A MODEL STRUCTURE FOR ENRICHED COLOURED OPERADS

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ABSTRACT. We prove that, under certain conditions, the model structure on a monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ can be transferred to a model structure on the category of $\mathcal{V}$-enriched coloured (symmetric) operads. As a particular case we recover the known model structure on simplicial operads.

CONTENTS

1. INTRODUCTION

Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a cofibrantly generated monoidal model category. In their paper [4], Berger and Moerdijk proved that under suitable assumptions on $\mathcal{V}$ the canonical model structure exists on $\mathcal{V}$-Cat, the category of (small) $\mathcal{V}$-enriched categories.

Symmetric coloured operads (also called symmetric multi-categories) can be regarded as generalized categories in which we allow the morphisms, now called *operations*, to have an arbitrary number of inputs.

Our goal in this paper is to generalize the results of [4] in order to get a canonical model structure on $\mathcal{V}$-Oper, the category of (small) $\mathcal{V}$-enriched coloured operads.

By the canonical model structure we mean the unique one (when it exists) for which the fibrant objects are the locally fibrant ones and the trivial fibrations are the local trivial fibrations which are surjective on the colours.

The main result of the paper is Theorem 4.22 which gives sufficient conditions on a right proper model category $\mathcal{V}$ in order to guarantee the existence of the canonical model structure on $\mathcal{V}$-Oper. For $\mathcal{V}$ equal to simplicial sets (with the Kan-Quillen model structure) one recovers the model structure on simplicial coloured operads presented in [9] and [26].

The existence of the canonical model structure on $\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper (the categories of non-symmetric coloured operads and symmetric constant-free operads respectively) is also established in Theorem 4.22, (3) under weaker assumptions on $\mathcal{V}$. Under the assumptions of Theorem 4.22, (3) also the category of $\mathcal{V}$-enriched unitary operads $\mathcal{V}$-UOper admits the canonical model structure; since this last case presents some (minor) differences with respect to the previous ones, we present it in a separate section (Theorem 7.10).

It turns out that our hypotheses on $\mathcal{V}$ have to be stricter than the ones made in [4] for a series of reasons. Given a set C let us denote by $\mathcal{V}\text{-}\mathsf{Cat}_C$ the category whose objects are the $\mathcal{V}$-enriched categories with C as set of objects and whose morphisms are the functors between them which are the identity at the level of objects. The category $\mathcal{V}\text{-}\mathsf{Oper}_C$ can be described as the category of algebras for a certain non-symmetric coloured operad Cat_C (whose set of colours is $C \times C$); if $\mathcal{V}\text{-}\mathsf{Cat}$ admits the canonical model structure then $\mathcal{V}\text{-}\mathsf{Cat}_C$ admits the model structure transferred from $\mathcal{V}^{C\times C}$ through the free-forgetful adjunction induced by Cat_C. In other words Cat_C has to be admissible in $\mathcal{V}$ for every $C \in \mathcal{S}et$.

In [5] and [24] sufficient conditions on $\mathcal{V}$ are given in order to guarantee that every non-symmetric operad is admissible in $\mathcal{V}$.

The case of $\mathcal{V}\text{-}\mathsf{Oper}$ is similar: for every set C we can consider the subcategory $\mathcal{V}\text{-}\mathsf{Oper}_C$ whose objects are the operads with C as set of colours and whose morphisms are the ones which are the identity on colours. This time $\mathcal{V}\text{-}\mathsf{Oper}_C$ is the category of algebras of a symmetric operad Op_C, that does not come from a non-symmetric one. For the canonical model structure over $\mathcal{V}\text{-}\mathsf{Oper}$ to exists, we need Op_C to be admissible in $\mathcal{V}$ for every $C \in \mathcal{S}et$, and this can not be guaranteed by the hypotheses in [4].

The fact that the operad is not non-symmetric forces us to restrict our hypotheses on $\mathcal{V}$ in order to apply [5, Theorem 2.1].

Something more can be said in the case in which we restrict our-selfs to the category of non-symmetric coloured operads or to the category of constant-free coloured operads. Again for every $C \in \mathcal{S}et$ we have an operad NSOp_C (resp. CfOp_C) whose algebras are non-symmetric (resp. constant-free) coloured operads with set of colours C. The operads NSOp_C and CfOp_C are not non-symmetric but they are Σ-free, or equivalently they give rise to tame polynomial monads.

Admissibility of tame polynomial monads was studied by Batanin and Berger in [2]. Their results allow us to enlarge the class of model categories $\mathcal{V}$ for which $\mathcal{V}\text{-}\mathsf{NSOper}$ (the category of $\mathcal{V}$-enriched non-symmetric operads) and $\mathcal{V}\text{-}\mathsf{cfOper}$ (the category of $\mathcal{V}$-enriched constant-free operads) admit the canonical model structure.

The outline of the paper is the following:

The first two sections are introductory; in Section 2 we recall the basic definitions of enriched (coloured) operad, algebra for an operad and results from the literature about admissibility of operads in monoidal model categories. In Section 3 we recall some basic facts about bifibrations.

Section 4 is the central part of the paper: the canonical model structure on $\mathcal{V}\text{-}\mathsf{Oper}$, $\mathcal{V}\text{-}\mathsf{NSOper}$ and $\mathcal{V}\text{-}\mathsf{cfOper}$ is presented and Theorem 4.22 is proven. The proof of Lemma 4.18, which is quite technical, is relegated to Appendix B.

In Section 5 it is proved that under our assumptions the canonical model structure on $\mathcal{V}\text{-}\mathsf{Oper}$, $\mathcal{V}\text{-}\mathsf{NSOper}$ and $\mathcal{V}\text{-}\mathsf{cfOper}$ is right proper as well.

In Section 6 we investigate left properness, which is a more subtle issue; in this case assuming that $\mathcal{V}$ is left proper is not enough to ensure that the canonical model structure on $\mathcal{V}\text{-}\mathsf{Oper}$ ($\mathcal{V}\text{-}\mathsf{NSOper}$, $\mathcal{V}\text{-}\mathsf{cfOper}$) is left proper. We show that if $\mathcal{V}$ is strongly h-monoidal (a concept introduced by Batanin and Berger [2]) then $\mathcal{V}\text{-}\mathsf{NSOper}$ and $\mathcal{V}\text{-}\mathsf{cfOper}$ are left proper. In Theorems 6.6 and 6.7 we also show that, under weaker assumptions on $\mathcal{V}$, the class of weak equivalences in $\mathcal{V}\text{-}\mathsf{NSOper}$ ($\mathcal{V}\text{-}\mathsf{cfOper}$, $\mathcal{V}\text{-}\mathsf{Oper}$) with domains and codomains which are close being cofibrant (in a sense to be defined depending on the context) is closed under push-out along cofibrations. We close the section with a counterexample that shows that sSets-Oper is not left proper.

In Section 7, using results of [11], we show that the canonical model structure on $\mathcal{V}$-cfOper can be transferred to $\mathcal{V}$-UOper.

To conclude, in Section 8 we present a different approach to our problem; as we explained , this article and [4] (in the case of $\mathcal{V}$-categories) investigate the existence of the *canonical model structure*; thus we could say that in order to find a meaningful model structure on the category of $\mathcal{V}$-operads (resp. $\mathcal{V}$-categories) we first focus on the "right" fibrant objects, namely the locally fibrant ones.

On the other hand there is also a class of morphisms of $\mathcal{V}$-operads ($\mathcal{V}$-categories) which is a natural candidate for the class of weak equivalences, namely the Dwyer-Kan weak equivalences (Definition 4.24). As in [4], we get that, under our hypotheses, the class of weak equivalences of the canonical model structure coincides with the Dwyer-Kan weak equivalences (Proposition 4.25).

Recently, in [23], Muro proved that under hypotheses different hypotheses on $\mathcal{V}$, that can be seen as less restrictive than the ones of [4], there exists a model structure on $\mathcal{V}$-Cat whose weak equivalences are the Dwyer-Kan weak equivalences (even though there is no control on the class of fibrant objects). In the last section, we show that the result of Muro can be extended to the case of $\mathcal{V}$-operads in order to get a model structure on $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper, $\mathcal{V}$-UOper) in which the weak equivalences are the Dwyer-Kan weak equivalences. This, together with results of [**SP14**], allows us to prove the existence of the Dwyer-Kan model structure onthe category of operads enriched in symmetric spectra (with the positive stable model structure).

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2. Preliminaries

Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category.

In this section we briefly recall the definitions of coloured operad enriched in $\mathcal{V}$ and of the category of algebras associated to it.

2.1. **Coloured operads.** For every $n \in \mathbb{N}$ let $[n] \in \mathcal{S}et$ be the set of all positive natural numbers smaller or equal to n;

Given a set C, the set of signatures in C is defined as

$$\mathrm{Seq}(C) = \{(c_1, \dots, c_n; c) \mid n \in \mathbb{N} \text{ and } c, c_1, \dots, c_n \in C\} = (\coprod_{n \in N} C^n) \times C\,.$$

For every element $s = (\bar{s}, r) \in \mathrm{Seq}(C)$, the sequence $\bar{s}$ will be denoted $\mathrm{in}(s)$ and r will be indicated $\mathrm{r}(s)$; there is a unique $n \in \mathbb{N}$ such that $\mathrm{in}(s) \in C^n$, this number is called the *valence* of s and denoted by $|s|$.

We will denote by $\mathrm{Seq}_0(C)$ the subset of $\mathrm{Seq}(C)$ formed by the signatures with valence different from 0.

For every $s = (s_1, \dots, s_n; r) \in \mathrm{Seq}(C)$ a sequence of signatures $(t_1, \dots, t_n)$ is said to be *composable* with s if $\mathrm{r}(t_i) = s_i$ for every $i \in [n]$; if this is the case we can produce the signature $s \circ (t_1, \dots, t_n)$. This new signature is such that $\mathrm{r}(s \circ (t_1, \dots, t_n)) = r$, its valence is $\sum_{i \in [n]} |t_i|$ and $\mathrm{in}(s \circ (t_1, \dots, t_n)) = (\mathrm{in}(t_1), \dots, \mathrm{in}(t_n))$.

Every map of sets $f\colon C \to D$ induces a map

$$\begin{aligned} f\colon \quad \mathrm{Seq}(C) &\longrightarrow \mathrm{Seq}(D) \\ (s_1, \dots, s_n; s) &\longmapsto (f(s_1), \dots, f(s_n); f(s)). \end{aligned}$$

A symmetric C-coloured operad enriched in $\mathcal{V}$ is an object $\mathcal{O}$ in the product category $\mathcal{V}^{\mathrm{Seq}(C)}$ with the following additional data:

- For every $s = (c_1, \dots, c_n; c) \in \mathrm{Seq}(C)$ and every sequence of signatures $(t_1, \dots, t_n)$ composable with s a morphism

$$\Gamma\colon (\bigotimes_{i\in[n]} \mathcal{O}(t_i)) \otimes \mathcal{O}(s) \longrightarrow \mathcal{O}(s \circ (t_1, \dots, t_n))$$

called *composition morphism*.
- For every $s \in \mathrm{Seq}(C)$ and every $\sigma \in \Sigma_{|s|}$ a morphism

$$\sigma^*\colon \mathcal{O}(s) \longrightarrow \mathcal{O}(s\sigma) \tag{2.1.1}$$

called *symmetric morphism* (here $s\sigma$ stands for $(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$).
- For every $c \in C$ a morphism

$$u_c\colon I \longrightarrow \mathcal{O}(c; c)$$

called the *identity operation* of c.

These morphisms are required to satisfy some associativity, unitality and equivariance conditions.

If we drop the actions of the permutation groups (2.1.1) (and hence the equivariance conditions) from the definition, we get *non-symmetric coloured operads* (sometimes called multi-categories). If we replace $\mathrm{Seq}(C)$ by $\mathrm{Seq}_0(C)$, i.e. we do not allowed the operad to have operations with empty input, then we get the definition of *constant-free operad* in the sense of [2] (also called non-unitary operad in the literature).

A morphism of C-coloured operads is, informally speaking, a morphism in $\mathcal{V}^{\mathrm{Seq}(C)}$ respecting composition, identity operations and symmetric morphisms. The category of C-coloured operads enriched in $\mathcal{V}$ will be indicated by $\mathcal{V}\text{-}\mathsf{Oper}_C$. In the same way we can define $\mathcal{V}\text{-}\mathsf{NSOper}_C$ and $\mathcal{V}\text{-}\mathsf{cfOper}_C$, the categories of non-symmetric C-coloured operads and constant-free C-coloured operads.

For more details about the definitions of coloured operads and their morphisms see for example [10] or [5].

It is also possible to define morphisms between operads with different sets of colours, we are now going to recall how. Every map of sets $f\colon C \to D$ induces functor

$$f^*\colon \mathcal{V}\text{-}\mathsf{Oper}_D \to \mathcal{V}\text{-}\mathsf{Oper}_C; \tag{2.1.2}$$

for every D-coloured operad $\mathcal{O}$ and every $s \in \mathrm{Seq}(C)$ the C-coloured operad $f^*\mathcal{O}$ is defined on the s-component as $f^*\mathcal{O}(s) = \mathcal{O}(f(s))$. Composition, identity operations and symmetries are defined in the obvious way using the ones in $\mathcal{O}$.

A morphism between a C-coloured operad $\mathcal{O}$ and a D-coloured operad $\mathcal{P}$ is defined as a couple $(f, \hat{f})$ where f is a map from C to D and $\hat{f}$ is a morphism of C-coloured operads $\hat{f}\colon \mathcal{O} \to f^*\mathcal{P}$. Given an E-coloured operad $\mathcal{Q}$ and a morphism $(g, \hat{g})$ from $\mathcal{P}$ to $\mathcal{Q}$ we can define the composition $(g, \hat{g}) \circ (f, \hat{f})$ as $(gf, \hat{f}f^*(\hat{g}))$.

In this way we can define the category $\mathcal{V}\text{-}\mathsf{Oper}$ of coloured operads (without a fixed set of colours). Given a coloured operad $\mathcal{O}$ its set of colours will be denoted by $\mathbf{Col}(\mathcal{O})$.

Given two monoidal model category $\mathcal{C}$, $\mathcal{D}$ and a monoidal functor between them $F\colon \mathcal{C} \to \mathcal{D}$ it is easy to see that it passes to the categories of operads, i.e. there is a functor $F_\sharp\colon \mathcal{C}\text{-}\mathsf{Oper} \to \mathcal{D}\text{-}\mathsf{Oper}$ such that for every $\mathcal{O} \in \mathcal{C}\text{-}\mathsf{Oper}$ one has $\mathbf{Col}(F_\sharp(\mathcal{O})) = \mathbf{Col}(\mathcal{O})$ and $F_\sharp\mathcal{O}(s) = F(\mathcal{O}(s))$ for every $s \in \mathrm{Seq}(\mathbf{Col}(\mathcal{O}))$.

In particular if $(\mathcal{V}, \otimes, I)$ is a bicomplete closed monoidal category there is always a strong monoidal functor

$$\begin{aligned} o^{\mathcal{V}} \colon (\mathcal{S}et, \times, *) &\longrightarrow (\mathcal{V}, \otimes, I), \\ S &\longmapsto \coprod_S I \end{aligned} \tag{2.1.3}$$

thus every $\mathcal{S}et$-operad $\mathcal{O}$ admits an incarnation $o_\sharp^{\mathcal{V}}(\mathcal{O})$ in $\mathcal{V}$.

2.2. **$\mathcal{O}$-Algebras.** Assume that $(\mathcal{V}, \otimes, \mathbf{I})$ is a bicomplete closed symmetric monoidal category. We briefly recall the definition of algebra for a coloured operad $\mathcal{O}$. Let $C = \mathbf{Col}(\mathcal{O})$, then an $\mathcal{O}$-algebra in $\mathcal{V}$ is an object of the product category $\mathcal{V}^C$ together with a morphism

$$\alpha_s \colon \mathcal{O}(s) \otimes A(c_1) \otimes \cdots \otimes A(c_n) \to A(c) \tag{2.2.1}$$

for every $s = (c_1, \ldots, c_n; c) \in \mathrm{Seq}(C)$. These *action maps* have to satisfy some obvious conditions of associativity, unitality and equivariance (*cf.* [10], [5]). A morphism of $\mathcal{O}$-algebras is a morphism in $\mathcal{V}^C$ which respects the action maps.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ be the category of $\mathcal{O}$-algebra in $\mathcal{V}$. There is a (finitary) monadic adjunction:

$$F_{\mathcal{O}} \colon \mathcal{V}^C \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \colon U_{\mathcal{O}} \tag{2.2.2}$$

If $\mathcal{Q}$ is another operad with set of colours D and $f \colon \mathcal{O} \to \mathcal{Q}$ is a morphism of operads then we have an adjunction:

$$f_! \colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathcal{Q}}(\mathcal{V}) \colon f^*$$

and a commutative diagram of adjunctions:

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \arrow[r, shift left, "f_!"] \arrow[d, shift left, "U_{\mathcal{O}}"] & \mathrm{Alg}_{\mathcal{Q}}(\mathcal{V}) \arrow[l, shift left, "f^*"] \arrow[d, shift right, "U_{\mathcal{Q}}"'] \\ \mathcal{V}^C \arrow[u, shift left, "F_{\mathcal{O}}"] \arrow[r, shift right, "f_!"'] & \mathcal{V}^D \arrow[l, shift right, "f^*"'] \arrow[u, shift right, "F_{\mathcal{Q}}"'] \end{tikzcd} \tag{2.2.3}$$

where the adjunction on the lower row is given by the inverse image functor f^* of the map $f \colon C \to D$ induced on the colours and its left adjoint.

If $\mathcal{O}$ is a $\mathcal{S}et$-operad we will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ the category of algebras of $o_\sharp^{\mathcal{V}}(\mathcal{O})$.

2.3. **Admissible monads.** For the rest of the section we fix $\mathcal{V}$, a cofibrantly generated model category with set of generating (trivial) cofibrations I (resp. J). Suppose that a finitary monad T over $\mathcal{V}$ is given. This amount of data produces a "free-forgetful" adjunction

$$F_T \colon \mathcal{V} \rightleftarrows \mathrm{Alg}_T(\mathcal{V}) \colon U_T$$

between $\mathcal{V}$ and the category of T-algebras. The category of T-algebras is complete and cocomplete and one can wonder if the model structure on $\mathcal{V}$ can be transferred along F_T, i.e. if there is a (necessarily unique) cofibrantly generated model structure on $\mathrm{Alg}_T(\mathcal{V})$ for which a morphism of T-algebras f is a weak equivalence (a fibration) if and only if $U_T(f)$ is a weak equivalence (resp. a fibration) in $\mathcal{V}$ and whose set of generating (trivial) cofibrations is $F_T(I)$ (resp. $F_T(J)$). If such a model structure exists, (F_T, U_T) is a Quillen adjunction and T is said to be *admissible*.

We recall that a class of morphisms S in a category $\mathcal{V}$ is *saturated* if and only if it is closed under push-outs, retracts and transfinite composition. A class of morphisms S in $\mathcal{V}$ which is closed under push-outs and transfinite composition is called *weakly saturated.* Given a set of morphisms K in $\mathcal{V}$ a *relative K-cell* is a morphism in the smallest weakly saturated class containing K.

Definition 2.1. Let T be a finitary monad over $\mathcal{V}$ and let K be a class of morphisms in K; a set of morphisms L in $\mathrm{Alg}_T(\mathcal{V})$ is *K-costable* if for every $l \in L$ and every pushout diagram in $\mathrm{Alg}_T(\mathcal{V})$:

(2.3.1)
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "l"'] & R \arrow[d, "m"] \\ B \arrow[r] & S \end{tikzcd}$$

the morphism $U_T(m)$ belongs to K.

Definition 2.2. Let T be a finitary monad on $\mathcal{V}$. For every $i\colon A \to B$ in $\mathcal{V}$ and every morphism of algebras $\alpha\colon F_T(A) \to R$, let i_α be the push-out of $F_T(i)$ along α:

(2.3.2)
$$\begin{tikzcd} F_T(A) \arrow[r, "\alpha"] \arrow[d, "F_T(i)"'] & R \arrow[d, "i_\alpha"] \\ F_T(B) \arrow[r] & R[i,\alpha]; \end{tikzcd}$$

i) T is *weakly K-admissible* if the set of morphisms $F_T(J)$ is $K \cap W$-costable (where W is the class of weak equivalences of $\mathcal{V}$);
ii) T is *K-admissible* if it is weakly K-admissible and the set of morphisms $F_T(I)$ is K-costable.

We recall that a *path object for a T-algebra X* is a factorization (in $\mathrm{Alg}_T(\mathcal{V})$) of the diagonal map

$$\begin{tikzcd} X \arrow[rr, "(\mathrm{id},\mathrm{id})"] \arrow[dr, "p"'] & & X \times X \\ & P(X) \arrow[ur, "q"'] & \end{tikzcd}$$

such that $U_T(p)$ is a fibration and $U_T(q)$ is a weak equivalence. A *fibrant replacement* functor for $\mathrm{Alg}_T(\mathcal{V})$ is an endofunctor $R\colon \mathrm{Alg}_T(\mathcal{V}) \to \mathrm{Alg}_T(\mathcal{V})$ together with a natural tranformation $r\colon \mathrm{id} \Longrightarrow R$ such that $U_T(R(X))$ is fibrant for every $X \in \mathrm{Alg}_T(\mathcal{V})$ and $U_T r$ is a natural weak equivalence.

Definition 2.3. A monad T on M *has path objects* if the following two conditions are satisfied:

- $\mathrm{Alg}_T(\mathcal{V})$ has a fibrant replacement functor;
- Every fibrant T-algebra admits a functorial path object.

Proposition 2.4. *If a finitary monad T on $\mathcal{V}$ has path objects then all relative $F_T(J)$-cells are weak equivalences. In particular a finitary monad T with path objects, such that the set of morphisms $F_T(I \cup J)$ (resp. $F(J)$) is K-costable is (weakly) K-admissible.*

For a proof of this proposition we refer the reader to [3, Section 2.6] and the references there.

Definition 2.5. Let K be a class of morphisms in $\mathcal{V}$, a cofibrantly generated model category. $\mathcal{V}$ is said to be *K-compactly generated* if:

- every object is small with respect to K;
- The class of weak equivalences is *K-perfect* (i.e. weak equivalences are closed under filtered colimits along morphisms of K).

If K is the whole class of morphisms of $\mathcal{V}$ we say that $\mathcal{V}$ is *strongly compactly generated.*

Remark 2.6. Every combinatorial model category which is pretty small in the sense of [25, Definition 2.0.1] is strongly compactly generated as Pavlov and Scholbach showed in [25, Lemma 2.0.2].

K-compactness and K-admissibility together are sufficient to guarantee admissibility:

Proposition 2.7. *([2, Theorem 2.11]) Every finitary K-admissible monad on a K-compactly generated category $\mathcal{V}$ is admissible.*

If $\mathcal{V}$ is a combinatorial model category, it is enough to check weak K-admissibility:

Proposition 2.8. *Every finitary weakly K-admissible monad on a combinatorial model category with K-perfect weak equivalences is admissible.*

Proof. The proof is identical to [2, Theorem 2.11] but since $\mathrm{Alg}_T(\mathcal{V})$ is a locally presentable category (Theorem A.4) the sets of morphism $F_T(I)$ and $F_T(J)$ automatically admit the small object argument, therefore condition ii of Definition 2.2 is unnecessary. □

2.4. **Admissible operads.** Suppose that $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category with set of generating (trivial) cofibrations I (resp. J). For every set C the product category $\mathcal{V}^C$ inherits a model structure in which the weak equivalences (cofibrations, fibrations) are the level-wise ones (*cf.* [19]).

For every $c \in C$ let $p_c \colon \mathcal{V}^C \to \mathcal{V}$ be the projection on the c-component. The functor p_c has a left adjoint ι_c that is defined on the objects in the following way: for every $A \in \mathcal{V}$ and $t \in C$

$$\iota_c(A)(t) = \begin{cases} A & \text{if } c = t \\ \emptyset & \text{otherwise} \end{cases}$$

where $\emptyset$ is the initial object of $\mathcal{V}$.

The model structure on $\mathcal{V}^C$ is cofibrantly generated and it has as set of generating cofibrations (resp. trivial cofibrations) $I_C = \{\iota_c(i) |$ for every $i \in I, c \in C\}$ (resp. $J_C = \{\iota_c(j) |$ for every $j \in J, c \in C\}$).

A coloured $\mathcal{V}$-enriched operad $\mathcal{O}$ with set of colours C is *admissible* if the corresponding monad (the one associated to adjunction (2.2.3)) is admissible, i.e. if the model structure on $\mathcal{V}^C$ can be transferred to the category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ through the (monadic) adjunction (2.2.3), i.e. if $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ can be endowed with a model structure in which a map f is a weak equivalence (fibration) if and only if $U_{\mathcal{O}}(f)$ is a weak equivalence (fibration) in $\mathcal{V}^C$. Note that if such a model structure is admissible then it is unique and it is cofibrantly generated: the set of generating (trivial) cofibrations is $F_{\mathcal{O}}(I_C)$ (resp. $F_{\mathcal{O}}(J_C)$). The transferred model structure on $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is also called the *transferred model structure.*

Definition 2.9. Fix a set C, a category $\mathcal{V}$ and a class of morphisms K in $\mathcal{V}$. We will say that a morphism $f \colon A \to B$ in $\mathcal{V}^C$ is a *local K-morphism* if for every $c \in C$ the morphism $f(c) \colon A(c) \to B(c)$ belongs to K.

Suppose that K is a saturated class of maps in $\mathcal{V}$, let K' be the class of local K-morphisms in $\mathcal{V}^C$; an operad $\mathcal{O}$ as above is *(weakly) K-admissible* in $\mathcal{V}$ if the corresponding monad is (weakly) K'-admissible.

Remark 2.10. If $\mathcal{V}$ is K-compactly generated model category, then $\mathcal{V}^C$ (with the induced model structure described above) is K'-compactly generated.

As a particular case of Propositions 2.7 and 2.8 we have:

Proposition 2.11. *Every K-admissible operad on a K-compactly generated category $\mathcal{V}$ is admissible.*

If $\mathcal{V}$ is combinatorial and the class of weak equivalences is K-perfect then every weakly K-admissible operad on $\mathcal{V}$ is admissible.

The admissibility of operads is investigated in [3], [5], [29], [17] , [25], [24] (in the non-symmetric case). In [2] the question of when a (finitary) polynomial monad is admissible is addressed. We recall that one can associate to every coloured operad $\mathcal{O}$ a monad whose category of algebras is the same as the one of $\mathcal{O}$ (that is the monad associated to the adjunction (2.2.3)). Not all the monads arise from operads but this is the case for finitary polynomial monads (in $\mathcal{S}et$). In fact there is an equivalence of categories between the category of finitary polynomial monads in $\mathcal{S}et$ and the category of Σ-free coloured operads in $\mathcal{S}et$ (*cf.* [21], [31] for definitions and proofs). A Σ-*free operad* is just an operad in which the action maps (2.2.1) act in a free way. If $\mathcal{O}_T$ is the $\mathcal{S}et$-operad associated to a certain polynomial monad T the latter is admissible in a monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ if and only if $o_\sharp^{\mathcal{V}}(\mathcal{O}_T)$ is admissible.

We now give some definitions in order to state some results of interested for us.

In a monoidal category, a saturated class of morphisms which is also closed under tensor products with arbitrary objects is called *monoidally saturated*; the *monoidal saturation* of a class of morphisms K is the minimal monoidally saturated class of morphisms containing K.

In $\mathcal{V}$ the morphisms belonging to the monoidal saturation of the class of cofibrations $I^{\otimes}$ will be called $\otimes$-*cofibrations.* An object in $\mathcal{V}$ is $\otimes$-*small* if it is small with respect to the class of $\otimes$-cofibrations. The class of weak equivalences is called $\otimes$-*perfect* if it is closed under filtered colimits along $\otimes$-cofibrations.

A coloured $\mathcal{V}$-enriched operad which is (weakly) $I^{\otimes}$-admissible is said *(weakly) $\otimes$-admissible.*

Definition 2.12. A cofibrantly generated monoidal model category $\mathcal{V}$ is *compactly generated* if the class of weak equivalences is $\otimes$-perfect and each object is $\otimes$-small (i.e. if it is $I^{\otimes}$-compactly generated).

We recall that every combinatorial monoidal model category whose class of weak equivalences is closed under filtered colimits is an example of compactly generated monoidal model category. Another example of compactly generated monoidal model category (which is not combinatorial) is given by the category of compactly generated (weak Hausdorff) topological spaces with the standard model structure and the cartesian product (*cf.* [4, p. 1.2]).

We have the following result:

Proposition 2.13. *(cf. [5, Theorem 2.1]) Suppose $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category with cofibrant unit. If it admits a monoidal fibrant replacement functor and contains a cocommutative comonoidal interval object, then every symmetric coloured $\mathcal{V}$-operad $\mathcal{O}$ has path objects.*

In particular if $\mathcal{O}$ is K-admissible and the domains of the generating (trivial) cofibrations are small relative to K for some saturated class of morphisms K in $\mathcal{V}$, then $\mathcal{O}$ is admissible.

We refer the reader to *loc. cit.* and [3] for the definition of interval object.

We give the following definition for convenience:

Definition 2.14. A cofibrantly generated model category is *strongly cofibrantly generated* if the domains of the generating cofibrations and trivial cofibrations are small (with respect to the whole category).

In its original version Theorem 2.1, [5] states that in a strongly cofibrantly generated category with a monoidal fibrant replacement functor and a cocommutative comonoidal interval object every operad is admissible.

Examples of stongly cofibrantly generated model category are abundant: for example any combinatorial model category is strongly cofibrantly generated.

Examples of cofibrantly generated monoidal model categories with cofibrant unit admitting a monoidal fibrant replacement functor and a cocommutative comonoidal interval object are:

- Simplicial Sets with the Kan-Quillen model structure;
- Chain Complexes over a field of characteristic zero with the projective model structure;
- Simplicial Modules over any ring, with the model structure transferred from Simplicial Sets;
- Compactly generated (weak Hausdorff) topological spaces with the Quillen model structure.

The first three examples are strongly cofibrantly generated (actually, they are all strongly compactly generated), hence every coloured symmetric operad is admissible. In the case of (weak Hausdorff) compactly generated topological space the objects are small only respect to $\mathcal{T}_1$, the class of T_1 closed inclusions (see Appendix C for the definition); as a consequence $\mathcal{T}_1$-admissible operads are admissible.

Recently also Pavlov and Scholbach found other conditions on $\mathcal{V}$ under which every every symmetric coloured operad is admissible in $\mathcal{V}$:

Theorem 2.15. *([25, Theorem 9.2.11]) Suppose $(\mathcal{V}, \otimes, \mathbf{I})$ is a combinatorial, pretty small symmetric monoidal model category which is symmetric i-monoidal, then every symmetric coloured operad is admissible in $\mathcal{V}$.*

We refer the reader to [25] for the definition of pretty small and symmetric i-monoidal and examples (Section 7, *loc. cit.*); the first three examples above fall in the scope of this theorem; symmetric spectra with the positive model structure are another example; one of the advantage of these conditions is that they are pretty robust, for example they are preserved under left Bousfield localization (*cf. loc. cit.*).

We recall also the monoid axiom, which was first introduced in [28] by Schwede-Shipley, in order to study the admissibility of the associative operad (the operad whose algebras are monoids):

Definition 2.16. A cofibrantly generated monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ satisfies the *monoid axiom* if any morphism in the monoidal saturated class of trivial cofibrations is a weak equivalence.

Remark 2.17. We would like to recall that the conditions of Pavlov and Scholbach (Theorem 2.15) implies the monoid axiom (*cf.* [25, Lemma 3.2.5]).

Proposition 2.18. *([2, Theorem 8.1]) Suppose $(\mathcal{V}, \otimes, \mathbf{I})$ is a compactly generated monoidal model category and satisfies the monoid axiom, then every tame (finitary) polynomial monad (in $\mathcal{S}et$) is $\otimes$-admissible in $\mathcal{V}$ (and hence admissible by Proposition 2.11).*

The following proposition is a slight variation of the preceding one and it can be proven in the same way

Proposition 2.19. *If $(\mathcal{V}, \otimes, \mathbf{I})$ is a combinatorial monoidal model category, satisfies the monoid axiom and the class of weak equivalences is $\otimes$-perfect then every tame (finitary) polynomial monad (in $\mathcal{S}et$) is weakly $\otimes$-admissible in $\mathcal{V}$.*

We will not give the definition of tame polynomial monad; we will content ourselves with few comments. Let us call an operad in $\mathcal{S}et$ *tame Σ-free* if its associated monad is tame polynomial. All tame polynomial monads arising from tame Σ-free operads and all non-symmetric operads are tame Σ-free.

We remark that proposition 2.18 was proved for non-symmetric one-coloured operads by Muro in [24].

Model categories satisfying the hypotheses of Proposition 2.18 are, among others, simplicial sets with Quillen's or Joyal's model structure, compactly generated spaces with the Quillen's model structure, chain complexes over a commutative ring with the projective model structure (*cf.* [2, Example 1.12, Proposition 2.5]).

3. Bifibrations

Let us fix a bicomplete symmetric monoidal category $\mathcal{V}$.
The functor $\mathbf{Ob}\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{S}et$ associates to each $\mathcal{V}$-enriched category its set of objects. Similarly the functor $\mathbf{Cl}\colon \mathcal{V}\text{-}\mathsf{Oper} \to \mathcal{S}et$ sends every $\mathcal{V}$-enriched coloured operads to its set of colours; we have a similar functor $\mathbf{Cl}$ for non-symmetric coloured operads and constant-free coloured operads as well.

The functors $\mathbf{Ob}$ and $\mathbf{Cl}$ are bifibrations (i.e. functors which are both fibrations and opfibrations), giving to $\mathcal{V}\text{-}\mathsf{Cat}$, $\mathcal{V}\text{-}\mathsf{Oper}$, $\mathcal{V}\text{-}\mathsf{NSOper}$ and $\mathcal{V}\text{-}\mathsf{cfOper}$ the structure of bifibred categories over $\mathcal{S}et$. We refer the reader to [16], [27] and [30] for the definition of bifibration, in the last two papers the presentation of $\mathcal{V}\text{-}\mathsf{Cat}$ is also given (note, [30] fixes an error of [27]). Classical references for fibred category are [13, expos. VI] and [8, §8].

Bifibrations are objects of a 2-category that we are now going to review.

Definition 3.1. Given two bifibrations $\alpha\colon X \to \mathcal{S}et$ and $\beta\colon Y \to \mathcal{S}et$ a *morphism of bifibrations* from α to β is an adjunction

$$F\colon X \rightleftarrows Y \colon G$$

with unit $\eta\colon \mathrm{id}_X \Longrightarrow GF$ and counit $\varepsilon\colon FG \Longrightarrow \mathrm{id}_Y$ such that:

(1) F is a cocartesian functor i.e. it preserves cocartesian arrows and $\beta F = \alpha$;
(2) G is a cartesian functor, i.e. it preserves cartesian arrows and $\alpha G = \beta$;
(3) $\beta\varepsilon = \mathrm{id}_\beta$ and $\alpha\eta = \mathrm{id}_\alpha$.

The symbol $\mathbf{BiFib}_{\mathcal{S}et}(\alpha, \beta)$ will denote the set of morphisms of bifibrations between α and β.

Definition 3.2. Given $h = (F_1, G_1)$ and $k = (F_2, G_2)$ in $\mathbf{BiFib}_{\mathcal{S}et}(\alpha, \beta)$ a *bicartesian natural transformation* between h and k is a natural transformation $\rho\colon F_1 \Longrightarrow F_2$ such that $\beta\rho = \mathrm{id}_\alpha$. The 2-category of bifibration over $\mathcal{S}et$, morphisms of bifibrations and bicartesian natural transformations (with composition defined in the obvious way), will be denoted by $\mathbf{BiFib}_{\mathcal{S}et}$.

From the theory of (bi)fibrations (see for example [16, Proposition 2.3]) we know that there is a 2-equivalence between $\mathbf{BiFib}_{\mathcal{S}et}$ and the 2-category of pseudo-functors from $\mathcal{S}et$ to $\mathsf{Cat}_{\mathrm{adj}}$ (the 2-category which has small categories as objects, adjunctions as 1-morphisms and natural transformations of right adjoint functors as 2-morphisms). Hence we can associate to each bifibration $\pi\colon \mathcal{C} \to \mathcal{S}et$ a pseudo-functor $\mathcal{F}\mathrm{ib}_\pi\colon \mathcal{S}et \to \mathsf{Cat}_{\mathrm{adj}}$, that determines π completely (up to equivalence). We recall here how this is done. Fix a bifibration $\pi\colon \mathcal{C} \to \mathcal{S}et$. For every $C \in \mathcal{S}et$ let $\mathcal{F}\mathrm{ib}_\pi(C)$ be the *fiber of* π *over* C, i.e. the minimal subcategory of $\mathcal{C}$ containing every object X such that $\pi(X) = C$ and every morphism f such that $\pi(f) = \mathrm{id}_C$.

A *cleavage* for π is a choice, for every map of sets $f\colon C \to D$ and every $A \in \mathcal{F}\mathrm{ib}_\pi(D)$, of an object $f^*(A) \in \mathcal{F}\mathrm{ib}_\pi(C)$ and a cartesian morphism $\phi_f\colon f^*(A) \to A$ such that $\pi\phi_f = f$.

A cleavege defines a unique functor $f^*\colon \mathcal{F}\mathrm{ib}_\pi(D) \to \mathcal{F}\mathrm{ib}_\pi(C)$ for every map $f\colon C \to D$, called the *inverse image functor* of f.

Dually, a *cocleavage* is a choice, for every map of sets $f\colon C \to D$ and every $B \in \mathcal{F}\mathrm{ib}_\pi(C)$, of an object $f_!(B) \in \mathcal{F}\mathrm{ib}_\pi(D)$ and a cocartesian morphism $\upsilon_f\colon B \to f_!(B)$ such that $\pi\upsilon_f = f$. A cocleavege defines a unique functor $f_!\colon \mathcal{F}\mathrm{ib}_\pi(C) \to \mathcal{F}\mathrm{ib}_\pi(D)$ for every map $f\colon C \to D$, called the *direct image functor* of f.

It follows that for every morphism $F\colon X\to Y$ in $\mathcal{C}$ there is a unique morphism $F^u\colon C\to Z$ in $\mathcal{F}\mathrm{ib}_\pi(\pi(X))$ such that $F=\phi_{\pi F}\circ F^u$. Dually there is a unique morphism $F_u\colon W\to Y$ in $\mathcal{F}\mathrm{ib}_\pi(\pi(Y))$ such that $F=F^u\circ \upsilon_{\pi F}$.

It can easily be proved that the inverse image functor is right adjoint to the direct image functor for every f. Equivalently a choice of a cleavage and a left adjoint for every inverse image functor provides a cocleavage. We recall that every bifibration admits a cleavage and a cocleavage.

We can now define the pseudo-functor $\mathcal{F}\mathrm{ib}_\phi\colon \mathcal{S}et\to \mathsf{Cat}_{\mathrm{adj}}$ associated to π: it sends every set C to the fiber category $\mathcal{F}\mathrm{ib}_\pi(C)$ and every map of set f in the adjunction $(f_!,f^*)$.

We now return to the bifibrations **Ob** and **Cl**. Given $C\in\mathcal{S}et$, the fiber of **Ob** over C is $\mathcal{V}\text{-}\mathsf{Cat}_C$, the category of $\mathcal{V}$-categories with set of objects C and functors between them which are the identity on C.

For every map of sets $f\colon C\to D$ and every $A\in\mathcal{V}\text{-}\mathsf{Cat}_D$ we can define the inverse image $f^*(A)\in\mathcal{V}\text{-}\mathsf{Cat}_C$ in the following way: for every $x,y\in C$, we set $f^*(A)(x,y)=A(f(x),f(y))$ and the composition is defined using the one of A. The unique map $\phi_f\colon f^*(A)\to A$, such that $\mathbf{Ob}(\phi_f)=f$ and $\phi_f(x,y)=\mathrm{id}_{A(f(x),f(y))}$ is cartesian. The collection of the ϕ_f's for every map of sets f and every $A\in\mathcal{V}\text{-}\mathsf{Oper}_D$ (where D is the target of f) provides a cleavage for **Ob**.

In a similar way the fiber of **Cl** over C coincides with $\mathcal{V}\text{-}\mathsf{Oper}_C$ as defined in 2.1. We can define a cleavage in such a way that the inverse image functor for a map $f\colon C\to D$ is defined as (2.1.2). The same can be done when the domain of **Cl** is $\mathcal{V}\text{-}\mathsf{NSOper}$ or $\mathcal{V}\text{-}\mathsf{cfOper}$, in these cases we denote the fiber over C by $\mathcal{V}\text{-}\mathsf{NSOper}_C$ and $\mathcal{V}\text{-}\mathsf{cfOper}_C$ respectively.

There are other (pseudo)functors from $\mathcal{S}et$ to $\mathsf{Cat}_{\mathrm{adj}}$ that we want to consider:

- $\mathsf{Gr}\colon\mathcal{S}et\to\mathsf{Cat}_{\mathrm{adj}}$ which sends every set C to the category $\mathcal{V}^{C\times C}$ and every map $f\colon C\to D$ to the adjunction
$$f_!\colon \mathcal{V}^{C\times C}\rightleftarrows \mathcal{V}^{D\times D}\colon f^*$$
induced by the morphism $f\times f\colon C\times C\to D\times D$. The bifibred category associated to it is $\mathcal{V}\text{-}\mathsf{Graph}$, the category of $\mathcal{V}$*-enriched graphs*.
- $\mathsf{MGr}\colon\mathcal{S}et\to\mathsf{Cat}_{\mathrm{adj}}$ which sends every set C to the category $\mathcal{V}^{\mathrm{Seq}(C)}$ and every map $f\colon C\to D$ to the adjunction
$$f_!\colon \mathcal{V}^{\mathrm{Seq}(C)}\rightleftarrows \mathcal{V}^{\mathrm{Seq}(D)}\colon f^*$$
induced by the morphism $\mathrm{Seq}(f)\colon\mathrm{Seq}(C)\to\mathrm{Seq}(D)$. The bifibred category associated to it is $\mathcal{V}\text{-}\mathsf{MultiGraph}$, the category of $\mathcal{V}$*-enriched multi-graphs*.
- $\mathsf{MGr}_0\colon\mathcal{S}et\to\mathsf{Cat}_{\mathrm{adj}}$ which sends every set C to the category $\mathcal{V}^{\mathrm{Seq}_0(C)}$ and every map $f\colon C\to D$ to the adjunction
$$f_!\colon \mathcal{V}^{\mathrm{Seq}_0(C)}\rightleftarrows \mathcal{V}^{\mathrm{Seq}_0(D)}\colon f^*$$
induced by the morphism $f\colon\mathrm{Seq}_0(C)\to\mathrm{Seq}_0(D)$. The bifibred category associated to it is $\mathcal{V}\text{-}\mathsf{RMultiGraph}$, the category of $\mathcal{V}$*-enriched constant-free multi-graphs*.

We want now to see objects of $\mathcal{V}\text{-}\mathsf{Cat}$ (resp. $\mathcal{V}\text{-}\mathsf{Oper}$) as $\mathcal{V}$-enriched graphs (resp. multi-graphs) with structure. We first need the following facts:

- there is a non-symmetric operad Cat_C such that $\mathrm{Alg}_{\mathsf{Cat}_C}(\mathcal{V})\cong\mathcal{V}\text{-}\mathsf{Cat}_C$. The set of colours of Cat_C is $C\times C$;
- there is a symmetric operad Op_C such that $\mathrm{Alg}_{\mathsf{Op}_C}(\mathcal{V})\cong\mathcal{V}\text{-}\mathsf{Oper}_C$. The set of colours of Op_C is $\mathrm{Seq}(C)$;
- there is a symmetric operad NSOp_C such that $\mathrm{Alg}_{\mathsf{NSOp}_C}(\mathcal{V})\cong\mathcal{V}\text{-}\mathsf{NSOper}_C$. The set of colours of NSOp_C is $\mathrm{Seq}(C)$;
- there is a symmetric operad CfOp_C such that $\mathrm{Alg}_{\mathsf{CfOp}_C}(\mathcal{V})\cong\mathcal{V}\text{-}\mathsf{cfOper}_C$. The set of colours of CfOp_C is $\mathrm{Seq}_0(C)$.

A description of Cat_C and Op_C can be found in [5]; Op_C is also described in detail in [14], the operads NSOp_C and CfOp_C are defined in a similar way.

As special cases of adjunction (2.2.2) we get the followings:

$$F_{\mathsf{Cat}_C}\colon \mathcal{V}^{C\times C} \rightleftarrows \mathcal{V}\text{-}\mathsf{Cat}_C \colon U_{\mathsf{Cat}_C} \tag{3.0.1}$$

$$F_{\mathsf{Op}_C}\colon \mathcal{V}^{\mathrm{Seq}(C)} \rightleftarrows \mathcal{V}\text{-}\mathsf{Oper}_C \colon U_{\mathsf{Op}_C} \tag{3.0.2}$$

$$F_{\mathsf{NSOp}_C}\colon \mathcal{V}^{\mathrm{Seq}(C)} \rightleftarrows \mathcal{V}\text{-}\mathsf{NSOper}_C \colon U_{\mathsf{NSOp}_C} \tag{3.0.3}$$

$$F_{\mathsf{CfOp}_C}\colon \mathcal{V}^{\mathrm{Seq}_0(C)} \rightleftarrows \mathcal{V}\text{-}\mathsf{cfOper}_C \colon U_{\mathsf{CfOp}_C}\ . \tag{3.0.4}$$

Furthermore for every map of sets $f\colon C \to D$ there are morphisms of operads $\mathsf{Cat}_f\colon \mathsf{Cat}_C \to \mathsf{Cat}_D$ and $\mathsf{Op}_f\colon \mathsf{Cat}_C \to \mathsf{Cat}_D$ (and the same is true if we replace $\mathsf{Op}\_$ with $\mathsf{NSOp}\_$ or $\mathsf{CfOp}\_$). Thanks to (2.2.3), from each of these morphisms we get an adjunction $(f_!, f^*)$ and a commutative diagram of adjunctions (we picture it only in the case of Op_f but the others are similar)

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Oper}_C \arrow[r, shift left, "f_!"] \arrow[d, shift left, "U_{\mathsf{Op}_C}"] & \mathcal{V}\text{-}\mathsf{Oper}_D \arrow[l, shift left, "f^*"] \arrow[d, shift right, "U_{\mathsf{Op}_D}"'] \\ \mathcal{V}^{\mathrm{Seq}(C)} \arrow[u, shift left, "F_{\mathsf{Op}_C}"] \arrow[r, shift right, "f_!"'] & \mathcal{V}^{\mathrm{Seq}(D)} \arrow[l, shift right, "f^*"'] \arrow[u, shift right, "F_{\mathsf{Op}_D}"'] \end{tikzcd} \tag{3.0.5}$$

and the horizontal maps turn out to be exactly the direct-inverse image adjunctions associated to the corresponding bifibration. This amounts to say that we have fibred adjunctions

$$F_{\mathsf{Cat}}\colon \mathcal{V}\text{-}\mathsf{Graph} \rightleftarrows \mathcal{V}\text{-}\mathsf{Cat}\colon U_{\mathsf{Cat}} \tag{3.0.6}$$

$$F_{\mathsf{Op}}\colon \mathcal{V}\text{-}\mathsf{MultiGraph} \rightleftarrows \mathcal{V}\text{-}\mathsf{Oper}\colon U_{\mathsf{Op}} \tag{3.0.7}$$

$$F_{\mathsf{NSOp}}\colon \mathcal{V}\text{-}\mathsf{MultiGraph} \rightleftarrows \mathcal{V}\text{-}\mathsf{NSOper}\colon U_{\mathsf{NSOp}} \tag{3.0.8}$$

$$F_{\mathsf{CfOp}}\colon \mathcal{V}\text{-}\mathsf{RMultiGraph} \rightleftarrows \mathcal{V}\text{-}\mathsf{cfOper}\colon U_{\mathsf{ROp}} \tag{3.0.9}$$

that fiber-wise are given by (3.0.1), (3.0.2), (3.0.3) and (3.0.4) respectively.

For every $C \in \mathcal{S}et$ there is also a morphism of operads $j_C\colon \mathsf{Cat}_C \to \mathsf{Op}_C$ which induces an adjunction between the categories of algebras

$$j_{C!}\colon \mathcal{V}\text{-}\mathsf{Cat}_C \rightleftarrows \mathcal{V}\text{-}\mathsf{Oper}_C \colon {j_C}^* \tag{3.0.10}$$

The functor $j_{C!}$ is the usual inclusion of the category of categories with C as set of objects in the category of C-coloured operads.

Furthermore for every map of sets $f\colon C \to D$ the following diagram of adjunctions is commutative

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Oper}_C \arrow[r, shift left, "f_!"] \arrow[d, shift left, "{j_C}^*"] & \mathcal{V}\text{-}\mathsf{Oper}_D \arrow[l, shift left, "f^*"] \arrow[d, shift right, "{j_D}^*"'] \\ \mathcal{V}\text{-}\mathsf{Cat}_C \arrow[u, shift left, "j_{C!}"] \arrow[r, shift right, "f^\sharp_!"'] & \mathcal{V}\text{-}\mathsf{Cat}_D \arrow[l, shift right, "f^{\sharp *}"'] \arrow[u, shift right, "j_{D!}"'] \end{tikzcd} \tag{3.0.11}$$

so we get an adjunction of fibred categories

$$j_!\colon \mathcal{V}\text{-}\mathsf{Cat} \rightleftarrows \mathcal{V}\text{-}\mathsf{Oper}\colon j^* \tag{3.0.12}$$

which fiber-wise is given by 3.0.10. The functor $j_!$ is just the usual fully faithful inclusion of $\mathcal{V}$-Cat in $\mathcal{V}$-Oper. Similarly we have morphisms of operads from Cat_C to NSOp_C, ROp_C and CfOp_C, which give us adjunctions:

(3.0.13) $$j_!\colon \mathcal{V}\text{-}\mathsf{Cat} \rightleftarrows \mathcal{V}\text{-}\mathsf{NSOper}\colon j^*$$

(3.0.14) $$j_!\colon \mathcal{V}\text{-}\mathsf{Cat} \rightleftarrows \mathcal{V}\text{-}\mathsf{cfOper}\colon j^*$$

3.1. **Fibered monadic adjunctions.** This section contains a technical result that will be needed to prove that $\mathcal{V}$-Oper is monadic over $\mathcal{V}$-Graph. The following proof was inspired by [7] and [20].

Proposition 3.3. *Given a morphism of bifibration over $\mathcal{S}et$:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, shift left, "F"] \arrow[dr, "\gamma"'] & & \mathcal{D} \arrow[ll, shift left, "U"] \arrow[dl, "\delta"] \\ & \mathcal{S}et & \end{tikzcd}$$

if the induced adjunction between the fibers

$$F_a\colon C_a \rightleftarrows D_a\colon U_a$$

is monadic for every $a \in \mathcal{S}et$ then (F,U) is monadic.

Proof. To prove that the adjunction is monadic it is sufficient to prove that U creates U-split coequalizers (*cf.* [8, Theorem 4.4.4]).

Suppose $f,g\colon A \to B$ are two morphisms in $\mathcal{D}$ such that Uf,Ug admits a split coequalizer C' in $\mathcal{C}$. Let $T = UF$, $a = \delta(A)$, $b = \delta(B)$ and $c = \gamma(C')$ and let T_a,T_b,T_c be the restrictions of T to the corresponding fibers.

We get two diagrams

$$\begin{tikzcd} FTUA \arrow[r, shift left, "FTUf"] \arrow[r, shift right, "FTUg"'] \arrow[d, shift left] \arrow[d, shift right] & FTUB \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & FTC' \arrow[d, shift left] \arrow[d, shift right] \\ FUA \arrow[r, shift left, "FUf"] \arrow[r, shift right, "FUg"'] \arrow[d] & FUB \arrow[r] \arrow[d] & FC' \arrow[d, dashed] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, dashed] & ? \end{tikzcd} \quad \begin{tikzcd} TTUA \arrow[r, shift left, "TTUf"] \arrow[r, shift right, "TTUg"'] \arrow[d, shift left] \arrow[d, shift right] & TTUB \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & TTC' \arrow[d, shift left] \arrow[d, shift right] \\ TUA \arrow[r, shift left, "TUf"] \arrow[r, shift right, "TUg"'] \arrow[d] & TUB \arrow[r] \arrow[d] & TC' \arrow[d] \\ UA \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UB \arrow[r] & C'. \end{tikzcd}$$

The diagram on the right (except the bottom-right corner) is obtained from the diagram on the left by applying U. In both diagrams the first two rows are split coequalizers; in the right one the last row is a split coequalizer too.

The three columns of both diagrams lie in the fiber of a,b and c respectively. The columns of the right diagram are split coequalizers; since T_a, T_b are monadic, the first two columns of the left diagram are also coequalizers. Furthermore, since T_c is monadic there exists $C \in \mathcal{C}_c$ such that it is a coequalizer for the third column and $U(C) = C'$. It follows that c is a coequalzer for the last row as well.

On the other hand if D us a coequalizer of f and g, then it is a coequalizer for the last column of the left diagram as well; hence $U(D)$ is a coequalizer for the last column of the right diagram (here we used again the fact that T_c is monadic); it follows that $U(D)$ is a coequalizer of Uf, Ug.

Therefore U creates U-split coequalizers, thus the adjunction (F,U) is monadic. $\square$

3.2. **Homotopy theory of the fibers.** Assume that $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category (*cf.* [19]) and let I, J be sets of generating cofibrations and generating trivial cofibrations respectively.

In order to prove the existence of the canonical model structure on $\mathcal{V}$-Cat, $\mathcal{V}$-Oper, $\mathcal{V}$-NSOper or $\mathcal{V}$-cfOper we want that all the fibers of the corresponding bifibration carry the transferred model structure. For this reason we give the following definitions:

Definition 3.4. A cofibrantly generated monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ with a distinguished saturated class of morphisms K:

(1) *admits transfer (weak K-transfer, K-transfer) for categories* if for every $C \in \mathcal{S}et$ the operad Cat_C is admissible (resp. weakly K-admissible, K-admissible) in $\mathcal{V}$;
(2) *admits transfer (weak K-transfer, K-transfer) for (symmetric) operads* if for every $C \in \mathcal{S}et$ the operad Op_C is admissible (resp. weakly K-admissible, K-admissible) in $\mathcal{V}$;
(3) *admits transfer (weak K-transfer, K-transfer) for non-symmetric operads* if for every $C \in \mathcal{S}et$ the operad NSOp_C is admissible (resp. weakly K-admissible, K-admissible) in $\mathcal{V}$;
(4) *admits transfer (weak K-transfer, K-transfer) for constant-free operads* if for every $C \in \mathcal{S}et$ the operad CfOp_C is admissible (resp. weakly K-admissible, K-admissible) in $\mathcal{V}$.

Remark 3.5. As a consequence of Proposition 2.11, if a K-compactly generated monoidal category $\mathcal{V}$ admits K-transfer (and transfer) for operads (non-symmetric operads, constant-free operads, categories) then it admits transfer for operads (resp. non-symmetric operads, constant-free operads, categories). Similarly if $\mathcal{V}$ is combinatorial and with K-perfect weak equivalences then weak K-transfer implies transfer.

Note that if $\mathcal{V}$ admits transfer (weak K-transfer, K-transfer) for (non-symmetric, constant-free) operads then it admits transfer (weak K-transfer,K-transfer) for categories.

Observe that if $\mathcal{V}$ admits transfer for (non-symmetric, constant-free) operads then all the adjunctions in diagrams (3.0.5) and (3.0.11) are Quillen adjunctions.

The operad Cat_C is non-symmetric. The operads Op_C, NSOp_C, and CfOp_C are only Σ-free, but only the last three are tame (*cf.* [2, section 9]).

As special cases of Propositions 2.13,2.18 and 2.19 we get:

Proposition 3.6. *Suppose $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category:*

(1) if $(\mathcal{V}, \otimes, \mathbf{I})$ is a strongly cofibrantly generated with cofibrant unit, admits a monoidal fibrant replacement functor and contains a cocommutative comonoidal interval object, then it admits transfer for symmetric operads.
(2) if $(\mathcal{V}, \otimes, \mathbf{I})$ is compactly generated monoidal model category satisfying the monoid axiom, then $\mathcal{V}$ admits $\otimes$-transfer for non-symmetric operads and constant-free operads.
(3) if $(\mathcal{V}, \otimes, \mathbf{I})$ is a combinatorial monoidal model category satisfying the monoid axiom, then $\mathcal{V}$ admits weak $\otimes$-transfer for non-symmetric operads and constant-free operads.

To conclude we would like to make few remarks about the generating (trivial) cofibrations of the fibers. Suppose that $\mathcal{V}$ admits transfer for symmetric operads. We know from section 2.2 that the sets of generating cofibrations and trivial cofibrations of $\mathcal{V}$-Oper_C are:

$$I_{\mathsf{Op}_C} = \{F_C(\iota_s(i))| \text{ for every } i \in I, s \in \mathrm{Seq}(C)\}$$

and

$$J_{\mathsf{Op}_C} = \{F_C(\iota_s(j))| \text{ for every } j \in J, s \in \mathrm{Seq}(C)\}$$

For every $n \in \mathbb{N}$ let $s_n \in \mathrm{Seq}([n+1])$ be the signature $(1, 2, \ldots, n; 0)$. As in section 2.2, we have a functor

$$\iota_{s_n} : \mathcal{V} \longrightarrow \mathcal{V}^{\mathrm{Seq}([n+1])} \tag{3.2.1}$$

left adjoint to the projection on the s_n-component. Observe that for every set C and every signature $s = (c_1, \dots, c_n; c_0)$ in C of valence n there is a map $f_s \colon [n] \to C$ (sending i to c_i) such that $f_s(s_n) = s$.

As a consequence $\iota_s = f_{s!}\iota_{s_n}$. Conversely for every map $f\colon [n]\to C$ there exists an $s \in \mathrm{Seq}(C)$ such that $\iota_s = f_!\iota_{s_n}$. This implies that the set of generating (trivial) cofibrations of $\mathcal{V}^{\mathrm{Seq}(C)}$ can be defined as $\{f_!^\sharp(\iota_{s_n}(i)) | \text{ for every } i \in I, n \in \mathbb{N}, f\colon [n] \to C\}$ (resp. $\{f_!^\sharp(\iota_{s_n}(j)) | \text{ for every } j \in J, n \in \mathbb{N}, f\colon [n] \to C\}$). So we can rewrite:

$$I_{\mathsf{Op}_C} = \{f_! F_{[n]}(\iota_{s_n}(i)) | \text{ for every } i \in I, n \in \mathbb{N}, f\colon [n] \to C\}$$

and

$$J_{\mathsf{Op}_C} = \{f_! F_{[n]}(\iota_{s_n}(i)) | \text{ for every } j \in J, n \in \mathbb{N}, f\colon [n] \to C\}$$

We will shorten the functor $F_{[n]}\iota_{s_n}$ in C_n.

Remark 3.7. Note that to give a commutative diagram of type:

$$\begin{tikzcd} C_n(X) \arrow[r, "a"] \arrow[d, "C_n(i)"'] & A \arrow[d, "f"] \\ C_n(Y) \arrow[r, "b"'] & B \end{tikzcd}$$

in $\mathcal{V}$-Oper is the same as to give a diagram:

$$\begin{tikzcd} a_! C_n(X) \arrow[r, "a_u"] \arrow[d, "a_! C_n(i)"'] & A \arrow[d, "f^u"] \\ a_! C_n(Y) \arrow[r, "\phi_a b^u"'] & f^*(B) \end{tikzcd}$$

in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(A)}$.

This implies that the map f^u is a fibration (trivial fibration) in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(A)}$ if and only if f has the right lifting property respect to $C_n(i)$ for every $n \in \mathbb{N}$ and $i \in I$ (resp. $i \in J$).

Similar considerations apply to $\mathcal{V}$-NSOper and $\mathcal{V}$-cfOper.

4. The model structure

4.1. **Weak equivalences and fibrations.** Let us fix a monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$. In this section we want to define the classes of weak equivalences and fibrations for the model structure that we want to establish on $\mathcal{V}$-Oper.

Let $\mathbb{I}$ be the $\mathcal{V}$-category with set of objects $\{0, 1\}$ representing a single isomorphism, i.e. $\mathbb{I}(0, 0) = \mathbb{I}(0, 1) = \mathbb{I}(1, 0) = \mathbb{I}(1, 1) = I$.

We recall the definition of $\mathcal{V}$-interval given in [4]:

Definition 4.1. Given a monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ which admits transfer for categories, a $\mathcal{V}$-*interval* is a cofibrant object in $\mathcal{V}$-$\mathsf{Cat}_{\{0,1\}}$ (with the transferred model structure), weakly equivalent to $\mathbb{I}$.

A set $\mathfrak{G}$ of $\mathcal{V}$-intervals is *generating* if each $\mathcal{V}$-interval $\mathbb{H}$ is a retract of a trivial extension $\mathbb{K}$ of a $\mathcal{V}$-interval $\mathbb{G}$ in $\mathfrak{G}$, i.e. there is a diagram in $\mathcal{V}$-$\mathsf{Cat}_{\{0,1\}}$

$$\begin{tikzcd} \mathbb{G} \arrow[r, tail, "j"] & \mathbb{K} \arrow[r, shift left, "r"] & \mathbb{H} \arrow[l, shift left, "i"] \end{tikzcd}$$

such that j is a trivial cofibration and $ri = \mathrm{id}_{\mathbb{H}}$.

Definition 4.2. Let $\partial_i\colon \{0,1\}\to\{0,1,2\}$ denote the order-preserving inclusion which omits i. The *amalgamation* of two objects $\mathbb{H}$ and $\mathbb{K}$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0,1\}}$ is defined as $\mathbb{H}*\mathbb{K}=\partial_1^*(\partial_{2!}\mathbb{K}\cup\partial_{0!}\mathbb{H})$, where the coproduct is taken in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0,1,2\}}$.

Note that $\partial_{2!}\mathbb{K}\cup\partial_{0!}\mathbb{H}$ is isomorphic to the colimit of the following diagram in $\mathcal{V}\text{-}\mathsf{Cat}$:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "i_0"] \arrow[d, "i_1"] & \mathbb{H} \\
\mathbb{K} &
\end{tikzcd}$$

where $\mathbf{1}$ is the $\mathcal{V}$-category representing a single object, i.e. the one with one object $*$ and $\mathbf{1}(*,*)=I$; The morphism i_j is the one determined by the object j of the target category.

It is clear that, given a $\mathcal{V}$-category C and two morphisms $h\colon\mathbb{H}\to C$ and $k\colon\mathbb{K}\to C$ such that $h(0)=k(1)$, there is a morphism $a\colon\mathbb{H}*\mathbb{K}\to C$ such that $a(0)=k(0)$ and $a(1)=h(1)$.

We also recall the following definitions from [4, p. 2.2]:

Definition 4.3. A $\mathcal{V}$-functor (i.e. a morphism in $\mathcal{V}\text{-}\mathsf{Cat}$) $F\colon A\to B$ is said to be:

- *path-lifting* if it has the right lifting property with respect to $i_k\colon\mathbf{1}\to\mathbb{H}$ for any $k\in\{0,1\}$ and any $\mathcal{V}$--interval $\mathbb{H}$ (see definition 4.2 and the discussion below for the definition of $\mathbf{1}$ and the i_k's).
- *essentially surjective* if for any object $b\colon\mathbf{1}\to B$ there is an object $a\colon\mathbf{1}\to A$, a $\mathcal{V}$- interval $\mathbb{H}$ and a commutative diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "a"] \arrow[dr, "i_0"] & & A \arrow[dd, "F"] \\
& \mathbb{H} \arrow[dr] & \\
\mathbf{1} \arrow[ur, "i_1"] \arrow[rr, "b"] & & B
\end{tikzcd}$$

Definition 4.4. Two objects a_0,a_1 of a $\mathcal{V}$- category A are

- *equivalent* if there exists a $\mathcal{V}$- interval $\mathbb{H}$ and a $\gamma\colon\mathbb{H}\to A$ such that $\gamma(0)=a_0$ and $\gamma(1)=a_1$;
- *virtually equivalent* if they become equivalent in some fibrant replacement A_f of A in $\mathcal{V}\text{-}\mathsf{Cat}_{\mathbf{Ob}(A)}$;
- *homotopy equivalent* if there exist maps

$$\alpha\colon I\to A_f(a_0,a_1) \text{ and } \beta\colon I\to A_f(a_1,a_0)$$

such that $\beta\alpha\colon I\to A_f(a_0,a_0)$ (resp. $\beta\alpha\colon I\to A_f(a_1,a_1)$) is homotopic to the arrow $I\to A_f(a_0,a_0)$ (resp. $I\to A_f(a_1,a_1)$) defining the identity of a_0 (resp. a_1).

Suppose now that $\mathcal{V}$ admits transfer for symmetric operads (non-symmetric, constant-free operads) and K is a class of morphisms in $\mathcal{V}$. We can give the following definitions:

Definition 4.5. a morphism $f\colon P\to Q$ in $\mathcal{V}\text{-}\mathsf{Oper}$ (resp. $\mathcal{V}\text{-}\mathsf{NSOper}$, $\mathcal{V}\text{-}\mathsf{cfOper}$) is said to be:

- a *local fibration (weak equivalence, trivial fibration, local K-morphism)* if the corresponding morphism $f^u\colon P\to f^*(Q)$ is a fibration (weak equivalence, trivial fibration, local K-morphism) in $\mathcal{V}\text{-}\mathsf{Oper}_{\mathbf{Col}(P)}$ (resp. $\mathcal{V}\text{-}\mathsf{NSOper}_{\mathbf{Col}(P)}$, $\mathcal{V}\text{-}\mathsf{cfOper}_{\mathbf{Col}(P)}$).
- *path-lifting* if it has the right lifting property with respect to $j_!(\{i\}\longrightarrow\mathbb{H})$, $i=0,1$ for any $\mathcal{V}$-interval $\mathbb{H}$ (equivalently $j^*(f)$ is path-lifting).

- *essentially surjective* if $j^*(f)$ is essentially surjective;
- a *fibration* if it is a path-lifting local fibration;
- a *weak equivalence* if it is an essentially surjective local weak equivalence;
- *fully faithful* if the corresponding morphism $f^u \colon P \to f^*(Q)$ is an isomorphism in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(P)}$ (resp. $\mathcal{V}$-$\mathsf{NSOper}_{\mathbf{Col}(P)}$, $\mathcal{V}$-$\mathsf{cfOper}_{\mathbf{Col}(P)}$).

Remark 4.6. Note that from Remark 3.7 and the characterization of fibrations in the transferred model structure on $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(P)}$ it follows that a morphism $f \colon P \to Q$ is a local fibration (trivial fibration) if and only if it has the right lifting property with respect to $J_{\mathrm{loc}} = \{C_n(j) | \ j \in J \ n \in \mathbb{N}\}$ (risp. $I_{\mathrm{loc}} = \{C_n(i) | \ i \in I \ n \in \mathbb{N}\}$).

We will give the proof of the following lemmas only in the case of symmetric operads, since the proofs in the non-symmetric and constant-free cases are identical.

Lemma 4.7. *A locally fibrant* $\mathcal{V}$-Oper *(resp.* $\mathcal{V}$-NSOper*,* $\mathcal{V}$-cfOper*) is fibrant.*

Proof. If P is locally fibrant in $\mathcal{V}$-Oper then $j^*(P)$ is locally fibrant in $\mathcal{V}$-Cat and hence it is path-lifting ([4, Lemma 2.3]). □

Lemma 4.8. *A morphism of* $\mathcal{V}$-Oper *(resp.* $\mathcal{V}$-NSOper*,* $\mathcal{V}$-cfOper*) is a trivial fibration if and only if it is a local trivial fibration which is surjective on colours.*

Proof. A morphism f in $\mathcal{V}$-Oper is a trivial fibration if and only if it is a local trivial fibration and $j^*(f)$ is path-lifting and essentially surjective. Since $j^*(f)$ is a local fibration if f is so and f is surjective on colours if and only if $j^*(f)$ is surjective on objects the statement follows from [4, Lemma 2.4]. □

4.2. **(Virtual) equivalence of colours.** We continue to assume that $\mathcal{V}$ admits transfer for symmetric operads (non-symmetric operads or constant-free operads, depending on the case we are interested in). The goal of this section is to prove Proposition 4.15, i.e. the 2-out-of-3 property for weak equivalences. We are going to give definitions and proofs only in the case of symmetric operads, since the arguments in the non-symmetric or constant-free case are almost identical.

Definition 4.9. Two colours c_0, c_1 of a $\mathcal{V}$-Operad P are:

- *equivalent* if they are equivalent as objects of $j^*(P)$ (in the sense of [4] and Definition 4.4);
- *virtually equivalent* if they become equivalent in some fibrant replacement P_f of P in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(P)}$;
- *homotopy equivalent* if they are homotopy equivalent as objects of $j^*(P)$. The relation of homotopy equivalence will be denoted by $\sim$.

From Lemma 2.8 in [4] applied to $j^*(P)$ we get immediately the following:

Lemma 4.10. *For any* $\mathcal{V}$*- operad* P*, equivalence and virtual equivalence are equivalence relations on* $\mathbf{Col}(P)$*.*

Lemma 4.11. *A local weak equivalence of operads* $F \colon P \to Q$ *reflects virtual equivalence of colours, i.e. for every* $c, d \in \mathbf{Col}(P)$ *if* $F(c)$ *and* $F(d)$ *are virtually equivalent in* Q*, then* c *and* d *are so in* P*.*

Proof. Since j^* preserves local weak equivalences, it follows directly from [4, Lemma 2.9]. □

Lemma 4.12. *If* $\mathcal{V}$ *is right proper then for any* $\mathcal{V}$*- operad* P*, virtually equivalent colours of* P *are equivalent.*

Proof. It follows from [4, Lemma 2.10] since if P_f is a fibrant replacement for P then $j^*(P_f)$ is a cofibrant replacement for $j^*(P)$ ($j^*_{\mathbf{Col}(P)}$ preserves weak equivalences and fibrant objects hence it preserves fibrant replacements). ☐

Lemma 4.13. *In any $\mathcal{V}$- operad P virtually equivalent colours are homotopy equivalent.*

Proof. if c_0, c_1 are virtually equivalent colours in P then they are virtually equivalent as objects of $j^*(P)$ thus they are homotopy equivalent by Lemma 2.11 in [4] ☐

Lemma 4.14. *Given $n \in \mathbb{N}$, $P \in \mathcal{V}\text{-}\mathsf{Oper}$ and $c_0, \dots c_n, d_0, \dots, d_n \in \mathbf{Col}(P)$ such that $c_0 \sim d_0, c_1 \sim d_1, \dots, c_n \sim d_n$ and $c \sim d$. Then $P(c_0, \dots, c_n; c)$ and $P(d_0, \dots, d_n; d)$ are related by a zig-zag of weak equivalences in $\mathcal{V}$. Moreover any morphism of $\mathcal{V}$- operads $f\colon P \to Q$ induces a functorially related zig-zag of weak equivalence between $Q(f(c_0), \dots, f(c_n); f(c))$ and $Q(f(d_0), \dots, f(d_n); f(d))$.*

Proof. We can prove the statement in the case in which $c_1 = d_1, \dots, c_n = d_n$, the general case will follow by iteration.

By assumption we have a fibrant replacement P_f of P in $\mathcal{V}\text{-}\mathsf{Oper}_{\mathbf{Col}(P)}$ and arrows $\alpha\colon \mathbf{I} \to P_f(c_0, d_0)$, $\beta\colon \mathbf{I} \to P_f(d_0, c_0)$ such that $\beta \circ \alpha\colon \mathbf{I} \to P_f(c_0, c_0)$ (resp. $\beta \circ \alpha\colon \mathbf{I} \to P_f(d_0, d_0)$) is homotopic to the arrow $\mathrm{id}_{c_0}\colon \mathbf{I} \to P_f(c_0, c_0)$ (resp. $\mathrm{id}_{d_0}\colon \mathbf{I}_{\mathcal{V}} \to P_f(d_0, d_0)$) given by the identity. Similarly we have arrows $\alpha'\colon \mathbf{I} \to P_f(c, d)$, $\beta'\colon \mathbf{I} \to P_f(d, c)$ such that $\beta' \circ \alpha'$ is homotopic to id_c and $\alpha' \circ \beta'$ is homotopic to id_d.

Using the internal composition of P_f we obtain two morphisms

$$(\alpha, \mathrm{id}_{c_1}, \dots, id_{c_n})^*\colon P_f(d_0, c_1, \dots, c_n; c) \longrightarrow P_f(c_0, c_1, \dots, c_n; c)$$

$$(\beta, \mathrm{id}_{c_1}, \dots, id_{c_n})^*\colon P_f(c_0, c_1, \dots, c_n; c) \longrightarrow P_f(d_0, c_1, \dots, c_n; c)$$

which become mutually inverse isomophisms in the homotopy category $\mathrm{Ho}(\mathcal{V})$, hence they are weak equivalences. Similarly the morphisms

$$\alpha'_*\colon P_f(d_0, c_1, \dots, c_n; c) \longrightarrow P_f(d_0, c_1, \dots, c_n; d)$$

$$\beta'_*\colon P_f(d_0, c_1, \dots, c_n; d) \longrightarrow P_f(d_0, c_1, \dots, c_n; c)$$

are also weak equivalences since they are mutually inverse isomorphisms in the homotopy category. The zig-zag of weak equivalences between $P(c_0, c_1, \dots, c_n; c)$ and $P(d_0, c_1, \dots, c_n; d)$ is given by

$$\alpha'_*(\beta, \mathrm{id}_{c_1}, \dots, id_{c_n})^*\colon P_f(c_0, c_1, \dots, c_n; c) \longrightarrow P_f(d_0, c_1, \dots, c_n; d)$$

concatenated with the weak equivalences

$$P(c_0, c_1, \dots, c_n; c) \xrightarrow{\sim} P_f(c_0, c_1, \dots, c_n; c)$$

and

$$P(d_0, c_1, \dots, c_n; d) \xrightarrow{\sim} P_f(d_0, c_1, \dots, c_n; d).$$

Every morphism of $\mathcal{V}$-operads $g\colon P \to Q$ induces a morphism between some fibrant replacements P_f and Q_f

$$\begin{tikzcd} P \arrow[rr, "\sim"] \arrow[d, "g^u"'] & & P_f \arrow[d, "g^u_f"] & P \arrow[r, "\sim"] \arrow[d, "f"'] & P_f \arrow[d, "g_f"] \\ g^*(Q) \arrow[rr, "\sim"'] & & g^*(Q_f) & Q \arrow[r, "\sim"'] & Q_f \end{tikzcd}.$$

The following diagram commutes (here $g(\alpha')$ is the composition of α' with $g_f\colon P_f(c,d)\to Q_f(g(c),g(d))$, etc.)

$$\begin{tikzcd}
P_f(c_0,c_1,\dots,c_n;c) \arrow[r, "\alpha'_*(\beta,\mathrm{id}_{c_1}\dots,id_{c_n})^*"] \arrow[d, "g_f"'] & P_f(d_0,c_1,\dots,c_n;d) \arrow[d, "g_f"] \\
Q_f(g(c_0),g(c_1),\dots,g(c_n);g(c)) \arrow[r, "g(\alpha')_*(g(\beta),\mathrm{id}_{g(c_1)},\dots,id_{g(c_n)})^*"'] & Q_f(g(d_0),g(c_1),\dots,g(c_n);g(d))
\end{tikzcd}$$

and it is easy to check that there is a functorially related zig-zag of weak equivalences between $Q(g(c_0),g(c_1),\dots,g(c_n);g(c))$ and $Q(g(d_0),g(c_1),\dots,g(c_n);g(d))$. □

Proposition 4.15. *If $\mathcal{V}$ is right proper, the class of weak equivalences in $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper) satisfies the 2-out-of-3 property.*

Proof. Let $f\colon P\to Q$ and $g\colon Q\to R$ be morphisms of $\mathcal{V}$- operads.

We have to prove that if two among f, g and fg are weak equivalences (i.e. essentially surjective local weak equivalences) then so is the remaining one. We are going to prove the three cases separately:

- Assume f and g are weak equivalences. It is easy to check that gf is a local weak equivalence. The fact that gf is essentially surjective follows from the fact that $j^*(gf)$ is (*cf.* [4, Proposition 2.13]);
- Assume f and gf are weak equivalences. It is immediate to check that g is essentially surjective. What is left to check is that g is a local weak equivalence. Given $d_0,d_1,\dots,d_n,d\in \mathbf{Col}(Q)$ since f is essentially surjective we can pick $c_0,c_1,\dots,c_n,c\in\mathbf{Col}(Q)$ such that $f(c_i)$ is equivalent to d_i for every $i\in\{0,\dots,n\}$ and $f(c)$ is equivalent to d. It follows from Lemma 4.13 that $f(c_i)\sim d_i$ for every $i\in\{0,\dots,n\}$ and $f(c)\sim d$ hence Lemma 4.14 gives us a zig-zag of weak equivalences (in $\mathcal{V}$) between $Q(f(c_0),\dots,f(c_n);f(c))$ and $Q(d_0,\dots,d_n;d)$. We get the following commutative diagrams:

$$\begin{tikzcd}
P(Q(c_0,\dots,c_1;c) \arrow[r, "f(c_0,\dots,c_1;c)"] \arrow[dr, "gf(c_0,\dots,c_1;c)"'] & Q(f(c_0),\dots,f(c_n);f(c)) \arrow[d, "g(f(c_0),\dots,f(c_n);f(c))"] \\
 & R(gf(d_0),\dots,gf(d_n);gf(d))
\end{tikzcd}$$

$$\begin{tikzcd}
Q(f(c_0),\dots,f(c_n);f(c)) \arrow[r, dash, "\sim"] \arrow[d, "g(f(c_0),\dots,f(c_n);f(c))"] & Q(d_0,\dots,d_n;d) \arrow[d, "g(d_0,\dots,d_n;d)"] \\
R(gf(d_0),\dots,gf(d_n);gf(d)) \arrow[r, dash, "\sim"] & R(g(d_0),\dots,g(d_n);g(d))
\end{tikzcd}$$

  (the zig-zag of weak equivalences on the bottom row of the last diagram is obtained from the one on the top row applying Lemma 4.14).
  The vertical arrow in the first diagram is a weak equivalence since the other two maps in the diagram are so (by assumption). Applying the 2-out-of-3 property of weak equivalences in $\mathcal{V}$ in the second diagram we get that $g(d_0,\dots,d_n;d)$ is a weak equivalence;
- Assume g and gf are weak equivalences. It is immediate to verify that f is a local weak equivalence too. It follows from Lemmas 4.11 and 4.12 that g reflects equivalences of objects thus the essentially surjectivity of f follows from the one of gf.

□

4.3. **Generating (trivial) cofibrations.** Lemma 4.8 implies that the class of the trivial fibrations (as defined in 4.3) in $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper) is characterized by the right lifting property respect to:

$$\mathcal{I} = I_{loc} \cup \{\emptyset \to j^*(\mathbf{1})\} \tag{4.3.1}$$

where $\mathbf{1}$ is the $\mathcal{V}$-category representing a single object (as in Definition 4.2) and I_{loc} is defined as in Remark 4.6. The morphisms which are surjective on the colours are exactly the ones having the right lifting property respect to the last map. The set $\mathcal{I}$ is therefore a good candidate for the set of generating cofibrations for the model structure we want to establish.

If there exists a set $\mathfrak{G}$ of generating $\mathcal{V}$-intervals we also have a good candidate for the set of generating trivial cofibrations, that is

$$\mathcal{J} = J_{loc} \cup \{j_!(i_1) \colon j_!(\mathbf{1}) \to j_!(\mathbb{G}) \mid \mathbb{G} \in \mathfrak{G}\} \; . \tag{4.3.2}$$

In fact, by definition, a morphism f in $\mathcal{V}$-Oper is path-lifting if and only if it has the right lifting property respect to $j_!(i_1) \colon j_!(\mathbf{1}) \to j_!(\mathbb{H})$ for every $\mathcal{V}$-interval $\mathbb{H}$, but if f is locally fibrant it is sufficient to check it only for the $j_!(i_1)$'s coming from generating $\mathcal{V}$-intervals. In fact, suppose that $\mathbb{H}$ is a retract of a trivial extension of a generating $\mathcal{V}$-interval $\mathcal{G}$ and that the following diagram commutes:

$$\begin{tikzcd}
 & & j_!(\mathbf{1}) \arrow[r, "a"] \arrow[lld, "j_!(i_1)"'] \arrow[d, "j_!(i_1)"] & A \arrow[d, "f"] \\
j_!(\mathbb{G}) \arrow[r, tail, "e"] \arrow[rrru, dashed, "l"] & j_!(\mathbb{K}) \arrow[r, shift left, "r"] & j_!(\mathbb{H}) \arrow[l, shift left, "i"] \arrow[r, "h"] & B
\end{tikzcd} .$$

If l exists, it can be extended to a map $l' \colon K \to A$ since e is a trivial cofibration (in $\mathcal{V}$-$\mathsf{Oper}_{\{0,1\}}$ and f is locally fibrant. The map $l'i$ gives the desired lifting for $\mathbb{H}$.

We want now to be sure that the domains of $\mathcal{I}$ (resp. $\mathcal{J}$) are small with respect to $\mathcal{I}$ (resp. $\mathcal{J}$), so that we can apply the small object argument. We now have to distinguish between the case of $\mathcal{V}$-Oper and the ones of $\mathcal{V}$-NSOper and $\mathcal{V}$-cfOper.

Lemma 4.16. *Let $\mathcal{V}$ be a model category admitting transfer for categories. In $\mathcal{V}$-Cat the transfinite composition of essentially surjective morphisms is essentially surjective.*

Proof. Let α be an ordinal (regarded as a category) and let $F \colon \alpha \to \mathcal{V}$-$\mathsf{Cat}$ be a functor defining a transfinite composition of essentially surjective morphisms. Let $A = F(0)$, $B = \varinjlim F$ and $f \colon A \to B$ be the transfinite composition obtained by F. Given an object $b \colon \mathbf{1} \to B$ we want to find an object $a \colon \mathbf{1} \to A$, an interval $\mathbb{H}$ and morphism $h \colon \mathbb{H} \to B$ such that $h(0) = f(a)$ and $h(1) = b$.

The proof will proceed by transfinite induction on α.

If α is the ordinal with one object (as a category) then there is nothing to prove (the transfinite composition is just the identity).

Suppose α is a limit ordinal and the statement is true for any $\beta < \alpha$.

In $\mathcal{V}$-Cat the object $\mathbf{1}$ is small so b factors through some $F(\beta)$ where β is a successor ordinal smaller then α, i.e. there is $b' \in F(\beta)$ such that $f_{\beta\alpha}(b') = b$, where $f_{\beta\alpha}$ is the canonical morphism from $F(\beta)$ to B.

F restricted to $\{\gamma \leq \beta\}$ defines a β-transfinite composition f_β of essentially surjective morphisms, which is essentially surjective by inductive hypothesis.

We can then find an interval $\mathbb{H}$, an object a in A and a morphism $h' \colon \mathbb{H} \to F(\beta)$ such that $h'(0) = f_\beta(a)$ and $h'(1) = b'$. The desired morphism is then $h = f_{\beta\alpha}\, h'$.

For the successor case suppose $\alpha = \beta + 1$ and that the statement is true for every transfinite composition indexed by β. Factor f through $F(\beta)$:

$$\begin{tikzcd} A \arrow[r, "f_\beta"] \arrow[rr, bend right, "f"'] & F(\beta) \arrow[r, "f_{\beta\alpha}"] & F(\alpha) = B \end{tikzcd}$$

By hypothesis $f_{\alpha\beta}$ is essentially surjective so we can find an object $b' \in F(\beta)$, an interval $\mathbb{H}'$ and a morphism $h' \colon \mathbb{H}' \to B$ such that $h'(0) = f_{\alpha\beta}(b')$ and $h'(1) = b$.

On the other hand f_β is also essentially surjective by inductive hypothesis so we can find $a \in A$, an interval $\mathbb{H}''$ and a morphism $h'' \colon \mathbb{H}'' \to F(\beta)$ such that $h''(0) = f_\beta(a)$ and $h''(1) = b'$.

Consider the amalgamation $\mathbb{H}'' * \mathbb{H}'$, since $f_{\beta\alpha}h''(1) = f_{\beta\alpha}h''(1) = h'(0)$ we get a morphism $h \colon \mathbb{H}'' * \mathbb{H}' \to B$ such that $h(0) = f_{\beta\alpha}h''(0) = f(a)$ and $h(1) = h'(1) = b$ by the universal property of push-outs. Consider a cofibrant replacement $\iota \colon \overline{\mathbb{H}'' * \mathbb{H}'} \to \mathbb{H}'' * \mathbb{H}'$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0,1\}}$. By Lemma 1.16 [4] the $\mathcal{V}$-category $\overline{\mathbb{H}'' * \mathbb{H}'}$ is an interval and $\iota h(0) = f(a)$ and $\iota h(1) = b$. The element b was arbitrarily chosen, so this proves that f is essentially surjective.

□

Corollary 4.17. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a monoidal model category which admits transfer for operads. In $\mathcal{V}$-Oper the transfinite composition of essentially surjective morphisms is essentially surjective.*

Proof. It follows directly from the definition of essential surjectivity in $\mathcal{V}$-Oper and Lemma 4.16. □

The lemma that follows is the enriched version of [9, Lemma 1.29]:

Lemma 4.18. *Consider a fully faithful embedding $i \colon A \to B$ in $\mathcal{V}$-Cat, in which A has one object 0 and B has as set of objects $\{0,1\}$ such that $i(0) = 0$. For every push-out in $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper) along $j_!(i)$:*

$$\begin{tikzcd} j_!(A) \arrow[r, "a"] \arrow[d, "j_!(i)"'] & P \arrow[d, "v"] \\ j_!(B) \arrow[r, "b"'] & Q \end{tikzcd}$$

the morphism of operads v is fully faithful.

Proof. This is a particular case of Proposition B.27.

□

Lemma 4.19. *Suppose $\mathcal{V}$ is compactly generated with cofibrant monoidal unit and satisfies the monoid axiom, then every relative $\mathcal{I}$-cell in $\mathcal{V}$-NSOper ($\mathcal{V}$-cfOper) is a local $\otimes$-cofibration.*

Proof. As stated in Proposition 3.6, $\mathcal{V}$ admits transfer for non-symmetric operads and constant-free operads. Since local $\otimes$-cofibrations are closed under transfinite composition it is sufficient to show that in any push-out square

(4.3.3)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "j"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

if i is in $\mathcal{I}$ then j is a local $\otimes$-cofibration. Since the unit is supposed to be cofibrant it is immediate to check that a push-out of $\emptyset \to j^*(\mathbf{1})$ is a local cofibration (hence a local $\otimes$-cofibration). Suppose now that $i \in I_{loc}$, i.e. $i = C_n(k)$ for some $k \colon a \to b$ in I (where I is the set of generating cofibrations of $\mathcal{V}$); then the statement follows from Proposition 2.18. In fact, in this case the push-out (4.3.3) can be obtained from the following push-out diagram in $\mathcal{V}$-NSOper_C (resp. $\mathcal{V}$-cfOper_C) where $C = \mathbf{Col}(X)$

$$\begin{tikzcd} f_! C_n(a) \arrow[r, "f_u"] \arrow[d, "f_!(i)"'] & X \arrow[d, "j"] \\ f_! C_n(b) \arrow[r, "g_u"'] & Y \end{tikzcd}$$

(see Appendix A for a description of colimits in these categories). Since $f_!(i)$ is a cofibration in $\mathcal{V}$-NSOper_C (resp. $\mathcal{V}$-cfOper_C), g is also a cofibration in the same category, hence a local $\otimes$-cofibration from Proposition 2.18. $\square$

Lemma 4.20. *Suppose $\mathcal{V}$ is a cofibrantly generated monoidal category with cofibrant unit:*

(1) if $\mathcal{V}$ admits transfer for operads, a set of generating $\mathcal{V}$-intervals $\mathfrak{G}$ and the class of weak equivalences is closed under transfinite composition, then every relative $\mathcal{J}$-cell in $\mathcal{V}$-Oper is a weak equivalence.

(2) if $\mathcal{V}$ is K-compactly generated, admits K-transfer for operads and a set of generating $\mathcal{V}$-intervals $\mathfrak{G}$, then every relative $\mathcal{J}$-cell in $\mathcal{V}$-Oper is a weak equivalence and a local K-morphism (Definition 2.9 and 4.5).

(3) if $\mathcal{V}$ is compactly generated (or combinatorial with $\otimes$-perfect weak equivalences), satisfies the monoid axiom and has a set of generating $\mathcal{V}$-intervals $\mathfrak{G}$, then every relative $\mathcal{J}$-cell in $\mathcal{V}$-NSOper and $\mathcal{V}$-cfOper is an essentially surjective local trivial $\otimes$-cofibration (in particular it is a weak equivalence).

Proof. The proof is similar to the one of Lemma 4.19. By Lemma 4.16 essentially surjective maps are closed under transfinite composition. Under the hypothesis of point (1) local weak equivalences are closed under transfinite composition so we are reduced to prove that, for every push-out diagram like (4.3.3), if $i \in \mathcal{J}$ then j is a weak equivalence. If $i \in J_{loc}$ then this follows from the fact that the push-out can be calculated in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(X)}$; this implies that j^u is a trivial cofibration in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(X)}$, thus j is local weak equivalence bijective on colours (in $\mathcal{V}$-Oper), hence a weak equivalence.

Let us consider now the case in which $i = j_!(i_1)\colon j_!(\mathbf{1}) \to j_!(\mathbb{G})$ for some $\mathbb{G} \in \mathfrak{G}$:

$$\begin{tikzcd} j_!(\mathbf{1}) \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d] \\ j_!(\mathbb{G}) \arrow[r] & Y . \end{tikzcd}$$

We can decompose it in two push-outs

$$\begin{tikzcd} j_!(\mathbf{1}) \arrow[r, "p"] \arrow[d, "j_!(\phi)"] & X \arrow[d, "\phi'"] \\ j_!(\mathbb{G}_{0,0}) \arrow[r] \arrow[d, "j_!(\psi)"] & X' \arrow[d, "\psi'"] \\ j_!(\mathbb{G}) \arrow[r] & Y \end{tikzcd}$$

where $\mathbb{G}_{0,0}$ is the full subcategory of $\mathbb{G}$ which has 0 as unique object, and ψ is the canonical inclusion. The functor ψ is fully faithful, hence Lemma 4.18 implies that ψ' is fully faithful (hence a local weak equivalence). Since ψ' is essentially surjective by construction, it is a weak equivalence in $\mathcal{V}$-Oper.

Let us consider now the push-out on top. It can be calculated in $\mathcal{V}\text{-}\mathsf{Oper}_{\mathbf{Col}(X)}$ (see Appendix A) as the push-out of p_u (which is a morphism from $p_!(j_!(\mathbf{1}))$ to X) along $p_!(j_!(\phi))$, in particular ϕ' is bijective on colours (so it is essentially surjective).

Under our hypotheses ϕ is a cofibration in $\mathcal{V}\text{-}\mathsf{Oper}_{\{*\}}$ (according to the [4, Interval Cofibrancy Theorem 1.15]), furthermore it is a trivial cofibration since $\mathbb{G}$ is weakly equivalent to $\mathbf{1}$.

Note that $p_!(j_!(\phi)) = p_!(j_{\{*\}!}(\phi))$ and $p_!$ and $j_{\{*\}!}$ are left Quillen functors, so $p_!(j_!(\phi))$ is a trivial cofibration in $\mathcal{V}\text{-}\mathsf{Oper}_{\mathbf{Col}(X)}$. This implies that ϕ' is a trivial cofibration in $\mathcal{V}\text{-}\mathsf{Oper}_{\mathbf{Col}(X)}$ as well; hence a local weak equivalence in $\mathcal{V}$-Oper.

Since ϕ' is furthermore bijective on colours, it is a weak equivalence in $\mathcal{V}$-Oper and this concludes the proof of point (1).

The proof of point (2) and (3) are almost identical, keeping in mind Proposition 2.18 and the fact that in a K-compactly generated model category the class of maps which are local weak equivalence and local K-morphisms is closed under transfinite composition. □

Lemma 4.21. *Let $\mathcal{V}$ be a cocomplete monoidal closed category and $n \in \mathbb{N}$ then:*

(1) If $X \in \mathcal{V}$ is small then $C_n(X)$ is small in $\mathcal{V}$-Oper.

(2) Suppose $\mathcal{V}$ is a K-compactly generated model category. If $X \in \mathcal{V}$ is K-small then $C_n(X)$ is small in $\mathcal{V}$-NSOper (resp. $\mathcal{V}$-cfOper) with respect to local K-morphisms.

Proof. Recall that $C_n(X) = F_{\mathsf{Op}_{[n]}} \iota_{s_n}(X)$. Both statements are a straight-forward consequence of Lemma A.2 and Lemma A.1. □

4.4. **Model structure.** We are now ready to prove that there exists a cofibrantly generated model structure on $\mathcal{V}$-Oper with (4.3.1) and (4.3.2) as generating cofibrations and trivial cofibrations and whose class of weak equivalences $\mathcal{W}$ is the one defined in Definition 4.5.

This is our main theorem:

Theorem 4.22. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a cofibrantly generated monoidal model category such that:*

- *The unit is cofibrant;*
- *The model structure is right proper;*
- *There exists a set $\mathfrak{G}$ of generating $\mathcal{V}$-intervals*

Then:

(1) If $\mathcal{V}$ is strongly cofibrantly generated, admits transfer for (non-symmetric, constant-free) operads Definition 3.4) and the class of weak equivalences is closed under transfinite composition, then there exists a cofibrantly generated model structure on $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper,$\mathcal{V}$-cfOper) such that fibrations and weak equivalences are the ones introduced in Definition 4.5.

(2) If $\mathcal{V}$ is K-compactly generated, admits K-transfer for (non-symmetric, constant-free) operads (Definition 3.4) , then there exists a cofibrantly generated model structure on $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper,$\mathcal{V}$-cfOper) such that fibrations and weak equivalences are the ones introduced in Definition 4.5.

(3) If $\mathcal{V}$ is compactly generated (or combinatorial with $\otimes$-perfect weak equivalences), satisfies the monoid axiom then there exists a cofibrantly generated model structure on $\mathcal{V}$-NSOper (resp. $\mathcal{V}$-cfOper) such that fibrations and weak equivalences are as in Definition 4.5.

Furthermore, in any of these cases, if $\mathcal{V}$ is combinatorial, the corresponding model structure on $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper,$\mathcal{V}$-cfOper) is combinatorial.

Proof. We pick (4.3.1) as set of generating cofibrations and (4.3.2) as set of generating trivial cofibrations.

Since the categories of $\mathcal{V}$-Oper, $\mathcal{V}$-NSOper and $\mathcal{V}$-cfOper are complete and cocomplete, to prove the existence of the model structure it is sufficient to prove (following Theorem 2.1.19 [19])) that:

(1) the class of weak equivalences has the 2-out-of-3 property and it is closed under retracts;
(2) the domains of $\mathcal{I}$ are small relative to $\mathcal{I}-$cell;
(3) the domains of $\mathcal{J}$ are small relative to $\mathcal{J}-$cell;
(4) $\mathcal{I}-\mathrm{inj} = \mathcal{W} \cap \mathcal{J}-\mathrm{inj}$
(5) $\mathcal{J}-\mathrm{cell} \subseteq \mathcal{W} \cap \mathcal{I}-\mathrm{cof}$.

(1) holds since $\mathcal{W}$ has the 2-out-of-3 property by Proposition 4.15 and it is closed under retract because local weak equivalence are closed under retracts and essentially surjectivity is closed under retracts (because it is so in $\mathcal{V}$-Cat, see proof of [4, Theorem 2.5]). Point 4 is Lemma 4.8. Points 2 and 3 follow from Lemma 4.21. Point 5 is Lemma 4.20.

The last statement is a consequence of Propostion A.5. □

Example 4.23. Model categories for which the first part of the theorem applies are, for example: simplicial sets with the Quillen model structure, (unbounded) chain complexes over a field of characteristic 0 with the projective model structure, simplicial modules over any ring (with the model structure transferred from the Quillen's one on simplicial sets) and $\mathcal{S}et$ with the model structure which has bijection as weak equivalences and surjections as fibrations. In the case of simplicial sets this was already proven in [9, Theorem 1.14] and [26]. For $\mathcal{V} = \mathcal{S}et$ one recovers the "folk" model structure on Operads (see [32]).

The category Top of (weak Hausdorff) compactly generated spaces satisfies the hypotheses of point (2) if we take $K = \mathcal{T}_1$, the class of T_1-closed inclusions.

All the categories listed above also satisfy the hypothesis of point (3). An example of model category which satisfy (3) but not (1) is the category of chain complexes over an arbitrary commutative ring with the projective model structure.

4.5. **Remark on weak equivalences.** In [26] weak equivalences are defined in a different way which is not a priori equivalent to ours:

Definition 4.24. A morphism $f\colon P \to Q$ of $\mathcal{V}$-enriched coloured operads is a *Dwyer-Kan weak equivalence* if:

(1) is a locally weak equivalence, i.e. for every $s \in \mathrm{Seq}(\mathbf{Col}(P))$ the morphism
$$f^u\colon P(s) \longrightarrow f^*Q(s)$$
is a weak equivalence in $\mathcal{V}$.
(2) The induced functor
$$\pi_0(j^*f)\colon \pi_0(j^*P) \longrightarrow \pi_0(j^*Q)$$
is essentially surjective.

We recall that for $A \in \mathcal{V}$-Cat the ordinary category $\pi_0(A) \in \mathsf{Cat}$ is the category which has as objects the objects of A and for every $x, y \in A$ set of morphisms $\pi_0(A)(x,y) = \mathrm{Ho}(\mathcal{V})(\mathbf{I}_{\mathcal{V}}, A(x,y))$; the composition in $\pi_0(A)(x,y)$ is defined in the evident way (*cf.* [4, Remark 2.7]).

It was already observed by Berger and Moerdijk that under our hypotheses (so also in the case of [26] where $\mathcal{V}$ is simplicial sets with the Kan-Quillen model structure) the two definitions coincide.

The following proposition is a straight-forward consequence of [4, Propositions 2.20,2.24]:

Proposition 4.25. *Suppose that $(\mathcal{V}, \otimes, \mathbf{I})$ is a right proper monoidal model category with a cofibrant unit admitting transfer for categories. Then the class of morphisms in $\mathcal{V}$-Oper($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper) which are essentially surjective locally weak equivalences coincides with the class of Dwyer-Kan weak equivalences.*

5. Right properness

We recall the following lemma, whose proof is trivial:

Lemma 5.1. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a right proper monoidal model category and $\mathcal{O}$ a coloured $\mathcal{V}$-operad. If the category of $\mathcal{O}$-algebras in $\mathcal{V}$ admits a transferred model structure then it is right proper.*

Taking $\mathcal{O} = \mathsf{Op}_C$ the previous lemma reads

Corollary 5.2. *If $\mathcal{V}$ is right proper and admits transfer for operads (non-symmetric operads, constant-free operads) then, for every $C \in \mathcal{S}et$ the transferred model structure on $\mathcal{V}$-Oper_C ($\mathcal{V}$-NSOper_C, $\mathcal{V}$-cfOper_C) is right proper.*

We can now prove the following:

Proposition 5.3. *Suppose that $\mathcal{V}$ satisfies the hypothesis of point (1) of Theorem 4.22. Then the model structure on $\mathcal{V}$-Oper is right proper.*

Proof. Suppose a pullback diagram is given in $\mathcal{V}$-Oper

$$\begin{tikzcd}
A \arrow[r, "w'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
B \arrow[r, "w"'] & Y
\end{tikzcd}$$

where f is a fibration and w is a weak equivalence. We have to prove that w' is a weak equivalence, i.e. it is an essentially surjective local weak equivalence.

Note that, the diagram

$$\begin{tikzcd}
A \arrow[r, "w'^u"] \arrow[d, "f'^u"'] & w'^*(X) \arrow[d, "w'^*(f^u)"] \\
f'^*(B) \arrow[r, "f'^*(w^u)"'] & (fw)^*(Y)
\end{tikzcd}$$

is a pullback diagram in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(A)}$, which is right proper by 5.2 , hence w' is a local weak equivalence.

We only have to prove that w' is essentially surjective. Suppose a colour x of X is given. Since w is essentially surjective we have an object b in B, an interval $\mathbb{H}$, a morphism $h\colon \mathbb{H} \to Y$, such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{j}_!(\mathbf{I}) \arrow[dr, "j_!(i_0)"] \arrow[dd, dashed, "a"] \arrow[ddd, bend right, "b"'] & & \mathbf{j}_!(\mathbf{I}) \arrow[dl, "j_!(i_1)"'] \arrow[dd, "x"] \\
& j_!(\mathbb{H}) \arrow[dr, dashed, "h'"] \arrow[ddr, "h"] & \\
A \arrow[rr, "w'"] \arrow[d, "f"] & & X \arrow[d, "f"] \\
B \arrow[rr, "w"] & & Y
\end{tikzcd}$$

Since $j_!(i_1)$ is a trivial cofibration and f is a fibration, h lifts to a morphism $h'\colon j_!(\mathbb{H}) \to X$ such that $h'(1) = x$ and $fh'(0) = w(b)$; from the universal property of A there is a colour $a \in \mathbf{Col}(A)$, such that $w'(a) = h'(0)$. Since x was chosen arbitrarily, this proves that w' is essentially surjective. $\square$

In an analogous way we can prove the same holds for the non-symmetric and the constant-free case:

Proposition 5.4. *Suppose that $\mathcal{V}$ satisfies the hypothesis of point (3) of Theorem 4.22. Then the model structures on $\mathcal{V}$-NSOper, and $\mathcal{V}$-cfOper are right proper.*

6. (Relative) Left properness

Given a set C and a left proper monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ admitting transfer for operads it is not true in general that the transferred model structures on $\mathcal{V}$-Oper_C, $\mathcal{V}$-NSOper_C or C-$\mathsf{cfOper}_{\mathcal{V}}$ are also left proper. Nevertheless, Batanin and Berger ([2]) found stronger conditions on $\mathcal{V}$ under which C-$\mathsf{NSOper}_{\mathcal{V}}$ or C-$\mathsf{cfOper}_{\mathcal{V}}$ are left proper. In this section we show how these results can be extended to the canonical model structure over non-symmetric and constant free operads.

Furthermore, in the cases in which it is not possible to prove that our model structure on (symmetric, non-symmmetric, constant-free) operads is left proper, we will try to isolate certain classes of weak equivalences which are closed under push-out along cofibrations.

Definition 6.1. Given a model category $\mathcal{V}$ a morphism $u\colon A \to B$ is a *h-cofibration* if for each weak equivalence $g\colon R \to S$ and each diagram of push-outs

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"] & R \arrow[r, "g"] \arrow[d] & S \arrow[d] \\ B \arrow[r] & R' \arrow[r, "g'"] & S' \end{tikzcd}$$

g' is a weak equivalence.

A monoidal model category $(\mathcal{V}, \otimes, \mathbf{I})$ is *h-monoidal* if for each (trivial) cofibration $f\colon X \to Y$ and each object Z, the morphism $f \otimes \mathrm{id}_Z$ is a (trivial) h-cofibration. A *strongly h-monoidal* model category is an h-monoidal model category in which the class of weak equivalences is closed under tensor product.

Remark 6.2. Every h-monoidal model category is left proper and every monoidal model category in which all objects are cofibrant is strongly h-monoidal; we refer the reader to [2] for examples of h-monoidal model categories.

Definition 6.3. Let T be admissible monad over a model category $\mathcal{V}$ with forgetful functor $U_T\colon \mathrm{Alg}_T(\mathcal{V}) \to \mathcal{V}$ the induced model structure over $\mathrm{Alg}_T(\mathcal{V})$ is called *relatively left proper* if the class of weak equivalences $f\colon X \to Y$ such that $U_T(X)$ and $U_T(Y)$ are cofibrant in $\mathcal{V}$ is closed under cobase change along cofibrations.

Theorem 6.4 ([2])**.** *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a compactly generated monoidal model category and let T be a tame polynomial monad over $\mathcal{V}$*

1. *if $\mathcal{V}$ is h-monoidal the induced model structure on $\mathrm{Alg}_T(\mathcal{V})$ is relatively left proper;*
2. *if $\mathcal{V}$ is strongly h-monoidal the induced model structure on $\mathrm{Alg}_T(\mathcal{V})$ is left proper.*

Lemma 6.5. *Let $i\colon K_0 \to K_1$ be a morphism in $\mathcal{V}$ and let*

$$\begin{tikzcd} C_n(K_0) \arrow[r, "f"] \arrow[d, "C_n(i)"'] & X \arrow[d, "j"] \\ C_n(K_1) \arrow[r, "g"'] & Y \end{tikzcd}$$

be a push-out square in $\mathcal{V}$-Oper *(*$\mathcal{V}$-NSOper,$\mathcal{V}$-cfOper*). Suppose that there exist a set* B *and functions* $k\colon [n+1] \to B$, $h\colon [n+1] \to \mathbf{Col}(X)$ *such that the underlying map of sets* $f\colon [n+1] \to \mathbf{Col}(X)$ *factors as* $f = hk$ *for some* $k\colon [n+1] \to B$*, then the induced diagram*

$$\begin{tikzcd} C_n(K_0) \arrow[r, "f^u"] \arrow[d, "C_n(i)"'] & h^*(X) \arrow[d, "j"] \\ C_n(K_1) \arrow[r, "g^u"'] & h^*(Y) \end{tikzcd} \tag{6.0.1}$$

is a push-out square.

Proof. Recall from Section 3.2 that $s_n = (1, \ldots, n; 0)$ and $C_n = F_{\mathsf{Op}_{[n]}} \iota_{s_n}$.

Let Y' be the push-out of the left-upper corner of (6.0.1); Y' has B as set of colours. There is a canonical map $p\colon Y' \to h^*(Y)$ which is the identity on the colours.

According to formula (C.1.4), for every $S \in \mathrm{Seq}(B)$

$$Y'(S) = \varinjlim_{T \in F\underline{\mathbf{T}}(B)_S^{\mathrm{op}}} Z_{k_! \iota_{s_n}(i), f^u}(T)$$

where $F\underline{\mathbf{T}}(B)_s$ is the category of *minimal* free-push-out trees with edges labeled by B, arity S and vertices marked by X (more precisely $h^*(X)$),K_0 or K_1.

Let $F\underline{\mathbf{T}}(B)_S^*$ be the subcategory of $F\underline{\mathbf{T}}(B)_S$ spanned by all the trees T such that for every vertex v of T marked by K_0 or K_1 the arity of v is $k(s_n)$. Since $k_! \iota_{s_n}(i)$ is trivial on entries with arity different from $k(s_n)$, the trees in $F\underline{\mathbf{T}}(B)_S^*$ are the relevant ones for the computation of $Y'(S)$, that is:

$$Y'(S) = \varinjlim_{T \in F\underline{\mathbf{T}}(B)_S^{*\mathrm{op}}} Z_{k_! \iota_{s_n}(i), f^u}(T).$$

Note that the labeling of a free-pushout tree in $F\underline{\mathbf{T}}(B)_S^*$ is completely determined by the underlying ordered tree, in fact

- the marking on the vertices is fully determined by the fact that the tree is minimal (Definition C.1);
- the label of an edge which belongs to a vertex v marked by K_0 or K_1 is determined by the requirement that $a(v) = k(s_n)$;
- each inner edge belongs to one vertex marked by K_0 or K_1 (because the tree is minimal);
- the labels on the outer edges are determined by the requirement that $a(T) = S$.

We have a similar expression for $h^*(Y)$:

$$h^*(Y)(S) = Y(h(S)) = \varinjlim_{T \in F\underline{\mathbf{T}}(C)_{h(S)}^{\mathrm{op}}} Z_{f_! \iota_{s_n}(i), f^u}(T)$$

(note that in this expression f^u stands for the map $f^u\colon C_n(K_0) \to f^*(X)$ and not for the map in diagram (6.0.1)).

Denote by C the set of objects of X (and Y). Let $F\underline{\mathbf{T}}(C)_{h(S)}^*$ be the subcategory of $F\underline{\mathbf{T}}(C)_{h(S)}$ spanned by all the trees T such that for every vertex v of T marked by K_0 or K_1 the arity of v is $f(s_n)$.

As in the previous case one can restrict himself to $F\underline{\mathbf{T}}(C)_{h(S)}^*$ in order to compute $Y(h(S))$:

$$Y(h(S)) = \varinjlim_{T \in F\underline{\mathbf{T}}(C)_{h(S)}^{*\mathrm{op}}} Z_{f_! \iota_{s_n}(i), f^u}(T).$$

Also in this case the labeling of a free-pushout tree in $F\underline{\mathbf{T}}(C)_{h(S)}^*$ is determined by the underlying ordered tree.

There is an obvious functor $\tilde{h}\colon F\underline{\mathbf{T}}(B)_S \to F\underline{\mathbf{T}}(C)_{h(S)}$ obtained by applying h to the labeling of the trees.

There is a natural isomorphism $\eta\colon Z_{k_!\iota_{s_n}(i),f^u} \to Z_{f_!\iota_{s_n}(i),f^u}\tilde{h}$; the induced morphism $\varinjlim \eta\colon Y'(S) \to h^*(Y)(S)$ is isomorphic to the S-component of p. The restriction of $\tilde{h}$ induces an isomorphism of categories between $F\underline{\mathbf{T}}(B)^*_S$ and $F\underline{\mathbf{T}}(C)^*_{h(S)}$, it follows that the morphism η (and hence $p(S)$) is an isomorphism. Since S was chosen in an arbitrary way, this proves that p is an isomorphism. □

Now we can prove the following (*cf.* [23]):

Theorem 6.6. *Suppose that $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category satisfying the hypothesis of Theorem 4.22 (3) and consider $\mathcal{V}$-NSOper ($\mathcal{V}$-cfOper) with the canonical model structure.*

(1) If $\mathcal{V}$ is h-monoidal, the class of weak equivalences in $\mathcal{V}$-NSOper ($\mathcal{V}$-cfOper) with locally cofibrant domain and codomain is closed under push-outs along cofibrations;
(2) if $\mathcal{V}$ is strongly h-monoidal, $\mathcal{V}$-NSOper ($\mathcal{V}$-cfOper) is left proper.

Proof. We have to check that, given a cofibration ϕ, a weak equivalence f and a push-out diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "\phi'"'] & Y, \end{tikzcd}$$

the morphism g is a weak equivalence; in case (1) we suppose that A and X are locally cofibrant.

It is immediate to check that g is essentially surjective (if a colour of Y is not in the image of g then it is in the one of ϕ' and the rest follows from the fact that f is essentially surjective).

Thus we just have to prove that g is a local weak equivalence. Since transfinite compositions can be computed in $\mathcal{V}$-MultiGraph (Lemma A.1), a cofibration in $\mathcal{V}$-NSOper is a local $\otimes$-cofibration (Lemma 4.19) and $\mathcal{V}$ is compactly generated, it is sufficient to check that the statement holds in the case in which ϕ is the push-out of a generating cofibration $l \in I_{\mathrm{loc}} \cup \{j_!(\emptyset \to \mathbf{1})\}$.

If l is $j_!(\emptyset \to \mathbf{1})$ the statement is trivially true: Y and B are just obtained from X and Y adding one colour.

Suppose that l is equal to $C_n(i)$ for some $i \in I$.

$$\begin{tikzcd} C_n(K_0) \arrow[r, "C_n(i)"] \arrow[d, "a"] & C_n(K_1) \arrow[d, "b"] \\ A \arrow[r, "\phi"] \arrow[d, "f"] & B \arrow[d, "g"] \\ X \arrow[r, "\phi'"] & Y. \end{tikzcd}$$

We can decompose the lower square as

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f^u"] & B \arrow[d, "g^u"] \\ f^*(X) \arrow[r, "f^*(\phi)"] \arrow[d, "\eta_f"] & f^*(Y) \arrow[d, "\eta_g"] \\ X \arrow[r, "\phi'"] & Y. \end{tikzcd}$$

The pasting lemma for push-outs and Lemma 6.5 imply that the upper square is a push-out.

The morphism η_g is fully faithful, thus it is sufficient to prove that g^u is a local weak equivalence. The upper square can be thought as a push-out square in $\mathcal{V}\text{-}\mathsf{NSOper}_C$ ($\mathcal{V}\text{-}\mathsf{cfOper}_C$) for $C = \mathbf{Col}(A) = \mathbf{Col}(B)$. Since ϕ is a cofibration in $\mathcal{V}\text{-}\mathsf{NSOper}_C$ ($\mathcal{V}\text{-}\mathsf{cfOper}_C$), f^u is a weak equivalence and $\mathcal{V}\text{-}\mathsf{NSOper}_C$ ($\mathcal{V}\text{-}\mathsf{cfOper}_C$) is (relatively) left proper, it follows that g^u is a weak equivalence and hence g is a local weak equivalence. □

6.0.1. *C-Collections.* Let Σ be the subcategory of $\mathcal{S}et$ whose set of objects is $\{[n] \mid n \in \mathbb{N}\}$ and whose morphisms are all the bijection between these sets. For every $C \in \mathcal{S}et$ let $\Sigma_C = (\Sigma/C) \times C$ the category of *symmetric C-signature*; the objects of this category are just triples $([n], f, d)$ where $n \in \mathbb{N}$, f is a map from $[n]$ to C and $d \in C$; there is thus a bijection between $\mathbf{Ob}(\Sigma_C)$ and $\mathrm{Seq}(C)$: each object $([n], f, d)$ is identified with the signature $(f(1), \ldots, f(n); d)$. A morphism $\sigma\colon ([n], f, d) \to ([m], g, d)$ is just a bijection $\sigma\colon [n] \to [m]$ such that $f = \sigma g$; if we identify objects with signatures, σ has the following form:

$$\sigma\colon (f(1), \ldots, f(n); d) \to (f(\sigma^{-1}(1)), \ldots, f(\sigma^{-1}(n)); d).$$

For every $C \in \mathcal{S}et$ let $\mathrm{Coll}_C(\mathcal{V}) = \mathcal{V}^{\Sigma_C^{\mathrm{op}}}$ be the category of *C-collections in $\mathcal{V}$*.

In other words a C-collection in $\mathcal{V}$ is an object $L \in \mathcal{V}^{\mathrm{Seq}(C)}$ together with a right action

$$\sigma^*\colon L(c_1, \ldots, c_n; c) \longrightarrow L(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)$$

for every $S = (c_1, \ldots, c_n; c) \in \mathrm{Seq}(C)$ and every permutation $\sigma \in \Sigma_n$ and a morphism of collections is just a morphism in $\mathcal{V}^{\mathrm{Seq}(C)}$ respecting these actions.

If $\mathcal{V}$ is cofibrantly generated model structure $\mathrm{Coll}_{\mathcal{V}}(C)$ admits a transferred model structure in which a morphism is a weak equivalence (fibration) if and only if the underlying morphism in $\mathcal{V}^{\mathrm{Seq}(C)}$ is a weak equivalence (fibration). Cofibrant C-collections always have cofibrant underlying objects in $\mathcal{V}^{\mathrm{Seq}(C)}$.

Every C-coloured operad has an underlying *C-collection.* If $\mathcal{V}$ admits transfer for operads then the free-forgetful adjunction

$$F^{\Sigma}_{\mathsf{Op}_C}\colon \mathrm{Coll}_C(\mathcal{V}) \rightleftarrows \mathcal{V}\text{-}\mathsf{Oper}_C\colon U^{\Sigma}_{\mathsf{Op}_C} \tag{6.0.2}$$

is a Quillen adjunction.

In this situation a $\mathcal{V}$-coloured operad P is said to be Σ-cofibrant if the underlying $\mathbf{Col}(P)$-collection $U^{\Sigma}_{\mathsf{Op}_C}(P)$ is cofibrant.

The following partial result holds for symmetric operads:

Theorem 6.7. *Suppose $(\mathcal{V}, \otimes, \mathbf{I})$ is a cofibrantly generated monoidal model category for which the canonical model structure over $\mathcal{V}$-Oper exists. Then the class of weak equivalences between Σ-cofibrant operads is closed under push-outs along cofibrations.*

Proof. The proof goes as in Theorem 6.6 using Proposition C.11 and the fact that weak equivalences in $\mathcal{V}$-Oper are closed under filtered colimits along local cofibrations with local cofibrant domain. □

6.1. **A counter-example for the left-properness of** sSets-Oper_***.** For completeness, and also to shed some light on why the left proper condition is true for sSets-cfOper_C but not for sSets-Oper_C, we present a counter-example that shows that the projective model structure on sSets-Oper_* is not left proper; here the category of simplicial set sSets is taken with the Kan-Quillen model structure. A similar counter-example due to Dwyer is presented in [15, Section 4].

As usual when talking about operads with one colour, we are going to identify $\mathrm{Seq}(*)$ with $\mathbb{N}$ (every signature is identified with the corresponding valence).

Let $\mathbb{T}$ be the maximal subgroupoid of $(\Omega_*^{\mathrm{ord}})^{\mathrm{op}}$ (Definition B.9). In other words $\mathbb{T}$ is the category of ordered trees and non-planar (ordered) isomorphism between them. Let $\mathbb{T}_n$ be the subset of $\mathbb{T}$ (resp. $\mathbb{T}^0$) spanned by the trees of arity n.

The free operad on a collection $A \in \mathrm{Coll}_*(\mathcal{V})$ has the following description: for every s in $\mathrm{Seq}(*) \cong \mathbb{N}$

$$F^{\Sigma}_{\mathsf{Op}_C}(A)(s) = \varinjlim_{T\in\mathbb{T}_n} \bigotimes_{v\in\mathrm{vert}(T)} A(a(v)) \cong \coprod_{[T]\in\pi_0(\mathbb{T}_n)} \varinjlim_{\mathrm{Aut}(T)} \bigotimes_{v\in\mathrm{vert}(T)} A(a(v)) \tag{6.1.1}$$

where $\pi_0(\mathbb{T}_n)$ is the set of path components of $\mathbb{T}_n$ and $\mathrm{Aut}(T)$ is the set of automorphism of T in $\mathbb{T}$.

An *ordered tree with-out stumps* is an ordered tree $T \in \mathbb{T}$ with no vertices of valence 0; note that the property of being with-out stumps is invariant under isomorphisms.

Proposition 6.8. *If T is a tree with-out stumps then* $\mathrm{Aut}(T) = \{\mathrm{id}_T\}$ *(in other words the path-component of $\mathbb{T}$ containing T is simply connected).*

Proof. Recall that, given an ordered tree T, an automorphism $f\colon T \to T$ in Ω_*^{ord} is an automorphism of the free-operad on the underlying tree $f\colon \Omega(T) \to \Omega(T)$ which respect the order τ_T on the leaves (*cf.* Definition B.9).

Since $\Omega(T)$ is the free edge(T)-coloured operad generated by the operations $(t_v, \lambda_v) \in \Sigma(T)(a(v))$ for every $v \in \mathrm{vert}(v)$ it will be enough to show that f is the identity on colours and $f((t_v, \lambda_v)) = (t_v, \lambda_v)$ for every $v \in \mathrm{vert}(T)$.

First we are going to show that f is the identity on the colours. Set

$$E = \{e \in \mathrm{edge}(T) \mid f(e) = e\}.$$

First observe that for every vertex $v \in \mathrm{vert}(T)$ if $\mathrm{in}(t_v) \subset E$ then $r(t_v) \subset E$, that is if the inputs of v are in E then the output of v is in E and $f((t_v, \lambda_v)) = (t_v, \lambda_v)$, thus $f(r(v)) = r(v)$; in fact there is only one operation of $\Omega(T)$ whose arity has input equal to $a(v)$, namely (t_v, λ_v), so $f((t_v, \lambda_v)) = (t_v, \lambda_v)$, thus $f(r(v)) = r(v)$.

It follows that, to prove our proposition it will be enough to prove that $E = \mathrm{edge}(T)$.

First notice that $\mathrm{in}(T) \subset E$ since $\tau_T f = \tau_T$ forces f to be the identity on the leaves.

Given $e \in \mathrm{edge}(T)$, a *branch* from e is a path that has e and a leave of T as extremes and that does not contain two inputs of the same vertex (that means that the path is going "up" from e to one of the leaves). Since T is with-out stumps, a branch from e exists for every $e \in \mathrm{edge}(T)$. For every $e \in \mathrm{edge}(T)$ let n_e be the maximal length of a branch from e.

To show that $e \in E$ for every $e \in \mathrm{edge}(T)$ We are going to reason by induction on n_e. If $n_e = 0$ then e is a leaves, thus $e \in E$. For the inductive step take $e \in \mathrm{edge}(T)$ such that $n_e = m > 0$ and suppose that $n_d < m$ implies that $d \in E$ for every $d \in \mathrm{edge}(T)$.

Since n_e is positive e is a root of a vertex v and $n_i < m$ for every input $i \in \mathrm{in}(v)$; it follows that e is in E. □

Therefore if $A \in \mathrm{Coll}_*(\mathrm{sSets})$ such that $A(0) \cong \emptyset$ formula 6.1.1 becomes

$$F^{\Sigma}_{\mathsf{Op}_*}(A)(s) \cong \coprod_{[T]\in\pi_0(\mathbb{T}^0_s)} \bigotimes_{v\in\mathrm{vert}(T)} A(a(v)) \tag{6.1.2}$$

where $\mathbb{T}^0_s$ is the full subcategory of $\mathbb{T}_s$ spanned by the trees with-out stumps.

Corollary 6.9. *Suppose $A, B \in \mathrm{Coll}_*(\mathrm{sSets})$ and $A(0) \cong B(0) \cong \emptyset$. If $f\colon A \longrightarrow B$ is a weak equivalence then $F^{\Sigma}_{\mathsf{Op}_*}(f)$ is a weak equivalence.*

Proof. From formula (6.1.2) it follows that

$$F^{\Sigma}_{\mathsf{Op}_*} f(n) \cong \coprod_{[T]\in\pi_0(\mathbb{T}^0_s)} (\prod_{v\in\mathrm{vert}(T)} f(a(v))).$$

Since weak equivalences in simplicial sets are stable under products and coproducts, the statement follows. □

The important point is that tree with stumps can have non-trivial isomorphisms; for example the tree represented in Figure 1 has automorphism group isomorphic to Σ_2.

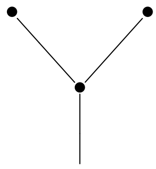


FIGURE 1. The stumped corolla S_2

For every $n \in \mathbb{N}$ let ι_n be the left adjoint to the n-projection $p_n \colon \mathrm{Coll}_*(\mathrm{sSets}) \longrightarrow \mathrm{sSets}^{\Sigma_n}$.

Corollary 6.10. *For every $n \geq 1$ the (left Quillen) functor $F^{\Sigma}_{\mathsf{Op}_*}\iota_n$ preserves weak equivalences*

Proof. The functor ι_n always preserves weak equivalences. $F\iota_n(A)(0) \cong \emptyset$ for every $n \geq 1$ and $A \in \mathrm{sSets}^{\Sigma_n}$. Therefore the statement follows from Corollary 6.9. □

We are now ready to describe our counter-example.

Consider the weak equivalence $j \colon E(\Sigma_2) \longrightarrow *$ in $\mathrm{sSets}^{\Sigma_2}$ in $\mathrm{sSets}^{\Sigma^2}$, where $E(\Sigma_2)$ is the total space of the universal Σ_2-bundle with the usual Σ_2-action and $*$ is a point (with the trivial Σ_2-action).

Let $I_0 = F^{\Sigma}_{\mathsf{Op}_*}\iota_0(*)$ be the operad with only one operation of arity 0. More explicitly $I_0(0) \cong *$ and $I_0(n) \cong \emptyset$ for $n \neq 0$. I_0 is a cofibrant operad (because both $F^{\Sigma}_{\mathsf{Op}_*}$ and ι_0 are left Quillen functor).

Consider the following push-out in sSets-Oper_*

$$\begin{tikzcd}
F^{\Sigma}_{\mathsf{Op}_*}\iota_2(E(\Sigma_2)) \arrow[r, "F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)"] \arrow[d] & F^{\Sigma}_{\mathsf{Op}_*}\iota_2(*) \arrow[d] \\
F^{\Sigma}_{\mathsf{Op}_*}\iota_2(E(\Sigma_2)) \sqcup I_0 \arrow[r, "F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)\sqcup I_0"] & F^{\Sigma}_{\mathsf{Op}_*}\iota_2(*) \sqcup I_0
\end{tikzcd}$$

The vertical maps (which are the canonical maps of the two coproducts) are cofibrations, and the top map is a weak equivalence by Corollary 6.9 hence if sSets-Oper_* would be left proper, $F\iota_2(j) \sqcup I_0$ should be a weak equivalence; we are going to show that this is not the case.

Let $\mathbb{S}$ be the full subcategory of $\mathbb{T}$ spanned by the stumped corollas (Remark 7.3).

Let $\mathbb{S}_n$ be the subcategory of $\mathbb{S}$ spanned by the elements of arity n.

For every $T \in \mathbb{S}$ let $\mathsf{st}(T)$ be the set of vertex of T of arity 0 and let us denote by e_T the unique vertices that is not a stump (if it exists).

If P is a sSets-operad such that $P(0) \cong \emptyset$ we have the following description of the coproduct $P \sqcup I_0$ (natural in P):

$$(P \sqcup I_0)(n) \cong \varinjlim_{T \in \mathbb{S}_n} \big(\big(\prod_{u \in \mathsf{st}(T)} *\big) \times P(e_T)\big) \cong \varinjlim_{T \in \mathbb{S}_n} P(e_T); \tag{6.1.3}$$

the previous formula can be obtained from the description of push-out of operads given in Appendix B. It follows that

$$(F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j) \sqcup I_0)(0) \cong \varinjlim_{T \in \mathbb{S}_0} (F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)(e_T)) \cong$$

$$\cong \varinjlim_{\mathrm{Aut}(S_2)} (F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)(e_{S_2})) \cong \varinjlim_{\Sigma_2} (F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)(2)),$$

where S_2 is the unique stumped corolla with arity 0 and one binary vertex. Thus

$$\varinjlim_{\Sigma_2} (F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j)(2)) \colon E(\Sigma_2)/\Sigma_2 \longrightarrow *$$

is not a weak equivalence of simplicial sets (since $E(\Sigma_2)/\Sigma_2 \simeq K(\Sigma_2, 1)$). It follows that $(F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j) \sqcup I_0)(0)$ (and thus $F^{\Sigma}_{\mathsf{Op}_*}\iota_2(j) \sqcup I_0$) is not a weak equivalence as well.

7. Unitary Operads

Fix a closed symmetric monoidal bicomplete category $(\mathcal{V}, \otimes, \mathbf{I})$ and a set C.

Definition 7.1. A C-coloured $\mathcal{V}$-operads P is *unitary* if $P(\emptyset; c) \simeq \mathbf{I}$ for every $c \in C$.

Let $\mathcal{V}$-UOper ($\mathcal{V}$-UOper$_C$) be the full subcategory of $\mathcal{V}$-Oper ($\mathcal{V}$-Oper_C) spanned by the unitary operads. Not surprisingly, the colour functor $\mathbf{Cl} \colon \mathcal{V}\text{-UOper} \to \mathcal{S}et$ is a bifibration, as we will see.

In this section we would like to show that under the same hypotheses on $\mathcal{V}$ of point (3) Theorem 4.22, the category $\mathcal{V}$-UOper admits the canonical model structure.

The category $\mathcal{V}$-UOper$_C$ has a final element Com_C such that $\mathsf{Com}_C(s) = \mathbf{I}$ for every $s \in \mathrm{Seq}(C)$; all the composition morphisms and symmetries of Com_C are isomorphic to the identity of $\mathbf{I}$. The one-coloured operad Com_* will be simply denoted by Com (it is, in fact the commutative monoids operad).

Depending on the context, we will also denote by Com_C the constant-free operad underlying Com_C or the reduced C-sequence underlying Com_C. Given a unitary operad $P \in \mathcal{V}\text{-UOper}_C$ and $s = (s_1, \ldots, s_n; s_0) \in \mathrm{Seq}(C)$ the unique morphism $c \colon P \to \mathsf{Com}_C$ is defined on the s-component as

$$P(s) \xrightarrow{\simeq} P(s) \otimes P(\emptyset; s_1) \otimes \cdots \otimes P(\emptyset; s_n) \xrightarrow{\Gamma_t} P(\emptyset; s_0) \simeq \mathbf{I}$$

where Γ_t is the composition morphism of Remark B.11 and t is the unique C-tree of arity $(\emptyset; s)$, with one vertex of arity s and all the other vertices of valence 0.

There is an obvious forgetful functor $U_{\mathsf{UOp}_C} \colon \mathcal{V}\text{-UOper}_C \to \mathcal{V}^{\mathrm{Seq}_0(C)}/\mathsf{Com}_C$.

One issue that stopped us from treating the case of unitary operads on the same foot of the non-symmetric and constant-free cases is that for every $C \in \mathcal{S}et$ there is no operad (thus no polynomial monad) whose category of algebras in $\mathcal{V}$ is $\mathcal{V}$-UOper$_C$ for every symmetric monoidal category $\mathcal{V}$.

This is really a minor issue since $\mathcal{V}$-UOper$_C$ is still the category of algebras for a finitary monad over $\mathrm{Seq}_0(C)/\mathsf{Com}_C$ so we are brought back to a similar setting as before.

Another important fact pointed out by Fresse [11, Part I, Ch. 3], that will be better explained in the next sections, is that unitary operad can be regarded as constant-free operad augmented over Com with extra structure (and some compatibility conditions); once the appropriate adjunctions are fixed, we will transfer the canonical model structure from $\mathcal{V}$-cfOper to $\mathcal{V}$-UOper.

7.1. **Slice categories and bifibrations.** Since we will make use of slice categories in the following sections we recall here some basic facts about them.

For every category $\mathcal{V}$ and every object $A \in \mathcal{V}$ let $\mathcal{V}/A$ be the slice category of objects over A; there is an obvious forgetful functor$\omega_A \colon \mathcal{V}/A \to \mathcal{V}$.

If $\mathcal{V}$ is a model category and $A \in \mathcal{V}$ then $\mathcal{V}/A$ admits a model structure in which a morphism f is a weak equivalence (fibration, cofibration) if and only if $\omega_A(f)$ is a weak equivalence (fibration, cofibration) in $\mathcal{V}$. If $\mathcal{V}$ is cofibrantly generated with set of generating (trivial) cofibrations I (J) then $\mathcal{V}/A$ is also cofibrantly generated with set of generating (trivial) cofibrations $I/A = \{f \mid \omega_A(f) \in I\}$ $(J/A = \{f \mid \omega_A(f) \in J\})$.

If $L \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon R$ is an adjunction of categories and $A \in \mathcal{D}$ then there is an induced adjunction

$$L/A \colon \mathcal{C}/RA \rightleftarrows \mathcal{D}/A \colon R/A$$

such that $R/A(f) = R(f)$ for every $f \in \mathcal{D}/A$. For every object $g \colon C \to RA$ of $\mathcal{C}/RA$ the object $L/A(g)$ is $\bar{g} \colon LC \to A$, the adjoint of g.

Suppose we have a bifibration $F \colon \mathcal{E} \to \mathcal{B}$ let $X \in \mathcal{E}$, then $F/X \colon \mathcal{E}/X \to \mathcal{B}/F(X)$ is a bifibration; for every object $g \colon B \to F(X)$ in $\mathcal{B}/F(X)$ the fiber of F/X over g is isomorphic to $E_B/g^*(X)$.

For example if the bifibration considered is $\mathbf{Ob} \colon \mathcal{V}\text{-}\mathsf{cfOper} \to \mathcal{S}et$ and Com is the commutative operad then $Ob/\mathsf{Com} \colon \mathcal{V}\text{-}\mathsf{Oper}/\mathsf{Com} \to \mathcal{S}et$ is a bifibration with fiber $\mathcal{V}\text{-}\mathsf{cfOper}_C/\mathsf{Com}_C$ for every $C \in Set$.

7.2. **Λ-sequences.** As we said at the beginning of this section, unitary operads can be seen as constant-free operads with additional structure, in particular their underlying collection has the structure of a Λ-sequence (*cf.* [11]), that we are now going to define. From now on we will assume that $(\mathcal{V}, \otimes, \mathbf{I})$ is a monoidal model category.

Let Λ be the subcategory of $\mathcal{S}et$ whose set of objects is $\{[n] \mid n \in \mathbb{N}\backslash\{0\}\}$ and whose morphisms are all the injective maps between these sets. Let Λ_+ be the subcategory of Λ with the same set of objects spanned by all the strictly monotone maps (respect to the usual linear order on $[n]$).

For every $C \in \mathcal{S}et$ let $\Lambda_C \simeq (\Lambda/C) \times C$ and $\Lambda_{+C} \simeq (\Lambda_+/C) \times C$ (where C is seen both as a set and a discrete category).

The set of objects of Λ_C and Λ_{+C} can be identified with $\mathrm{Seq}_0(C)$.

Let $\Lambda_C\mathrm{Sq}(\mathcal{V}) \simeq \mathcal{V}^{\Lambda_C^{\mathrm{op}}}$ and $\Lambda_{C+}\mathrm{Sq}(\mathcal{V}) \simeq \mathcal{V}^{\Lambda_{+C}^{\mathrm{op}}}$ be the categories of *symmetric* and *non-symmetric* Λ_C*-sequences.*

Let Σ_{0C} be the full subcategory Σ_C (section 6.0.1) spanned by the reduced signature $\mathrm{Seq}_0(C)$ and let $\mathrm{Coll}_{0C}(\mathcal{V})$ be the category $\mathcal{V}^{\Sigma_{0C}^{\mathrm{op}}}$; if we regard the set $\mathrm{Seq}_0(C)$ as a discrete category there is an obvious commutative diagram of inclusions:

$$\begin{tikzcd} \Sigma_{0C}^{\mathrm{op}} \arrow[r, "m"] & \Lambda_C^{\mathrm{op}} \\ \mathrm{Seq}_0(C) \arrow[u, "s"] \arrow[r, "l"] & \Lambda_{+C}^{\mathrm{op}} \arrow[u, "r"'] \end{tikzcd}$$

which induces a commutative diagram of adjunctions

(7.2.1)
$$\begin{tikzcd} \mathrm{Coll}_{0C}(\mathcal{V}) \arrow[r, shift left, "m_!"] \arrow[d, shift left, "s^*"] & \Lambda_C\mathrm{Sq}(\mathcal{V}) \arrow[l, shift left, "m^*"] \arrow[d, shift left, "r^*"] \\ \mathcal{V}^{\mathrm{Seq}_0(C)} \arrow[u, shift left, "s_!"] \arrow[r, shift left, "l_!"] & \Lambda_{C+}\mathrm{Sq}(\mathcal{V}) \arrow[l, shift left, "l^*"] \arrow[u, shift left, "r_!"] \end{tikzcd}$$

where the upper-star functors, i.e. the obvious forgetful functors, are the right adjoints.

Lemma 7.2. *In diagram (7.2.1) the mate $\eta\colon s_! l^* \Longrightarrow m^* r_!$ is a natural isomorphism.*

Proof. Both $s_!$ and $r_!$ are defined by left Kan extension along s and r respectively.

More explicitly for every $A \in \mathcal{V}^{\mathrm{Seq}_0(C)}$ and every $t \in \mathrm{Seq}_0(C)$ with valence n

$$s_! A(t) \cong \coprod_{\sigma \in \Sigma_n} A(\sigma t) \tag{7.2.2}$$

Note that every morphism in Λ_C can be factored as a bijection followed by an order preserving map (i.e. a map in Λ_{+C}) in a unique way; as a consequence of this fact $r_!$ admits the following simple expression: for every $B \in \Lambda_{\mathcal{V}+}\mathrm{Sq}(C)$ and every $t \in \mathrm{Seq}_0(C)$

$$r_! B(t) = \coprod_{\sigma \in \Sigma_n} B(\sigma(t)) \tag{7.2.3}$$

where n is the valence of t.

Expressions (7.2.2) and (7.2.3) show that for every $B \in \Lambda_{\mathcal{V}+}\mathrm{Sq}(C)$

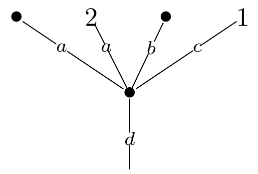
$$\eta_B\colon s_! l^* B \longrightarrow m^* r_! B$$

is an isomorphism. $\square$

The previous lemma is basically saying that given a Λ_+-sequence X, the underlying symmetric collection of the free Λ-sequence $r_!(X)$ is just the free symmetric collection over (the underlying collection of) X.

Remark 7.3. A *stumped C-corolla* is a C-tree (in the sense of Appendix B) with at most one vertex with non-empty input. An *ordered stumped C-corolla* is an ordered C-tree whose underlying C-tree is a stumped C-corolla, that is a stumped corolla with a linear order on the leaves.

For every $s, t \in \Lambda_C$, each morphism $f\colon s \to t$ is uniquely determined by an ordered stumped C-corollas T_f with a non-empty set of leaves, such that $a(T) = s$ and $a(v) = t$ and $f(u_i) = v_j$ if and only if they determine the same edge in T_f. For example if $t = (a, a, b, c; d)$, $s = (c, a; d)$ and f is such that $f(1) = 4$, $f(2) = 2$ then T_f is the following tree



For every unitary C-coloured $\mathcal{V}$-operad P the underlying reduced C-sequence $\{P(s)\}_{s \in \mathrm{Seq}_0(C)}$ inherits a Λ_C-sequences structure defined in the following way: for every morphism $f\colon (s_1, \dots, s_n; d) \to (t_1, \dots, t_m; d)$ in Λ_C the associated map $f^*\colon P(t) \to P(s)$ is the composition

$$P(t) \xrightarrow{\ \simeq\ } P(t) \otimes \bigotimes_{i \in K_f} P(\emptyset; t_i) \xrightarrow{\ \Gamma_{T_f}\ } P(s)$$

where $K_f = [m] - f([n])$ is the complement of the image of f (as a map in $\mathcal{S}et$) and Γ_{T_f} is the composition morphism of P associated to T_f in Remark B.11. This construction is cleary functorial, that is there is a "forgetful" functor

$$u^*\colon \mathcal{V}\text{-UOper}_C \longrightarrow \Lambda_{\mathcal{V}}\mathrm{Sq}(C)$$

that associates to each unitary operads its underlying reduced collection (with the Λ-structure previously defined). Furthermore, since every unitary C-coloured operads has a unique morphism towards Com_C, u^* restricts to a functor

$$u^*\colon \mathcal{V}\text{-UOper}_C \longrightarrow \Lambda_{\mathcal{V}}\mathrm{Sq}(C)/\mathsf{Com}_C$$

where $\Lambda_C\mathrm{Sq}(\mathcal{V})/\mathsf{Com}_C$ is the category of Λ-*sequences augmented over* Com_C.

The obvious inclusion functor $l\colon \mathrm{Seq}_0(C)\to\Lambda_{C+}$ induces a restriction functor

$$l^*\colon \Lambda_{C+}\mathrm{Sq}(\mathcal{V}) \longrightarrow \mathcal{V}^{\mathrm{Seq}_0(C)}$$

which has a left adjoint $l_!$ and a right adjoint l_*.

For every $A\in\mathcal{V}^{\mathrm{Seq}_0(C)}$ the object $l_!(A)$ is the left Kan extension of A along l^{op} therefore

$$l_!(A)(s)\simeq \coprod_{t\in\mathrm{Seq}_0(C)}\ \coprod_{\Lambda_{+C}(s,t)} A(t)\simeq \coprod_{T\in\mathbb{S}_s} A(a(v_T)) \tag{7.2.4}$$

where $\mathbb{S}_s$ is the set of stumped C-corollas with arity s.

Proposition 7.4. *Let $\mathcal{V}$ be a cofibrantly generated model category. The adjunction*

$$l_!\colon \mathcal{V}^{\mathrm{Seq}_0(C)} \rightleftarrows \Lambda_{C+}\mathrm{Sq}(\mathcal{V})\colon l^*$$

is a Quillen adjunction between the projective model structure. Furthermore l^* preserves cofibrations.

Proof. It is immediate to check that l^* is right Quillen and the projective model structure on $\Lambda_{C+}\mathrm{Sq}(\mathcal{V})$ is transferred from $\mathcal{V}^{\mathrm{Seq}_0(C)}$. Therefore, if I is a set of generating cofibrations for $\mathcal{V}$, a set of generating cofibrations for $\Lambda_{C+}\mathrm{Sq}(\mathcal{V})$ is given by

$$I_\Lambda=\{l_!\iota_s(i)\mid s\in\mathrm{Seq}_0(C),\ i\in I\}.$$

Since l^* preserve colimits, to check that it preserve cofibrations it is sufficient to show that $l^*(k)$ is a cofibration in $\mathcal{V}^{\mathrm{Seq}(C)}$ for every $k\in I_\Lambda$, that is $l^*(k)(t)$ is a cofibration for every $t\in\mathrm{Seq}(C)$. Suppose $k=l_!\iota_s(i)$ for some $s\in\mathrm{Seq}(C)$ and $i\in I$, then

$$l^*(k)(t)=l_!\iota_s(i)(t)=\coprod_{T\in\mathbf{S}_t}\iota_s(i)(a(v_T))=\coprod_{T\in\mathbf{S}_t,a(v_T)=t} i$$

which is a cofibration in $\mathcal{V}$, since the class of cofibrations is closed under coproducts. □

There is an obvious forgetful functor $e^*\colon \mathcal{V}\text{-UOper}_C\to\mathcal{V}\text{-}\mathsf{cfOper}_C/\mathsf{Com}_C$.

Proposition 7.5. *([11]) The functor e^* creates all limits and colimits.*

Proof. In [11, Part I, Ch. 3, Proposition 7.4] the case for $C=*$ (i.e. the un-coloured case) is proved. The case for a different set of colours is similar. □

Proposition 7.6. *(*cf. *[11]) The functor u^* admits a left adjoint $u_!$ and is monadic and finitary. Furthermore in the following diagram*

$$\begin{tikzcd} \mathcal{V}\text{-UOper}_C \arrow[r, "e^*"] \arrow[d, shift left, "u^*"] & \mathcal{V}\text{-}\mathsf{cfOper}_C/\mathsf{Com}_C \arrow[d, shift left, "U_{\mathsf{CfOp}_C}"] \\ \Lambda_{C+}\mathrm{Sq}(\mathcal{V})/\mathsf{Com}_C \arrow[u, shift left, "u_!"] \arrow[r, shift right, "l^*"'] & \mathcal{V}^{\mathrm{Seq}_0(C)}/\mathsf{Com}_C \arrow[l, shift right, "l_!"'] \arrow[u, shift left, "F_{\mathsf{CfOp}_C}"] \end{tikzcd} \tag{7.2.5}$$

the following relations hold:

(1) $l^*u^*\simeq U_{\mathsf{CfOp}_C}e^*$;
(2) $e^*u_!\simeq F_{\mathsf{CfOp}_C}l^*$.

Proof. Diagram (7.2.5) decomposes in two squares

$$\begin{tikzcd}
\mathcal{V}\text{-UOper}_C \arrow[r, "e^*"] \arrow[d, shift left, "v^*"] & \mathcal{V}\text{-}\mathsf{cfOper}_C/\mathsf{Com}_C \arrow[d, shift left, "U'_{\mathsf{CfOp}_C}"] \\
\Lambda_C\mathrm{Sq}(\mathcal{V})/\mathsf{Com}_C \arrow[u, shift left, "v_!"] \arrow[r, shift right, "m^*"'] \arrow[d, shift left, "s^*"] & \mathrm{Coll}_{\mathcal{V}}(C)/\mathsf{Com}_C \arrow[u, shift left, "F'_{\mathsf{CfOp}_C}"] \arrow[l, shift right, "m_!"'] \arrow[d, shift left, "r^*"] \\
\Lambda_{C+}\mathrm{Sq}(\mathcal{V})/\mathsf{Com}_C \arrow[u, shift left, "s_!"] \arrow[r, shift right, "l^*"'] & \mathcal{V}^{\mathrm{Seq}_0(C)}/\mathsf{Com}_C \arrow[u, shift left, "r_!"] \arrow[l, shift right, "l_!"']
\end{tikzcd}$$

where v^* (called ω in [11, Part I, Ch. 3])and U'_{CfOp_C} are the obvious forgetful functors. The existence of $v_!$ was proven by Fresse [11, Part I, Theorem 3.3.2]. In there ([11, Part I, Theorem 3.3.3]) it is also proved that v^* is finitary and monadic. The functor s^* creates all colimits, thus it is finitary and monadic; it follows that u^* admits a left adjoint and it is finitary and monadic.

Furthermore in [11, Part I, Theorem 3.3.2] it is proved that $e^*v_! \simeq F'_{\mathsf{CfOp}_C} m^*$; therefore

$$e^*u_! \simeq e^*v_!s_! \simeq F'_{\mathsf{CfOp}_C} m^*s_! \simeq F'_{\mathsf{CfOp}_C} r_!l^* \simeq F_{\mathsf{CfOp}_C} l^*$$

by Lemma 7.2. □

Corollary 7.7. *The functor e^* has a left adjoint $e_!$.*

Proof. This is an application of the adjoint lifting theorem [8, Theorem 4.5.6]. □

Remark 7.8. Recall that the category of $\mathcal{V}$-enriched categories with set of objects C is embedded in $\mathcal{V}$-cfOper_C via the functor $j_! \colon \mathcal{V}\text{-}\mathsf{Cat}_C \to \mathcal{V}\text{-}\mathsf{cfOper}_C$, left adjoint to j^*. This adjunction passes to the slices categories

$$j_! \colon \mathcal{V}\text{-}\mathsf{Cat}_C/\mathsf{Com}_C \rightleftarrows \mathcal{V}\text{-}\mathsf{cfOper}_C/\mathsf{Com}_C \colon j^*$$

(one can either see the source category as a comma category or interpret Com_C as $j^*(\mathsf{Com}_C)$ in that expression).

The composition $e_!j_!$ is isomorphic to the inclusion of $\mathcal{V}$-Cat_C in $\mathcal{V}$-UOper$_C$ and it is therefore fully-faithful.

Now we are ready to prove the following theorem that guarantees the existence of the projective model structure on $\mathcal{V}$-UOper$_C$ under mild hypotheses on $\mathcal{V}$.

Proposition 7.9. *Suppose $\mathcal{V}$ admits transfer for constant-free operads. $\mathcal{V}$-UOper$_C$ admits the projective model structure, which is transferred along $(e_!, e^*)$. Furthermore e^* preserves cofibrations.*

Proof. Let I and J be the sets of generating cofibrations and generating trivial cofibrations in $\mathcal{V}/\mathbf{I}$.

We know that a choice of sets of generating cofibrations and trivial cofibrations in $\mathcal{V}$-cfOper_C is

$$I_{\mathsf{CfOp}_C} = \{F_{\mathsf{CfOp}_C}\iota_s(i) \mid i \in I,\ s \in \mathrm{Seq}_0(C)\}$$
$$J_{\mathsf{CfOp}_C} = \{F_{\mathsf{CfOp}_C}\iota_s(j) \mid j \in J,\ s \in \mathrm{Seq}_0(C)\}.$$

(*cf.* Section 3.2).

Therefore sets of generating (trivial) cofibrations for $\mathcal{V}$-$\mathsf{cfOper}_C/\mathsf{Com}_C$ are

$$I_{\mathsf{CfOp}_C}/\mathsf{Com}_C = \{(F_{\mathsf{CfOp}_C}\iota_s)(i) \mid i \in I/\mathbf{1},\ s \in \mathrm{Seq}_0(C)\}$$
$$J_{\mathsf{CfOp}_C}/\mathsf{Com}_C = \{(F_{\mathsf{CfOp}_C}\iota_s)(j) \mid j \in J/\mathbf{1},\ s \in \mathrm{Seq}_0(C)\}$$

(where F_{CfOp_C} and ι_s actually stands for $F_{\mathsf{CfOp}_C}/\mathsf{Com}_C$ and ι_s/Com_C).

Since e^* preserves all limits and colimits, to check that the projective model structure transfers along $(e_!, e^*)$ and e^* preserves cofibration it is sufficient to check that the set of morphisms $e^*e_!(I_{\mathsf{CfOp}_C}/\mathsf{Com}_C)$ ($e^*e_!(J_{\mathsf{CfOp}_C}/\mathsf{Com}_C)$) is contained in the class of (trivial) cofibrations of $\mathcal{V}\text{-}\mathsf{cfOper}_C$.

Suppose $i \in I/\mathbf{1}$ ($i \in J/\mathbf{1}$) and $s \in \mathrm{Seq}_0(C)$, then

$$e^*e_!F_{\mathsf{CfOp}_C}\iota_s(i) \simeq e^*u_!l_!\iota_s(i) \simeq F_{\mathsf{CfOp}_C}l^*l_!\iota_s(i).$$

The functors F_{CfOp_C}, l^*, $l_!$ and ι_s are left Quillen functors, therefore $e^*e_!F_{\mathsf{CfOp}_C}\iota_s(i)$ is a (trivial) cofibration. □

7.3. **The model structure on unitary operads.** For every map of sets $f\colon C \to D$ the restriction functor

$$f^*\colon \mathcal{V}\text{-}\mathsf{Oper}_D \to \mathcal{V}\text{-}\mathsf{Oper}_C$$

restricts to unitary operads:

$$f^*\colon \mathcal{V}\text{-}\mathrm{UOper}_D \to \mathcal{V}\text{-}\mathrm{UOper}_C.$$

It is an easy application of the adjoint lifting theorem [8, Theorem 4.5.6] to check that f^* admits a left adjoint $f_!$. This assignment is functorial in f, that is it defines a (pseudo)functor

$$\mathcal{V}\text{-}\mathrm{UOper}_{\_}\colon \mathcal{S}et \longrightarrow \mathsf{Cat}_{\mathrm{adj}}.$$

Applying the Grothendieck construction to $\mathcal{V}\text{-}\mathrm{UOper}_{\_}$ we get a bifibration

$$\mathbf{Cl}\colon \mathcal{V}\text{-}\mathrm{UOper} \to \mathcal{S}et;$$

the total category $\mathcal{V}$-UOper is isomorphic to the full subcategory of $\mathcal{V}\text{-}\mathsf{Oper}$ spanned by the unitary operads.

The adjunction $(e_!, e^*)$ is natural in C and therefore, via the Grothendieck construction we get an adjunction

$$e_!\colon \mathcal{V}\text{-}\mathsf{cfOper}/\mathsf{Com} \rightleftarrows \mathcal{V}\text{-}\mathrm{UOper}\colon e^* \tag{7.3.1}$$

where, furthermore, e^* preserves colimits.

Proposition 7.10. *Suppose that $\mathcal{V}$ satisfies the hypotheses of point (1),(2) or (3) of Theorem 4.22 for constant-free operads. Then the canonical model structure on $\mathcal{V}\text{-}\mathsf{cfOper}$ can be transferred to $\mathcal{V}$-UOper along $(e_!, e^*)$. Furthermore $e_!$ preserves cofibrations.*

If $\mathcal{V}$ is combinatorial then the transferred model structure on $\mathcal{V}$-UOper is combinatorial.

Proof. Since $e_!$ preserves colimits it is sufficient to prove that $e^*e_!(i)$ is a (trivial) cofibration in $\mathcal{V}\text{-}\mathsf{cfOper}$ for every $i \in \mathcal{I}/\mathsf{Com}$ ($i \in \mathcal{J}/\mathsf{Com}$).

If $i \in I_{loc}/\mathsf{Com}$ ($i \in J_{loc}/\mathsf{Com}$) the statement follows from Proposition 7.9. Otherwise $i = j_!(k)$ for some (trivial) cofibration $k \in \mathcal{V}\text{-}\mathsf{Cat}_C$ and $e^*e_!(i) \simeq i$; in fact the counit of $(e_!, e^*)$ is is an isomorphism in the image of $j_!$.

The last statement is a consequence of Proposition A.5. □

Corollary 7.11. *Suppose that $\mathcal{V}$ satisfies the hypotheses of point (1),(2) or (3) of Theorem 4.22 for constant-free operads (in particular $\mathcal{V}$ is right proper), then the transferred model structure on $\mathcal{V}$-UOper is right proper.*

If $\mathcal{V}$ satisfies the hypotheses of point (3) of Theorem 4.22 and it is strongly h-monoidal then the the transferred model structure on $\mathcal{V}$-UOper is left proper.

Proof. The model structure on $\mathcal{V}$-UOper is transferred from $\mathcal{V}\text{-}\mathsf{cfOper}$ which is right proper, therefore it is right proper. If $\mathcal{V}$ is strongly h-monoidal then $\mathcal{V}\text{-}\mathsf{cfOper}$ is left proper by Theorem 6.6; since e^* preserves push-outs, cofibrations and preserves and reflects weak equivalences $\mathcal{V}$-UOper is left proper too. □

Remark 7.12. In this section we transfer the projective model structure from $\Lambda_C \mathrm{Sq}(\mathcal{V})$ to C-UOper$_{\mathcal{V}}$. Λ_C can be endowed with a generalized Reedy stucture, so a generalized Reedy model structure can be put on $\Lambda_C \mathrm{Sq}(\mathcal{V})$, as shown in [11, Part II,8]. Fresse showed that this generalized Reedy model structure can also be transferred to C-UOper$_{\mathcal{V}}$.

We choose to use the projective model structure on C-UOper$_{\mathcal{V}}$ in order to get the canonical model structure on $\mathcal{V}$-UOper. In any case we believe that similar technique can be used to get a model structure on $\mathcal{V}$-UOper with the same weak equivalence (but more cofibrations) such that the restriction on each fiber C-UOper$_{\mathcal{V}}$ coincides with the model structure transferred from the generalized Reedy model structure.

8. The Dwyer-Kan Model Structure

Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a monoidal model category. In section 4.5 we proved that, under our hypothesis (in particular $\mathcal{V}$ is right proper and with cofibrant unit), the class of weak equivalences of the canonical model structure on $\mathcal{V}$-Oper coincides with the class of Dwyer-Kan weak equivalences (Definiton 4.24).

Dwyer-Kan weak equivalences are a quite natural class of weak equivalences to consider in $\mathcal{V}$-Oper and one could wonder if under (more general) conditions on $\mathcal{V}$, there is still a model structure on $\mathcal{V}$-Oper having the Dwyer-Kan weak equivalences as class of weak equivalences.

Definition 8.1. A *Dwyer-Kan model structure* on $\mathcal{V}$-Oper ($\mathcal{V}$-Cat) a model structure on $\mathcal{V}$-Oper ($\mathcal{V}$-Cat) which has the Dwyer-Kan weak equivalences as weak equivalences and the local trivial fibration surjective on colours (objects) as trivial fibrations.

Note that if a Dwyer-Kan model structure exists on $\mathcal{V}$-Oper ($\mathcal{V}$-Cat) then it is unique.

Muro showed that under less restrictive hypothesis on $\mathcal{V}$ than [4] the Dwyer-Kan model structure on $\mathcal{V}$-Cat exists.

Theorem 8.2. *([23, Theorem 1.1]) Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a combinatorial monoidal model category satisfying the monoid axiom. Then $\mathcal{V}$-Cat admits the Dwyer-Kan model structure.*

In this section we would like to show how we can use Muro's result to prove the existence of the Dwyer-Kan model structure on $\mathcal{V}$-Oper.

In order to do so, we need to recall some definitions from [23].

8.1. Generating trivial cofibrations for the Dwyer-Kan model structure.

Definition 8.3. A *weak interval* is a categories $\mathbb{K} \in \mathcal{V}$-$\mathsf{Cat}_{\{0,1\}}$ such that 0 and 1 are equivalent in $\pi_0(\mathbb{K})$.

Weak intervals are called just intervals in [23], we choose this terminology in order to distinguish them from the intervals of Berger and Moerdijk ([4]) defined in section 4.1.

Definition 8.4. A set of weak intervals $\mathfrak{W}$ is a *generating set of weak intervals* if for every $\mathcal{V}$-category $\mathcal{D}$ and every $x, y \in \mathcal{D}$ such that x is equivalent to y in $\pi_0(\mathcal{D})$, there exists a weak interval $\mathbb{K} \in \mathfrak{W}$ and a morphism $f\colon \mathbb{K} \to \mathcal{D}$ such that $f(0) = x$ and $f(1) = y$.

Proposition 8.5. *([23]) If $(\mathcal{V}, \otimes, \mathbf{I})$ is a combinatorial monoidal model category and satisfies the monoid axiom a generating set of weak intervals always exists.*

Fix $\mathfrak{W}$, a set of generating weak intervals for $\mathcal{V}$. For every $\mathbb{K} \in \mathfrak{W}$ we are going to define a map that will be part of the generating trivial cofibrations for the Dwyer-Kan model structure.

Let $i\colon \{0\} \to \{0,1\}$ the obvious inclusion of sets. Let $k\colon \widetilde{i^*(\mathbb{K})} \to i^*(\mathbb{K})$ be a cofibrant replacement for $i^*(\mathbb{K})$ in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0\}}$. We can factor the composition of $i_!(k)$ and the counit of $(i_!, i^*)$

$$\begin{tikzcd} i_!\widetilde{i^*(\mathbb{K})} \arrow[r, "i_!(k)"] \arrow[dr, "\theta_{\mathbb{K}}"'] & i_!i^*(\mathbb{K}) \arrow[r] & \mathbb{K} \\ & \widetilde{\mathbb{K}} \arrow[ur, "g"'] & \end{tikzcd}$$

into a cofibration $\theta_{\mathbb{K}}$ followed by a trivial fibration g. Let $\theta_{\mathbb{K}}\colon \widetilde{i^*(\mathbb{K})} \to \widetilde{\mathbb{K}}$ the composition of $\theta'_{\mathbb{K}}$ with a cocartesian arrow $\widetilde{i^*(\mathbb{K})} \to i_!\widetilde{i^*(\mathbb{K})}$ above i.

The following set is part of a set of generating cofibrations for the Dwyer-Kan model structure on $\mathcal{V}\text{-}\mathsf{Cat}$

$$J' = \{\theta_{\mathbb{K}}\colon \mathbb{K} \in \mathfrak{W}\}.$$

More precisely, under the hypotheses of Theorem 8.2, a morphism in $\mathcal{V}\text{-}\mathsf{Cat}$ is a Dwyer-Kan fibration if and only if it is a local fibration and has the right lifting property with respect to J'.

Theorem 8.6. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a combinatorial monoidal model category satisfying the monoid axiom such that*

- *the class of weak equivalences is closed under transfinite composition;*
- *$\mathcal{V}$ admits transfer for symmetric operads.*

Then $\mathcal{V}$-Oper admits the Dwyer-Kan model structure; moreover it is combinatorial.

Proof. Let J' be a set of generating trivial cofibrations for $\mathcal{V}\text{-}\mathsf{Cat}$; as set of generating trivial cofibrations we take

$$\mathcal{J}_D K = j_!(J') \cup J_{loc}$$

and as set of generating trival cofibrations we take $\mathcal{I}_D K = \mathcal{I}$ as in (4.3.1).

Since the categories of $\mathcal{V}$-Oper is complete and cocomplete, to prove the existence of the model structure it is sufficient to prove (following Theorem 2.1.19 [19])) that:

(1) the class of weak equivalences has the 2-out-of-3 property and it is closed under retracts;
(2) the domains of $\mathcal{I}$ are small relative to $\mathcal{I}-$cell;
(3) the domains of $\mathcal{J}$ are small relative to $\mathcal{J}-$cell;
(4) $\mathcal{I}_{DK}-\mathrm{inj} = \mathcal{W} \cap \mathcal{J}_{DK}-\mathrm{inj}$
(5) $\mathcal{J}_{DK}-\mathrm{cell} \subseteq \mathcal{W} \cap \mathcal{I}-\mathrm{cof}$.

Point (1) is obvious to check for Dwyer-Kan weak equivalences, point (2) and (3) are automatically satisfied because $\mathcal{V}$-Oper is locally presentable (Propostion A.5).

Recall that a morphism f in $\mathcal{V}$-Oper

- belongs to $\mathcal{J}_{DK}-$inj if and only if it is a local fibration and a $j^*(f)$ is a fibration in $\mathcal{V}\text{-}\mathsf{Cat}$;
- belongs to W if and only if it is a local weak equivalence and $j^*(f)$ is a Dwyer-Kan weak equivalence;
- belongs to $\mathcal{I}_{DK}-$inj if and only if it is a local trivial fibration surjective on colours.

Recall also that a morphism g in $\mathcal{V}\text{-}\mathsf{Cat}$ is a trivial Dwyer-Kan fibration if and only if it is a local trivial fibration surjective on objects. It follows that f is in $\mathcal{I}_{DK}-$inj if and only if f is a local trivial fibration and $j^*(f)$ is a local trivial fibration in $\mathcal{V}\text{-}\mathsf{Cat}$, that is if and only if f belongs to $\mathcal{J}_{DK}-$inj. Therefore point (4) is proven.

For point (5) the proof goes as for Lemma 4.20. Since the class of weak equivalences in $\mathcal{V}$ is closed under filtered colimits, Dwyer-Kan weak equivalences are closed under transfinite composition (*cf.* [23, Proposition 9.2]). This implies that to check that $\mathcal{J}_{DK}-$cell is contained in the class of Dwyer-Kan weak equivalences it is sufficient to show that for every $i \in \mathcal{J}_{DK}$ and every push out diagram

(8.1.1)
$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "j"'] & X \arrow[d, "h"] \\ B \arrow[r, "q"'] & Y \end{tikzcd}$$

the morphism h is a weak equivalence.

If $i \in J_{loc}$ this was already proven in Lemma 4.20 (the hypothesis that the unit is cofibrant is not used for that passage).

If $i = j_!(\theta_{\mathbb{K}})$ then for some $\mathbb{K} \in \mathfrak{W}$ then we can factor k as $\psi\phi$ where ϕ is in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0\}}$ and ψ is fully faithful. The push-out is then decomposed in two push-out square:

$$\begin{tikzcd} j_!\widetilde{i_!i^*(\mathbb{K})} \arrow[r, "p"] \arrow[d, "j_!(\phi)"] \arrow[dd, bend right, "j_!(k)"'] & X \arrow[d, "\phi'"] \arrow[dd, bend left, "h"] \\ j_!(\widetilde{\mathbb{K}})(0,0) \arrow[r] \arrow[d, "j_!(\psi)"] & X' \arrow[d, "\psi'"] \\ j_!(\widetilde{\mathbb{K}}) \arrow[r, "q"] & Y \end{tikzcd}$$

Since ψ is fully faithful ψ' is fully faithful. It is proven in [23, Theorem 7.13,7.14] that ϕ is a trivial cofibration in $\mathcal{V}\text{-}\mathsf{Cat}_{\{0\}}$, thus ϕ' is a trivial cofibration in $\mathcal{V}\text{-}\mathsf{Oper}_{\{0\}}$, hence a local weak equivalence. This prove that $h = \psi'\phi'$ is a local weak equivalence. The functor $\pi_0(j^*(h))$ is essentially surjective, in fact the only object not in the image of $\pi_0(j^*(h))$ is $\pi_0(q)(1)$, which is isomorphic to $\pi_0(q)(0)$. This prove that h is a Dwyer-Kan weak equivalence. $\square$

In [25] Pavlov and Scholbach proved that if $\mathcal{V}$ is combinatorial, pretty small and symmetric i-monoidal, it admits transfer for symmetric operads. Since pretty smallness and symmetric i-monoidality implies the monoid axiom we get the following corollary

Corollary 8.7. *If $\mathcal{V}$ is combinatorial, pretty small and symmetric i-monoidal (in the sense of Pavlov and Scholbach), the Dwyer-Kan model structure on $\mathcal{V}$-Oper exists.*

For example the category of symmetric spectra with the positive model structure satisfies the hypotheses Corollary 8.7 but not of Theorem 4.22. We refer the reader to [25] for other examples of model categories satisfying this assumptions.

Remark 8.8. If the Dwyer-Kan model structure and the canonical model structure exist then they coincides if and only if the fibrant objects of the Dwyer-Kan model structure are the locally fibrant ones.

Proposition 4.25 grants us that under the hypotheses of Theorem 4.22 the canonical model structure on $\mathcal{V}$-Oper coincides with the Dwyer-Kan model structure. If this is true under more general conditions remains to be investigated.

For completeness we state the analogous of Theorem 8.6 for non-symmetric, constant-free and unitary operads:

Theorem 8.9. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a combinatorial monoidal model category satisfying the monoid axiom and such that the class of weak equivalences is $\otimes$-perfect. Then $\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper, $\mathcal{V}$-UOper admit the Dwyer-Kan model structure; moreover it is combinatorial.*

Proof. Under these hypotheses $\mathcal{V}$ admits $\otimes$-transfer for non-symmetric and constant-free operads. The proof now goes exactly as in Theorem 8.6. The Dwyer-Kan model structure on $\mathcal{V}$-UOper can be transferred from the one on $\mathcal{V}$-cfOper as in Proposition 7.10. $\square$

Appendix A. Filtered colimits in $\mathcal{V}$-Oper

A.1. **Colimits in bifibrations.** Suppose we have a bifibration $\pi\colon \mathcal{C} \to \mathcal{S}et$ such that for every $C \in \mathcal{S}et$ the fiber $\mathcal{F}\mathrm{ib}_\pi(C)$ is cocomplete. We want to show that $\mathcal{C}$ is cocomplete as well, expressing the colimits in $\mathcal{C}$ as colimits in certain fibers.

Fix a small category I. Let $\Delta[1]$ be the category representing morphisms, i.e. the category with two object 0 and 1 and only one morphism from 0 to 1.

Let $\mathcal{D}$ be any category. A functor $F\colon I \times \Delta[1] \to \mathcal{D}$ is just a natural transformation from $F_{|I\times\{0\}}$ to $F_{|I\times\{1\}}$. Given $i \in I$ we will denote with F_i the morphism in $\mathcal{D}$ given by $F_{\{i\}\times\Delta[1]}$ (in this way we recover the usual notation for natural transformations).

Given two natural transformations $F, G\colon I \times \Delta[1] \to \mathcal{C}$ such that $F_{|I\times\{1\}} = G_{|I\times\{0\}}$, we denote by $G * F\colon I \times \Delta[1] \to \mathcal{C}$ their (horizontal) composite.

Given a morphism $f\colon a \to b$ in $\mathcal{C}$ (that we can see as a functor $f\colon \Delta[1] \to C$) let $\bar{f}\colon I \times \Delta[1] \to$ be the constant natural transformation from a to b, i.e. $\bar{f} = f \circ p_2$ where $p_2\colon I \times \Delta[1] \to \Delta[1]$ is the second projection; in other words $\bar{f}_i = f$ for every $i \in I$.

Suppose we want to compute the colimit of a certain functor $L\colon I \to \mathcal{C}$. We can first compute the colimit of πL in $\mathcal{S}et$, that we can see as a functor from $l\colon I \times \Delta[1] \to \mathcal{S}et$ such that $l_{|I\times\{0\}} = \pi L$ and $l_{|I\times\{1\}}$ is constant of value l_1. We can lift l to a "cocartesian cylinder" $c\colon I \times \Delta[1] \to \mathcal{C}$, this means that $\pi c = l$ and $c_{|I\times\{0\}} = L$, and for every other $k\colon I \times \Delta[1] \to \mathcal{C}$ such that $\pi k = f * l$ and $k_{|I\times\{0\}} = L$, there exist a unique $f'\colon I \times \Delta[1] \to \mathcal{C}$ such that $f'_{|I\times\{0\}} = c_{|I\times\{1\}}$, $f'_{|I\times\{1\}} = k_{|I\times\{1\}}$, its projection on sets is $\pi(f') = f$ and $f' * c = k$.

Note that the image of $c_{|I\times\{1\}}$ lies in $\mathcal{F}\mathrm{ib}_\pi(l_1)$; we claim that the colimit of L is $q * c$ where $q\colon I \times \Delta[1] \to \mathcal{F}\mathrm{ib}_\pi(l_1)$ is the colimit of $c_{|I\times\{1\}}$.

In fact, take another cocone $k\colon I \times \Delta[1] \to \mathcal{C}$ such that $k_{|I\times\{0\}} = L$ and $k_{|I\times\{1\}}$ is constant of value k_1. By the universal property of l (it is the colimit of πL) the functor πk factors in a unique way as $(\pi k) * \bar{z} = l$ where $z\colon l_1 \to \pi(k_1)$ is the morphism given by the universal property of l_1.

The cocone $\bar{z}$ can be lifted in a unique way to a cocone $f'\colon I \times \Delta[1] \to \mathcal{C}$ from $c_{|I\times\{1\}}$ to k_1 such that $f' * c = k$. Since π is a fibration f' factors in a unique way as $f' = \bar{\phi}_z q'$ where q' is a cocone in $\mathcal{F}\mathrm{ib}_\pi(l_1)$ and $\phi_z\colon z^*(k_1) \to k_1$ is the map of a chosen cleavage. By the universal property of q there is unique map $w\colon q_1 \to z^*(k_1)$ in $\mathcal{F}\mathrm{ib}_\pi(l_1)$ such that $\bar{w} * q = q'$. It is easy to see that $\overline{\phi_z w} * (q * c) = k$ and $\phi_z w$ is the unique map with this property.

Summarizing, to compute $\varinjlim L$ we first compute $\varinjlim(\pi L) = l_1$; We consider the "direct image" of L in $\mathcal{F}\mathrm{ib}_\pi(l_1)$:

$$\begin{aligned} L_! \colon I &\longrightarrow \mathcal{F}\mathrm{ib}_\pi(l_1) \\ i &\longmapsto l_i!(L(i)) \end{aligned}$$

(this is nothing but $c_{|I\times\{1\}}$ in the previous description). What we have proved is that every cocone from L to an object k_1 of C factor in a unique way through the "cocartesian cylinder" and a cone from $L_!$ to k_1, and every cone from $L_!$ to k_1 factors in a unique way trough $\varinjlim(L_!)$. Hence $\varinjlim L \simeq \varinjlim(L_!)$.

If $\mathcal{V}$ is a cocomplete symmetric monoidal category then for every coloured operad $\mathcal{O}$ the category of algebras $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ is cocomplete. This implies in particular that the fibers of the bifibred categories $\mathcal{V}$-Cat, $\mathcal{V}$-Oper, $\mathcal{V}$-NSOper and $\mathcal{V}$-cfOper are cocomplete. The fibers of $\mathcal{V}$-MultiGraph,$\mathcal{V}$-MultiGraph and $\mathcal{V}$-RMultiGraph are also cocomplete; hence we can use the previous method for the description of colimits in these categories (and obtain another proof that they are all cocomplete).

A.2. **Filtered colimits in $\mathcal{V}$-Graph and $\mathcal{V}$-Oper.** Even though the description of the colimits given in the previous section is very general, it might not be very effective for actual computations of colimits.

In the case in which the bifibration we consider is $\pi\colon \mathcal{V}\text{-MultiGraph} \to \mathcal{S}et$ and I is filtered a more explicit description can be given. Let $L\colon I \to \mathcal{V}\text{-MultiGraph}$; let $L^\sharp = \pi L$ and let $l\colon I \times \Delta[1] \to \mathcal{S}et$ be its colimit cocone with vertex $l_1 \in Set$.

Every signature $s \in \mathrm{Seq}(l_1)$ is of the form $(l_i(d))$ for some $i \in I$ and $d \in \mathrm{Seq}(L^\sharp(i))$. The couple (i,d) will be called a *representative* for s. The colimit of L is a multi-graph C with set of colours l_1. For every $s \in \mathrm{Seq}(l_1)$ the s-component of C is:

$$C(s) = \varinjlim_{(j,f_j)\in i/I} L_j(f_j(d)) \tag{A.2.1}$$

where (i,d) is some representative for s (it can be shown that this definition does not depend on the choice of the representatives).

There is an obvious cocone $\eta\colon I \times \Delta[1] \to \mathcal{V}\text{-Graph}$ from L to C such that $\eta = \pi l$. It is easy to check that η is a colimit for L.

Note that for every $n \in \mathbb{N}$ and every $s, s_1, \dots, s_n \in \mathrm{Seq}(c)$ for which it make sense to write $s \circ (s_1, \dots, s_n)$, one can find an $i \in I$ and $d, d_1, \dots, d_n \in \mathrm{Seq}(L^\sharp(i))$ such that

- (i,d) is a representative of s;
- for every $j \in [n]$ the couple (i, d_j) is a representative of s_j;
- it makes sense to write $d \circ (d_1, \dots, d_n)$;
- $(i, d \circ (d_1, \dots, d_n))$ is a representative for $s \circ (s_1, \dots, s_n)$.

If s is of the form $(a;a)$ for some $a \in c$ then d can be chosen of the form $(v;v)$ for some $v \in \pi(L(i))$.

We remark also that given $\sigma \in \Sigma_{|s|}$ if (i,d) is a representative for s, then $(i, \sigma d)$ is a representative for σs.

The same description of filtered colimits can be given in $\mathcal{V}$-RMultiGraph.

We want to show that the right adjoint functors in the adjunctions (3.0.7),(3.0.8) and (3.0.9) create filtered colimits.

Lemma A.1. *Let $\mathcal{V}$ be a cocomplete monoidal closed category.*

(1) The right adjoint functor U_{Op} between $\mathcal{V}$-MultiGraph and $\mathcal{V}$-Oper creates filtered colimits.
(2) The right adjoint functor U_{NSOp} between $\mathcal{V}$-MultiGraph and $\mathcal{V}$-Oper creates filtered colimits.
(3) The right adjoint functor U_{CfOp} between $\mathcal{V}$-RMultiGraph and $\mathcal{V}$-cfOper creates filtered colimits.

Proof. We are going to prove point 1 only, the other proofs are almost identical.

Let I be a small filtered category and let $L'\colon I \to \mathcal{V}\text{-Oper}$. If $L = U_{\mathsf{Op}} L'$ and $\eta\colon I \times \Delta[1] \to \mathcal{V}\text{-MultiGraph}$ be colimit cocone of L with apex C described as above. It will be convenient to denote $L(i)$ by L_i.

we have to prove that C can be endowed of an operad structure such that for every $i \in I$ $n_i\colon L_i \to C$ becomes a morphism of operads.

The structure of operad on C is induced in a natural way from the ones on the L_i's thanks to the fact that $\mathcal{V}$ is monoidal closed.

In fact, for every $n \in \mathbb{N}$ and every $s, s_1, \dots, s_n \in \mathrm{Seq}(c)$ for which it makes sense to write $s \circ (s_1, \dots, s_n)$, we can find $i \in I$ and $d, d_1, \dots, d_n \in \mathrm{Seq}(\pi(L_i))$ such that

$$C(s) = \varinjlim_{(j,f_j)\in i/I} L_j(f_j(d))$$
$$C(s_k) = \varinjlim_{(j,f_j)\in i/I} L_j(f_j(d_k))$$

for every $k \in [n]$, and such that to write $d \circ (d_1, \dots, d_n)$ makes sense.

Since i/I is filtered and $\mathcal{V}$ is monoidal closed:

$$(\bigotimes_{k\in[n]} C(s_k)) \otimes C(s) \simeq \varinjlim_{(j,f_j)\in i/I} (\bigotimes_{k\in[n]} L_j(f_j(d_k))) \otimes L_j(f_j(d))$$

The composition Γ is defined as the unique map that makes the following diagram commute:

$$\begin{tikzcd}
(\bigotimes_{k\in[n]} L_j(f_j(d_k))) \otimes L_j(f_j(d)) \arrow[r, "\Gamma_j"] \arrow[d] & L_j(f_j(d \circ (d_1, \dots, d_n))) \arrow[d] \\
(\bigotimes_{k\in[n]} C(s_k)) \otimes C(s) \arrow[r, "\Gamma"'] & C(d \circ (d_1, \dots, d_n))
\end{tikzcd}$$

where the vertical maps are the canonical ones defined by the colimits.

The morphism Γ exists (and is unique) because the following diagram commutes for every morphism $g\colon (j, f_j) \to (z, f_z)$ in i/I:

$$\begin{tikzcd}
(\bigotimes_{k\in[n]} L_j(f_j(d_k))) \otimes L_j(f_j(d)) \arrow[r, "\Gamma_j"] \arrow[d, "\bigotimes_{k\in[n]} L(g)\otimes L(g)"'] & L_j(f_j(d \circ (d_1, \dots, d_n))) \arrow[d, "L(g)"] \\
(\bigotimes_{k\in[n]} L_z(f_z(d_k))) \otimes L_z(f_j(d)) \arrow[r, "\Gamma_z"'] & L_z(f_z(d \circ (d_1, \dots, d_n)))
\end{tikzcd}$$

since all $L(g)$'s are morphisms of operads.

Let $s \in \mathrm{Seq}(c)$ and (i, d) one of its representatives. For every $\sigma \in \Sigma_{|s|}$ the right action of σ is defined as the unique map σ^* (on the second row) that makes the following diagram commute for every $(j, f_j) \in i/I$:

$$\begin{tikzcd}
L_j(f_j(d)) \arrow[r, "\sigma^*"] \arrow[d] & L_j(f_j(d)\sigma) \arrow[d] \\
C(s) \arrow[r, "\sigma^*"'] & C(s\sigma)
\end{tikzcd}$$

where the σ^* on the first row is given by the action of σ on L_j and the vertical arrows are the canonical arrows of the colimits defining $C(s)$ and $C(s\sigma)$.

The only things left to define are the identity operations. Given a colour $a \in c$ let $(i, (v; v))$ be a representative for $(c; c)$. The identity operation for a is defined as the unique map u_a such that the following diagram commutes for every $(j, f_j) \in i/I$:

$$\begin{tikzcd}
 & L_j(f_j(v; v)) \arrow[d] \\
I \arrow[ur, "u_{f_j(v)}"] \arrow[r, "u_a"] & C(a; a)
\end{tikzcd}$$

where $u_{f_j(v)}$ is the identity for $f_j(v)$ in L_j and the vertical arrow is the one that comes from the definition of $C(a; a)$ as colimit.

It is left to the reader to verify that this is a good definition of operad structure and that, with this structure, all η_i's become morphisms of operads.

□

The explicit description of filtered colimits of multi-graphs allows to prove the following:

Lemma A.2. *Let $\mathcal{V}$ be a cocomplete monoidal closed category. If $X \in \mathcal{V}$ is small respect to transfinite compositions of morphisms in a certain class of maps $\mathcal{S}$ then, $\iota_{s_n}(X)$ (see Section 2.4) is small in $\mathcal{V}$-$\mathsf{MultiGraph}$ (in $\mathcal{V}$-$\mathsf{RMultiGraph}$) respect to transfinite compositions of $\mathcal{S}$-local morphisms, for every $n \in \mathbb{N}$.*

Proof. Let λ be a regular ordinal such that X is λ-small with respect to transfinite compositions in $\mathcal{S}$ and let $L\colon \lambda \to \mathcal{V}$-$\mathsf{MultiGraph}$ be a continuous functor. Suppose a morphism $f\colon \iota_{s_n}(X) \to \varinjlim L$ is given. We have to prove that f factors through some $L_j = L(j)$ for some $j \in \lambda$. This morphism is completely determined by the induced map on the colours $\mathbf{Col}(f)\colon [n+1] \to \mathbf{Col}(\varinjlim L)$ and its evaluation at the s_n-level:

$$f\colon \iota_{s_n}(X)(s_n) = X \to \varinjlim L(\mathbf{Col}(f)(s_n)). \tag{A.2.2}$$

Since $[n+1]$ is finite, $\mathbf{Col}(f)$ factors through some $h\colon [n+1] \to \mathbf{Col}(L(i))$ for some $i \in \lambda$. The couple $(i, h(s_n))$ is then a representative for $f(s_n)$, and the codomain of (A.2.2) can be rewritten as $C(s) = \varinjlim_{(j,g_j)\in i/\lambda} L_j(g_j(h(s_n)))$ (according to formula A.2.1), which is a λ-transfinite composition of morphism in $\mathcal{S}$.

Since X is λ-small respect to $\mathcal{S}$, the morphism (A.2.2) factors through a morphism $f'\colon X \to L_j(g_j(h(s_n))$ for some $(j, g_j) \in i/\omega$. This morphism together with the map on the colours $g_j h\colon [n+1] \to \mathbf{Col}(L_j)$ determines a morphism of multi-graphs $f'\colon \iota_{s_n}(X) \to L_j$ such that $f = k_j f'$, where $k_j\colon L_j \to \varinjlim L$ is the canonical map associated to the colimit.

The proof for the case of constant-free multi-graphs is identical. □

A.3. **$\mathcal{V}$-Oper is locally presentable.** Fix a regular cardinal λ. In this section we are going to prove the not so surprising result that $\mathcal{V}$-$\mathsf{MultiGraph}$ and $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper) are λ-locally presentable whenever $\mathcal{V}$ is. For the definition of λ-locally presentable category we refer the reader to [1].

We start with the case of multi-graphs

Proposition A.3. *Suppose $\mathcal{V}$ is a λ-locally presentable symmetric monoidal category, then $\mathcal{V}$-$\mathsf{MultiGraph}$ ($\mathcal{V}$-$\mathsf{RMultiGraph}$) is λ-locally presentable.*

Proof. Since we have already shown that $\mathcal{V}$-$\mathsf{MultiGraph}$ is cocomplete, it is sufficient to prove that $\mathcal{V}$-$\mathsf{MultiGraph}$ has a set of strong generators which are λ-presentable (*cf.* [1, Theorem 1.20]). The proof is completely analogous to the one for $\mathcal{V}$-Graph given in [20], therefore we will just give a set of strong generators for $\mathcal{V}$-$\mathsf{MultiGraph}$. Suppose that $\mathcal{G}$ is a set of λ-presentable strong generators for $\mathcal{V}$, then

$$\{\iota_{s_n}(G) \mid G \in \mathcal{G} \cup \{\emptyset\},\ n \in \mathbb{N}\}$$

(where ι_{s_n} is as in (3.2.1) and $\emptyset$ is the initial object of $\mathcal{V}$) is a set of strong generators for $\mathcal{V}$-$\mathsf{MultiGraph}$ (*cf.* [20, Lemma 4.2]), the reader can check that all its elements are λ-presentable (*cf.* [20, Lemma 4.3]). □

Recall the following result from the literature

Theorem A.4. *[12, Satz 10.3] Let $\mathcal{C}$ be a λ-locally presentable category and let $\mathbf{T}$ be a finitary monad over $\mathcal{C}$. Then the category of algebras $\mathrm{Alg}_{\mathbf{T}}(\mathcal{C})$ is λ-locally presentable.*

We are now ready to prove the following

Proposition A.5. *Suppose $\mathcal{V}$ is a λ-locally presentable symmetric monoidal category. Then $\mathcal{V}$-Oper, $\mathcal{V}$-NSOper, $\mathcal{V}$-cfOper and $\mathcal{V}$-UOper are λ-locally presentable categories.*

Proof. From Proposition 3.3 we get that the adjunctions (3.0.7), (3.0.8), (3.0.9) are monadic; the functors U_{Op}, U_{NSOp} and U_{CfOp} are also finitary by Lemma A.1. We can now apply Theorem A.4 combined with Proposition A.3 to get the desired result. For the case of unitary operads we can just notice that the adjunction (7.3.1) is also monadic and finitary, hence we can apply Theorem A.4 again to obtain that $\mathcal{V}$-UOper is λ-locally presentable. □

Appendix B. Push-out of C-coloured Operads

The aim of this appendix is to give a proof of Proposition B.27. An explicit proof of this proposition will be given only in the case of $\mathcal{V}$-Oper. The main ingredient of this proof is the explicit description of push-outs in $\mathcal{V}$-Oper_C given in Proposition B.23. A similar description can be given for push-outs in $\mathcal{V}$-NSOper_C, using planar trees instead of non-planar ones. Once such a description is given, the proof of Proposition B.27 in the case of $\mathcal{V}$-NSOper becomes almost identical to the one for symmetric operads. Since the push-outs of constant-free operads are still constant-free operads, the case of $\mathcal{V}$-cfOper follows from the one of $\mathcal{V}$-Oper.

B.1. **Left Kan Extension along opfibrations.** Before giving the desired description of push-outs coloured operads we would like to make a remark on left Kan extension along opfibration that will be useful in order to decompose certain colimits (typically indexed by the total category of a certain opfibration). Given a functor $f\colon \mathcal{E}\to\mathcal{B}$ between small categories and a cocomplete category $\mathcal{V}$, the restriction functor

$$f^*\colon \mathcal{V}^{\mathcal{B}} \longrightarrow \mathcal{V}^{\mathcal{E}}$$

always admits a left adjoint

$$\begin{aligned} f_!\colon & \mathcal{V}^{\mathcal{E}} \longrightarrow \mathcal{V}^{\mathcal{B}} \\ & D \longmapsto \mathrm{Lan}_f D \end{aligned}$$

which associates to each $\mathcal{E}$-diagram in $\mathcal{V}$ its left Kan extension along f. Note that if $\mathcal{V}$ is a cofibrantly generated model category both $\mathcal{V}^{\mathcal{E}}$ and $\mathcal{V}^{\mathcal{B}}$ admits the projective model structure and $(f_!, f^*)$ becomes a Quillen adjunction between this two model structures.

If f is an opfibration then for every E-diagram D in $\mathcal{V}$ and every $b\in\mathcal{B}$

(B.1.1) $$(\mathrm{Lan_f D})(b) \simeq \varinjlim_{\mathcal{E}_b} D_{\mathcal{E}_b}$$

where $\mathcal{E}_b$ is the fiber of f over b. Note that the colimit of D now decomposes as

(B.1.2) $$\varinjlim_{\mathcal{E}} D \simeq \varinjlim_{\mathcal{B}} f_!(D) \simeq \varinjlim_{b\in\mathcal{B}} (\mathrm{Lan_f D})(b) \simeq \varinjlim_{b\in\mathcal{B}} \varinjlim_{\mathcal{E}_b} D_{\mathcal{E}_b}$$

If f is an opfibration, to empathize the fact that $f_!$ is component-wise the colimit along the fibers, we will denote the functor $f_!$ by $\varinjlim^f$.

B.2. **Trees.** Let C be a set and $(\mathcal{V},\otimes,\mathbb{I})$ a cocomplete monoidal category. We want to give an explicit description of the push-outs in $\mathcal{V}$-Oper_C.

Let Γ be the category with three objects o, x, y, generated by the following graph:

$$\begin{tikzcd} o \arrow[r, "\iota_1"] \arrow[d, "\iota_2"] & x \, . \\ y & \end{tikzcd}$$

Given a category $\mathcal{C}$, a push-out diagram is just a Γ-diagram $F\colon \Gamma \to \mathcal{C}$ together with a colimit for it.

(B.2.1)
$$\begin{tikzcd} F(o) \arrow[r, "F(\iota_2)"] \arrow[d, "F(\iota_1)"'] & F(x) \arrow[d, "p"] \\ F(y) \arrow[r, "q"'] & Z \end{tikzcd} .$$

For a description of this colimit in $\mathcal{V}$-Oper_C we need to introduce trees. We are going to define for every $S \in \mathrm{Seq}(C)$ a category of trees (with a certain marking on vertices) $\mathbf{T}(C)_S$ which will be helpful to define $Z(S)$; more precisely $Z(S)$ will be a colimit indexed by $\mathbf{T}(C)_S$.

Talking about operads the natural "category of trees" to work with is the dendroidal category Ω introduced by Moerdijk and Weiss ([22],[32]).

We are now going to briefly recall the definitions of Ω and other categories derived from it.

B.2.1. *The dendroidal category.* Following Weiss exposition a *graph* will be for us a couple (E, V) where E is a non-empty set, called the set of *edges*, and $V \subset P(E)$ such that every $e \in E$ belongs to at most two element of V; The set V is called the set of *vertices*. Given a graph G the set of vertices will be denoted $\mathrm{vert}(G)$ and the set of edges will be indicated with $\mathrm{edge}(G)$.

The edges that belong to only one vertex are called *outer edges* and the others are *inner edges*.

Given a graph $G = (E, V)$, a *subgraph* of G is a graph (E', V') such that $E' \subset E$ and $V' \subset V$. A *path* of length n in G is a f sequence $(e_1, \dot{,} e_n)$ of distinct element of E and a sequence $(v_1, \dots, v_{n-1})$ of distinct element of V such that $\{e_i, e_{i+1}\} \subset v_i$ for every $1 \leq i < n$; the edges e_1 and e_n are called the *extremes of the path.* A *loop* in G is a path of length greater then 1 with the extremes belonging to a common vertex. Two edges e, f of G are *connected* if they are extremes of a common path; this path, if it exists, can be chosen of minimal length, this minimal length is called the *distance* between e and f. A graph is *connected* if every pair of edges is *connected.*

A *tree* is a finite connected graph with no loops and a chosen outer edge called the *root.* The outer edges different from the root are called *leaves.* The root of a tree T will be denoted by $\mathrm{r}(T)$ and the set of leaves will be called $\mathrm{in}(T)$.

A *subtree* of a tree T is just a subgraph of T equipped with the structure of a tree whose root is the outer edge with minimal distance from $\mathrm{r}(T)$.

Trees with only one vertex are called *corollas*; in a corolla the unique vertex is always the whole set of edges. The tree with one edge and no vertices will be called the *empty tree* and denoted by $|$.

The *arity* of a non-empty tree T is defined as the corolla $a(T)$ whose set of edges is the set of outer edges of T and the root is $\mathrm{r}(T)$.

For every vertex v of a tree T we will denote by t_v the corolla which has v as set of edges and as root the edge connected to the root of T with the shortest path (one can check that this is a good definition). The set $\mathrm{in}(t_v)$ is also denoted by $\mathrm{in}(v)$.

Every tree T freely generates a coloured operad (in $\mathcal{S}et$) $\Omega(T)$ whose set of colours is $\mathrm{edge}(T)$. We will not give a detailed description of $\Omega(T)$, but we want to recall that there is a bijective correspondence between $\coprod_{S \in \mathrm{Seq}(\mathrm{edge}(T))} \Omega(T)(S)$ (the set of operations of $\Omega(T)$) and the couples (t, τ) where t is a subtree of T and τ is a total order on $\mathrm{in}(t)$ and each $\Omega(T)(S)$ has at most one element.

The objects of Ω are trees and a morphism between T and T' is a morphism of operads between $\Omega(T)$ and $\Omega(T')$.

Given a map $f\colon T \to T'$ in Ω and a vertex v, suppose that

$$o_v \in \coprod_{S \in \mathrm{Seq}(\mathrm{edge}(T))} \Omega(T)(S)$$

is the operation associated to (t_v, τ) (here we can choose τ freely). The tree $f(v)$ is defined as the subtree of T' associated to $f(o_v)$ (this definition does not depend on τ).

Given a tree T and an inner edge $e \in \mathrm{edge}(T)$ belonging to two vertices v and u the tree T/e is defined as the tree having $\mathrm{edge}(T/e) = \mathrm{edge}(T) - \{e\}$ and $\mathrm{vert}(T/e) = (\mathrm{vert}(T) - \{u, v\}) \cup \{(u \cup v) - \{e\}\}$ and $\mathrm{r}(T)$ as root. There is a unique map in Ω from T/e to T which is the inclusion $\mathrm{edge}(T/e) \to \mathrm{edge}(T)$ at the level of colours:

$$\partial_e\colon T/e \longrightarrow T,$$

which is called an *inner face (map)*.

Given a vertex $v \in T$ with only one inner edge e attached to it, the tree T/v is defined as the one with $\mathrm{edge}(T/v) = \mathrm{edge}(T) - (v - \{e\})$ and $\mathrm{vert}(T/v) = \mathrm{vert}(T) - \{v\}$, the root of T is e if $\mathrm{r}(T) \in v$ and $\mathrm{r}(T)$ otherwise. There is a unique map in Ω from T/e to T which is the inclusion $\mathrm{edge}(T/v) \to \mathrm{edge}(T)$ at the level of colours:

$$\partial_v\colon T/v \longrightarrow T,$$

such a map is called an *outer face (map)*. If T is a corolla, given an edge $e \in \mathrm{edge}(T)$ the unique map from the empty tree to T which maps the unique element of $|$ in e

$$\partial_e\colon | \longrightarrow T,$$

is also called an outer face.

Given a vertex v with only two edges d, e, let us denote with E' the set obtained as a quotient of $\mathrm{edge}(T)$ identifying d and e and let $\pi\colon \mathrm{edge}(T) \to E'$ be the canonical projection. The tree $T.v$ is defined in the following way: E' is the set of edges and the vertices are the images of the vertices of T along π. There is a unique map in Ω from T to $T.v$ which is π on the colours:

$$\sigma_v\colon T \longrightarrow T.v,$$

such a map is called a *degeneracy (map)*.

We have then the following important characterization of morphisms in Ω:

Proposition B.1 (Lemma 3.1 [22], Theorem 2.2.6 [32])**.** *Morphisms in Ω are generated by inner face maps, outer face maps, degeneracies and isomorphisms. More specifically every morphism f can be factorized in $f = \sigma\theta\partial$ where ∂ is a composition of degeneracy map, θ is an isomorphism and σ is a composition of face maps.*

Definition B.2. A *planar structure* on a tree T is a collection $\lambda = \{\lambda_v\}_{v \in \mathrm{vert}(T)}$ where λ_v is a total order on $\mathrm{in}(v)$.

A *planar tree* is a couple (T, λ) where T is a tree and λ a planar structure on it.

A planar structure on T induces a planar structure on every subtree t of T that will be denote by λ_t, and a planar structure on $a(T)$ that will be denoted by τ_λ.

Definition B.3. A morphism of planar trees $f\colon (T, \lambda) \to (T', \lambda')$ is a morphism $f\colon T \to T'$ in Ω such that for every vertex $v \in \mathrm{vert}(T)$ the image of the operation associated to (t_v, λ_v) is the operation associated to $(f(v), \tau_{\lambda'_{f(v)}})$.

Remark B.4. Recall that for every finite set F of cardinality n the set of total orders over F is in bijective correspondence with the set of bijections from F to $[n]$.

With this identification, the symmetric group Σ_n acts from the left (by post-composition) on the set of total orders over F (in a free and transitive way). As a consequence the group $G_T = \prod_{v\in \mathrm{vert}(T)} \Sigma_{|v|}$ acts freely and transitively on the set of planar structures over T.

The following is easy to prove:

Proposition B.5. *For every map of trees $f\colon S\to T$ and every planar structure λ on T there is a unique planar structure λ' on S that makes f a morphism of planar trees, i.e. $\lambda f = \lambda'$.*

Definition B.6. Consider a tree T with $|\mathrm{in}(T)| = n$, a total order τ on $\mathrm{in}(T)$ and other trees $T_1,\dots,T_n$. Let E be the quotient of $E' = \mathrm{edge}(T)\cup\mathrm{edge}(T_1)\cup\cdots\cup\mathrm{edge}(T_n)$ obtained identifying $\mathrm{r}(T_i)$ with $\tau^{-1}(i)$ for every $i\in[n]$ and let $\pi\colon E\to E'$ be the canonical projection. The tree $T\circ(T_1,\dots,T_n)$ (called the *grafting* of $(T_1,\dots,T_n)$ over T) is defined as the tree having E as set of edges and $\{\pi(v)\mid v\in \mathrm{vert}(T)\cup\mathrm{vert}(T_1)\cup\cdots\cup\mathrm{vert}(T_n)\}$ as set of vertices. In practice $T\circ(T_1,\dots,T_n)$ should be regarded as the tree obtained by gluing the roots of $T_1\dots,T_n$ to the corresponding leaves of T.

Let $\bar\Omega$ be the subcategory of Ω which has as objects all the objects of Ω and as morphisms only the ones generated by inner face maps, degeneracies and isomorphisms.

Note that in $\bar\Omega$ every morphism $f\colon T\to T'$ induces a morphism on the arities $f\colon a(T)\to a(T')$ (which is always an isomorphism).

B.2.2. *C-trees.*

Definition B.7. A C-tree is a (T,s) where T is in $\bar\Omega$ and $s\colon \mathrm{edge}(T)\to C$ is a map of sets (called a *C-labeling* for T).

A morphism of C-trees $f\colon (A,s)\to(B,t)$ is a morphism $f\colon A\to B$ of underlying trees such that $tf = s$ (in this equation f is regarded as the underlying map on the edges).

The category of C-trees will be denoted $\bar\Omega_C$.

Remark B.8. Given a C-tree (T,s), the tree $a(T)$ and every subtree of a C-tree inherits a C-labeling from s. These trees will always be considered with their induced C-tree structures.

Given C-trees $(T,s),(T_1,s_1),\dots,(T_n,s_n)$ such that $n = |\mathrm{in}(T)|$, and an order τ on $\mathrm{in}(T)$ if $s_i(\mathrm{r}(T_i)) = s(\tau^{-1}(i))$ for every $i\in[n]$ the map $s\cup s_1\cup\cdots\cup s_n$ factors in a C-labeling for $T\circ(T_1,\dots,T_n)$; under these conditions it then makes sense to talk about grafting of C-trees.

B.2.3. *Ordered trees.*

Definition B.9. An *ordered C-tree* is a triple (A,λ_A,τ_A) such that $A\in\bar\Omega_C$, λ_A is a planar structure on A and τ_A is a planar structure on $a(A)$ (i.e. a linear order on the leaves). Note that no compatibility conditions between λ_A and τ_A are required.

An *(ordered) morphism* of ordered C-trees $f\colon (A,\lambda_A,\tau_A)\to(B,\lambda_B,\tau_B)$ is just a morphism of the underlying C-trees such that the order on the leaves is respected, i.e. $\tau_b f = \tau_a$ (here f stands for maps on the edges, restricted to the leaves).

An *unordered morphism* of ordered C-trees is just a morphism of the underlying C-trees with no further conditions.

The category of ordered C-trees and (ordered) morphisms will be denoted by Ω_C^{ord}, while $\Omega_C^{\mathrm{unord}}$ will denote the category of ordered C-trees and unordered morphisms.

Let $\Sigma(C)$ be the category of planar corollas and non-planar morphisms between them.

There is a functor $a\colon \Omega_C^{\mathrm{unord}}\to\Sigma(C)$ sending each tree (T,λ_A,τ_A) to its arity $(a(T),\tau_A)$. The functor a is in fact a bifibration and $\coprod_{S\in\Sigma(C)} a^{-1}(S)$, the disjoint union of the fibers of a, is isomorphic to Ω_C^{ord}.

Remark B.10. Every signature in C can be seen as a planar C-corolla. This identification associates to $S = (c_1, \dots, c_n; c) \in \mathrm{Seq}(C)$ the triple (C_n, s_S, τ) where C_n is the n-corolla, $s_S\colon \mathrm{in}(C_n) \to C$ is the C-labeling such that $s_S(i) = c_i$ for every $n \in [n]$ and $s_S(*) = c$ and τ is the usual order on $[n] = \mathrm{in}(C_n)$. This corolla will be also denoted by S. With S_x,S_y,S_o we will indicate S together with a marking x, y or o on the unique vertex. Conversely to every planar

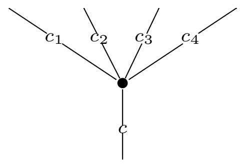
$(c_1, \dots, c_n; c) \quad \rightsquigarrow$ 

Figure 2. Assignement of a planar C-corolla to a signature in C.

C-corolla (A, τ) we can associate a signature in C.

Under this identification the category of collections $\mathrm{Coll}_C(\mathcal{V})$ is isomorphic to $\mathcal{V}^{\Sigma(C)^{\mathrm{op}}}$.

Remark B.11. An ordered C-tree (t, λ_t, τ_t) should be regarded as a scheme of composition for operations in C-coloured operads. This means that given a C-coloured operad $\mathcal{O}$ (in $\mathcal{S}et$) and an operation in $\mathcal{O}$ of signature $a(a(v))$ (i.e. an element of $\mathcal{O}(a(v))$) for every $v \in \mathrm{vert}(t)$, we should be able to compose them following the shape of t and get an operation in $\mathcal{O}$ of signature $a(t)$. More formally given a C-coloured operad $\mathcal{O}$ we can build a map

$$\Gamma_t\colon \bigotimes_{v \in \mathrm{vert}(t)} \mathcal{O}(a(v)) \longrightarrow \mathcal{O}(a(t)).$$

that will be called *composition along* t.

It is defined by induction on m, the number of vertices of t:

- If $m = 0$ then $t = |_c$, then $\mathcal{O}(a(t)) = I$ and the map is just the identity of c in $\mathcal{O}$:
$$u_c\colon I \longrightarrow \mathcal{O}(c; c);$$
- If $m = 1$ then t is a corolla and the map is the symmetry map
$$\sigma^*\colon \mathcal{O}(a(v)) \longrightarrow \mathcal{O}(a(t)),$$
where $\sigma \in \Sigma_{|v|}$ is the unique one such that $\sigma\lambda_{t,v} = \tau_t$;
- If $m > 1$ suppose that Γ_t was already defined for all the trees such that $|\mathrm{vert}(t)| < m$. Suppose that the decomposition of t is $t = s \circ (s_1, \dots, s_n)$ and that σ is the twisting of (t, λ_t, τ_t), then
$$\bigotimes_{v \in verxt} \mathcal{O}(a(v)) \simeq \bigotimes_{i \in [n]} (\bigotimes_{v \in \mathrm{vert}(s_i)} \mathcal{O}(a(v))) \otimes \bigotimes (\bigotimes_{u \in \mathrm{vert}(s)} \mathcal{O}(a(u)))$$
(the isomorphism is given by symmetric isomorphisms of $\mathcal{V}$); the morphism Γ_t is defined as $\sigma^{-1*}\gamma \circ (\bigotimes_{i \in [n]} \Gamma_{s_i}) \otimes \Gamma_s$, where
$$\gamma\colon (\bigotimes_{i \in [n]} \mathcal{O}(a(s_i))) \otimes \mathcal{O}(a(s)) \longrightarrow \mathcal{O}(a(t, \tau\lambda_t, \tau\lambda_t))$$
is the usual composition in $\mathcal{O}$.

It is easy to check that for every morphism of C-coloured operads $f\colon \mathcal{O} \to \mathcal{P}$ the equality $f_{a(t)}\Gamma_t = \Gamma_t \circ (\bigotimes_{v\in \mathrm{vert}(t)} f_{a(v)})$ holds.

B.2.4. *Push-out trees.* For describing the push-out (B.2.1) we need trees that represent shapes of formal compositions of operations coming from $F(o)$, $F(x)$ and $F(y)$; for this reason we are going to put a marking on the vertices of the trees.

Definition B.12. An ordered push-out C-tree is a couple (T, M_T), where $T \in \bar{\Omega}_C^{\mathrm{ord}}$ and $M_T\colon \mathrm{vert}(T) \to \mathrm{Ob}(\Gamma)$ is a map of sets (Γ is the category defined at the beginning of this Appendix). The map M_T will be called the *marking map of* T.

A morphism of push-out C-trees $f\colon (T, M_T) \to (T', M_{T'})$ is a morphism of underlying ordered C-trees $f\colon T \to T'$ such that for every $v \in \mathrm{vert}(T)$ and for every $u \in \mathrm{vert}(f(v))$ there is a morphism in Γ from $M_T(u)$ to $M_T(v)$ (N.B. this morphism goes in the "opposite direction" of f since $v \in T$ and $v' \in T'$); if this morphism exists, it is unique and it will be denoted $\iota_{f,u}$.

A morphism of push-out C-trees will be called *marking-preserving* if for every vertex u of the target $\iota_{f,u} = \mathrm{id}$.

A morphism of push-out C-trees will be called *tree-preserving* if the underlying morphism of trees is the identity.

The category of ordered push-out C-trees will be denoted $\mathbf{T}(C)$.

Alternatively, observe that every morphism of trees $f\colon T \to V$ induces a map of sets $f\colon |\mathrm{vert}(V)| \to |\mathrm{vert}(T)|$; one can consider the functor:

$$
\begin{aligned}
\phi\colon \bar{\Omega}_C^{\mathrm{ord}} &\longrightarrow \mathsf{Cat} \\
T &\longmapsto \Gamma^{\mathrm{op}\,|\mathrm{vert}(T)|}
\end{aligned}
\tag{B.2.2}
$$

The category $\mathbf{T}(C)$ is then isomorphic to the total category of the opfibration $\int \phi$, i.e. the Grothendieck construction over ϕ. In this interpretation marking-preserving morphisms are cocartesian morphisms while tree-preserving morphisms are morphisms lying in the fibers of ϕ.

The following proposition is then straight-forward:

Proposition B.13. *A morphism of push-out trees $f\colon (T, M_T) \to (S, M_S)$ can always be factorized in a marking-preserving morphism followed by a tree-preserving morphism.*

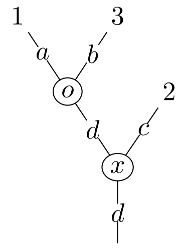
If we replace $\bar{\Omega}_C^{\mathrm{unord}}$ we obtain an unordered version of $\mathbf{T}(C)$, that we will denote by $\mathbf{T}^{\mathrm{un}}(C)$, together with an opfibration $\pi\colon \mathbf{T}^{\mathrm{un}}(C) \to \bar{\Omega}_C^{\mathrm{unord}}$. Clearly $\mathbf{T}^{\mathrm{un}}(C)$ is opfibered over $\Sigma(C)$ via the functor $a\pi$; since $\Sigma(C)$ is a groupoid $a\pi$ is a bifibration.

For every $S \in \mathbf{Ob}(\Sigma(C)) = \mathrm{Seq}(C)$ the fiber $(a\pi)^{-1}(S)$ will be denoted by $\mathbf{T}(C)_S$; note that $\mathbf{T}(C) \cong \coprod_{S\in \mathrm{Seq}(C)} \mathbf{T}(C)_S$. In other word the following diagram is a pull-back of categories (where $\mathrm{Seq}(C)$ is seen as a discrete category and i is the set of objects inclusion):

$$
\begin{tikzcd}
\mathbf{T}(C) \arrow[r] \arrow[d] & \mathbf{T}^{\mathrm{un}}(C) \arrow[d, "\pi a"] \\
\mathrm{Seq}(C) \arrow[r, "i"'] & \Sigma(C)
\end{tikzcd}
$$

Given $T \in \mathbf{T}^{\mathrm{un}}(C)$(or $\mathbf{T}(C)$) we will often write $a(T)$ instead of $a\pi(T)$ (since this is the arity of T).

We will picture ordered push-out C-trees in the following way:



where $a, b, c, d \in C$ represent the labeling on the edges, the symbols on the vertices represent the marking and the numbers on the leaves indicate the total order chosen. The edges of each vertex are ordered from the left to the right and this determines the planar structure.

Remark B.14. Note that we can associate to every morphism of ordered push-out trees $f\colon (A, \lambda_A, \tau_A) \to (B, \lambda_B, \tau_B)$ a change of planar structure $\sigma_f \in G_A$ called the *planar change of f*. It is defined as the unique one such that $\sigma_f \cdot \lambda_A = \lambda'_A$ where λ'_A is the unique planar structure on A that makes f a morphism of planar trees (see Remark B.4).

A morphism f such that $\sigma_f = \mathrm{id}$ is called a *planar morphism.*

We can also associate to f an element $\pi_f \in G_{a(A)} \simeq G_{a(B)} \simeq \Sigma_n$ (where n is the number of leaves of A), called the *leaves permutation of f*; it is defined as the planar change of the induced map $f\colon (a(A), \tau_A, \tau_A) \to (a(B), \tau_B, \tau_B)$.

Given an ordered push-out tree (t, λ, τ) there is a unique planar structure τ_λ induced on $a(t)$ (see Definition B.3). The planar change of the map $(a(t), \tau_\lambda, \tau) \to (a(t), \tau, \tau)$ is called the *twisting* of (t, λ, τ).

Using Proposition B.13 and B.1 it is easy to prove the following:

Proposition B.15. *A map of ordered push-out trees f can be factorized as $f = gh\theta$ where h is planar tree-preserving, g is a composition of planar marking-preserving inner-face and degeneracy morphisms and θ is a marking-preserving isomorphism.*

Definition B.16. For every $S \in \mathrm{Seq}(C)$, by $\mathbf{T}(C)_S$ we will denote the subcategory of $\mathbf{T}(C)$ spanned by all objects $(A, \lambda_A, \tau_A) \in \mathbf{T}(C)$ such that $(a(A), \tau_A) = S$.

B.3. **Description of push-outs.** Recall that we want to describe push-outs in the category $\mathcal{V}$-Oper_C, i.e. colimits of functors $F\colon \Gamma \to \mathcal{V}$-$\mathsf{Oper}_C$, as explained at the beginning of Section B.2. We will denote $F(o)$,$F(x)$ and $F(y)$ by F_o,F_x and F_y respectively.

B.3.1. *Digression on* $\mathsf{Cat}//\mathcal{V}$. Let $\mathsf{Cat}//\mathcal{V}$ be the category defined as follows: the objects are the functors of the kind $F\colon \mathcal{C} \to \mathcal{V}$ where $\mathcal{C}$ is a (not fixed) small category and a morphism between $F\colon \mathcal{C} \to \mathcal{V}$ and $G\colon \mathcal{D} \to \mathcal{V}$ is a couple (h, α) where $h\colon \mathcal{C} \to \mathcal{D}$ and α is a natural transformation between F and $G \circ h$. Compositions and identities are defined in the evident way.

Since $\mathcal{V}$ is a symmetric monoidal category, $\mathsf{Cat}//\mathcal{V}$ comes endowed with a tensor product $F \otimes G\colon \mathcal{C} \times \mathcal{D} \to \mathcal{V}$ such that $F \otimes G((C, D)) = F(C) \otimes F(D)$. The unit of $\otimes$ is the functor $\hat{I}\colon * \to \mathcal{V}$ whose image is I, the unit of $\mathcal{V}$.

There is a functor

$$\begin{aligned} \varinjlim\colon \mathsf{Cat}//\mathcal{V} &\longrightarrow \mathcal{V} \\ F \quad &\longmapsto \varinjlim F \end{aligned},$$

which has a right adjoint $(\tilde{-})$ which associates to every $V \in \mathcal{V}$ the functor

$$\begin{aligned} \tilde{V}\colon * &\longrightarrow \mathcal{V} \\ * &\longmapsto V\,. \end{aligned}$$

The functors $\varinjlim$ and $\tilde{(-)}$ are both strong monoidal since $\mathcal{V}$ is monoidal closed.

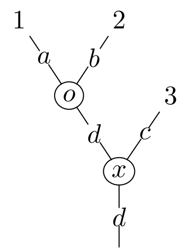
From a (C-coloured) operad $\mathcal{O} = \{\mathcal{O}(\mathcal{S})\}_{S\in \mathrm{Seq}(C)}$ in $\mathsf{Cat}//\mathcal{V}$ we get a (C-coloured) operad $\varinjlim \mathcal{O} = \{\varinjlim \mathcal{O}(\mathcal{S})\}_{S\in\mathrm{Seq}(C)}$ in $\mathcal{V}$.

B.3.2. *Description of Z.* We are now going to describe an operad $\hat{Z}$ in $\mathsf{Cat}//\mathcal{V}$ the colimit of which will be the desired push-out Z (see diagram B.2.1).

For every $S \in \mathrm{Seq}(C)$ the S-component of $\hat{Z}$ is given by the functor:

$$\begin{aligned}\hat{Z}(S)\colon\ \mathbf{T}(C)^{\mathrm{op}}_S &\longrightarrow \mathcal{V}\\ (T,\lambda_T,\tau_T) &\longmapsto \bigotimes_{v\in \mathrm{vert}(T)} F_{M_T(v)}(a(v))\end{aligned} \tag{B.3.1}$$

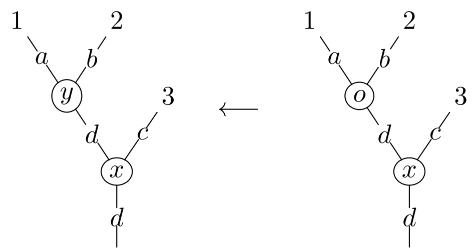
Some pictures at this moment might be helpful. For example the functor $\hat{Z}$ associate to the tree on the left the tensor product on the right:



$$\longmapsto \quad \begin{matrix} F_o(a,b;d)\\ \otimes \\ F_x(d,c;d).\end{matrix}$$

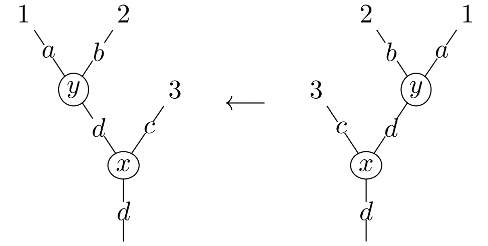
For the definition of $\hat{Z}(S)$ on a morphism $f\colon (A,\lambda_A,\tau_A)\to(B,\lambda_B,\tau_B)$ let us consider several cases:

- if f is planar and tree-preserving then $\hat{Z}(S)(f)$ is $\otimes F(\iota_{f,u})_{a(u)}$ (recall that the $\iota_{f,u}$'s were defined in Definition B.12) ;



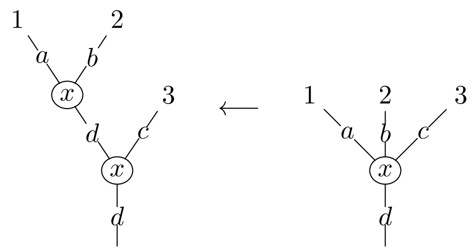
$$\begin{matrix} F_o(a,b;d)\\ \otimes \\ F_x(d,c;d)\end{matrix} \quad \longrightarrow \quad \begin{matrix} F_y(a,b;d)\\ \otimes \\ F_x(d,c;d)\end{matrix}$$

- If f is marking-preserving and the underlying morphism $f\colon A\to B$ is an isomorphism, $\hat{Z}(S)(f)$ is defined as $\theta(\bigotimes_{v\in\mathrm{vert}(A)} \sigma^*_{f,v})$ where $\sigma^*_{f,v}$ is the symmetry morphism on the $a(v)$ component of the operad F_{M_v} associated to $\sigma_{f,v}\in\Sigma_{|v|}$ (see Remark B.14); θ is just the appropriate symmetric isomorphism of $\mathcal{V}$ (that comes with the structure of symmetric monoidal category);



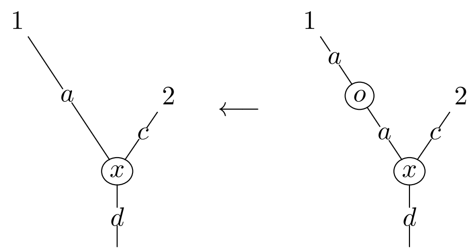
$$\begin{array}{c} F_y(a,b;d) \\ \otimes \\ F_x(d,c;d) \end{array} \longrightarrow \begin{array}{c} F_y(b,a;d) \\ \otimes \\ F_x(c,d;d) \end{array}$$

- If f is planar ,marking-preserving and the underlying morphism of trees $f\colon A \to B$ is an inner face ∂_e, then $\hat{Z}(S)(f)$ is defined using the composition of the operad which marks the extremes of e;



$$\begin{array}{c} F_x(a,b;d) \\ \otimes \\ F_x(d,c;d) \end{array} \longrightarrow F_x(a,b,c;d)$$

- If f is planar ,marking-preserving and the underlying morphism of trees $f\colon A \to B$ is a degeneracy map σ_v then $\hat{Z}(S)$ is defined using the natural unit isomorphism of $\mathcal{V}$ and the identity of $F_{M_v}(a(v))$ (note that $a(v) = (c;c)$ for some $c \in C$).



$$F_x(a,c;d) \longrightarrow \begin{array}{c} I \\ \otimes \\ F_x(a,c;d) \end{array} \longrightarrow \begin{array}{c} F_o(a;a) \\ \otimes \\ F_x(a,c;d) \end{array}$$

These four cases fully determined the definition of $\hat{Z}(S)$ on morphisms (Proposition B.15). It is routine to check that this is a good definition.

We want to define a C-operad structure on the collection $\{\hat{Z}(S)\}_{S\in \mathrm{Seq}(C)}$.

First notice that for every signature $S \in \mathrm{Seq}(C)$ and every $\sigma \in \Sigma_{|S|}$ there is a morphism $\sigma^*\colon \hat{Z}(S) \to \hat{Z}(\sigma^{-1}(S))$ given by the couple (F_σ, μ_σ). The functor $F_\sigma\colon \mathbf{T}(C)^{\mathrm{op}}_S \to \mathbf{T}(C)^{\mathrm{op}}_{\sigma^{-1}S}$ sends a tree (T, λ_T, τ_T) to $(T, \lambda_T, \sigma^{-1}\tau_T)$ and sends a morphism to the unique one having the same underlying morphism of (push-out C-labeled) trees. The natural transformation $\mu_\sigma\colon \hat{Z}(S) \Longrightarrow \hat{Z}(\sigma^{-1}S)F_\sigma$ is just the identity on every object.

Remark B.17. Note that this amount of data defines a functor

$$\mathbf{Z}\colon \mathbf{T}^{\mathrm{un}}(C)^{\mathrm{op}} \longrightarrow \mathcal{V}$$

such that $\mathbf{Z}|_{\mathbf{T}(S)^{\mathrm{op}}} = \hat{Z}(S)$; for every morphism $\sigma \in \Sigma(C)^{\mathrm{op}}$ every cartesian morphism $f\colon (T, \sigma\tau_T) \to (T, \tau_T)$ over σ (in the bifibration $(a\pi)^{\mathrm{op}}$ defined in section B.2.4) is sent to $\mathbf{Z}(f) = \mu_{\sigma,(T,\sigma\tau_T)}$.

Proposition B.18. *There is an operad structure on the collection* $\{\hat{Z}(S)\}_{S\in\mathrm{Seq}(C)}$.

Proof. For every $n \in \mathbb{N}$, for every $S \in \mathrm{Seq}(C)$ such that $|S| = n$, and for every $S_1, \ldots, S_n \in \mathrm{Seq}(C)$ such that $(\mathrm{r}(S_1), \ldots, \mathrm{r}(S_n)) = \mathrm{in}(S)$ we can define the composition morphism:

$$\circ\colon \hat{Z}(S_1) \otimes \cdots \otimes \hat{Z}(S_n) \otimes \hat{Z}(S) \longrightarrow \hat{Z}(S \circ (S_1, \ldots, S_n))$$

as the couple (γ, μ), where γ is the functor

$$\begin{aligned} \gamma\colon \mathbf{T}(C)^{\mathrm{op}}_{S_1} \times \cdots \times \mathbf{T}(C)^{\mathrm{op}}_{S_n} \times \mathbf{T}(C)^{\mathrm{op}}_S &\longrightarrow \mathbf{T}(C)^{\mathrm{op}}_{S\circ(S_1,\ldots,S_n)} \\ (T_1, \ldots, T_n) &\longmapsto T \circ (T_1, \ldots, T_n) \end{aligned}$$

(we leave to the reader the definition of γ on the morphisms). The natural transformation $\mu\colon \hat{Z}(S_1) \otimes \cdots \otimes \hat{Z}(S_n) \otimes \hat{Z}(S) \Longrightarrow \hat{Z}(S \circ (S_1, \ldots, S_n))$ is defined using symmetric isomorphisms of $\mathcal{V}$.

The identity on the colour $c \in C$ is given by the morphism:

$$\mathrm{id}_C = (G, \iota)\colon \tilde{I} \longrightarrow \hat{Z}(c; c),$$

where $G\colon * \to \mathbf{T}(C)^{\mathrm{op}}_{(c;c)}$ has as image $|_c$, the empty tree labeled by c. The natural transformation ι is just an arrow $I = \tilde{I}(*) \to \hat{Z}(c; c)(|_c) = I$, which is defined to be the identity of I.

For every signature $S \in \mathrm{Seq}(C)$ the right action of $\Sigma_{|S|}$ is given for every $\sigma \in \Sigma_{|S|}$ by the morphism:

$$\sigma^*\colon \hat{Z}(S) \longrightarrow \hat{Z}(\sigma^{-1}(S)),$$

described above. We leave to the reader to check that this defines an operad structure, i.e. to check associativity, equivariance and unitality. $\square$

Corollary B.19. *The collection* $Z = \{\varinjlim \hat{Z}(S)\}_{S\in\mathrm{Seq}(C)}$ *has an operad structure.*

Remark B.20. The functor $(a\pi)^{\mathrm{op}}$ is an opfibration. By construction the underlying symmetric collection of the operad Z coincides with $\varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}$ (*cf.* Remark B.17 and Section B.1 for the notation).

Remark B.21. From Remark B.11, for every planar tree with an order on the leaves we have "composition" morphisms:

$$(H_t, \nu_t) =: \Gamma^{\hat{Z}}_t\colon \bigotimes_{v\in\mathrm{vert}(t)} \hat{Z}(a(v)) \longrightarrow \hat{Z}(a(t))$$

and

$$\varinjlim \Gamma_t^{\hat{Z}} = \Gamma_t^{Z} \colon \bigotimes_{v\in \mathrm{vert}(t)} Z(a(v)) \longrightarrow Z(a(t)).$$

It is routine to check that the following diagram commutes:

$$\begin{tikzcd} \bigotimes_{v\in\mathrm{vert}(t)} \hat{Z}(a(v))(t_v) \arrow[r, "\nu_t"] \arrow[d] & \hat{Z}(a(t))(t) \arrow[d] \\ \bigotimes_{v\in\mathrm{vert}(t)} Z(a(v)) \arrow[r, "\Gamma_t^Z"'] & Z(a(t)). \end{tikzcd} \tag{B.3.2}$$

There are morphisms of collections from $p' \colon \tilde{F_x} \to \hat{Z}$ and $q' \colon \tilde{F_y} \to \hat{Z}$ (recall that $(\tilde{-})$ is the right adjoint of $\varinjlim$). On the S-operation p is defined as $p'_s = (P_S, \mu_S)$ where $P(*) = S_x$ and μ_S is just the identity. In a similar way $q'_S = (Q_S, \nu_S)$ with $Q(*) = S_y$ and ν_S which is just the identity.

These maps are not maps of operads (at least not in a strict sense), but they become so in $\mathcal{V}$.

In fact, applying $\varinjlim$ to p' and q' we obtain maps $p \colon F_x \to Z$ and $q \colon F_y \to Z$. These maps make diagram B.2.1 commute and are in fact maps of operads.

Proposition B.22. *The morphisms $p \colon F_x \to \hat{Z}$ and $q \colon F_y \to \hat{Z}$ are morphisms of operads.*

Proof. Consider $s, s_1, \ldots, s_n \in \mathrm{Seq}(C)$ such that $s \circ (s_1, \ldots, s_n)$ makes sense and let $t = s \circ (s_1, \ldots, s_n)$ (where the grafting is view as a grafting of trees) then the following diagram commutes:

$$\begin{tikzcd} (\bigotimes_{i\in[n]} F_x(s_i)) \otimes F_x(s) \arrow[r, "p'"] \arrow[d, "\Gamma_t^{F_x}"] & (\bigotimes_{i\in[n]} \hat{Z}(s_i)(s_{i,x})) \otimes \hat{Z}(s)(s_x) \arrow[r] \arrow[d, "\nu_t"] & (\bigotimes_{i\in[n]} Z(s_i)) \otimes Z(s) \arrow[d, "\Gamma^{\hat{Z}} t"] \\ F_x(s \circ (s_1, \ldots, s_n)) \arrow[r, "p'"] & \hat{Z}(a(t))(a(t)_x) \arrow[r] & Z(a(t)) \ . \end{tikzcd}$$

In fact the square on the right is just (B.3.2) and the outer square shows that p respects the composition.

For every signature $S \in \mathrm{Seq}(C)$ and every $\sigma \in \Sigma_{|S|}$ if in the diagram above one takes as tree the corolla associated to S with a twisting σ (see Remark B.14) instead of t one obtains the compatibility of p with the action of σ.

In the same way taking $|_c$, the empty tree labeled with $c \in C$, instead of t one obtains the preservation of the identity of c.

This proves that p is a map of operads. The case of q is analogous. $\square$

Proposition B.23. *Diagram (B.2.1) with Z, p and q defined as above, is a push-out diagram in the category $\mathcal{V}$-Oper_C.*

Proof. Suppose another C-coloured operad L is given, with morphism $h \colon F_x \to L$ and $k \colon F_y \to L$ such that $hF(\iota_2) = kF(\iota_1)$. We have to define a morphism of C-coloured operads $l \colon Z \to L$ such that $lp = h$ and $lq = k$.

For every $S \in \mathrm{Seq}(C)$, to define a morphism $l_S \colon Z(S) \to L(S)$ is equivalent to give a morphism from $l_{S,t} \colon \hat{Z}(S)(t) \to L(S)$ for every $t \in \mathbf{T}(C)_S$ in a compatible way (for every morphism $i \colon t \to t'$ the relation $l_{S,t} = l_{S,t'} Z(S)(i)$ has to hold).

Recall that $\hat{Z}(S)(t) = \bigotimes_{v\in \mathrm{vert}(t)} M_t(v)(a(v))$; the morphism $l_{S,t}$ is then defined as the composition of

$$\bigotimes_{v\in \mathrm{vert}(t)} w_v \colon Z(S)(t) \longrightarrow \bigotimes_{v\in \mathrm{vert}(t)} L(a(v))$$

and

$$\Gamma_t \colon \bigotimes_{v\in \mathrm{vert}(t)} L(a(v)) \longrightarrow L(S),$$

where:

$$w_v = \begin{cases} h_{a(v)} & \text{if } M_t(v) = x \\ k_{a(v)} & \text{if } M_t(v) = y \\ yk_{a(v)} & \text{if } M_t(v) = o \end{cases}$$

and Γ_t is the composition of L obtained following the shape of t (seen as a planar tree without marking on the vertices) as defined in Remark B.11.

It can be checked that this is a good definition and the obtained map is a morphism of operads.

It remains to check that l is unique. Suppose that there is another l' such that $l'p = h$ and $l'q = k$; this implies that for every $S \in \mathrm{Seq}(C)$ one has to define $l'_{S,S_x} = h_S = l_{S,S_x}$ and $l'_{S,S_y} = k_S = l_{S,S_y}$.

For every $S \in \mathrm{Seq}(C)$ and $t \in Z(S)$ the following diagram has to commute:

$$\begin{tikzcd} \bigotimes_{v\in \mathrm{vert}(t)} \hat{Z}(a(v))(t_v) \arrow[r, "\simeq", "\nu_t"'] \arrow[d, "\otimes_v l'_{a(v),t_v}"'] & \hat{Z}(S)(t) \arrow[d, "l'_{S,t}"] \\ \bigotimes_{v\in \mathrm{vert}(t)} L(a(v)) \arrow[r, "\Gamma_t"] & L(S), \end{tikzcd}$$

where ν_t is defined as in Remark B.21.

As we see, the vertical arrow on the left has to be equal to $\otimes_v l_{a(v),a(v)_{M_t(v)}}$ so $l'_{S,t}$ is forced to be defined as $l_{S,t}$.

$\square$

B.4. **Push-outs along fully-faithful inclusions.** Before proving the main result of the appendix, Proposition B.27, we need a couple of techincal lemmas.

Definition B.24. For a map of sets $f\colon A \to B$ and a sequence $S \in \mathrm{Seq}(B)$ we denote by T_f the category of couples $(t, \{l_v\}_{v\in \mathrm{vert}(t)})$ where t is a planar ordered trees and $\{l_v\}_{v\in\mathrm{vert}(t)}$ is a collection of maps $l_v \colon \mathrm{edge}(a(v)) \to A$, such that if $e \in \mathrm{edge}(T)$ belong to two distinct vertices v, v' then $l_v(e) \neq l_{v'}(e)$ and $fl_v(e) = fl_{v'}(e)$; in other words each outer edge of t is labeled by an element of A and each inner edge is labeled by a couple $(a, a') \in A \times A$ such that $f(a) = f(a')$ and $a \neq a'$ and $a(t) = S$. The morphisms in this category are given by non-planar isomorphisms preserving the order on the leaves and the labeling. For every tree $t \in T_f(S)$ and for every $v \in \mathrm{vert}(t)$ the corollas $a(t)$ and $a(v)$ has a natural planar A-tree structure. For every $S \in \mathrm{Seq}(B)$ let $T_f(S)$ be the full subcategory of T_f spanned by the trees t such that $f(a(t)) = S$.

For every $\mathcal{O} \in \mathcal{V}\text{-}\mathsf{Oper}_A$ there is a functor $\bar{\mathcal{O}}$

$$\begin{aligned} \bar{O} \colon T_f^{\mathrm{op}} &\longrightarrow \mathcal{V} \\ t &\longmapsto \textstyle\bigotimes_{v\in \mathrm{vert}(t)} \mathcal{O}(a(v)) \end{aligned}$$

The following lemma is straightforward to prove, and can be regarded as a particular case of [2, Theorem 6.16].

Lemma B.25. *For every map of sets $f\colon A \to B$, every $\mathcal{O} \in \mathcal{V}\text{-}\mathsf{Oper}_A$ and $S \in \mathrm{Seq}(B)$:*

$$f_!\mathcal{O}(S) = \varinjlim_{t \in T_f(S)^{\mathrm{op}}} \bar{\mathcal{O}}(t)$$

Suppose a push-out square is given in $\mathcal{S}et$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "h"] \\ C \arrow[r, "g"'] & X \end{tikzcd} \tag{B.4.1}$$

and suppose futhermore that i is an injection. This induces a commutative square of adjunctions

$$\begin{tikzcd} \mathcal{V}\text{-}\mathsf{Oper}_A \arrow[r, shift left, "f_!"] \arrow[d, shift left, "i_!"] & \mathcal{V}\text{-}\mathsf{Oper}_B \arrow[l, shift left, "f^*"] \arrow[d, shift right, "h_!"'] \\ \mathcal{V}\text{-}\mathsf{Oper}_C \arrow[u, shift left, "i^*"] \arrow[r, shift right, "g_!"'] & \mathcal{V}\text{-}\mathsf{Oper}_X \arrow[l, shift right, "g^*"'] \arrow[u, shift right, "h^*"'] \end{tikzcd}$$

with a comparison natural transformation $\phi\colon f_!i^* \Longrightarrow h^*g_!$.

Proposition B.26. *Given a diagram as (B.4.1) the comparison natural tranformation*

$$\phi\colon f_!i^* \Longrightarrow h^*g_!$$

is a natural isomorphism.

Proof. For every $\mathcal{O} \in \mathcal{V}\text{-}\mathsf{Oper}_C$ and $S \in \mathrm{Seq}(B)$

$$f_!i^*\mathcal{O}(S) = \varinjlim_{t \in T_f(S)^{\mathrm{op}}} \bar{\mathcal{O}}(i(t))$$

and

$$h^*g_!\mathcal{O}(S) = \varinjlim_{t' \in T_g(h(S))^{\mathrm{op}}} \bar{\mathcal{O}}(t').$$

The comparison map $\phi_{\mathcal{O}}(S)\colon f_!i^*\mathcal{O}(S) \to h^*g_!\mathcal{O}(S)$ is induced by the functor

$$\begin{aligned} \bar{i}\colon T_f(S) &\longrightarrow T_g(h(S)) \\ t &\longmapsto i(t) \end{aligned}$$

the statement follows from the fact that, under our hypotheses, $\bar{i}$ is an isomorphism of categories. □

Let now set ourself in the category of coloured operads, with no set of colours fixed. We want to prove the following fact:

Proposition B.27. *Suppose we have a push-out diagram in $\mathcal{V}$-Oper ($\mathcal{V}$-NSOper or $\mathcal{V}$-cfOper):*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & P \arrow[d, "h"] \\ B \arrow[r, "g"'] & Q. \end{tikzcd}$$

If i is injective on colours and fully-faithful then h is injective on colours and fully-faithful.

Proof. (Proposition B.27) As we remarked at the beginning of the Appendix we will give a proof only for $\mathcal{V}$-Oper.

Recall that since the colours functor from $\mathcal{V}$-Oper to $\mathcal{S}et$ preserve colimits, at the level of colours the diagram

$$\begin{tikzcd} \mathbf{Col}(A) \arrow[r, "f"] \arrow[d, "i"'] & \mathbf{Col}(P) \arrow[d, "h"] \\ \mathbf{Col}(B) \arrow[r, "g"'] & \mathbf{Col}(Q) \end{tikzcd}$$

is a push-out diagram. The first part of the statement then follows from the fact that in $\mathcal{S}et$ the push-out of an injective function is injective. For simplicity let $C = \mathbf{Col}(Q)$, since h is injective we can suppose that $\mathbf{Col}(P) \subset C$.

The only thing we have to prove is that h is fully-faithful, i.e. $h^u\colon P \to h^*(Q)$ (notation of Section 3) is an isomorphism in $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(P)}$.

Recall from the previous discussion about colimits in the category of $\mathcal{V}$-Oper that our diagram is a push-out if and only if the following one

$$\begin{tikzcd} g_!i_!A \arrow[r, "h_!(f_u)"] \arrow[d, "g_!(i_u)"'] & h_!P \arrow[d, "h_u"] \\ g_!B \arrow[r, "g_u"'] & Q \end{tikzcd} \tag{B.4.2}$$

is a push-out diagram in $\mathcal{V}$-Oper_C.

In $\mathcal{V}$-$\mathsf{Oper}_{\mathbf{Col}(P)}$ we have the equality $h^u = \eta_P h^*(h_u)$, where η is the unit of the adjunction $(h_!, h^*)$. Since h is injective on colours the unit is a natural isomorphism, so we are reduced to prove that $h^*(h_u)$ is an isomorphism.

To prove that $h^*(h_u)$ is an isomorphism is exactly to prove that for every $S = (s_1, \dots, s_n; s) \in \mathrm{Seq}(\mathbf{Col}(P))$ the morphism $h_S\colon h_!P(S) = P(S) \to Q(S)$ is an isomorphism.

Note that by Propostion B.26 $h^*g_!(i_u) \cong f_!i^*(i_u) \cong f_!i^u$; but i^u is an isomorphism by hypothesis, thus $h^*g_!(i_u)$ is an isomorphism, in other words

$$g_!(i_u)(S)\colon g_!i_!A(S) \longrightarrow \ g_!B(S)$$

is an isomorphism for every $S \in \mathrm{Seq}(\mathbf{Col}(P))$.

Fix $S = (s_1, \dots, s_n; s) \in \mathrm{Seq}(\mathbf{Col}(P))$. From the description of push-outs given in section B.3 we get that $Q(S) = \varinjlim \hat{Z}(S)$ for a functor $\hat{Z}(S)\colon \mathbf{T}(C)_S^{op} \to \mathcal{V}$, where $\mathbf{T}(C)_S$ is the category of ordered push-out trees with arity S (in which this time we can suppose that the vertices are marked in $\{A, B, P\}$).

The map $h_S\colon h_!P(S) = P(S) \to Q(S)$ is exactly the canonical map from $\hat{Z}(S)(S_P) \to Q(S)$.

We are now going to do a series of changes of the index category of the colimit $\varinjlim \hat{Z}(S)$ in order to show that it is canonically isomorphic to $P(S)$.

First, consider $\mathbf{T}(C)'_S$ the full subcategory of $\mathbf{T}(C)_S$ spanned by all the trees t such that for every $v \in \mathrm{vert}(t)$:

- if $M_t(v) = A$ then $a(v) \in \mathrm{Seq}(hf(\mathbf{Col}(A)))$ or $a(v) = (c; c)$ for some $c \in C$;
- if $M_t(v) = B$ then $a(v) \in \mathrm{Seq}(g(\mathbf{Col}(B)))$ or $a(v) = (c; c)$ for some $c \in C$;
- if $M_t(v) = P$ then $a(v) \in \mathrm{Seq}(\mathbf{Col}(P))$ or $a(v) = (c; c)$ for some $c \in C$;

Note that if t does not belong to $\mathbf{T}(C)'_S$ then $\hat(Z)(S)(t) = \varnothing$ and if $t \in \mathbf{T}(C)'_S$ and there exists a morphism $t \to t'$ then also $t' \in \mathbf{T}(C)'_S$ so if $\hat{Z}'(S)$ is the restriction of $\hat{Z}(S)$ to $\mathbf{T}(C)'^{\mathrm{op}}_S$ we have $\varinjlim \hat{Z}(S) = \varinjlim \hat{Z}'(S)$.

Now we want to add an inverse to every morphism l in of $\mathbf{T}(C)'_S$ of the following kinds:

(i) $l\colon (T, \lambda_T, M_T) \to (T, \lambda_T, M'_T)$ such that T has a vertex v such that $a(v) \in \mathrm{Seq}(hf(\mathbf{Col}(A)))$, l is tree-preserving and $M_T(v) = B$, $M'_T(v) = A$ and $M_T(u) = M'_T(u)$ for every $u \neq v$.

(ii) $l\colon (T, \lambda_T, M_T) \to (T, \lambda_T, M'_T)$, where T has a vertex v such that $a(v) = (c; c)$ for $c \in C - \mathbf{Col}(P)$, l is the identity, $M_T(v) = P$, $M'_T(v) = A$ and $M_T(u) = M'_T(u)$ for every $u \neq v$.

The category $\mathbf{T}(C)''_S$ is the one obtained by $\mathbf{T}(C)'_S$ inverting all morphisms of kind (i) and (ii). It exists, since for every $T, T' \in \mathbf{T}(C)''_S$ the set of morphisms $\mathbf{T}(C)''_S(T, T')$ is a subset of the set of morphisms between the underlying trees.

Let $i\colon \mathbf{T}(C)'_S \to \mathbf{T}(C)''_S$ be the obvious inclusion. All the morphisms of kind (i) and (ii) are sent to isomorphisms by $\hat{Z}'$. For morphisms of kind (i) this follows from the fact that if $S \in \mathrm{Seq}(hf(\mathbf{Col}(A))) \subset \mathrm{Seq}(\mathbf{Col}(P))$ then $g_!(i)_S\colon g_!i_!A(S) \to g_!B(S)$ is an isomorphism. For morphisms of kind (ii) similarly this is a consequence of the fact that if $c \in \mathbf{Col}(Q) - \mathbf{Col}(P)$ then

$$g_!(i)_{(c;c)}\colon g_!i_!A(c;c) \longrightarrow h_!P(c;c)$$

is an isomorphism (both terms are actually isomorphic to the unit of $\mathcal{V}$).

The functor $\hat{Z}'$ factors then through i and a functor $\hat{Z}''\colon \mathbf{T}(C)''^{\mathrm{op}}_S \to \mathcal{V}$, moreover $\varinjlim \hat{Z}' = \varinjlim \hat{Z}''$.

The object S_P (the corolla associated to S marked with P) is initial in $P\mathbf{T}(C)''_S$. To prove this fact, notice first that for every $t \in \mathbf{T}(C)''_S$ there is at most one map from S_P to t; in fact the maps in $P\mathbf{T}(\mathbf{Col}(Q))''$ are uniquely determined by the underlying maps of C-trees, and for every C-tree t with $a(t) = S$ and an order on the leaves τ there is a unique map from $a(S)$ to t respecting τ.

To prove that S_P is initial in $P\mathbf{T}(C)''_S$ is then sufficient to exhibit a map from S_P to t for every $t = (t, M_t, \lambda_t, \tau_t) \in P\mathbf{T}(C)''_S$; this map is given by the following composition:

$$S_P \xrightarrow{\theta} a(t) \xrightarrow{q} t''' \xrightarrow{r} t'' \xrightarrow{r} t' \xrightarrow{s} t,$$

where

- $t' = (t, M_{t'}, \lambda_t, \tau_t)$ and s is a tree-preserving morphism such that for every vertex $v \in \mathrm{vert}(t)$:
  - if $a(v) \in \mathrm{Seq}(hf(\mathbf{Col}(A)))$ and $M_t(v) = A$ then $M_{t'}(v) = B$;
  - if $a(v) \notin \mathrm{Seq}(g(\mathbf{Col}(B)))$ then $M_{t'}(v) = P$;
  - otherwise $M_{t'}(v) = M_t(v)$.

  The map s is obtained using inverses of the maps of kind (ii). Notice that t' has no vertex marked by A.
- r is an iteration of marking-preserving inner faces of inner edges with both vertices marked by B. This process is carried out until t'' has no inner edges of this kind. As a consequence all the vertices v in t'' that are marked by B have arity in $\mathrm{Seq}(hf(\mathbf{Col}(A)))$;
- r is a tree preserving morphism such that on every vertex $v \in \mathrm{vert}(t''')$ the marking in P. This map is obtained by compositions of inverses of morphism of kind i;
- q is obtained by composition of marking-preserving inner faces;
- θ is the unique (marking-preserving) isomorphism from S_P to $a(t)$.

Since S_P is initial (final in the opposite category) the map

$$h_S\colon h_!P(S) = Z(S)(S_P) \longrightarrow Q(S) = \varinjlim Z(S)$$

is an isomorphism, and this concludes the proof. □

Appendix C. Push-out alongs free maps and Topological Operads

The aim of this section is to prove that for every $C \in \mathcal{S}et$ the operad Op_C is $\mathcal{T}_1$-admissible in Top, where Top the category of compactly generated (weak Hausdorff) spaces and $\mathcal{T}_1$ is the class of T_1 closed inclusions.

Recall that a T_1 *closed inclusion* is a map $f\colon X \to Y$ in Top such that it is a closed inclusion and every point in $Y \backslash f(X)$ is closed. The class of T_1 closed inclusions is monoidally saturated in Top and will be denoted by $\mathcal{T}_1$.

The monoidal model category $(\mathsf{Top}, \times, *)$ admits a monoidal fibrant replacement functor and a contains a cocommutive comonoidal interval thus, thanks to Proposition 2.4 and 2.13, it is sufficient to check condition ii of Definition 2.2 is satisfied by Op_C. In other words we want to check that given a generating (trivial) cofibration $i\colon K_0 \to K_1$ in $\mathsf{Top}^{\mathrm{Seq}(C)}$ and a map of C-coloured operads $\alpha\colon F_{\mathsf{Op}_C}(K_0) \to X$ the map i_α in the push-out diagram

(C.0.3)
$$\begin{tikzcd} F_{\mathsf{Op}_C}(K_0) \arrow[r, "\alpha"] \arrow[d, "F_{\mathsf{Op}_C}(i)"'] & X \arrow[d, "i_\alpha"] \\ F_{\mathsf{Op}_C}(K_1) \arrow[r] & Y \end{tikzcd}$$

is a local $\mathcal{T}_1$-morphism.

For this reason we begin by describing push-outs along free maps in $\mathcal{V}$-Oper_C.

C.1. **Description of push-out along free maps.** Consider a a bicomplete closed monoidal category $\mathcal{V}$ and a set C. Suppose a map $i\colon K_0 \to K_1$ in $\mathcal{V}^{\mathrm{Seq}(C)}$, a map $\alpha\colon F_{\mathsf{Op}_C}(K_0) \to X$ in $\mathcal{V}$-Oper_C and a push-out diagram as (C.0.3) are given.

Consider the category of ordered push-out C-trees $\mathbf{T}(C)$ introduced in Section B.2.4, for simplicity this time the vertices can be marked by K_0 (instead of o), K_1 (instead of y) or X (in place of x).

Let us define $F\mathbf{T}(C)$ ($F\mathbf{T}^{\mathrm{un}}(C)$), a (non-full) subcategory of $\mathbf{T}(C)$ (resp. $\mathbf{T}^{\mathrm{un}}(C)$): it contains all objects of $\mathbf{T}(C)$ and all morphisms $f\colon T \to S$ in $\mathbf{T}(C)$ which are composition of:

- inner-faces of inner edges between vertices marked by X;
- degeneracies map of vertices marked by X;
- isomorphism whose change of planar structure (*cf.* Remark B.14) is the identity on the vertices marked by K_0 and K_1.

We will refer to $F\mathbf{T}(C)$ as the *category of free-push-out C-trees.*

As in Section B.2.4, there is a bifibration $a\pi\colon F\mathbf{T}^{\mathrm{un}}(C) \to \Sigma(C)$ and pull-back diagram of categories

(C.1.1)
$$\begin{tikzcd} F\mathbf{T}(C) \arrow[r] \arrow[d] & F\mathbf{T}^{\mathrm{un}}(C) \arrow[d, "\pi a"] \\ \mathrm{Seq}(C) \arrow[r, "i"'] & \Sigma(C) \end{tikzcd}$$

For every $S \in \mathrm{Seq}(C)$ let us denote by $F\mathbf{T}(C)_S$ the fiber $(a\pi)^{-1}(S)$, i.e. the full subcategory of $F\mathbf{T}(C)$ spanned by the objects $(T, \lambda_T, M_T, \tau_T)$ such that $a(T) = S$. Note that $F\mathbf{T}(C) \simeq \coprod_{S \in \mathrm{Seq}(C)} F\mathbf{T}(C)_S$.

For every $S \in \mathrm{Seq}(C)$, using diagram (C.0.3) we can define a functor

(C.1.2)
$$Z_{i,\alpha}(S)\colon F\mathbf{T}(C)_S^{\mathrm{op}} \longrightarrow \mathcal{V}$$

in the same way in which we define functor (B.3.1), that is

$$Z_{i,\alpha}(T, M_T) \simeq \bigotimes_{v\in \mathrm{vert}(T)} M_T(v)(a(v)).$$

Furthermore for every $\sigma \in \Sigma_{|S|}$ there is a morphism in $\mathsf{Cat}//\mathcal{V}$

$$(F_\sigma, \mu_\sigma)\colon Z_{i,\alpha}(S) \longrightarrow Z_{i,\alpha}(\sigma^{-1}S) \tag{C.1.3}$$

This amount of data can be collected into a functor

$$\mathbf{Z}_{i,\alpha}\colon F\mathbf{T}^{\mathrm{un}}(C)^{\mathrm{op}} \to \mathcal{V}$$

as was done in Remark B.17.

Using techniques similar to the ones used in Appendix B one can check that in diagram (C.0.3)

$$Y(S) \simeq \varinjlim Z_{i,\alpha}(S). \tag{C.1.4}$$

more precisely, one can check that the underlying symmetric collection of Y coincides with $\varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}_{i,\alpha}$ (Section B.1).

Furthermore if we denote by $Z_{i,\alpha,0}(S)$ the restriction of $Z_{i,\alpha}(S)$ to the full subcategory of $F\mathbf{T}(C)_S$ spanned by the trees with vertices marked only by X then $X(S) \simeq \varinjlim Z_{i,\alpha,0}(S)$ and, under these identifications, $i_\alpha(S)$ is the canonical map $\varinjlim Z_{i,\alpha,0}(S) \to \varinjlim Z_{i,\alpha}(S)$.

We remark that $F\mathbf{T}(C)^{\mathrm{op}}$ is in fact the *internal classifier for free algebras extensions* of Op_C described in [2], thus (C.1.2) and (C.1.4) can be regarded as a particular case of the results of [2, Section 7.1].

The category $F\mathbf{T}(C)_S$ is quite big so, in the case $\mathcal{V} = \mathsf{Top}$, it might be hard to prove that i_α is a T_1 closed inclusion. Since we know that T_1 closed inclusions are closed under push-out and transfinite compositions, we will try to decompose this colimit as a transfinite composition of iterated push-outs along colimits of maps which are simpler to understand.

Definition C.1. Let us call a tree $T \in F\mathbf{T}^{\mathrm{un}}(C)$ *minimal* if for each $e \in \mathrm{edge}(T)$ there exist one and only one $v \in \mathrm{vert}(T)$ such that $e \in v$ and $M_T(v) = X$. The full subcategory of $F\mathbf{T}^{\mathrm{un}}(C)$ ($F\mathbf{T}(C)_S$) spanned by minimal trees will be denoted by $F\underline{\mathbf{T}}^{\mathrm{un}}(C)$ (resp. $F\underline{\mathbf{T}}(C)_S$). It is important to notice that $F\underline{\mathbf{T}}(C)_S$ is cofinal in $F\mathbf{T}(C)_S$.

Note that every map between minimal trees can be factored as a composite of inner-faces and isomorphisms (whose domains and codomains are not necessarily minimal), in other words no degeneracies are involved.

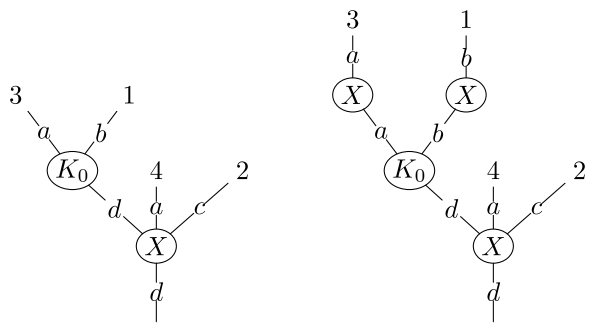


FIGURE 3. The tree on the right is minimal, while the tree on the left is not.

From now on, to keep the notation a little bit lighter, we will denote by $\mathbf{Z}$ the functor (C.1.2), leaving the subscripts implicit.

For every $n \in \mathbb{N}$ we can define the following full subcategories of $F\mathbf{T}^{\mathrm{un}}(C)$ $(F\underline{\mathbf{T}}(C)_S)$:

- $F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{\leq n}$ (resp. $F\underline{\mathbf{T}}(C)_{S,\leq n}$) spanned by all the trees with at most n vertices not marked by X;
- $F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n$ (resp. $F\underline{\mathbf{T}}(C)_{S,n}$) spanned by all the trees with exactly n vertices not marked by X;
- $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^0_n$ (resp. $F\underline{\mathbf{T}}(C)^0_{S,n}$) spanned by all the trees with exactly n vertices not marked by X and at least one marked by K_0;
- $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ (resp. $F\underline{\mathbf{T}}(C)^1_{S,n}$) spanned by all the trees with exactly n vertices not marked by X and no vertices marked by K_0;
- $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^+_{\leq n} = F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{S,\leq n} \cup F\underline{\mathbf{T}}^{\mathrm{un}}(C)^0_{S,n+1}$ (resp. $F\underline{\mathbf{T}}(C)^+_{S,\leq n} = F\underline{\mathbf{T}}(C)_{S,\leq n} \cup F\underline{\mathbf{T}}(C)^0_{S,n+1}$).

All this subcategory of $F\mathbf{T}^{\mathrm{un}}(C)$ are bifibered over $\Sigma(C)$ via the (appropriate restriction of) $a\pi$.

Let $\mathbf{Z}_{\leq n}$, $\mathbf{Z}_n$, $\mathbf{Z}^0_n$,$\mathbf{Z}^1_n(S)$,$\mathbf{Z}^+_{\leq n}(S)$ be the restrictions of $\mathbf{Z}$ the corresponding subcategory of $F\mathbf{T}^{\mathrm{un}}(C)$.

It it is clear that we have fully faithful inclusions $j_n \colon F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n \to F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{n+1}$ and that $F\underline{\mathbf{T}}^{\mathrm{un}}(C) = \bigcup_{n\in\mathbb{N}} F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n$. This produces a transfinite composition in $\mathcal{V}^{\Sigma(C)^{\mathrm{op}}} \cong \mathrm{Coll}_C(\mathcal{V})$:

$$X \simeq \varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}_{\leq 0} \xrightarrow{j_0} \varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}_{\leq 1} \xrightarrow{j_1} \dots \xrightarrow{j_{n-1}} \varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}_{\leq n} \xrightarrow{j_{n+1}} \dots \tag{C.1.5}$$

whose colimit is i_α.

We would like to express this transfinite composition as an iteration of push-outs.

We notice at first that $F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{\leq n} = F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{\leq n-1} \cup F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n = F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n \cup F\underline{\mathbf{T}}^{\mathrm{un}}(C)^+_{\leq n-1}$. Since there are no morphisms between objects in $F\underline{\mathbf{T}}^{\mathrm{un}}(C)_{\leq n-1}$ and $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ the following diagram is a push-out square in $\mathrm{Coll}_C(\mathcal{V})$:

$$\begin{tikzcd} \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}^0_n \arrow[r, "u_n"] \arrow[d] & \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}_n \arrow[d] \\ \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}^+_{\leq,n-1} \arrow[r] & \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}_{\leq,n} \end{tikzcd} \tag{C.1.6}$$

Furthermore we have the following:

Lemma C.2. *For every $n \in \mathbb{N}$ the subcategory $F\underline{\mathbf{T}}(C)_{\leq n}$ is cofinal in $F\underline{\mathbf{T}}(C)^+_{\leq n}$ (or equivalently $F\underline{\mathbf{T}}(C)^{\mathrm{op}}_{\leq n}$ is final in $F\underline{\mathbf{T}}(C)^{+\,\mathrm{op}}_{\leq n}$).*

As a consequence diagram (C.1.6) can be rewritten as:

$$\begin{tikzcd} \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}^0_n \arrow[r, "u_n"] \arrow[d] & \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}_n \arrow[d] \\ \varinjlim \mathbf{Z}_{\leq,n-1} \arrow[r] & \varinjlim^{a\pi^{\mathrm{op}}} \mathbf{Z}_{\leq,n} \end{tikzcd} \tag{C.1.7}$$

This means that the n-th map of transfinite composition (C.1.5) can be expressed as a push-out along the canonical map:

$$u_n \colon \varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}^0_n \to \varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}_n \tag{C.1.8}$$

Note that $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^0_n$ and $F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n$ are fibred over $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$.

Indeed a functor $p \colon F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n \to F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ is defined as follows:

- for every object $(T, M_T) \in F\underline{\mathbf{T}}^{\mathrm{un}}(C)_n$ the image $p((T, M_T))$ is (T, M'_T), where for every $v \in \mathrm{vert}(T)$ the marking $M'_T(v)$ is X if $M_T(v) = X$ and is K_1 otherwise.
- for every morphism $f\colon (T, M_T) \to (S, M_S)$ the image $p(f)$ is the unique morphism with the same underlying morphism of trees.

It is easy to verify that this is a good definition and that p is indeed a fibration or equivalently p^{op} is a opfibration.

For every $n \in \mathbb{N}$ we define the n-cube category $\square_n$ as the cartesian product $\Delta[1]^n$; the punctured n-cube category $\boxtimes_n$ is its full subcategory spanned by all the objects different from the final object $(1, 1, \dots, 1)$.

For every $T \in F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ the fiber $p^{-1}(T)$ is isomorphic to $\square_n^{\mathrm{op}}$ (which is actually isomorphic to $\square_n$).

The restriction p_0 of p to $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^0_n$ is fibred over $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ and the fiber $p_0^{-1}(T)$ is isomorphic to $\boxtimes_n^{\mathrm{op}}$.

As a consequence the fibered colimit of $\mathbf{Z}^0_n$ and $\mathbf{Z}_n$ can be calculated as the colimit indexed by $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^{1\,\mathrm{op}}_n$ of the colimits calculated on the fibers of p (*cf.* Section B.1 and [18]):

$$\varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}_n \simeq \varinjlim_{T \in F\underline{\mathbf{T}}(C)^{1\,\mathrm{op}}_{S,n}}{}^{a\pi^{\mathrm{op}}} \varinjlim_{t \in p^{-1}(T)^{\mathrm{op}}} \mathbf{Z}(t)$$

$$\varinjlim{}^{a\pi^{\mathrm{op}}} \mathbf{Z}^0_n \simeq \varinjlim_{T \in F\underline{\mathbf{T}}(C)^{1\,\mathrm{op}}_{S,n}}{}^{a\pi^{\mathrm{op}}} \varinjlim_{t \in p_0^{-1}(T)^{\mathrm{op}}} \mathbf{Z}(t)$$

thus we rewrite the map (C.1.8) as

$$u_n \simeq \varinjlim{}^{a\pi^{\mathrm{op}}} u_{n,T} \tag{C.1.9}$$

where for every $T \in F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ the map $u_{n,T}\colon \varinjlim_{t \in p_0^{-1}(T)^{\mathrm{op}}} \mathbf{Z}(t) \to \varinjlim_{t \in p^{-1}(T)^{\mathrm{op}}} (t)$ is the canonical one induced by the fully faithful inclusion $p_0^{-1}(T) \to p^{-1}(T)$.

C.2. **Description of** u_n, T**.** Let us fix an element $(T, M_T) \in F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ (here we explicit the marking map for convenience). Let $v_1, v_2, \dots, v_n \in \mathrm{vert}(T)$ be the n vertices marked by K_1.

The element of $p^{-1}(T, M_T)$ are all $(V, M_V) \in F\underline{\mathbf{T}}(C)_{V,n}$ such that $V = T$ and for every $v \in \mathrm{vert}(T)$ one has $M_V(v) = X$ if and only if $M_T(v) = X$; in other words all the n vertices that are marked by K_1 in T have to be marked by K_0 or K_1.

For every $a = (a_1 \dots, a_n) \in \square_n$ let $M_{T,a}\colon \mathrm{vert}(T) \to \{X, K_0, K_1\}$ be the unique marking such that $M_{T,a}(v_m) = K_{a_m}$ for every $m \in [n]$ and $M_{T,a}(v) = X$ for every other vertex. Then there is an isomorphism of categories

$$\begin{aligned} s_T\colon \square_n^{\mathrm{op}} &\longrightarrow p^{-1}(T, M_T) \\ a &\longmapsto (T, M_{T,a}) \end{aligned}$$

which assigns to each morphism $g\colon a \to b$ in $\square_n^{\mathrm{op}}$ the unique tree-preserving morphism $s_T(g)\colon (T, M_{T,a}) \to (T, M_{T,b})$.

Let $Z_T\colon \square_n \to \mathcal{V}$ be the composite $\mathbf{Z} \circ s_T^{\mathrm{op}}$; then for every $b \in \square_n$

$$Z_T(b) \simeq \Big(\bigotimes_{m \in [n]} K_{b_m}(a(v_m))\Big) \otimes \Big(\bigotimes_{v \in \mathrm{vert}(T) \setminus \{v_1, \dots, v_n\}} X(a(v))\Big)$$

and for every map $l\colon b \to c$ in $\square_n$ the morphism $Z_T(l)$ is a tensor product of some $i(a(v_m))$'s and identities.

The category $\square_n$ has a final object $\bar{1} = (1, \dots, 1)$ hence the codomain of u_n is isomorphic to $Z_T(\bar{1}) = Z(T, M_T)$. Furthermore

$$u_{n,T} \colon \varinjlim_{b \in \boxtimes_n} \mathbf{Z}_T(b) \longrightarrow \mathbf{Z}(T, M_T) \tag{C.2.1}$$

can be calculated as an iterated pushout-product along the $i(a(v_m))$'s tensored with $\bigotimes_{v \in \mathrm{vert}(T) \setminus \{v_1, \dots, v_n\}} X(a(v))$ by a well known argument. More explicitly:

$$u_{n,T} \simeq \bigodot_{m \in [n]} i(a(v_m)) \otimes \bigotimes_{v \in \mathrm{vert}(T) \setminus \{v_1, \dots, v_n\}} X(a(v)) \tag{C.2.2}$$

where $\odot$ is the pushout-product of morphisms induced by the monoidal product of $\mathcal{V}$.

This implies in particular that:

Lemma C.3. *For every $T \in F\underline{\mathbf{T}}(C)^1_{S,n}$ and $n \in \mathbb{N}$ the morphism $u_{n,T}$ belongs to the monoidally saturated class of the set of maps $\{i(S) \mid S \in \mathrm{Seq}(C)\}$.*

C.3. u_n **as quotient of equivariant maps.** The category $F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1$ is a groupoid, in facts all its morphisms are marking-preserving isomorphisms of trees. Furthermore the bifibration $a\pi$ (diagram (C.1.1)) restricts to a bifibration of groupoids

$$\begin{tikzcd} \coprod_{S \in \mathrm{Seq}(C)} F\underline{\mathbf{T}}(C)^1_{S,n} \arrow[r] \arrow[d] & F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n \arrow[d, "a\pi"] \\ \mathrm{Seq}(C) \arrow[r] & \Sigma(C) \end{tikzcd} \tag{C.3.1}$$

For every $T \in F\underline{\mathbf{T}}(C)^1_{S,n}$ let $\mathrm{Aut}_{\mathrm{ord}}(T)$ be the group of automorphism of T and let $\mathrm{Aut}(T)$ be the group of unordered automorphism of T. Diagram (C.3.1) restricts to the following exact sequence of finite groups

$$\begin{tikzcd} \mathrm{Aut}_{\mathrm{ord}}(T) \arrow[r] \arrow[d] & \mathrm{Aut}(T) \arrow[d, "a\pi"] \\ * \arrow[r, "\mathrm{id}"'] & \mathrm{Aut}(S) \end{tikzcd}$$

Let $\mathcal{G}_S$ be the set of connected components of $F\underline{\mathbf{T}}(C)^1_{S,n}$ and suppose that a representative $T_G \in G$ is chosen for every $G \in \mathcal{G}_S$. The S-component of the morphism u_n (formula C.1.9) is naturally an $\mathrm{Aut}(S)^{\mathrm{op}}$-objects and is isomorphic to the coproduct of maps between orbits:

$$u_n(S) \simeq \coprod_{G \in \mathcal{G}_S} \varinjlim{}^{a\pi_G^{\mathrm{op}}} u_{n,T_G} \simeq \coprod_{G \in \mathcal{G}_S} (u_{n,T_G} / \mathrm{Aut}_{\mathrm{ord}}(T_G)^{\mathrm{op}}) \tag{C.3.2}$$

where $a\pi_G$ is the restriction of $a\pi$ to $\mathrm{Aut}(T_G)$.

Remark C.4. Every path component of $G \in F\underline{\mathbf{T}}(C)^1_{S,n}$ is equivalent to the group of automorphisms of its representative T_G. Suppose $T_G = (T, \lambda_T, M_T, \tau_T)$, then an automorphism of T_G is just a (non-planar) automorphism of T (a C-tree) preserving the order on the leaves τ_T, the marking M_T (which takes values in $\{X, K_1\}$) and whose change of planar structure is trivial in vertices marked by K_1. Note that if T does not have vertices of arity 0 then there are no non-trivial automorphism of T_G, since in this case the group of automorphism of T acts freely on the set of linear orders on the leaves of T.

We remark that our decomposition of i_α in a transfinite composition of push-outs is similar to the one obtained in [3, Sections 5.8-5.11] where the push-out along a free map generated by a map of collections is considered.

C.4. $\mathcal{T}_1$ **admissibility of** Op_C **in** Top. We can now prove that Op_C is $\mathcal{T}_1$ is admissible in Top. Thanks to Proposition 2.4 and Proposition 2.13 we are reduced to prove the following:

Lemma C.5. *Suppose a diagram as (C.0.3) is given in* $\mathsf{Top}\text{-}\mathsf{Oper}_C$. *If i is a local cofibration then i_α is a local $\mathcal{T}_1$-morphism.*

Proof. For every $S \in \mathrm{Seq}(C)$ the map $i_\alpha(S)$ can be expressed as the transfinite composite of (C.1.5). For every $n \in \mathbb{N}$ the map j_n is a push-out of u_n (*cf.* (C.1.6). Whereas the class of T_1 closed inclusions is saturated it is sufficient to show that u_n belongs to this class for every $n \in \mathbb{N}$. From Lemma C.3 we deduce that $u_{n,T}$ is a T_1 closed inclusion for every $T \in F\underline{\mathbf{T}}(C)^1_{S,n}$, since the class of T_1 closed inclusions is monoidally saturated. We can conclude that u_n is a T_1 closed inclusion by applying Lemma C.6. □

Lemma C.6. *Let G be a finite discrete group. The quotient of a G-map in* Top *whose underlying map is in $\mathcal{T}_1$ is in $\mathcal{T}_1$.*

Proof.

$$\begin{tikzcd} X \arrow[r, "\pi_X"] \arrow[d, "f"'] & X/G \arrow[d, "f/G"] \\ Y \arrow[r, "\pi_Y"'] & Y/G \end{tikzcd}$$

Note that since G is finite π_X and π_Y are both closed. It is easy to check that f/G is a closed inclusion. If $[y] \in Y/G$ belongs to $(Y/G)\backslash(X/G)$ then y belongs to $Y\backslash X$ hence it is closed. Since π_Y is a closed map we conclude that $[y]$ is closed as well.

□

C.5. **Tameness.** We would like to stress that the category $F\underline{\mathbf{T}}(C)^{1\,\mathrm{op}}$ is final in the category $\mathbf{T}^{T+1}$ defined in [2, Section 7] for the (polynomial) monad T associated to Op_C. More in general suppose we have a bicomplete monoidal category $\mathcal{V}$, a polynomial monad T with set of colours C and a class of morphisms A in $\mathcal{V}$ monoidally saturated and closed respect to colimits indexed by $\mathbf{T}^{T+1}$. Suppose that i is a local A-morphism in $\mathcal{V}^C$. It can be proven that in a push-out diagram in the category $\mathrm{Alg}_T(\mathcal{V})$ along the free map $F_T(i)$ like (C.0.3) the map i_α is a local A-morphism.

By definition a polynomial monad is tame if $\mathbf{T}^{T+1}$ has a final object in every connected component so the condition on A to be closed respect to colimits over $\mathbf{T}^{T+1}$ becomes empty. In the case of Op_C we have seen that in $\mathbf{T}^{\mathsf{Op}_C+1}$ each component has a final subcategory which is equivalent to a group.

C.6. Σ**-cofibrant Operads.** We continue to assume that $(\mathcal{V}, \otimes, \mathbf{I})$ is a monoidal model category with cofibrant unit admitting transfer for operads.

Proposition C.7. *Suppose that a push-out diagram as (C.0.3) is given. If X is a Σ-cofibrant C-coloured $\mathcal{V}$-operad and i is a cofibration in $\mathcal{V}^{\mathrm{Seq}(C)}$ then the morphism u_n (C.1.7) is a cofibration in* $\mathrm{Coll}_C(\mathcal{V})$ *for every $n \in \mathbb{N}$. Furthermore, if i has a cofibrant domain, then the domain of u_n is cofibrant in* $\mathrm{Coll}_C(\mathcal{V})$ *for every $n \in \mathbb{N}$.*

Proof. As we prove in section the morphism i_α can be built as a transfinite composition in $\mathrm{Coll}_C(\mathcal{V})$ of push-outs along maps that we denoted by u_n (*cf.* diagrams (C.1.5) and (C.1.7)).

To prove that u_n is a cofibration for every $n \in \mathbb{N}$ is equivalent to prove that $u_n(S)$ is an $\mathrm{Aut}(S)^{\mathrm{op}}$-cofibration for every $S \in \mathrm{Seq}(C)$. The functor

$$-/\mathrm{Aut}_{\mathrm{ord}}(T)^{\mathrm{op}} \colon \mathcal{V}^{\mathrm{Aut}(T)^{\mathrm{op}}} \to \mathcal{V}^{\mathrm{Aut}(S)^{\mathrm{op}}}$$

is a left Quillen functor for every $T \in F\mathbf{T}(C)$ (its right adjoint is the restriction functor $a\pi^*$).

According to formula (C.3.2), to prove that $u_n(S)$ is an $\mathrm{Aut}(S)^{\mathrm{op}}$-cofibration is therefore sufficient to check that $u_{n,T}$ is an $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibration for every $T \in F\underline{\mathbf{T}}(C)^1_n$. To prove that the domain of u_n is $\mathrm{Aut}(S)^{\mathrm{op}}$-cofibrant (if the domain of i is cofibrant) it is also sufficient to check that the domain of $u_{n,T}$ is $\mathrm{Aut}(T)$-cofibrant. Both statement are proven in Lemma C.10.

□

Corollary C.8. *Suppose that a push-out diagram as (C.0.3) is given. If X is a Σ-cofibrant C-coloured $\mathcal{V}$-operad and i is a cofibration in $\mathcal{V}^{\mathrm{Seq}(C)}$ then i_α is a Σ-cofibration.*

Corollary C.9. *Suppose that $f\colon X \to Y$ is a cofibration in $\mathcal{V}$-Oper_C and X is Σ-cofibrant, then f is a Σ-cofibration.*

The following Lemma has a proof almost identical to the one of [3, Lemma 5.9].

Lemma C.10. *For every $T \in F\underline{\mathbf{T}}(C)^1_n$ the morphism $u_{n,T}$ (formula (C.2.2) is an $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibration. Furthermore, if i has cofibrant domain, then the domain and codomain of $u_{n,T}$ are $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibrant.*

Proof. Let $F\mathbf{T}^{\mathrm{un}}(C)^1_n$ be the full subcategory of $F\mathbf{T}^{\mathrm{un}}(C)$ spanned by all the (non necessarily minimal) tree with no vertices marked by K_0 and exactly n vertices marked by K_1 and not necessarily minimal. Note that the definition of $u_{n,T}$ make sense for every $T \in F\mathbf{T}(C)^1_n$. We are going to prove by induction on $|\mathrm{vert}(T)| = m$ that $u_{n,T}$ is an $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibration for every $T \in F\mathbf{T}(C)^1_n$. For $m = 1$ there are two cases:

- If the only vertex v is marked by K_1 then $\mathrm{Aut}(T) = *$ and $u_{n,T} = i(a(v))$ is a cofibration in $\mathcal{V}$ by hypothesis; if the domain of i is cofibrant the domain of $u_{n,T}$ is cleary cofibrant;
- If the only vertex v is marked by K_1 then $\mathrm{Aut}(T) = \mathrm{Aut}(a(T))$ and $u_{n,T} = \mathrm{id}_{X(a(v))}$.

For $m > 1$ suppose the statement is true for every tree T' with $\mathrm{vert}(T') < m$. Since T has more than one vertex it can be decomposed as $V \circ (T_1, \dots, T_l)$ (definition B.6). where V is a corolla. Suppose that the set $\{T_1, \dots, T_h\}$ is subdivided by the isomorphism relation in exactly k classes $L_1,\dots,L_k$, let $l_j = |L_j|$ and pick a $T_{h_j} \in L_j$ for every $j \in [k]$. Let v be the unique vertex of V. There are now two case:

- if $M_T(v) = K_1$ then $\mathrm{Aut}(T) \simeq \mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_h)$ and $u_{n,T} \simeq i(a(v)) \odot u_{n_1,T_1} \odot \cdots \odot u_{n_h,T_h}$. The map $i(a(v))$ is a cofibration in $\mathcal{V}$ by hypothesis, while for every $j \in [h]$ the morphism u_{n_j,T_j} is an $\mathrm{Aut}(T_j)^{\mathrm{op}}$ by inductive hypothesis, By an iterated application of [6, Lemma 2.5.3] it follows that $u_{n,T}$ is an $\mathrm{Aut}(T)$-cofibration. A similar argument shows that if i has a cofibrant domain then the domain of $u_{n,T}$ is cofibrant.
- if $M_T(v) = X$ then

  $$\mathrm{Aut}(T) \simeq (\Sigma^{\mathrm{op}}_{l_1} \times \cdots \times \Sigma^{\mathrm{op}}_{l_k}) \rtimes (\mathrm{Aut}(T_{h_1})^{l_1} \times \cdots \times \mathrm{Aut}(T_{h_k})^{l_k})$$

  and $u_{n,T} \simeq X(a(v)) \otimes (\bigodot_{j\in[k]} u^{\odot l_j}_{n,T_{h_j}})$. $X(a(v))$ is $\mathrm{Aut}(a(v))$-cofibrant by hypothesis and each $u_{n,T_{h_j}}$ is $\mathrm{Aut}(T_{h_j})^{\mathrm{op}}$-cofibrant by inductive hypothesis.There for $u_{n,T}$ is an $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibrant by an iterated application of [6, Lemma 2.5.3]. To show that the domain of $u_{n,T}$ is $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibrant if the domain of i is cofibrant is again an application of [6, Lemma 2.5.3].

□

The following proposition establishes a kind of relative left properness for the category $\mathcal{V}$-Oper_C that allows us to prove Theorem 6.7. A similar result was independently proven by Hackney, Yau and Robertson ([15, Theorem 3.1.8]).

Proposition C.11. *Let $(\mathcal{V}, \otimes, \mathbf{I})$ be a cofibrantly generated monoidal model category which admits transfer for operads and let C be a set. In $\mathcal{V}$-Oper_C the class of weak equivalences between Σ-cofibrant objects is closed under push-out along cofibrations.*

Proof. It is sufficient to prove that if it is given a commutative diagram in $\mathcal{V}$-Oper_C

(C.6.1)
$$\begin{tikzcd} F_{\mathsf{Op}_C}(K_0) \arrow[r, "\alpha"] \arrow[d, "F_{\mathsf{Op}_C}"] & X \arrow[r, "\beta"] \arrow[d, "i_\alpha"] & X' \arrow[d, "i_{\beta\alpha}"] \\ F_{\mathsf{Op}_C}(K_1) \arrow[r] & Y \arrow[r, "\gamma"] & Y' \end{tikzcd}$$

in which both square are push-outs, i is a cofibration in $\mathcal{V}^{\mathrm{Seq}(C)}$ and β is a weak equivalence between Σ-cofibrant operads, then γ is a weak equivalence between Σ-cofibrant operads.

It is not restrictive to suppose that i has cofibrant domain. In fact we can decompose the left diagram in two push-out squares

$$\begin{tikzcd} F_{\mathsf{Op}_C}(K_0) \arrow[r, "F_{\mathsf{Op}_C}(\alpha)"] \arrow[d, "F(i)"] & F_{\mathsf{Op}_C}U_{\mathsf{Op}_C}(X) \arrow[r, "\varepsilon"] \arrow[d, "F(i')"] & X \arrow[d, "i_\alpha"] \\ F_{\mathsf{Op}_C}(K_1) \arrow[r] & F_{\mathsf{Op}_C}(L) \arrow[r] & Y \end{tikzcd}$$

where the map ε is the counit of the adjunction $(F_{\mathsf{Op}_C}, U_{\mathsf{Op}_C})$ and the left square is obtained by the push-out square

$$\begin{tikzcd} K_0 \arrow[r, "\alpha"] \arrow[d, "i"'] & U_{\mathsf{Op}_C}(X) \arrow[d, "i'"] \\ K_1 \arrow[r] & L \end{tikzcd}$$

by applying F_{Op_C}. We can thus replace i with i' and assume that K_0 is cofibrant.

We saw that the maps i_α and $i_{\beta\alpha}$ can be constructed as transfinite compositions (*cf.* (C.1.5))

$$X \simeq Y_0 \xrightarrow{j_0} Y_1 \longrightarrow \dots \xrightarrow{j_{n-1}} Y_n \xrightarrow{j_{n+1}} \dots$$

and

$$X' \simeq Y'_0 \xrightarrow{j'_0} Y'_1 \longrightarrow \dots \xrightarrow{j'_{n-1}} Y'_n \xrightarrow{j'_{n+1}} \dots$$

where j_n (j'_n) is a push-out of maps along maps u_n^α (resp. $u_n^{\beta\alpha}$); here we add a superscript to emphasize the dependency on α of u_n.

This construction is functorial in X thus we get a ladder of morphisms

$$\begin{tikzcd} Y_0 \arrow[r, "j_0"] \arrow[d, "\gamma_0"] & Y_1 \arrow[r, "j_1"] \arrow[d, "\gamma_1"] & \dots \arrow[r, "j_{n-1}"] & Y_n \arrow[r, "j_{n+1}"] \arrow[d, "\gamma_n"] & \dots \\ Y'_0 \arrow[r, "j'_0"] & Y'_1 \arrow[r, "j'_1"] & \dots \arrow[r, "j'_{n-1}"] & Y'_n \arrow[r, "j'_{n+1}"] & \dots \end{tikzcd}$$

and $\gamma \simeq \varinjlim_{n\in\mathbb{N}} \gamma_n$.

Since under our hypotheses j_n and j'_n are Σ-cofibrations between cofibrant objects for every $n \in \mathbb{N}$ (Proposition C.7) to prove that γ is a weak equivalence it is sufficient to prove that γ_n is a weak equivalence for every $n \in \mathbb{N}$. Let us look how this γ_n's are constructed.

For every $T \in F\underline{\mathbf{T}}^{\mathrm{un}}(C)^1_n$ the map β induces a morphism $\beta_T \colon u^{\alpha}_{n,T} \to u^{\beta\alpha}_{n,T}$ (*cf.* (C.2.1)) in the arrow category of $\mathrm{Coll}_C(\mathcal{V})$, that is there is a commutative diagram in $\mathrm{Coll}_C(\mathcal{V})$

(C.6.2)
$$\begin{tikzcd} s(u^{\alpha}_{n,T}) \arrow[r, "u^{\alpha}_{n,T}"] \arrow[d, "s(\beta_T)"'] & t(u^{\alpha}_{n,T}) \arrow[d, "t(\beta_T)"] \\ s(u^{\beta\alpha}_{n,T}) \arrow[r, "u^{\beta\alpha}_{n,T}"'] & t(u^{\beta\alpha}_{n,T}) \end{tikzcd}$$

(where $s(-)$ and $t(-)$ are the functions that associate to every morphism its source and target respectively) and this assignment is functorial in T.

More explicity, as in formula (C.2.2)

$$s(\beta_T) \simeq s(\bigodot_{m\in[n]} i(a(v_m))) \otimes \bigotimes_{v\in \mathrm{vert}(T)\setminus\{v_1,\dots,v_n\}} \beta(a(v))$$
$$t(\beta_T) \simeq t(\bigodot_{m\in[n]} i(a(v_m))) \otimes \bigotimes_{v\in \mathrm{vert}(T)\setminus\{v_1,\dots,v_n\}} \beta(a(v)).$$

Under our hypotheses these two maps are both weak equivalences between $\mathrm{Aut}(T)^{\mathrm{op}}$-cofibrant objects (Lemma C.10) or in other words the morphisms $s(\beta_-)$ and $t(\beta_-)$ in $\mathcal{V}^{F\underline{\mathbf{T}}^{\mathrm{un}}(C)^{1,\mathrm{op}}_n}$ are weak equivalences between cofibrant objects. We can apply the functor

$$\varinjlim{}^{a\pi^{\mathrm{op}}} \colon \mathcal{V}^{F\underline{\mathbf{T}}^{\mathrm{un}}(C)^{1\mathrm{op}}_n} \to \mathrm{Coll}_C(\mathcal{V})$$

(Section B.1) to diagram (C.6.2) and get a commutative diagram

(C.6.3)
$$\begin{tikzcd} s(u^{\alpha}_{n}) \arrow[r, "u^{\alpha}_{n}"] \arrow[d, "s(\bar\beta_n)"'] & t(u^{\alpha}_{n}) \arrow[d, "t(\bar\beta_n)"] \\ s(u^{\beta\alpha}_{n}) \arrow[r, "u^{\beta\alpha}_{n}"'] & t(u^{\beta\alpha}_{n}) \end{tikzcd}$$

where $\bar{\beta} = \varinjlim^{a\pi^{\mathrm{op}}} \beta_-$. Since $\varinjlim^{a\pi^{\mathrm{op}}}$ is left Quillen (*cf.* Section B.1) $s(\bar{\beta}) = \varinjlim^{a\pi^{\mathrm{op}}} s(\beta_-)$ and $t(\bar{\beta}_n) = \varinjlim^{a\pi^{\mathrm{op}}} t(\beta_-)$ are both weak equivalences between cofibrant objects.

For every $n \in \mathbb{N}$ there is a commutative diagram

(C.6.4)
$$\begin{tikzcd} Y_n \arrow[d, "\gamma_n"'] & s(u^{\alpha}_{n}) \arrow[l] \arrow[r, "u^{\alpha}_{n}"] \arrow[d, "s(\bar\beta_n)"] & t(u^{\alpha}_{n}) \arrow[d, "t(\bar\beta_n)"] \\ Y'_n & s(u^{\beta\alpha}_{n}) \arrow[l] \arrow[r, "u^{\beta\alpha}_{n}"] & t(u^{\beta\alpha}_{n}) \end{tikzcd}$$

in $\mathrm{Coll}_C(\mathcal{V})$. $\gamma_0 = \beta$ and for every $n \in \mathbb{N}$ the push-out of the upper-row is Y_{n+1}, the push-out of the lower-row is Y'_{n+1} and γ_{n+1} is the canonical map between them.

Now we can reason by induction to prove that every γ_i is a weak equivalence. For $n = 0$ this is true by hypothesis. If the statement is true for for γ_n then the vertical maps in diagram (C.6.4) are weak equivalences, the horizontal maps on the right are Σ-cofibrations and all the objects in the diagram are Σ-cofibrant, therefore γ_n is also a weak equivalence between Σ-cofibrant objects by Reedy's patching lemma. $\square$

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