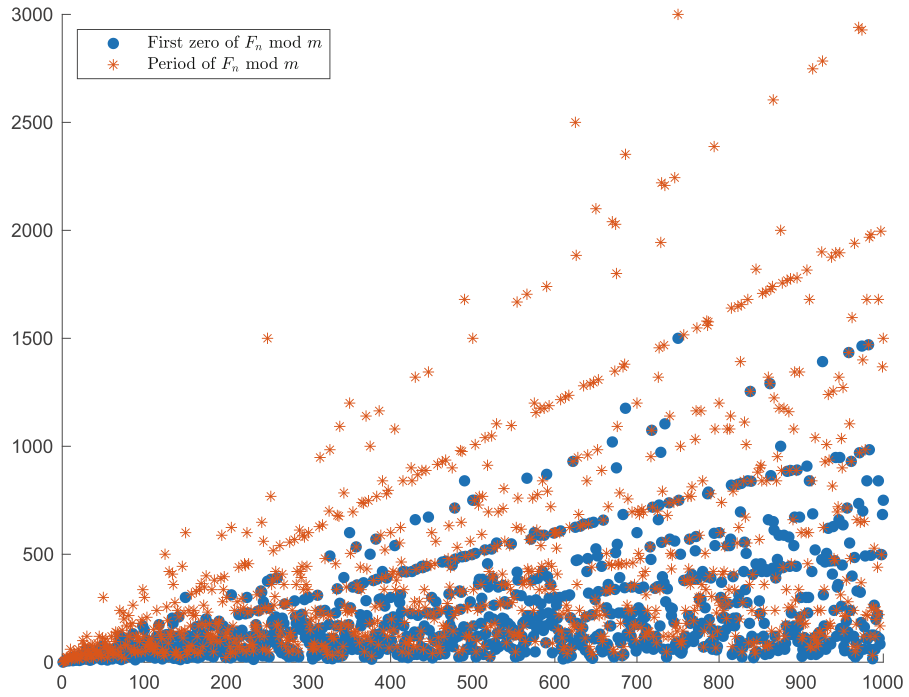

Fig. 1. The index of the smallest n such that $F_n \equiv 0 \bmod m$ and the period of $(F_n \bmod m)_{n \geq 0}$; m is on the horizontal axis.

4 Fibonacci Morphism

We now turn to properties of the morphic Fibonacci sequence. Let s be any word in $\{0,1\}^+$; by $\text{len}(s)$ we denote its length, and by $\text{wt}(s)$ its weight, that is, the number of letters equal to 1 in s. As usual we can apply the morphism $\mathcal{F}$ recursively to any word.

Lemma 3. *For every $s \in \{0,1\}^+$ of length m and weight w, and every $j \geq 0$:*

$$len(\mathcal{F}^j(s)) = F_{j+2} \cdot m - F_j \cdot w.$$

Proof. Note that by construction the lengths of the iterates of $\mathcal{F}$ on the letters are Fibonacci numbers:

$$\text{len}(\mathcal{F}^j(0)) = F_{j+2} \quad \text{and} \quad \text{len}(\mathcal{F}^j(1)) = F_{j+1},$$

justifying the name Fibonacci for this morphism. Also $\text{len}(\mathcal{F}(s)) = 2(m-w) + w = 2m - w$ and $\text{wt}(\mathcal{F}(s)) = m - w$. For $j \geq 2$ we find likewise that

$$\text{len}(\mathcal{F}^j(s)) = 2 \cdot \text{len}(\mathcal{F}^{j-1}(s)) - \text{wt}(\mathcal{F}^{j-1}(s)).$$

Since

$$\mathrm{wt}(\mathcal{F}^{j-1}(s)) = \mathrm{len}(\mathcal{F}^{j-1}(s)) - \mathrm{len}(\mathcal{F}^{j-2}(s)),$$

we get

$$\mathrm{len}(\mathcal{F}^{j}(s)) = \mathrm{len}(\mathcal{F}^{j-1}(s)) + \mathrm{len}(\mathcal{F}^{j-2}(s)).$$

So $\mathrm{wt}(\mathcal{F}^{j}(s)) = \mathrm{len}(\mathcal{F}^{j-2}(s))$, and also

$$\mathrm{wt}(\mathcal{F}^{j}(s)) = \mathrm{wt}(\mathcal{F}^{j-1}(s)) + \mathrm{wt}(\mathcal{F}^{j-2}(s)).$$

That is, both the lengths and the weights satisfy the Fibonacci recurrence.

As $\mathrm{len}(\mathcal{F}^{0}(s)) = m = F_2 \cdot m - F_0 \cdot w$ and $\mathrm{len}(\mathcal{F}^{1}(s)) = 2m - w = F_3 \cdot m - F_1 \cdot w$, the result follows by induction.

Corollary 1. *For every $m \geq 2$ there exists $j \geq 2$ such that for every $s \in \{0,1\}^+$:*

$$\mathit{len}(s) = m \quad \Rightarrow \quad \mathit{len}(\mathcal{F}^{j}(s)) \equiv 0 \bmod m.$$

Proof. Choose j such that $F_j \equiv 0 \bmod m$, as in Lemma 1; then

$$\mathrm{len}(\mathcal{F}^{j}(s)) = F_{j+2} \cdot m - F_j \cdot \mathrm{wt}(s) \equiv 0 \bmod m$$

by Lemma 3.

A morphism with this property is called an *m-block stable substitution* in [5]. This paper discusses fixed points of m-block substitutions and $\mathcal{F}^3$ is given as an example of a 2-block stable substitution. See also [12].

We also use the following lemma, which says that every subword of σ occurs in each residue class of indices for every modulus. A subword of an infinite word here means a contiguous finite subsequence.

Lemma 4. *Let w be a subword of σ of length $m \geq 2$. For every k with $0 \leq k < m$, there exists $n \equiv k \bmod m$ such that $w = \sigma[n] \cdots \sigma[n+m-1]$.*

Proof. Suppose that w starts at index n_0, so $w = \sigma[n_0] \cdots \sigma[n_0+m-1]$. Note that $\sigma = \mathcal{F}^{\infty}(0) = \mathcal{F}^{\infty}(1)$, so there exists j_0 such that for all $j \geq j_0$,

$$w = \mathcal{F}^{j}(1)[n_0] \cdots \mathcal{F}^{j}(1)[n_0+m-1].$$

Since $(F_n \bmod m)_{n \geq 0}$ is purely periodic (Lemma 1), and since $F_1 \equiv 1 \bmod m$, we can find arbitrary large j such that $F_{j+2} \equiv 1 \bmod m$. Take such $j \geq j_0$. Then write

$$\sigma = \mathcal{F}^{j}(\sigma) = \mathcal{F}^{j}(0)\mathcal{F}^{j}(1) \cdots$$

Using that $\mathrm{len}(\mathcal{F}^{j}(0)) = F_{j+2}$, we find that

$$w = \sigma[F_{j+2} + n_0] \cdots \sigma[F_{j+2} + n_0 + m - 1].$$

Let $n_1 := F_{j+2} + n_0$. Clearly $n_1 \equiv F_{j+2} + n_0 \equiv n_0 + 1 \bmod m$. By iterating, we obtain for all k such that $0 \leq k \leq m-1$ existence of $n \equiv k \bmod m$ such that $w = \sigma[n] \cdots \sigma[n+m-1]$, which is the statement of the lemma.

By V_m we will denote the set of words of length m over $\{0,1\}$ that occur when we 'chop up' σ in consecutive words of length m:

$$V_m = \{\sigma[0]\cdots\sigma[m-1], \sigma[m]\cdots\sigma[2m-1], \sigma[2m]\cdots\sigma[3m-1], \ldots\}.$$

Note that many of the 2^m possible words over $\{0,1\}$ do not occur in σ. In fact, the Fibonacci sequence is known to be a Sturmian sequence. This means that only $m+1$ different words of length m occur ([1] Ch. 9, 10.5.8). Using Lemma 4, we conclude that $|V_m| = m+1$.

5 Proof of the Theorem

We are now ready to give our constructive proof of Theorem 1. For any $m \geq 2$ and $0 \leq k < m$, we will find an alphabet together with a morphism and coding which generate the arithmetic subsequence $(\sigma[k+n\cdot m])_{n\geq 0}$.

Proof. Let k, m be given. Choose j, as in Lemma 1, such that m divides F_j. Denote the words in V_m by $w_0, \ldots, w_m$. Define a new alphabet $\Sigma = \{a_0, \ldots, a_m\}$ and a map g by setting $g(w_i) = a_i$ for $w_i \in V_m$ and inductively extending g to concatenations of words in V_m: if $g(w)$ is defined, then $g(ww_i) = g(w)a_i$. On Σ, we define a morphism h by

$$h(a_i) = g(\mathcal{F}^j(g^{-1}(a_i))), \qquad 0 \leq i \leq m.$$

Note that this is well-defined, since the argument of g is of the form $\mathcal{F}^j(w_i)$, which is a word in $\{0,1\}^+$ of length a multiple of m by Lemma 3.

Assume without loss of generality that $w_0 = \sigma[0]\cdots\sigma[m-1]$. Define the infinite pure morphic sequence $h^\infty(a_0)$, then

$$\begin{aligned} h^\infty(a_0) &= (g \circ \mathcal{F}^j \circ g^{-1})^\infty(a_0) \\ &= (g \circ \mathcal{F}^\infty \circ g^{-1})(a_0) \\ &= (g \circ \mathcal{F}^\infty)(w_0) = g(\sigma). \end{aligned}$$

This means that there is a one-to-one correspondence between $h^\infty(a_0)$ and σ. Indeed, the n-th letter of $h^\infty(a_0)$ can be used to recover the n-th block of length m in σ as follows:

$$g^{-1}(h^\infty(a_0)[n]) = \sigma[n\cdot m]\cdots\sigma[(n+1)\cdot m-1].$$

This allows to create a coding $\tau : \Sigma \to \{0,1\}$ defined by

$$\tau(a_i) = (g^{-1}(a_i))[k] = w_i[k],$$

mapping the nth letter in $h^\infty(a_0)$ to $\sigma[n\cdot m+k]$, so $\tau(h^\infty(a_0))$ equals $(\sigma[n\cdot m+k])_{n\geq 0}$. This completes the construction of the morphism generating $(\sigma[n\cdot m+k])_{n\geq 0}$.

Note that $|\Sigma| = |V_m| = m+1$. For $a_i \in \Sigma$, using that $\mathrm{len}(w_i) = m$ for all i, we obtain by Lemma 3

$$\mathrm{len}(h(a_i)) = \mathrm{len}(g(\mathcal{F}^j(g^{-1}(a_i)))) = \frac{1}{m}\mathrm{len}(\mathcal{F}^j(w_i)) \leq F_{j+2}. \tag{1}$$

The smallest j that works is $z(m)$, which is at most $2m$ by Lemma 2. It follows that the complexity of the morphism is at most $(m+1)F_{2m+2}$.

6 Examples

Example 1. By way of example we will construct the subsequence $(\sigma[4n+3])_{n\geq 0}$. First note that Theorem 1 gives the upper bound $5 \cdot F_{2\cdot 4+2} = 5 \cdot 55 = 275$ for the complexity of this subsequence. This bound is based on the general Lemma 2. If the first zero $z(m)$ of $(F_n \bmod m)_{n\geq 0}$ is known, we could possibly do better, since we can replace the bound in Theorem 1 by $(m+1)F_{z(m)+2}$. In our current example, the first nonzero Fibonacci number divisible by 4 is $F_6 = 8$. This implies that the upper bound for the complexity can already be improved to $5 \cdot F_{6+2} = 105$.

There are 5 subwords w_0, w_1, w_2, w_3, w_4, of length 4 in V_4:

$$w_0 = 0100, \qquad w_1 = 1010, \qquad w_2 = 0101, \qquad w_3 = 0010, \qquad w_4 = 1001.$$

In fact, they are all present already in the initial segment of σ shown above. They correspond to the letters a_0, a_1, a_2, a_3, a_4 of our new alphabet Σ by the map $g(w_i) = a_i$. Since F_6 is the first positive Fibonacci number divisible by 4, the length of $\mathcal{F}^6(w)$ is a multiple of 4 for all $w \in V_4$ (by Corollary 1). Computing the images $\mathcal{F}^6(w_i)$ and defining the morphism h by $h(a_i) = g(\mathcal{F}^6(w_i))$ gives

$$\begin{aligned}
a_0 &\mapsto a_0a_1a_0a_1a_2a_3a_2a_3a_2a_3a_4a_3a_4a_0a_4a_0a_4a_0a_1,\\
a_1 &\mapsto a_0a_1a_0a_1a_2a_3a_2a_3a_4a_3a_4a_3a_4a_0a_4a_0a_1,\\
a_2 &\mapsto a_0a_1a_0a_1a_2a_3a_2a_3a_2a_3a_4a_3a_4a_0a_4a_0a_4,\\
a_3 &\mapsto a_0a_1a_0a_1a_2a_3a_2a_3a_2a_3a_4a_3a_4a_3a_4a_0a_4a_0a_1,\\
a_4 &\mapsto a_0a_1a_0a_1a_2a_3a_2a_3a_4a_3a_4a_3a_4a_0a_4a_0a_4,
\end{aligned}$$

so images of this substitution are words of length 19 and 17. Finally, the coding τ sends each letter a_i to the final 'bit' of w_i:

$$\tau(a_i) = (g^{-1}(a_i))[3] = \begin{cases} 0 & \text{for} \quad i = 0,1,3, \\ 1 & \text{for} \quad i = 2,4. \end{cases}$$

Now we can read off the initial segment easily from these:

$$(\sigma[4n+3])_{n\geq 0} = \tau(h^\infty(a_0)) = 0000101010101010100\ 00001010101010100\ 00\cdots.$$

The complexity of the morphism h is $3 \times 17 + 2 \times 19 = 89$, considerably smaller than the upper bound of 105.

Table 1 gives some statistics for $m = 2, \ldots, 30$:

- the index of the first zero $z(m)$,
- an upper bound for the complexity of $\sigma([n \cdot m + k])_{n \geq 0}$, not using $z(m)$,
- an improved upper bound using $z(m)$,
- an approximation for the complexity of h (see Remarks in the next section),
- the actual complexity of h (with the exception of the case $m = 30$, which is too big to handle).

Table 1. Upper bounds, approximation and true complexity of h for $m = 2, \ldots, 30$

m	$z(m)$	$(m+1)F_{2m+2}$	$(m+1)F_{z(m)+2}$	$[(1-\phi^4)(m+1)F_{z(m)+2}]$	compl(h)
2	3	24	15	13	13
3	4	84	32	27	27
4	6	275	105	90	89
5	5	864	78	67	67
6	12	2639	2639	2254	2255
7	8	7896	440	376	377
8	6	23256	189	161	161
9	12	67650	3770	3220	3210
10	15	194821	17567	15004	15005
11	10	556416	1728	1476	1473
12	12	1578109	4901	4186	4181
13	7	4449354	476	407	407
14	24	12480600	1820895	1555230	1555935
15	20	34852944	283376	242032	242335
16	12	96949079	6409	5474	5473
17	9	268746336	1602	1368	1368
18	12	742675211	7163	6118	6123
19	18	2046683100	135300	115560	115580
20	30	5626200216	45744489	39070458	39088169
21	8	15430992126	1210	1033	1033
22	30	42235173769	50101107	42791454	42764027
23	24	115380647424	2913432	2488368	2488056
24	12	314656725625	9425	8050	8045
25	25	856733282574	5106868	4361786	4359619
26	21	2329224424344	773739	660852	660911
27	36	6323840144076	1094468732	934787896	934658668
28	24	17147315166491	3520397	3006778	3007037
29	14	46440262677600	29610	25290	25281
30	60	125634925674311	125634925674311	107305037048062	?

Theorem 1 gives an upper bound on the complexity of the arithmetic subsequences of Fibonacci; also note that this upper bound is the same for all residue classes with the same modulus. Strong conjectures for the true complexity are obtained by exhaustively searching all possible morphisms of a given complexity. However, strictly speaking such experiments only give lower bounds. Indeed, even for the smallest cases, the true value of the morphic complexity is not rigorously proved. These true values might be different for different residue classes. Our computer experiments give rise to the conjecture that this indeed is the case for modulus $m = 2$.

Example 2. A brute force search (checking morphisms with words up to 3 letters on an alphabet of size 5 and up to 2 letters on an alphabet of size 6) strongly suggested complexity 8 for $(\sigma[2n])_{n\geq 0}$, and complexity 10 for $(\sigma[2n+1])_{n\geq 0}$. Morphisms of these complexities can be defined as follows:
$\sigma[2n] = f_0^\infty(0)$ where

$$f_0(0) = 01, \quad f_0(1) = 2, \quad f_0(2) = 31, \quad f_0(3) = 04, \quad f_0(4) = 0,$$

with coding τ_0 given by

$$\tau_0(0) = 0, \quad \tau_0(1) = 0, \quad \tau_0(2) = 1, \quad \tau_0(3) = 1, \quad \tau_0(4) = 1$$

and $\sigma[2n+1] = f_1^\infty(0)$ where

$$f_1(0) = 01, \quad f_1(1) = 21, \quad f_1(2) = 3, \quad f_1(3) = 24, \quad f_1(4) = 51, \quad f_1(5) = 1$$

with coding τ_1 given by

$$\tau_1(0) = 1, \quad \tau_1(1) = 0, \quad \tau_1(2) = 0, \quad \tau_1(3) = 1, \quad \tau_1(4) = 1, \quad \tau_1(5) = 1.$$

Morphisms of the same complexity were independently found by Zantema [17].

We had no proof for $(\sigma[2n])_{n\geq 0} = \tau_0(f_0^\infty(0))$ and $(\sigma[2n+1])_{n\geq 0} = \tau_1(f_1^\infty(0))$, but merely an agreement on (more than a million) initial terms. Jeffrey Shallit informed us that equality can be proved with the automated theorem prover Walnut [11,14].

7 Further Remarks

As can be seen from Table 1, the bound $(m+1)F_{z(m)+2}$ seems reasonably sharp, and the approximation is pretty good.

A few remarks can be made here.

(a) In our proof of Theorem 1, we take the smallest j such that F_j is a multiple of m. This guarantees (Lemma 3) that $\mathrm{len}(\mathcal{F}^j(w)) = F_{j+2}\cdot m - F_j\cdot \mathrm{wt}(w)$ is a multiple of m for all $w \in V_m$. One could wonder about divisibility of $\mathrm{wt}(w)$ by m. Note that if $w_1 = \sigma[n]\cdots\sigma[n+m-1]$ and $w_2 = \sigma[n+1]\cdots\sigma[n+m]$, then $|\mathrm{wt}(w_1) - \mathrm{wt}(w_2)| \leq 1$. The difference cannot be always zero, since that would imply periodicity of σ. All subwords of σ of length m are in V_m (Lemma 4). It follows that there exist $w_1, w_2 \in V_m$ with $|\mathrm{wt}(w_1) - \mathrm{wt}(w_2)| = 1$. In particular, not all weights can be divisible by m. Therefore the condition that F_j is a multiple of m is necessary, and we cannot take a smaller j.

(b) In (1), we bound $\text{len}(\mathcal{F}^j(w))$ by $F_{j+2} \cdot m$, ignoring that Lemma 3 gives a smaller bound. If j and m are not very small, then $F_j \approx \phi \cdot F_{j+1}$ and $\text{wt}(w) \approx \phi^2 \cdot \text{len}(w)$, where $\phi = (\sqrt{5}-1)/2$ is the positive solution of $\phi^2 = 1 - \phi$. So

$$\text{len}(\mathcal{F}^j(w)) = F_{j+2} \cdot m - F_j \cdot \text{wt}(w) \approx (1-\phi^4) \cdot F_{j+2} \cdot m.$$

This means that the error in (1) is relatively small ($(1-\phi^4) \approx 0.854$). This observation can be used to approximate the complexity of the morphism h by $(1-\phi^4)(m+1)F_{z(m)+2}$ without explicitly determining h. If m itself is a Fibonacci number, $\text{wt}(w)$ often is a Fibonacci number as well, and in this case the approximation is particularly good. The table above gives these approximations rounded to the nearest integer.

(c) As noted before, our morphism gives upper bounds for the complexity of arithmetic subsequences of Fibonacci. Example 2 indicates that morphisms of lower complexity sometimes do exist, but we do not know if less complex morphisms exist for all m, let alone how to systematically construct them.

(d) To find all words in V_m, one could just examine the subwords of σ starting from the beginning until $m+1$ different words of length m have been found (cf. Lemma 4). In the cases for m up to 30 we considered for Table 1 this did not take long; compare for example [2], Sect. 3.1.
Alternatively, one could start with $w_0 = \sigma[0] \cdots \sigma[m-1]$ and $V = \{w_0\}$. Then compute the image $\mathcal{F}^{z(m)}(w_0)$ and add all its subwords to V. Iterate by computing the images of all newly added words and adding their subwords to V as well. Stop if V contains $m+1$ words. In every iteration, at least one new word will be added to V, so at most m images have to be computed before this procedure terminates.

(e) Jeffrey Shallit reports that the upper bound 2^{4m^2+1} can be deduced from [3] and from the theory behind Walnut [14]. This is much weaker than our upper bound $(m+1)F_{2m+2}$. On the other hand, Walnut gives examples showing that our bound probably is still far from the truth.

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