Global hyperbolicity revisited

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Why (still) talk about global hyperbolicity?

- beautiful and very rich concept
- new foundational results can still be obtained

Definition(s) of global hyperbolicity

Global hyperbolicity and general relativity

Global hyperbolicity and the null distance

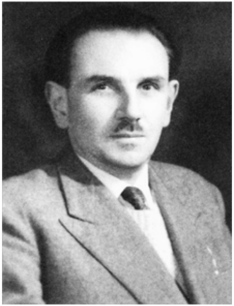
Analytic definition (1952)

Original definition

A spacetime (M, g) is globally hyperbolic if for any points $p, q \in M$ the space of all causal curves joining p and q is compact (in a suitable topology).

First applications:

- uniqueness of solutions to hyperbolic PDEs on a manifold
- Cauchy problem in general relativity

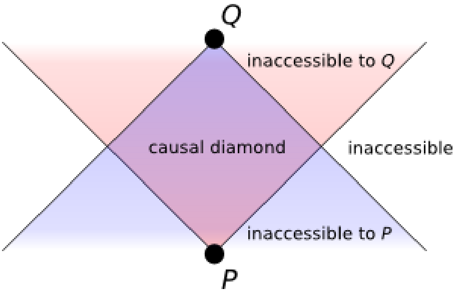


Jean Leray

Modern definition

A spacetime (M, g) is **globally hyperbolic** if ...

- strongly causal + all causal diamonds are compact (e.g., Hawking–Ellis 1973)
- **causal + all causal diamonds** $J^+(p) \cap J^-(q)$ **are compact** (Bernal–Sánchez 2007)
- noncompact, dim $M \geq 3$ + all causal diamonds are compact (Hounnonkpe–Minguzzi 2019)



Basic properties: as (causally) good as it gets

Global hyperbolicity

⇓

Causal simplicity

⇓

Causal continuity

⇓

Causal easiness

⇓

Stable causality
(K-causality, existence of time)

⇓

Strong causality

⇓

Weak distinction

⇓

Non-total imprisonment

⇓

Causality

⇓

Chronology

Lorentzian distance

$$\mathcal{A}_\vee \quad \ldots \text{ class of piecewise smooth future-directed causal paths}$$

$$L_g(\gamma) := \int_a^b \sqrt{-g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt \ \ldots \text{ length of } \gamma \in \mathcal{A}_\vee$$

$$d_g(p,q) := \begin{cases} \sup\{L_g(\gamma) \,|\, \gamma \in \mathcal{A}_\vee \text{ between } p \text{ and } q\} & q \in J^+(p) \\ 0 & q \notin J^+(p) \end{cases}$$

For **globally hyperbolic spacetimes**

- d_g is finite and continuous on $M \times M$
- $\exists$ **length-maximizing causal geodesic** from p to $q \in J^+(p)$ (Avez 1963, Seifert 1967)

Importance of global hyperbolicity for GR

Global hyperbolicity is crucial for

- **initial value formulation** of the Einstein equations:

admissible initial data $(\mathcal{S}, h, k) \Rightarrow \exists!$ maximal GH solution (M, g)

- **singularity theorems** of Penrose and Hawking, e.g.,:

GH spacetime with trapping, $Ric \geq 0 \Rightarrow$ geodesically incomplete

- **splitting theorems** for spacetimes (Eschenburg, Galloway, ...)

Bartnik's splitting conjecture (1988)

GH spacetime with $Ric \geq 0$, geod. compl. $\Longrightarrow (\mathbb{R} \times \mathcal{S}, -d\tau^2 \oplus h)$

- **cosmic censorship** (no/unstable Cauchy horizons)

Importance of global hyperbolicity for GR

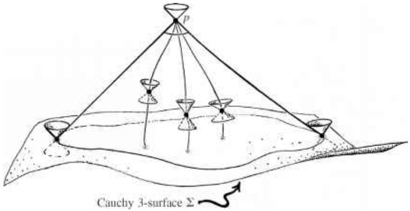
Why is global hyperbolicity relevant for these results?

- causal properties of $g \leftrightsquigarrow$ topology of M
- allows globalization: local $\rightarrow$ global

How do we see and use the globalness of global hyperbolicity?

Characterization via

- Cauchy time functions
- Cauchy hypersurfaces
- null distance completeness



Cauchy time functions

Global hyperbolicity

⇓

Causal simplicity

⇓

Causal continuity

⇓

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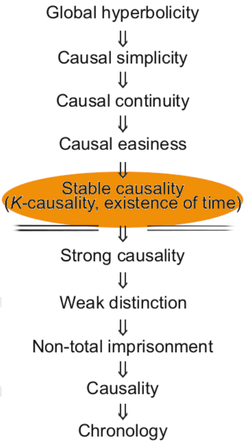
Cauchy time functions



A function $\tau\colon M \to \mathbb{R}$ is

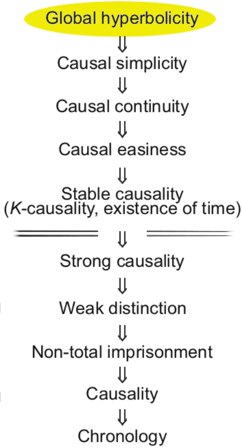
- **time function** if continuous and $p < q \Longrightarrow \tau(p) < \tau(q)$
- **temporal function** if smooth (or C^1) and $\nabla\tau$ is past-directed timelike

Theorem (Hawking, Bernal–Sánchez)

(M, g) stably causal $\Longleftrightarrow \exists$ time/temporal function τ

- level sets $\tau^{-1}(\{s\})$ are acausal
- orthog. decomposition $g = -\alpha d\tilde{\tau}^2 + \bar{g}$ with $\alpha > 0$, $\bar{g}$ positive semi-definite

Cauchy time functions

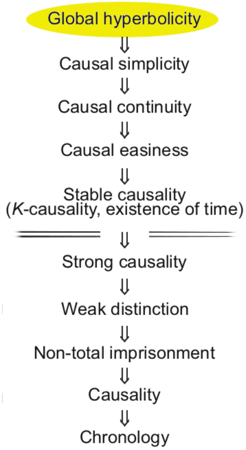


Theorem (Geroch 1970, Bernal–Sánchez 2005)

(M, g) *globally hyperbolic* $\Longleftrightarrow$
$\exists$ **Cauchy time/temporal function** τ

- all level sets $\tau^{-1}(\{s\})$, $s \in \mathbb{R}$, are **Cauchy surfaces**
- smooth **Cauchy splitting** $(M, g) \cong (\mathbb{R} \times \mathcal{S}, -\alpha d\tau^2 + \bar{g}_\tau)$

Recent extensions



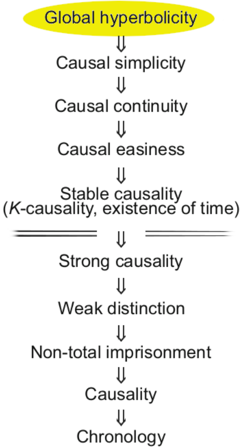
Most of these results have been extended to lower regularity situations ...

- **closed cone structures** (Fathi–Siconolfi 2012, Bernard–Suhr 2018, Minguzzi 2019)
- C^0 **metrics** (Sämann 2016, ...)
- **Lorentzian length spaces** (Kunzinger–Sämann 2018, Aké Hau–Cabrera Pacheco–Solis 2020, B.–García-Heveling 2021, ...)

... and spacetimes with timelike boundaries

- (Solis 2006, Aké–Flores–Sánchez 2021, ...)

Recent novel time function characterizations



Theorem (Bernard–Suhr 2018)

(M, g) *globally hyperbolic* $\iff$
$\exists$ **completely uniform temporal fct.** τ

- $\exists$ complete Riemannian metric h such that for all causal vectors v

$$d\tau(v) \geq \|v\|_h$$

Theorem (B.–García-Heveling 2024)

(M, g) *globally hyperbolic* $\iff$
$\exists$ *time function* τ *such that the corresp.* **null distance** $\hat{d}_\tau$ *is* **complete** *metric on* M

Metric completeness

Theorem (B.–García-Heveling 2024)

(M, g) globally hyperbolic $\Longleftrightarrow \exists$ complete null distance $\hat{d}_\tau$

- Extends part of the Riemannian **Hopf–Rinow theorem**:

  (Σ, σ) is geodesically complete~~(Σ, σ) is geodesically complete~~

  $\Leftrightarrow$ (Σ, d_σ) is metrically complete

  $\Leftrightarrow$ (Σ, d_σ) is proper (closed + bounded sets $\Rightarrow$ compact)

- Theorem be extended to proper cone structures and "semi-Riemannian spacetimes" (B. 2023)

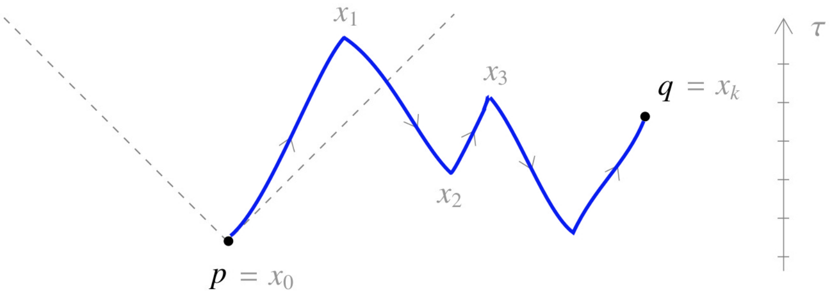
What is $\hat{d}_\tau$?

Null distance

Let (M, g) be a spacetime with time function τ.

Null distance (Sormani–Vega 2016)

$$\begin{aligned}\mathcal{B} &\quad \text{... class of piecewise causal paths} \\ \hat{L}_\tau(\beta) &= \sum_{i=1}^{k} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))| \quad \text{... null length of } \beta \in \mathcal{B} \\ \hat{d}_\tau(p, q) &= \inf\{\hat{L}_\tau(\beta) \mid \beta \in \mathcal{B} \text{ from } p \text{ to } q\}\end{aligned}$$



Basic properties of the null distance

- $\hat{d}_\tau$ is symmetric
- $\hat{d}_\tau$ satisfies the $\triangle$-inequality
- $\hat{d}_\tau$ is <u>not</u> necessarily positive definite
  (e.g., $\tau(p) := t^3$ in Minkowski space is not)

Theorem (Sormani–Vega 2016)

For sufficiently nice (e.g., temporal, locally anti-Lipschitz) time functions τ the null distance $\hat{d}_\tau$ is a length **metric** *on (M, g).*

Basic properties of the null distance

Moreover, the null distance $\hat{d}_\tau$

- induces the manifold **topology**
- is **conformally invariant**
- **scales** for $\lambda > 0$: $\hat{d}_\tau = \lambda \hat{d}_{\tilde{\tau}} \iff \tau = \lambda \tilde{\tau} + C$
- is **bounded on causal diamonds**

How much does $\hat{d}_\tau$ depend on τ?

Quite similar to Riemannian situation:

- **locally bi-Lipschitz** for class of "weak temporal" functions (and on compact sets)
- **globally** there is quite some difference ...
  - depends on causal properties of (M, g) (step on causal ladder)
  - depends on choice of time function τ (whether it reflects the step on the causal ladder)

When does the null distance encodes causality?

By definition also: $p \leq q \Longrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$

What about the converse?
When is $p \leq q \Longleftrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$ possible?

Initial results:

+ true for **warped products** $g = -dt^2 + f(t)^2\sigma$ and $\tau(t,p) = \phi(t)$ with $\phi' > 0$ (Sormani–Vega 2016)
− incompleteness of $\hat{d}_\tau$ is an obstruction to $\Longleftarrow$
− even on a globally hyperbolic subset of Minkowski space a bad choice of time function shows $\not\Longleftarrow$ (B.–García-Heveling 2024)

When the null distance encodes causality

Locally always true:

Theorem (Sakovich–Sormani 2023)

If τ is **locally anti-Lipschitz** *then locally around every $p \in M$ there is a neighborhood U such that for all $q \in U$:*

$$p \leq q \iff \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$$

Globally only in **special** cases:

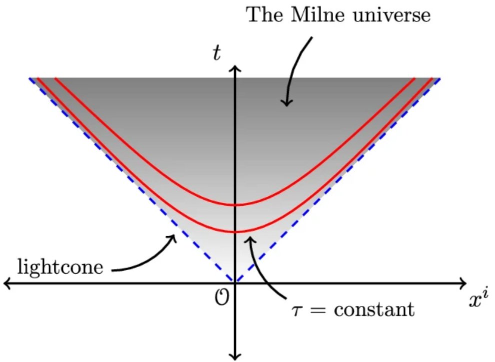
Theorem (B.–García-Heveling 2024)

If (M,g) is globally hyperbolic and τ is locally anti-Lipschitz time function whose nonempty **level sets are** *(future/past)* **Cauchy**, *then for all $p,q \in M$:*

$$p \leq q \iff \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$$

- (future/past) causally complete enough (Galloway 2023)

Application: cosmology



Definition (Andersson–Galloway–Howard 1998, Wald–Yip)

Cosmological time function: $\tau_c(p) = \sup d_g(J^-(p), p)$

- regular τ_c are locally anti-Lipschitz
- level sets of τ_c are future Cauchy

$\Rightarrow$ null distance $\hat{d}_{\tau_c}$ encodes causality globally

Characterization of global hyperbolicity

Have seen:

(M, g) globally hyperbolic $\Longrightarrow \exists$ globally well-behaved $\hat{d}_\tau$

Now show also related:

Theorem (B.–García-Heveling 2024)

(M, g) globally hyperbolic $\Longleftrightarrow \exists$ complete null distance $\hat{d}_\tau$

Sketch of proof

(⇒) globally hyperbolic $\Longrightarrow \exists$ completely uniformly temporal τ (Bernard–Suhr 2018), i.e., complete Riemannian metric h s.t.

$$\tau(q) - \tau(p) = \int_0^1 \underbrace{d\tau(\dot\gamma(s))}_{\geq \|\dot\gamma(s)\|_h} ds \geq L_h(\gamma) \geq d_h(p,q)$$

$\Longrightarrow \hat{d}_\tau$ complete (Allen–B. 2022)

(⇐) If $\hat{d}_\tau$ complete and τ not Cauchy
$\Longrightarrow \exists$ w.l.o.g. future-directed future-inext. causal curve γ with $\lim_{s\to\infty} \tau(\gamma(s)) < \infty$
$\Longrightarrow ((\tau \circ \gamma)(n))_n$ is Cauchy sequence in $\mathbb{R}$ and $\hat{d}_\tau(\gamma(n), \gamma(m)) = |\tau(\gamma(m)) - \tau(\gamma(n))|$
$\Longrightarrow (\gamma(n))_n$ Cauchy sequence in $(M, \hat{d}_\tau)$, thus converges
$\Longrightarrow \gamma$ extendible, contradiction. Thus τ Cauchy. □

Globally hyperbolic spacetimes

- in most cases compact causal diamonds enough to define
- characterization via Cauchy surface crucial for GR
- characterized by $\exists$ completely uniform time function τ
- characterized by $\exists$ complete null distance $\hat{d}_\tau$
- also relevant for nice causal properties of $\hat{d}_\tau$

Thank you for your attention!