On autoequivalences of the \((\infty, 1)\)-category of \(\infty\)-operads

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Higher category theory and higher operad theory can be formalized by means of different approaches.

\((\infty, 1)\)-categories

- Quasi-categories (Joyal, Lurie).
- Simplicial categories (Bergner).
- Segal categories (Hirschowitz–Simpson).
- Complete Segal spaces (Rezk).

All these models admit *model structures* and all of them are *Quillen equivalent* (Bergner, Joyal–Tierney).
\(\infty\)-operads

- Dendroidal sets (Moerdijk–Weiss).
- Simplicial operads (Cisinski–Moerdijk).
- Complete dendroidal Segal spaces (Cisinski–Moerdijk).
- \(\infty\)-operads (Lurie).

All these models admit \textit{model structures} and all of them are \textit{Quillen equivalent} (Cisinski–Moerdijk, Heuts–Hinich–Moerdijk).
**Question:** What is the $(\infty, 1)$-category of autoequivalences of these models? In how many ways can we compare these models?

**Theorem (Toën)**
\[
\text{Aut((}\infty, 1)\text{-categories)} \cong \mathbb{Z}/2\mathbb{Z} \text{ and the non-trivial element corresponds to passage to the opposite category.}
\]

Any two possibly different ways of comparing two models for $(\infty, 1)$-categories differ at most by passage to opposites.

**Goal:** Compute $\text{Aut}(\infty\text{-operads})$. 
Main strategy

Let $\mathcal{C}$ be a quasi-category. We want to compute $\text{Aut}(\mathcal{C})$. Find a small category $A$ inside $\mathcal{C}$ such that:

(i) $A \hookrightarrow \mathcal{C}$ is dense.

(ii) The autoequivalences of $\mathcal{C}$ restrict to autoequivalences of $A$.

Then it follows that $\text{Aut}(\mathcal{C}) \rightarrow \text{Aut}(A)$ is fully faithful.

To compute $\text{Aut}(\mathcal{C})$ it is enough to compute $\text{Aut}(A)$ and to check that the previous functor is essentially surjective.

If $\mathcal{C}$ is a localization of a category of simplicial presheaves on $A$, then (under good conditions) it satisfies the two conditions above.
Main strategy

For a small category $A$, we denote by:

- $\text{Pr}(A)$ the category of preheaves on $A$.
- $\text{sPr}(A)$ the category of simplicial preheaves on $A$.
- $\mathcal{P}(A)$ the quasi-category of preheaves on $A$.

Proposition

Let $A$ be a small category and $S$ a set of morphisms of $\text{Pr}(A)$. Suppose that:

(i) Representable presheaves in $\text{Pr}(A)$ are $S$-local.

(ii) The autoequivalences of $S^{-1}\text{Pr}(A)$ restrict to autoequivalences of $A$.

Then $A \to S^{-1}\mathcal{P}(A)$ induces a fully faithful functor

$$\text{Aut}(S^{-1}\mathcal{P}(A)) \to \text{Aut}(A).$$
The only model proposed so far for $\infty$-operads based on simplicial presheaves is $\Omega$-spaces.

Let Oper denote the category of symmetric coloured operads.

The category of trees $\Omega$ is the category whose objects are trees and whose morphisms are given by

$$\Omega(S, T) = \text{Oper}(\Omega(S), \Omega(T)),$$

where $\Omega(T)$ denotes the operad generated by the tree $T$.

There is an inclusion $\Omega \to \text{Oper}$ that induces a fully faithful dendroidal nerve functor $N_d : \text{Oper} \to \text{Pr}(\Omega)$. The category $\text{Pr}(\Omega)$ is called the category of dendroidal sets.
Let $C$ denote the full subcategory of $\Omega$ consisting of $\eta$ and the corollas.

For every tree $T$ let $\mathcal{D}_T$ denote the functor $C/T \to C \to \Omega$. Then the canonical morphism

$$\text{colim} \mathcal{D}_T = \colim_{(C, C \to T) \in C/T} C \to T$$

is an isomorphism in $\Omega$. 
Canonical decomposition of trees

$T$

$T_1$

$T_2$

$T_3$

$\eta_{e_1}$

$\eta_{e_2}$

$\eta_{e_3}$

$\eta_{e_4}$

$\eta_{e_5}$

$\eta_{e_6}$
Canonical decomposition of trees

Let $T$ be a tree. The spine of $T$ is the dendroidal set

$$I_T = \colim_{(C, C \to T) \in C/T} C,$$

where the colimit is taken in Pr(Ω). There is a canonical morphism of dendroidal sets $i_T : I_T \to T$. We will denote by $I$ the set

$$I = \{i_T \mid T \in \Omega\}.$$

Let $J$ be the simply connected groupoid on two objects and let $J \to \eta$ be the unique map of operads. For any tree $T$ we obtain an induced map of operads

$$j_T : J \otimes T \to \eta \otimes T \xrightarrow{\simeq} T.$$

We will denote by $J$ the set

$$J = \{N_d(j_T) \mid T \in \Omega\}.$$
Dendroidal spaces

The category of *dendroidal spaces* is $\text{sPr}(\Omega) \cong \text{Pr}(\Omega \times \Delta)$.

**Theorem (Cisinski–Moerdijk)**

*There is a (generalized Reedy) model structure on $\text{sPr}(\Omega)$ whose weak equivalences are the objectwise simplicial weak homotopy equivalences. The model category of complete dendroidal Segal spaces is the left Bousfield localization of $\text{sPr}(\Omega)_{\text{Reedy}}$ with respect to the set $I \cup J$.***

The *quasi-category* of $\Omega$-spaces $\Omega\text{-Sp}$ is a localization of the quasi-category $\mathcal{P}(\Omega)$ by the set $I \cup J$.  

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On autoequivalences of $\infty$-operads*
An operad is *rigid* if every invertible unary operation is an identity. For example, operads induced by trees are rigid.

**Proposition**

A dendroidal set is \((\mathcal{J} \cup \mathcal{J})\)-local if and only if it is the dendroidal nerve of a rigid operad.

**Proposition**

If \(F\) is an autoequivalence of rigid operads, then \(F(T) \cong T\). In particular, \(F\) induces an autoequivalence of \(\Omega\).
Let $\Sigma_\Omega = \prod_{T \in \Omega} \text{aut}_\Omega(T)$ and let $(\Sigma_\Omega)_{sc}$ be the simply connected groupoid on $\Sigma_\Omega$.

Given an element $\sigma = (\sigma_T)_{T \in \Omega}$ in $\Sigma_\Omega$, we define an autoequivalence $F_\sigma$ by setting:

(i) $F_\sigma(T) = T$.

(ii) For a map $f: S \to T$, we set $F_\sigma(f) = \sigma_T f \sigma_S^{-1}$.

This assignment defines a functor $\Phi: (\Sigma_\Omega)_{sc} \to \text{Aut}(\Omega)$.
Computing $\text{Aut}(\Omega)$

Let $F$ be an autoequivalence of $\Omega$. We define $\sigma(F)$ in $\Sigma_\Omega$ in the following way: $\sigma(F)_T$ is the unique automorphism of $T$ such that

$$\sigma(F)_T \circ c = F(c)$$

for every morphism $c : \eta \to T$ of $\Omega$.

**Theorem**

The functor $\Phi : (\Sigma_\Omega)_{sc} \to \text{Aut}(\Omega)$ is an isomorphism of categories. In particular, $\text{Aut}(\Omega)$ is a contractible groupoid.

**Theorem**

The quasi-category $\text{Aut}(\Omega-Sp)$ is a contractible Kan complex.
### Summary

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