# Nonsmoothness in General Relativity: why and how

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# Outline

# A large-scale model for our universe

**Einstein's key idea**

force $\neq$ **gravitation** $=$ **geometric** property of space and time

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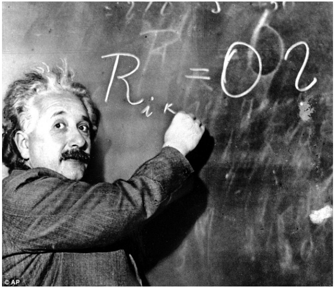
force $\neq$ **gravitation** $=$ **geometric** property of space and time

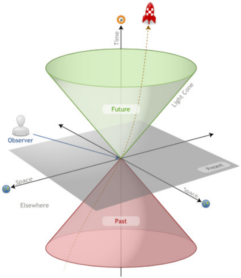
$(M, \mathbf{g})$ 4-dim. **spacetime**
$=$ connected time-oriented
Lorentzian manifold

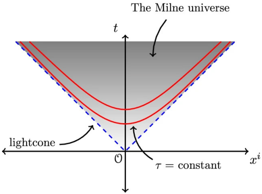
**Einstein equations (1915)**

$$\underbrace{\mathbf{Ric} - \tfrac{1}{2} R\,\mathbf{g}}_{\text{curvature}} + \Lambda \mathbf{g} = \underbrace{\frac{8\pi G}{c^4}\,\mathbf{T}}_{\text{matter}}$$



# Important exact solutions




## Minkowski spacetime

$g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$

## Schwarzschild spacetime

$g = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$

with $f(r) = 1 - \frac{2m}{r}$

## FLRW spacetimes

$g = -dt^2 + a(t)^2 \sigma_\Sigma$

with $\Sigma$ constant curvature 3-space
& 2nd order ODE for $a$

# Major achievements and problems in mathematical GR

- **global geometry and analysis**

- **black holes**

- **two-body problem and gravitational waves**

- **connections to quantum theory**

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  - black hole thermodynamics (Beckenstein, Hawking et al)
  - entropy generalizing energy conditions using optimal transportion (McCann) and synthetic approaches to spacetimes

# Outline

# 1. Realistic matter models are not smooth

**Perfect fluids** $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$ are used in astrophysics


in models of stars and planets (fluid/gas balls) and clusters (dust)

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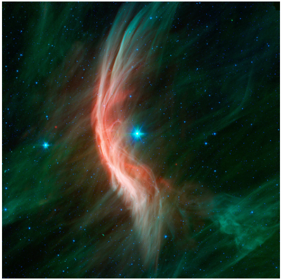

in models of stars and planets (fluid/gas balls) and clusters (dust)

- matter-vacuum **boundaries** $\rightsquigarrow$ e.g. **g** only Lipschitz
- formation of **shock waves** (grav. collapse) $\rightsquigarrow$ BV regularity

# 2. Problems with large curvature and loss of predictability

curvature scalars blow up $\Rightarrow$ $\mathbf{g}$ cannot be $C^2$ extended

**Black hole interior**

**Big bang**

# 2. Problems with large curvature and loss of predictability

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**Black hole interior**

- Schwarzschild is $C^0$-inextendible (Hawking, Sbierski 2018)
- generic black hole interiors are (not uniquely?) $C^0$-extendible beyond Cauchy horizon (Dafermos–Luk 2025) but likely $C^{0,1}_{\mathrm{loc}}$-inextendible (Sbierski 2024+)

**Big bang**

- $C^2$-inextendibility for scalar fields (Fournodavlos et al 2023, Oude Groeninger–Petersen–Ringström 2023+)
- closely related to stable big bang formation
- $C^0$-(in)extendibility for FLRW etc. (Galloway–Ling 2017, Sbierksi 2023+, Graf–van den Beld-Serrano 2024+)

# 3. Asymptotic behavior depends on regularity

**Problem at future null infinity $\mathscr{I}^+$**

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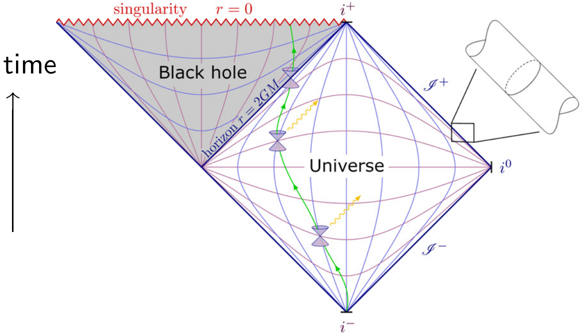

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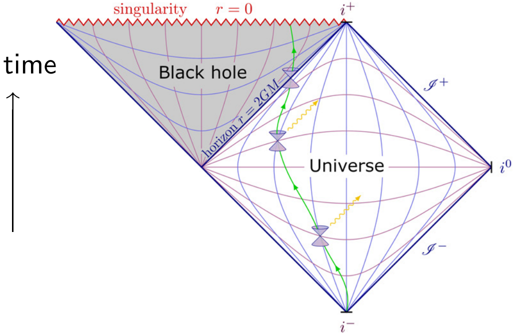
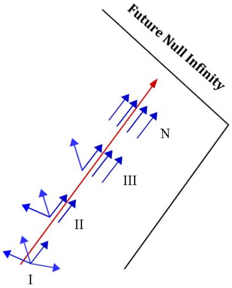
- *assume* $\exists$ of **smooth conformal compactifications** (Penrose) $\rightsquigarrow$ "peeling" of Weyl tensor: along light rays, as $s \to \infty$,
$$\mathbf{W} = W^{(N)} s^{-1} + W^{(III)} s^{-2} + W^{(II)} s^{-3} + W^{(I)} s^{-4} + O(s^{-5})$$



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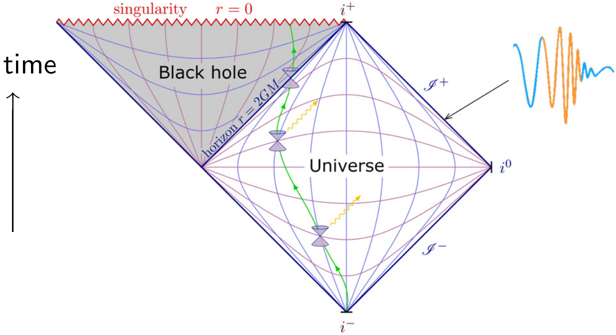


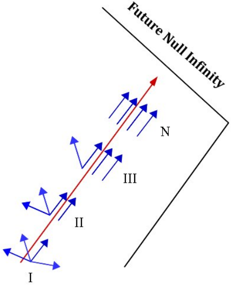

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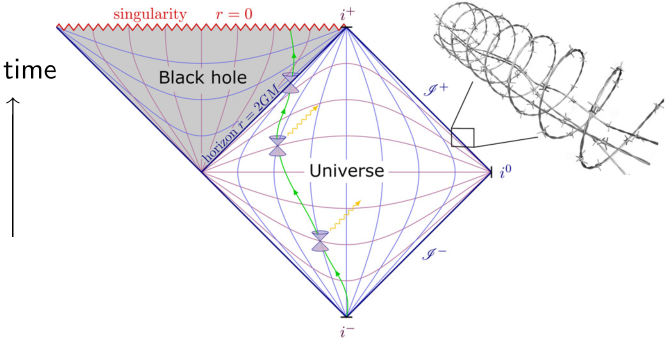
- PDE theory: **not achieved** for physical systems! (observable)



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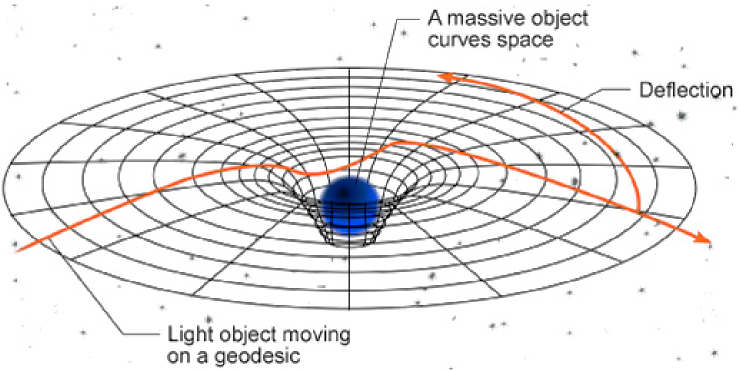
**Massless "test" particles**

- move along geodesics in background spacetime
- approximate motions for extreme 2-body systems (sun–earth)



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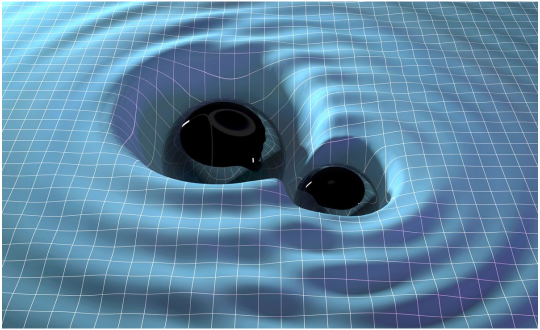

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**Massive particles**

- heavily interact with and change spacetime geometry
- early approaches to describe all matter by $\delta$'s and use second Bianchi identity $0 = \nabla \mathbf{G} = \nabla \mathbf{T}$ to model motion (Einstein–Infeld–Hoffmann 1940s) $\rightsquigarrow$ leads to inconsistencies
- new approach uses singular timelike boundaries of zero area with $m_{\text{Bray}} < 0$ (B.–Kiessling–Tahvildar-Zadeh 2021)

singular int. boundaries $\Rightarrow$ $\mathbf{g}$ does not extend (smoothly) to $\partial M$

# Outline

# Treshold regularity and below

**Problem for $\mathbf{g}$ below $C^{1,1}$**

**geodesics are badly behaved** (e.g., already locally nonunique)

BUT causality theory ok for $\mathbf{g} \in C^{0,1}_{\mathrm{loc}}$ (Chruściel–Grant, Minguzzi)

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Different approaches for nonsmooth $\mathbf{g}$

- **sequences** of smooth $\mathbf{g}_n \to \mathbf{g}$, possibly in combination with uniform sectional/Ricci curvature bounds
- **distributional curvature** used for proving singularity theorems for $\mathbf{g} \in C^1$ (Graf 2020) and below
- **metric (measure) spacetimes** based on interaction of causality + topology, together with a distance (and measure)

# Notions of metric spacetimes

**All synthetic approaches have in common**

instead of Lorentzian manifold $(M, \mathbf{g})$ work with:
**causality** ($\leq$ and $\ll$), **topology** on $X$, **distance**(s) $d$ (and curves)

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We zoom closer into two of the last approaches based on $d$ being

1. the **Lorentzian distance** $d_g$
2. the **null distance** $\hat{d}_\tau$

and recall their origin, properties and future

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But first: recap of **causality theory**...

# Recap: Causal character

Let $(M, \mathbf{g})$ be a Lorentzian manifold (without boundary), convention $(- + \cdots +)$.

**Theorem (Poincaré, Hopf 1926 & Markus 1955)**

$\exists$ *Lorentzian* $\mathbf{g} \iff M$ *noncompact or compact with* $\chi(M) = 0$

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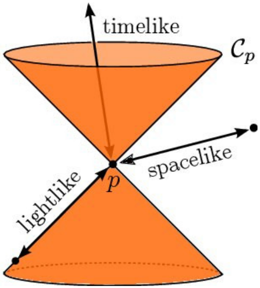


A tangent vector $v \in T_pM$ is called

- **timelike** if $g(v, v) < 0$,
- **spacelike** if $g(v, v) > 0$ or $v = 0$,
- **lightlike** if $g(v, v) = 0$ and $v \neq 0$,
- **null** if $g(v, v) = 0$,
- **causal** if timelike or lightlike.

# Recap: Causal relations on spacetimes

Can distinguish past and future globally if $\exists T \in \mathfrak{X}(M)$ timelike.

**Definition**

**spacetime** $=$ time-oriented Lorentzian manifold $(M, \mathbf{g}, [T])$

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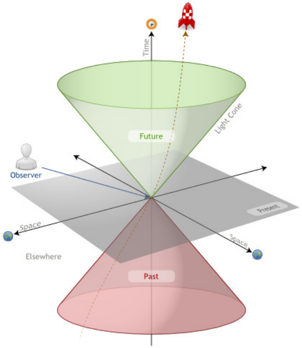


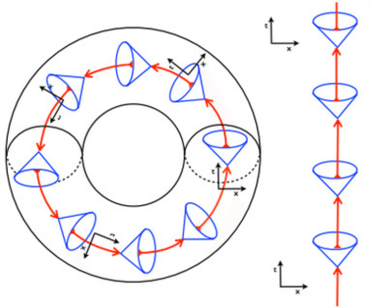

Then can define $v \in T_pM$ future-directed if $g(v, T) < 0$, and

- **timelike relation** $p \ll q$ if $\exists$ future-directed timelike curve from $p$ to $q$
- **causal relation** $p \leq q$ if $\exists$ future-directed causal curve from $p$ to $q$ ($p < q$) or $p = q$

# Recap: Global hyperbolicity

Global hyperbolicity
⇓
Causal simplicity
⇓
Causal continuity
⇓
Causal easiness
⇓
Stable causality
($K$-causality, existence of time)
⇓
Strong causality
⇓
Weak distinction
⇓
Non-total imprisonment
⇓
Causality
⇓
Chronology

$(M, \mathbf{g})$ **causal** if $\leq$ is antisymmetric



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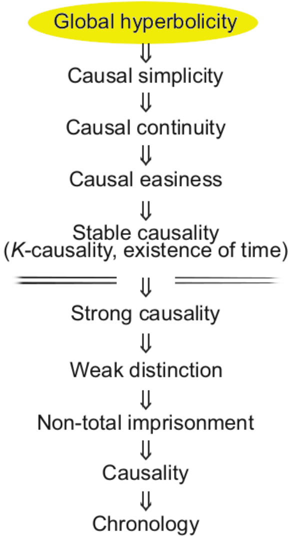
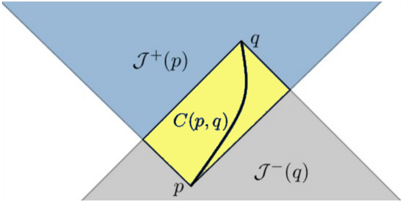



$(M, g)$ **globally hyperbolic** if causal and $\{p \leq x \leq q\}$ compact for all $p, q \in M$



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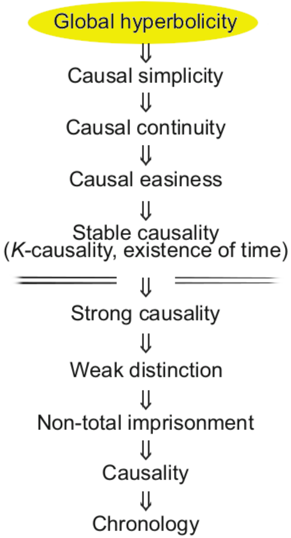


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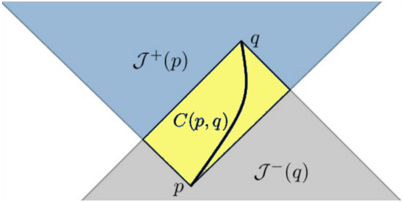


$\Leftrightarrow$ $\exists$ Cauchy time function $\tau \colon M \to \mathbb{R}$

$\Leftrightarrow$ $\exists$ Cauchy surface in $M$

# Distances: ingredients needed for constructing $d_g$ and $\hat{d}_\tau$

1. a class of curves
2. a length functional
3. a sup or inf
4. "good" properties and theorems

# Definition of the Lorentzian distance

Let $(M, \mathbf{g})$ be a spacetime.

**Lorentzian distance**

$$\begin{aligned}
\mathcal{A}_{\vee} &\ \dots \text{ class of piecewise smooth future-directed causal paths} \\
L_g(\gamma) &:= \int_a^b \sqrt{-g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\,dt \ \dots \text{ length of } \gamma \in \mathcal{A}_{\vee} \\
d_g(p,q) &:= \begin{cases} \sup\{L_g(\gamma) \,|\, \gamma \in \mathcal{A}_{\vee} \text{ between } p \text{ and } q\} & p \leq q \\ 0 & p \not\leq q \end{cases}
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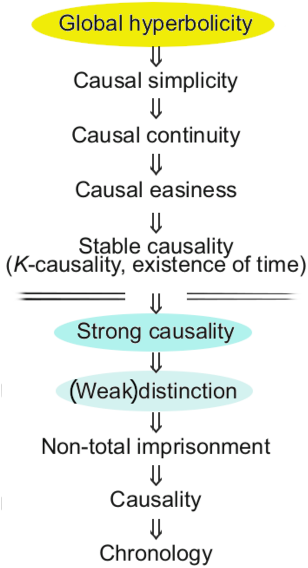
Other notations and conventions used:

- $d_g = \tau$ and called time sep. function in (Kunzinger–Sämann 2018)
- $d_g = \ell^+$ with $\ell(p,q) = -\infty$ if $p \not\leq q$ (McCann 2020, ...)

# Properties of the Lorentzian distance

- $d_g \colon M \times M \to [0, +\infty]$
- $d_g(p,p) = 0$ or $d_g(p,p) = +\infty$
- $d_g(p,q) = +\infty \ \forall p, q \iff M$ is totally viscious
- reverse $\triangle$-ineq. for $p \leq r \leq q$: $d_g(p,q) \geq d_g(p,r) + d_g(r,q)$
- $d_g$ is lower semicontinuous
- $d_g(p,q) > 0 \iff p \ll q$

# Special properties for more special $(M, g)$



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For **distinguishing spacetimes**

$\supseteq$ **strongly causal spacetimes**

$\supseteq$ **globally hyperbolic spacetimes**

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For **distinguishing spacetimes**

- future/past "balls" $B_\varepsilon^\pm(p)$ form subbasis for manifold topology
- $d_g$ continuous $\Rightarrow$ $(M, g)$ causally continuous

$\supseteq$ **strongly causal spacetimes**

- $d_g$ is locally finite, and continuous in a neighborhood of $\triangle M$ (Beem–Ehrlich 1979)

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- $\exists$ length-maximizing causal geodesic from $p$ to $q \in J^+(p)$ (Avez 1963, Seifert 1967)
- $(M, d_g)$ timelike Cauchy complete $\Leftrightarrow$ $(M, d_g)$ finitely compact (Beem 1976)

# Overview of properties $d_g$

| | Lorentzian distance $d_g$ |
|---|---|
| metric |  |
| finite & continuous |  if glob. hyp. |
| Hopf–Rinow type result |  if glob. hyp. |
| good with lengths and $g$ |  |
| good with lower curvature bounds |  sectional |
| |  Ricci |

For synthetic $d_g$ framework, see work of Alexander, Beran, Braun, B., Calisti, Cavalletti, Ebrahimi, García-Heveling, Gigli, Graf, Grant, Ketterer, Kunzinger, McCann, Minguzzi, Mondino, Ohanyan, Rott, Sämann, Solis, Soultanis, Steinbauer, Suhr . . .

# Recap: time and functions

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Non-total imprisonment
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Causality
$\Downarrow$
Chronology

$(M, g)$ **stably causal** if (unique) smallest transitive closed relation containing $\leq$ is antisymmetric

$\Leftrightarrow \exists$ time function $\tau \colon M \to \mathbb{R}$
(Hawking 1968, Minguzzi 2009)

# Recap: time functions

A function $\tau \colon M \to \mathbb{R}$ is

- **isotone/causal function** if $p \leq q \Rightarrow \tau(p) \leq \tau(q)$
  (e.g., $\tau \equiv 1$ or $\tau \equiv 0$)

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$\supseteq$ **temporal function** if $C^1$ and $\nabla\tau$ past-directed timelike
($\exists \Leftrightarrow (M, g)$ stably causal; Bernal–Sánchez 2005;
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Remember also: time $\subseteq$ isotone

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$\supseteq$ **rushing function** if $p \ll q \Rightarrow \tau(q) - \tau(p) \geq d_g(p,q)$
  ($\exists \implies d_g$ finite)

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$\supseteq$ **steep function** if $C^1$ and $g(\nabla\tau, \nabla\tau) \leq -1$
  ($\exists \iff (M,g) \hookrightarrow \mathbb{L}^{N+1}$ isometrically; Müller–Sanchez 2011)

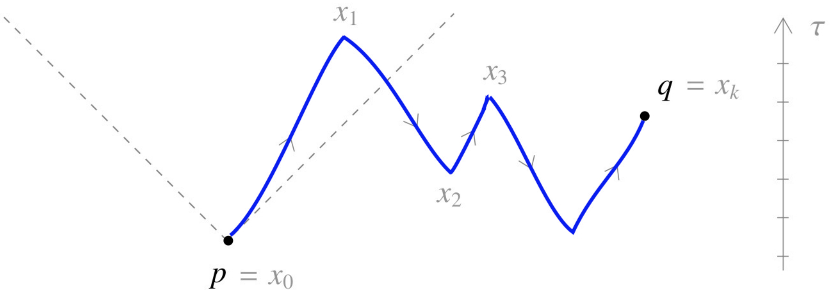
Remember also: time $\subseteq$ isotone, steep $\subseteq$ rushing

# Definition of the null distance

Let $(M, g)$ be a spacetime

**Null distance** (Sormani–Vega 2016)

$\mathcal{B}$ ... class of piecewise causal paths

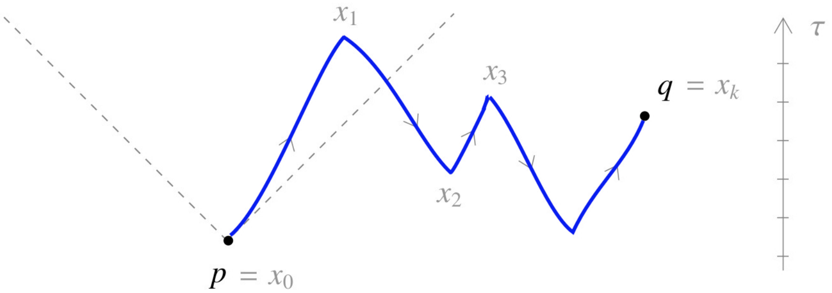


# Definition of the null distance

Let $(M, g)$ be a spacetime with time function $\tau$.

**Null distance** (Sormani–Vega 2016)

$$\begin{aligned} \mathcal{B} &\ \ldots \text{ class of piecewise causal paths} \\ \hat{L}_\tau(\beta) &= \textstyle\sum_{i=1}^{k} |\tau(\beta(s_i)) - \tau(\beta(s_{i-1}))| \ \ldots \text{ null length of } \beta \in \mathcal{B} \\ \hat{d}_\tau(p,q) &= \inf\{\hat{L}_\tau(\beta) \mid \beta \in \mathcal{B} \text{ from } p \text{ to } q\} \end{aligned}$$



# Properties of the null distance

- $\hat{d}_\tau$ is finite (and bounded on causal diamonds)
- $\hat{d}_\tau$ is conformally invariant and scales with $\tau$
- $\hat{d}_\tau$ is symmetric
- $\hat{d}_\tau$ satisfies the $\triangle$-inequality
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**Theorem (Sormani–Vega 2016 )**

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**Theorem (Sormani–Vega 2016, Allen–B. 2022)**

*$\tau$ nice (e.g., weak temporal) time function*
$\Longrightarrow$ *$\hat{d}_\tau$ is* **length metric** *that induces manifold topology*

# When does the null distance encodes causality?

By definition also: $p \leq q \Longrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$

What about the converse?
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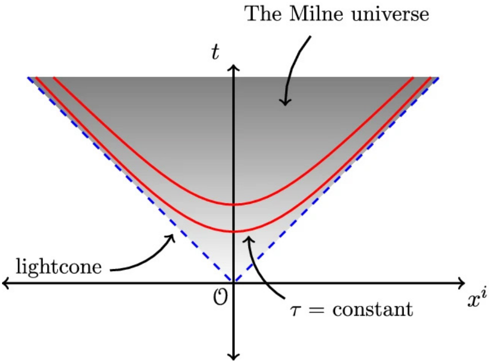
What about the converse?
When is $p \leq q \Longleftrightarrow \hat{d}_\tau(p,q) = \tau(q) - \tau(p)$ possible?

- **locally always** true for locally anti-Lipschitz time functions (Sakovich–Sormani 2023)
- **globally** only if $(M,g)$ globally hyperbolic and (B.–García-Heveling 2024)
  - all nonemtpy $\tau$-level sets future/past Cauchy
  - $\Leftrightarrow$ all future/past causally complete (Galloway 2024)

# Application: cosmology




**Definition (Andersson–Galloway–Howard 1998, Wald–Yip)**

Cosmological time function: $\tau_c(p) = \sup d_g(J^-(p), p)$

- regular $\tau_c$ are locally anti-Lipschitz
- level sets of $\tau_c$ are future Cauchy

$\Rightarrow$ null distance $\hat{d}_{\tau_c}$ encodes causality globally (B.–G.-H. 2024)

# How much does $\hat{d}_\tau$ depend on $\tau$?

Similar to Riemannian situation (B. 2015; Hopf–Rinow 1939):

- **locally bi-Lipschitz** for class of weak temporal functions

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**Theorem (B.–García-Heveling 2024)**

*$(M, g, \tau)$ and $(M, \tilde{g}, \tilde{\tau})$ spacetimes, $K$ compact*
$\Longrightarrow \exists C > 1 \forall p, q \in K : \frac{1}{C}\hat{d}_\tau(p,q) \leq \hat{d}_{\tilde{\tau}}(p,q) \leq C\hat{d}_\tau(p,q)$

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**Theorem (B.–García-Heveling 2024)**

$\exists \tau$ *such that* $(M, \hat{d}_\tau)$ *complete* $\Longleftrightarrow (M, g)$ *globally hyperbolic*

# Sketch of proof

(⇒) globally hyperbolic $\Longrightarrow \exists$ completely uniformly temporal $\tau$ (Bernard–Suhr 2018), i.e., complete Riemannian metric $h$ s.t.

$$\tau(q) - \tau(p) = \int_0^1 \underbrace{d\tau(\dot\gamma(s))}_{\geq \|\dot\gamma(s)\|_h} ds \geq L_h(\gamma) \geq d_h(p,q)$$

$\Longrightarrow \hat{d}_\tau$ complete (Allen–B. 2022)

(⇐) If $\hat{d}_\tau$ complete and $\tau$ not Cauchy
$\Longrightarrow \exists$ w.l.o.g. future-directed future-inext. causal curve $\gamma$ with $\lim_{s\to\infty} \tau(\gamma(s)) < \infty$
$\Longrightarrow ((\tau \circ \gamma)(n))_n$ is Cauchy sequence in $\mathbb{R}$ and $\hat{d}_\tau(\gamma(n), \gamma(m)) = |\tau(\gamma(m)) - \tau(\gamma(n))|$
$\Longrightarrow (\gamma(n))_n$ Cauchy sequence in $(M, \hat{d}_\tau)$, thus converges
$\Longrightarrow \gamma$ extendible, contradiction. Thus $\tau$ Cauchy. □

# Summary: comparison

| | $d_g$ | $\hat{d}_\tau$ |
| --- | --- | --- |
| metric |  |  |
| finite & continuous |  |  |
| Hopf–Rinow type result |  |  |
| good with lengths and $g$ |  |  |
| good with lower curvature bounds |  sectional |  |
| |  Ricci | ? |

For synthetic $\hat{d}_\tau$ framework, see work of Allen, Burgos, B., Ebrahimi, Flores, Galloway, García-Heveling, Kunzinger, Sakovich, Sánchez, Sormani, Steinbauer, Vega . . .

# Largely open: connections between $d_g$ and $\hat{d}_\tau$

When does a sensible $\hat{d}_\tau$ (and $\tau$) exist in a $d_g$-synthetic theory?

Potential connections to explore:

- rushing functions $p \ll q \Rightarrow \tau(q) - \tau(p) \geq d_g(p, q)$ (Rennie–Whale 2016, Minguzzi 2019, ...)
- cosmological time function $\tau_c(p) = \sup d_g(J^-(p), p)$ (Andersson–Galloway–Howard 1998, Wald–Yip 1981, ...)
- cosmological volume function $\tau_v(p) = \mathrm{vol}_g(I^-(p))$ (García-Heveling 2024)
- global hyperbolicity
- closed cone structures $(M, C)$ with nonempty open $I^\pm$ & $d_g$

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- wider approach to study **convergence and stability** of certain classes of spacetimes (Allen–B. 2022, Minguzzi–Suhr 2024, Sakovich–Sormani 2024+, Braun–Sämann 2025+)
- potential further applications: connections to quantum gravity, simplifications in numerical relativity?