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Hyperboloidal evolution for the cubic wave equation

Asymptotic behavior of global solutions

Annegret Y. Burtscher

University of Bonn, Mathematical Institute, Sofja Kovalevskaja Research Group

burtscher@math.uni-bonn.de

Abstract

We identify a codimension-1 Lipschitz manifold of initial data leading to solutions for the cubic wave equation which remain $t^{-\frac{1}{2}+}$-close to the selfsimilar attractor $\sqrt{2}/t$ and its Lorentz boosts. These global solutions thus exhibit a nondispersive decay, contrary to small data evolutions that disperse to infinity.

Introduction

Solutions to the cubic focusing wave equation

$$(-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0, \quad x \in \mathbb{R}^3. \tag{1}$$

with small initial data exist globally and scatter to zero. On the other hand, large data generically leads to finite time blowup. The selfsimilar solution $v_0(t,x) = \sqrt{2}/t$ plays a special role, because it exists globally for $t \geq 1$ and decays in a nondispersive manner.

Main Objectives

1. reveal the role of the selfsimilar solution $v_0(t,x) = \sqrt{2}/t$ for the Cauchy problem of (1)
2. compare the asymptotic behavior to solutions with small initial data

Methods

The main difficulty arises from the fact that v_0 has infinite energy. This problem can be avoided by studying the Cauchy problem for (1) in hyperboloidal coordinates and by considering a suitable energy space which is equivalent to $H^1 \times L^2(\mathbb{B}^3)$ as Banach space.

1 Hyperboloidal foliation

For $T \in (-\infty, 0)$ we consider spacelike hyperboloidal slices Σ_T of the future light cone emanating from the origin. Each slice Σ_T is unbounded and parametrized by the Kelvin transform. Initial data are prescribed on Σ_{-1}.

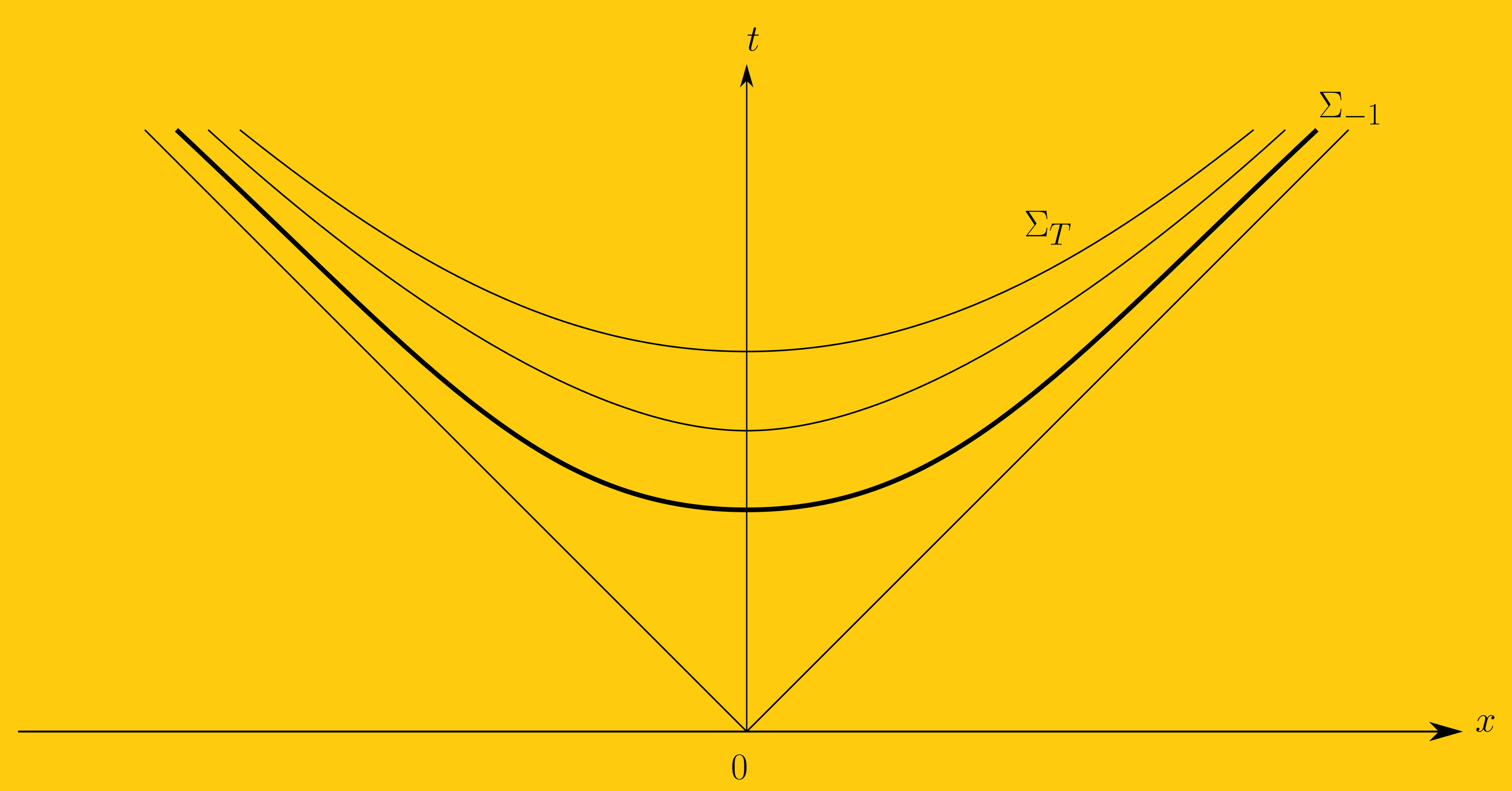

Figure 1: The hyperboloidal foliation $(\Sigma_T)_T$ of the future light cone with initial hyperboloid Σ_{-1}.

2 Modulation ansatz

Equation (1) is conformally invariant and we allow for hyperbolic rotations by considering the 3-parameter family $v_a := \Lambda_a(v_0)$, $a \in \mathbb{R}^3$, generated by Lorentz boosts applied to v_0. In similarity coordinates (τ, ξ) equation (1) is equivalent to an evolution system

$$\partial_\tau \Psi(\tau) = \mathbf{L}\Psi(\tau) + \mathbf{N}(\Psi(\tau)), \tag{2}$$

where $\mathbf{L}$ is an linear operator and $\mathbf{N}$ is nonlinear. Static solutions Ψ_a correspond to v_a. We allow for the rapidity a to depend on τ, with $a_\infty := \lim_{\tau\to\infty} a(\tau) \in \mathbb{R}^3$, and with the modulation ansatz

$$\Psi(\tau) = \Psi_{a(\tau)} + \Phi(\tau)$$

we arrive at an evolution equation for the perturbation term Φ:

$$\partial_\tau \Phi(\tau) - \mathbf{L}\Phi(\tau) - \mathbf{L}'_{a_\infty}\Phi(\tau) = [\mathbf{L}'_{a(\tau)} - \mathbf{L}'_{a_\infty}]\Phi(\tau) + \mathbf{N}_{a(\tau)}(\Phi(\tau)) - \partial_\tau \Psi_{a(\tau)}. \tag{3}$$

3 Spectral analysis for the linearized equation

The homogeneous problem of (3) is the linear equation $\partial_\tau \Phi(\tau) = (\mathbf{L} + \mathbf{L}'_{a_\infty})\Phi(\tau)$. For small $|a_\infty|$, the compact perturbation $\mathbf{L} + \mathbf{L}'_{a_\infty}$ of the operator $\mathbf{L}$ generates a strongly continuous 1-parameter semigroup $\mathbf{S}_{a_\infty}(\tau)$ on $H^1 \times L^2(\mathbb{B}^3)$. Perturbation theory allows us to estimate the spectrum, i.e.,

$$\sigma(\mathbf{L} + \mathbf{L}'_{a_\infty}) \subseteq \{z \in \mathbb{C} \,|\, \Re z < -\tfrac{1}{2} + \varepsilon\} \cup \{0, 1\}.$$

Spectral projections $\mathbf{P}_{a_\infty,j}, \mathbf{Q}_{a_\infty}, \widetilde{\mathbf{P}}_{a_\infty}$ onto the 3- and 1-dimensional eigenspaces of 0 and 1 and the remaining unbounded part of the spectrum, yield growth estimates for the linearized evolution:

$$\begin{aligned} \mathbf{S}_{a_\infty}(\tau)\mathbf{P}_{a_\infty,j} &= \mathbf{P}_{a_\infty,j}, \\ \mathbf{S}_{a_\infty}(\tau)\mathbf{Q}_{a_\infty} &= e^\tau \mathbf{Q}_{a_\infty}, \\ \|\mathbf{S}_{a_\infty}(\tau)\widetilde{\mathbf{P}}_{a_\infty} f\| &\lesssim e^{(-\frac{1}{2}+\varepsilon)\tau}\|\widetilde{\mathbf{P}}_{a_\infty} f\|, \quad f \in H^1 \times L^2(\mathbb{B}^3). \end{aligned}$$

4 Fixed point arguments for the full nonlinear equation

The right hand side is included by rewriting (3) as an integral equation for given initial value $\Phi(0) = u$, i.e.,

$$\begin{aligned} \Phi(\tau) &= \mathbf{S}_{a_\infty}(\tau)u - \int_0^\tau \mathbf{S}_{a_\infty}(\tau - \sigma)\left[(\mathbf{L}'_a - \mathbf{L}'_{a_\infty})\Phi(\sigma) + \mathbf{N}_{a(\sigma)}(\Phi(\sigma)) - \partial_\sigma \Psi_{a(\sigma)}\right] d\sigma \\ &=: \mathbf{K}_u(\Phi, a)(\tau). \end{aligned} \tag{4}$$

All terms in the integrand become small and satisfy Lipschitz estimates with respect to Φ and the rapidity a. By choosing $a(\tau)$ in a suitable way we suppress the instability of $\mathbf{P}_{a_\infty,j}$ that arises from the Lorentz symmetry of (1). To isolate the time-translation instability we add a correction term $\mathbf{C}_u(\Phi, a)$ and first solve the modified weak equation

$$\Phi(\tau) = \mathbf{K}_u(\Phi, a)(\tau) - \mathbf{S}_{a_\infty}(\tau)\mathbf{C}_u(\Phi, a) \tag{5}$$

by contraction arguments. Solutions to (5) with vanishing correction term thus satisfy the original integral equation (4). The condition $\mathbf{C}_u(\Phi, a) = 0$ defines a codimension-1 manifold $\mathcal{M}$ of initial data.

Results

Theorem 1 (Codimension-1 initial data [2])**.** *There exists a codimension-1 Lipschitz manifold $\mathcal{M}$ of initial data in $H^1 \times L^2(\Sigma_{-1})$, with $(0,0) \in \mathcal{M}$, for which the hyperboloidal initial value problem*

$$\begin{aligned} (-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 &= 0, \\ v|_{\Sigma_{-1}} &= v_0|_{\Sigma_{-1}} + f, \\ \nabla_n v|_{\Sigma_{-1}} &= \nabla_n v_0|_{\Sigma_{-1}} + g, \end{aligned}$$

with $(f,g) \in \mathcal{M}$, has a unique solution (in the Duhamel sense) v defined on the future development of Σ_{-1}. For a unique $a \in \mathbb{R}^3$ and Lorentz boost $v_a = \Lambda_a(v_0)$, and any $\delta \in (0,1)$, we have

$$\|v - v_a\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}+}.$$

Proceeding along the same lines (without perturbations) yields a decay estimate for small data evolutions.

Theorem 2 (Small initial data [2])**.** *There exists $\varepsilon > 0$ such that the initial value problem*

$$\begin{aligned} (-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 &= 0, \\ v|_{\Sigma_{-1}} &= f, \\ \nabla_n v|_{\Sigma_{-1}} &= g. \end{aligned}$$

for initial data $(f,g) \in H^1 \times L^2(\Sigma_{-1})$ with $\|(f,g)\| < \varepsilon$ has a unique global solution v (in the Duhamel sense) which satisfies, for any $\delta \in (0,1)$, the localized Strichartz norms

$$\|v\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}}.$$

Conclusions

- Solutions to small initial data exhibit dispersive decay (Theorem 2).
- The selfsimilar solution $v_0(t,x) = \sqrt{2}/t$ and its Lorentz boosts v_a have constant Strichartz norm, hence the solutions described in Theorem 1 decay in a nondispersive manner.

References

[1] P. Bizoń and A. Zenginoğlu, Universality of global dynamics for the cubic wave equation, *Nonlinearity* **22** (2009), 2473–2485.

[2] A. Burtscher and R. Donninger, Preprint (2015).

[3] R. Donninger and A. Zenginoğlu, Nondispersive decay for the cubic wave equation, *Analysis & PDE* **7** (2014), 461–495.