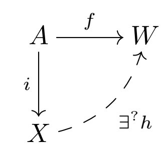
LECTURE 8: COFIBRATIONS

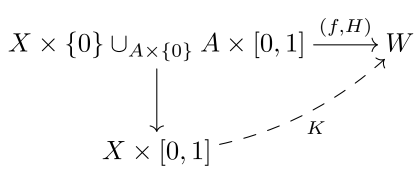
In this section we introduce the class of cofibration which can be thought of as nice inclusions. There are inclusions of subspaces which are 'homotopically badly behaved' and these will be excluded by considering cofibrations only. To be a bit more specific, let us mention the following two phenomenona which we would like to exclude. First, there are examples of contractible subspaces $A \subseteq X$ which have the property that the quotient map $X \to X/A$ is *not* a homotopy equivalence. Moreover, whenever we have a pair of spaces (X, A), we might be interested in extension problems of the form:



Thus, we are looking for maps h as indicated by the dashed arrow such that $h \circ i = f$. In general, it is not true that this problem 'lives in homotopy theory'. There are examples of homotopic maps $f \simeq g$ such that the extension problem can be solved for f but not for g. By design, the notion of a cofibration excludes this phenomenon.

Definition 8.1.

(i) Let $i\colon A \to X$ be a map of spaces. The map i has the *homotopy extension property* with respect to a space W if for each homotopy $H\colon A \times [0,1] \to W$ and each map $f\colon X \times \{0\} \to W$ such that $f(i(a), 0) = H(a, 0)$, $a \in A$, there is map $K\colon X \times [0,1] \to W$ such that the following diagram commutes:



(ii) A map $i\colon A \to X$ is a *cofibration* if it has the homotopy extension property with respect to all spaces W.

The terminology 'homotopy *extension* property' is of course motivated by the case of an inclusion of a subspace. And, in fact, it turns out that an arbitrary cofibration $i\colon A \to X$ is always injective.

The space showing up in the definition of a cofibration is a pushout. In the case of the inclusion of a closed subspace there is the following simplification.

Lemma 8.2. *Let $i\colon A \to X$ be the inclusion of a closed subspace. Then we have a homeomorphism $\phi\colon X \times \{0\} \cup_{A\times\{0\}} A \times [0,1] \to X \times \{0\} \cup A \times [0,1]$ which is compatible with the maps* *to* $X \times [0,1]$*, that is, which makes the following diagram commute:*

$$\begin{array}{ccc} X \times \{0\} \cup_{A\times\{0\}} A \times [0,1] & \xrightarrow{\phi} & X \times \{0\} \cup A \times [0,1] \\ & \searrow & \downarrow \\ & & X \times [0,1]. \end{array}$$

Proof. The universal property of the pushout allows us to construct a map as follows:

$$\begin{array}{ccccc} & X \times \{0\} & & & \\ \nearrow & & \searrow & & \\ A \times \{0\} & & & X \times \{0\} \cup_{A\times\{0\}} A \times [0,1] & \overset{\phi}{\dashrightarrow} X \times \{0\} \cup A \times [0,1] \\ \searrow & & \nearrow & & \\ & A \times [0,1] & & & \end{array}$$

Here the square on the left is the pushout diagram while the two bent arrows are just the inclusions. The universal property gives us the continuous map ϕ which is easily seen to be a bijection. Now, it suffices to observe that ϕ is also a closed map so that we actually have a homeomorphism. By definition of the quotient topology it is enough to check that the two inclusions (the bent arrows) are closed maps which is always the case for the upper one. Since A is assumed to be a closed subspace of X also the lower inclusion is a closed map. Thus, we have shown that ϕ is a homeomorphism and it is immediate that it is compatible with the two maps to $X \times [0,1]$. □

Using this lemma we can now easily establish the following convenient criterion which allows us to identify certain maps as being cofibrations.

Proposition 8.3. *Let* $A \subseteq X$ *be a closed subspace. Then the following are equivalent:*

(i) *The inclusion* $i\colon A \to X$ *is a cofibration.*

(ii) *All extension problems of the form*

$$\begin{array}{ccc} X \times \{0\} \cup A \times [0,1] & \xrightarrow{f} & W \\ {\scriptstyle j}\downarrow & \nearrow & \\ X \times [0,1] & & \end{array}$$

admit a solution as indicated by the dashed arrow.

(iii) *The map* $j\colon X \times \{0\} \cup A \times [0,1] \to X \times [0,1]$ *admits a retraction.*

Proof. The equivalence of the first two statements follows immediately from the previous lemma. It is also easy to see that (ii) implies (iii) since it suffices to consider the lifting problem given by the identity:

$$\begin{array}{ccc} X \times \{0\} \cup_{A\times\{0\}} A \times [0,1] & \xrightarrow{id} & X \times \{0\} \cup_{A\times\{0\}} A \times [0,1] \\ {\scriptstyle j}\downarrow & \overset{r}{\nearrow} & \\ X \times [0,1] & & \end{array}$$

Finally, if j admits a retraction r then any extension problem as in (ii) admits a solution given by $f \circ r$. $\square$

Recall from a previous lecture that the inclusion

$$j_n \colon J^n = I^n \times \{0\} \cup \partial I^n \times I \longrightarrow I^n \times I$$

admits a retraction. It is immediate that our preferred homeomorphism $D^n \cong I^n$ (which restricts to a homeomorphism $S^{n-1} \cong \partial I^n$) shows that also the inclusion

$$D^n \times \{0\} \cup S^{n-1} \times I \longrightarrow D^n \times I$$

admits a retraction. Thus, an application of the previous proposition gives us the following example.

Example 8.4. The inclusion $S^{n-1} \to D^n$ is a cofibration.

Our next aim is to show that if (X, A) is a relative CW complex, then the inclusion $A \to X$ is a cofibration. The above example gives us one of the basic building blocks. Since relative CW complexes are built inductively using certain constructions it is convenient to first establish some 'closure properties' of the class of cofibration.

We begin by observing that there is the following reformulation. Recall that associated to an arbitrary space W there is the path space W^I (endowed with the compact-open topology) which comes with natural evaluation maps $W^I \to W$.

Lemma 8.5. *A map $i \colon A \to X$ is a cofibration if and only if for all spaces W and commutative diagrams of the form*

$$\begin{array}{ccc} A & \longrightarrow & W^I \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle \epsilon_0} \\ X & \longrightarrow & W \end{array}$$

there is a diagonal filler as indicated, i.e., such a map making both triangles commutative.

Proof. This follows immediately from the fact that we have a natural bijection between the set of maps $X \times I \to W$ and maps $X \to W^I$ (see Proposition 2.1, Lecture 2). The naturality of these bijections is essential to conclude the proof and the details are left as an exercise. $\square$

This lemma allows us to establish the following closure properties of cofibrations.

Proposition 8.6.

(i) *Homeomorphisms are cofibrations. Similarly, if we have maps $i \colon A \to X$, $i' \colon A' \to X'$, and homeomorphisms $A \cong A'$, $X \cong X'$ such that*

$$\begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[\cong]{} & X' \end{array}$$

commutes, then i is a cofibration if and only if i' is one.

(ii) *Cofibrations are closed under composition, that is, if $i \colon A \to X$ and $j \colon X \to Y$ is a cofibration then so is $j \circ i \colon A \to Y$.*

(iii) *Cofibrations are closed under coproducts, that is, if we have a family $i_j \colon A_j \to X_j$, $j \in J$, of cofibrations then also the map $\sqcup_j i_j \colon \sqcup_j A_j \to \sqcup_j X_j$ is a cofibration.*

(iv) *Cofibrations are stable under pushouts, that is, if we have a pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \longrightarrow & Y \end{array}$$

such that i *is a cofibration, then also* j *is a cofibration.*

(v) *For two spaces* X *and* Y*, the inclusion* $X \to X \sqcup Y$ *is a cofibration. In particular, taking* X *to be the empty space, the map* $\emptyset \to Y$ *is cofibration for every space* Y*.*

Proof. All of these facts follow more or less directly from the definition or the above lemma. We will give a proof of the stability under pushouts and leave the remaining ones as exercises. So, in the the notation of (iv) let us consider the situation:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & W^I \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} & \overset{d}{\nearrow} & \big\downarrow{\scriptstyle \epsilon_0} \\ X & \xrightarrow[k]{} & Y & \xrightarrow[l]{} & W \end{array}$$

Thus, we are given the solid arrow diagram and we try to find a diagonal filler d as indicated. Using the fact that i is a cofibration we can find a solution e to the following problem on the left:

$$\begin{array}{ccc} A & \xrightarrow{gf} & W^I \\ {\scriptstyle i}\big\downarrow & \overset{e}{\nearrow} & \big\downarrow{\scriptstyle \epsilon_0} \\ X & \xrightarrow[kl]{} & W \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[e]{} & W^I \end{array}$$

The commutativity of the upper triangle tells us that we have maps $e\colon X \to W^I$ and $g\colon B \to W^I$ such that the above square on the right commutes. The universal property of the pushout implies that there is a unique map $d\colon Y \to W^I$ such that $d \circ j = g$ and $d \circ k = e$. We leave it to the reader to check that this map d does the job (which follows by using once more the universal property of pushouts). $\square$

Remark 8.7.

(i) Recall from the section on fibrations, that we introduced two such classes, namely the Hurewicz fibrations and the Serre fibrations. The cofibrations introduced in this section are often also referred to as *Hurewicz cofibrations*. Since we will not consider 'Serre cofibrations' in this course, we instead decided to simplify the terminology and drop the name Hurewicz.

(ii) For readers knowing about model categories we want to include this warning. The class of cofibrations introduced in this section is *not* the class of cofibrations in the Hurewicz model structure on the category of all spaces. Instead, the cofibrations in that model structure are given by the *closed* Hurewicz cofibrations. Nevertheless, it turns out that every object in the Hurewicz model structure is cofibrant, i.e., the unique map from the empty space to the given one in a closed Hurewicz cofibration. With this respect, this model category thus behaves vastly different than the Serre model structure on spaces. (It is even true that every object in the Hurewicz model structure is both cofibrant and fibrant.)

With these closure properties of cofibrations we can now deduce the following important result.

Theorem 8.8. *Let (X, A) be a relative CW complex. Then the inclusion $i\colon A \to X$ is a cofibration.*

Proof. We thus have to show that every problem of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & W^I \\ {\scriptstyle i}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle \epsilon_0} \\ X & \xrightarrow[k]{} & W \end{array}$$

admits a solution. By definition of a relative CW complex we have a filtration of X,

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

such that the following two properties are satisfied:

(i) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells for $n \geq 0$.
(ii) The space X is the union $\bigcup_{n \geq -1} X^{(n)}$ endowed with the weak topology and so comes, in particular, with continuous inclusions $i_n\colon X^{(n)} \to X$.

Thus, for every n we have a pushout diagram

$$\begin{array}{ccc} J_n \times \partial e^n = \coprod_{\sigma \in J_n} \partial e^n_\sigma & \longrightarrow & X^{(n-1)} \\ \downarrow & & \downarrow{\scriptstyle i_{n,n-1}} \\ J_n \times e^n = \coprod_{\sigma \in J_n} e^n_\sigma & \longrightarrow & X^{(n)}. \end{array}$$

We know already that $\partial e^n \to e^n$ is a cofibration (Example 8.4). Since cofibrations are stable under coproducts and pushouts we conclude that also the maps $i_{n,n-1}\colon X^{(n-1)} \to X^{(n)}$ are cofibrations. But this means that we can inductively find solutions to the following problems

$$\begin{array}{ccc} A & \xrightarrow{f} & W^I \\ {\scriptstyle i_{0,-1}}\downarrow & \overset{d_0}{\nearrow} & \downarrow{\scriptstyle \epsilon_0} \\ X^{(0)} & \xrightarrow[ki_0]{} & W, \end{array} \qquad \begin{array}{ccc} X^{(0)} & \xrightarrow{d_0} & W^I \\ {\scriptstyle i_{1,0}}\downarrow & \overset{d_1}{\nearrow} & \downarrow{\scriptstyle \epsilon_0} \\ X^{(1)} & \xrightarrow[ki_1]{} & W, \end{array} \qquad \begin{array}{ccc} X^{(1)} & \xrightarrow{d_1} & W^I \\ {\scriptstyle i_{2,1}}\downarrow & \overset{d_2}{\nearrow} & \downarrow{\scriptstyle \epsilon_0} \\ X^{(2)} & \xrightarrow[ki_2]{} & W, \end{array}$$

where in each step we use that the map on the left is a cofibration. Note that we use the inductively constructed solution $d_n\colon X^{(n)} \to W^I$ as an input for the problem in the next dimension. Using the weak topology on X, there is a unique map $d\colon X \to W^I$ such that $d \circ i_n = d_n\colon X^{(n)} \to W^I$, and hence, in particular, $d \circ i = f\colon A \to W^I$. Thus the upper triangle in the initial problem commutes. We leave it to the reader to check that this map is a solution for the initial problem, that is, that also the lower triangle commutes. For that purpose, you will again have to use that X is endowed with the weak topology. □

This theorem tells us that plenty of cofibrations show up in nature. In fact, we now want to show that every map of spaces can be factored in a cofibration followed by a homotopy equivalence. This factorization uses the *mapping cylinder construction* which is obtained as follows. Let $f\colon X \to Y$

be a map of spaces. Then the *mapping cylinder* M_f of f is defined by the following pushout:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle j} \\ X \times I & \xrightarrow[k]{} & M_f \end{array}$$

(Note that the mapping cylinder construction was already used in the definition of a cofibration.) Thus, M_f is obtained by gluing a cylinder $X \times I$ on Y by identifying points $(x,1) \sim f(x)$. The constant homotopy $f \circ p\colon X \times I \to X \to Y$ and the identity $\mathrm{id}\colon Y \to Y$ are two maps with the same target and which satisfy $\mathrm{id} \circ f = (f \circ p) \circ i_1\colon X \to Y$. Thus, by the universal property of the pushout we obtain a map $r\colon M_f \to Y$ which just 'collapses the cylinder'. By design, this map r satisfies the equations

$$r \circ j = \mathrm{id}\colon Y \longrightarrow Y \qquad \text{and} \qquad r \circ k = f \circ p.$$

The first equation of course tells us that r is a retraction. We also have a map from X to the mapping cylinder, given by the 'inclusion as the top of the cylinder', namely the map

$$i = k \circ i_0\colon X \xrightarrow{i_0} X \times I \xrightarrow{k} M_f.$$

The following proposition justifies our intuition that this map i is a nice inclusion. Before we state it, let us recall the following definition.

Definition 8.9. Let (Z, C) be a pair of spaces with inclusion $\iota\colon C \to Z$. The map ι is the inclusion of a *strong deformation retract* if it is the inclusion of a retract, that is, such that $r \circ \iota = \mathrm{id}_C\colon C \to C$ together with a homotopy $H\colon \iota \circ r \simeq \mathrm{id}_Z$ relative to C. The map $r\colon Z \to C$ is then called a *strong deformation retraction.*

Proposition 8.10. *Let $f\colon X \to Y$ be a map of spaces as above. Then the map f factors as $f = r \circ i\colon X \to M_f \to Y$. Moreover, $r\colon M_f \to Y$ is a strong deformation retraction and $i\colon X \to M_f$ a cofibration.*

Proof. To check that we have such a factorization it suffices to make the following calculation:

$$r \circ i = r \circ k \circ i_0 = f \circ p \circ i_0 = f.$$

We know that $j\colon Y \to M_f$ is the inclusion of a retract with retraction given by $r\colon M_f \to Y$. Thus, to conclude that we have a deformation retraction it suffices to show that the map $j \circ r\colon M_f \to M_f$ is homotopic to the identity (relative to Y). The idea is of course to 'linearly collapse' the cylinder and keep the rest fixed. In formulas, consider the map $H\colon M_f \times I \to M_f$ by setting

$$H([y,t]) = [y] \qquad \text{and} \qquad H([x,s],t) = [x, ts + (1-t)].$$

We leave it to the reader to check that this map is well-defined and continuous. Then it is immediate from the formula that $H(-,0) = j \circ r$ and $H(-,1) = \mathrm{id}$. By construction, the homotopy is constant on Y so that we know that r is a strong deformation retraction.

It remains to show that $i\colon X \to M_f$ is a cofibration. With our preparation, this is now easily established. In fact, note that there is a diagram of the following form

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ {\scriptstyle\iota_1}\downarrow & & \downarrow{\scriptstyle\iota_1} \\ X \sqcup X & \xrightarrow[id\sqcup f]{} & X \sqcup Y \\ \downarrow & & \downarrow{\scriptstyle(i,j)} \\ X \times I & \xrightarrow[k]{} & M_f \end{array}$$

in which the maps ι are the inclusions of the first summands. It is easy to check that the composition in the right column is our map $i\colon X \to M_f$ so that by Proposition 8.6 it is enough to check that both maps in right column are cofibrations. That same proposition already takes care of the first map. To show that also the second map is a cofibration we can again apply Proposition 8.6 to deduce that it is enough to show that $X \times \partial I = X \sqcup X \to X \times I$ is a cofibration. Since this map is the inclusion of a closed subspace we can apply Proposition 8.3 to conclude that we only have to show that the map

$$X \times \partial I \times \{0\} \cup X \times I \times \{0\} \to X \times I \times I$$

admits a retraction. But this is immediate since we know that $\partial I \times I \cup I \times \{0\} = J^1 \to I^2$ admits a retraction. □

Now, a strong deformation retraction is, in particular, a homotopy equivalence so that we have managed factoring an arbitrary map into a cofibration followed by a homotopy equivalence. The mapping cylinder construction is helpful for other purposes as well. For example, it can be used to establish the following result.

Exercise 8.11. Let $i\colon A \to X$ be a cofibration. Then i is injective.

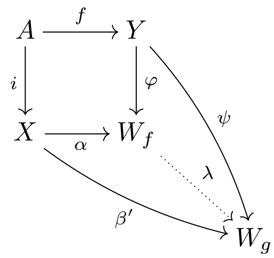
Next, we will show that the pushout of a homotopy equivalence along a cofibration is again a homotopy equivalence.

Proposition 8.12. *Let $i\colon A \to X$ be a cofibration and let $f, g\colon A \rightrightarrows Y$ be two maps. If f and g are homotopy, then the two pushouts $X \cup_f Y$ and $X \cup_g Y$ (of i and f, and of i and g, respectively) are homotopy equivalent spaces. And if the given homotopy between f and g respects the base point, so does the resulting homotopy equivalence.*

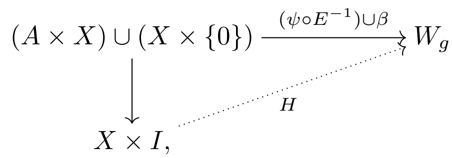
Proof. Let us write

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[\alpha]{} & W_f \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[\beta]{} & W_g \end{array}$$

for the two pushouts (so $W_f = X \cup_f Y$ and $W_g = X \cup_g Y$), and $E\colon A \times I \to Y$ for the given homotopy, from $f = E_0$ to $g = E_1$. We are going to construct maps $\lambda\colon W_f \to W_g$ and $\mu\colon W_g \to W_f$, and show that they are inverse up to homotopy. To get λ, we use the pushout property in the diagram

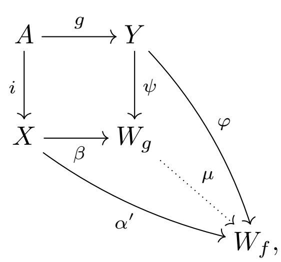


for a suitable β'. We cannot use β, because we may not have $\beta \circ i = \psi \circ f$. However, $\beta \circ i = \psi \circ g \simeq \psi \circ f$, and by the homotopy extension property, we will be able to find a map $\beta' \simeq \beta$, with $\beta' \circ i = \psi \circ f$. Indeed, to this end consider the diagram

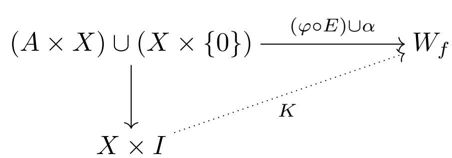


where E^{-1} is the homotopy E running in the opposite direction, $E^{-1}(a,t) = E(a, 1-t)$, from g to f, so that $(\psi \circ E^{-1})_0 = \psi \circ E_1 = \psi \circ g = \beta \circ i$. This shows that the map $(\psi \circ E^{-1}) \cup \beta$ on top of the diagram is well defined. The homotopy extension property provides a homotopy $H \colon X \times I \to W_g$ with $H|_{A \times I} = \psi \circ E^{-1}$ and $H_0 = \beta$, and we define $\beta' = H_1$.

Similarly, we find a map μ in the diagram

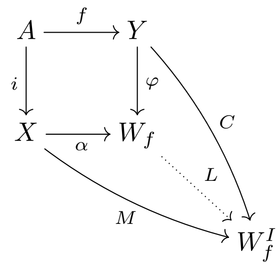


where the map α' satisfies that $\alpha' \circ i = \varphi \circ g$ as in the diagram, and α' is found by applying the homotopy extension property in



and then defining $\alpha' = K_1$.

We now claim that $\mu \circ \lambda \simeq \mathrm{id}_{W_f}$ and $\lambda \circ \mu \simeq \mathrm{id}_{W_g}$. These statements are, of course, proved in the same way, so let us prove the first. To find a homotopy $L \colon \mu \circ \lambda \simeq \mathrm{id}_{W_f}$, it is natural to try and use the pushout property, as in



(1)

and get L from two homotopies $\mu \circ \lambda \circ \varphi \simeq \varphi$ and $\mu \circ \lambda \circ \alpha \simeq \alpha$, which agree on $A \times I$. Now $\mu \circ \lambda \circ \varphi = \varphi$, so for C we can take the *constant* homotopy from φ to itself. It remains to find a homotopy M between $\mu \circ \lambda \circ \alpha = \mu \circ \beta'$ and α, such that $M|_{A \times I}$ is the constant homotopy $A \times I \to A \xrightarrow{f} Y \xrightarrow{\varphi} W_f$.

Now, we *do* have a homotopy between $\mu \circ \beta'$ and α, namely

$$\alpha = K_0 \overset{K}{\Longrightarrow} K_1 = \alpha' = \mu \circ \beta = \mu \circ H_0 \overset{\mu \circ H}{\Longrightarrow} \mu \circ H_1 = \mu\beta',$$

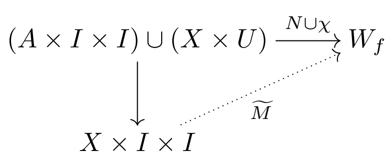
but when restricted to $A \times I$ this is

$$\varphi(E * E^{-1})\colon \alpha \circ i \overset{\varphi \circ E}{\Longrightarrow} \alpha' \circ i = \varphi \circ E_1 = \varphi \circ g \overset{\varphi \circ E^{-1}}{\Longrightarrow} \varphi \circ f = \alpha \circ i.$$

This homotopy, first $\varphi \circ E$ and then $\varphi \circ E$ in the opposite direction, is obviously *homotopic* to the constant homotopy from $\alpha \circ i$ to itself. Write N for this homotopy. So $N\colon A \times I \times I \to W_f$ satisfies the following identities:

$$\begin{cases} N(a,s,0) = \varphi(E * E^{-1})(a,s), \\ N(a,s,1) = \alpha \circ i(a), \\ N(a,0,t) = \alpha \circ i(a) = N(a,1,t). \end{cases}$$

Now apply the homotopy extension property again as in the diagram



(cf. Exercise 8.13 below) to find an M a required in diagram (1) above: here U is the U-shape $\{0,1\} \times I \cup I \times \{0\} \to I \times I$, and the map $\chi\colon X \times U \to W_f$ is defined as follows:

$$\begin{cases} \alpha \text{ on } X \times \{0\} \times I, \\ \mu \circ \beta' \text{ on } X \times \{1\} \times I, \\ K * (\mu \circ H) \text{ on } X \times I \times \{0\}. \end{cases}$$

Now $M = \widetilde{M}_1$ is a homotopy as required in (1), and the pushout property of (1) will give the homotopy L form the identity to $\mu \circ \lambda\colon W_f \to W_f$.

We leave the proof of the statement concerning the basepoints to the reader. □

Exercise 8.13. Prove that if $i\colon A \to X$ is a cofibration, then it has the homotopy extension property in diagrams of the form

$$\begin{array}{ccc} A \times I \times I \cup X \times U & \longrightarrow & Z \\ \big\downarrow & \nearrow & \\ X \times I \times I. & & \end{array}$$

Remark 8.14. Consider again the two pushout squares in the first sentence of the proof above which define $W_f = X \cup_f Y$ and $W_g = X \cup_g Y$. The homotopy equivalences $\lambda\colon W_f \xrightarrow{\simeq} W_g$ and $\mu\colon W_g \xrightarrow{\simeq} W_f$ are compatible up to homotopy with these squares, in the sense that

$$\lambda \circ \varphi \simeq \psi \text{ and } \lambda \circ \alpha \simeq \beta, \qquad \mu \circ \psi \simeq \varphi \text{ and } \mu \circ \beta \simeq \alpha$$

(and in fact two of these homotopies are equalities).

Corollary 8.15. *Let the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & Y \end{array}$$

be a pushout in Top *(or* Top_**) in which i (and hence j) is a cofibration. If f is a homotopy equivalence, then so is g.*

Proof. Let $e\colon B \to A$ be a homotopy inverse for f, and form consecutive pushouts as in the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{e} & A & \xrightarrow{f} & B \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle l} \\ X & \xrightarrow[h]{} & Y & \xrightarrow[m]{} & Z & \xrightarrow[n]{} & W. \end{array}$$

Then all the vertical maps are cofibrations. Now $e \circ f$ and $f \circ e$ are homotopic to the identities, so by Proposition 8.12 and Remark 8.14, there are homotopy equivalences $\lambda\colon Z \xrightarrow{\simeq} X$ with $\lambda \circ m \circ h \simeq \mathrm{id}_X$ (and $\lambda \circ k \simeq i$), and $\lambda'\colon W \xrightarrow{\simeq} Y$ with $\lambda' \circ n \circ m \simeq \mathrm{id}_Y$ (and $\lambda' \circ l \simeq j$). Since λ and λ' are homotopy equivalences, so are $m \circ h$ and $n \circ m$. But then m itself must be a homotopy equivalence, by evident properties of isomorphisms in the homotopy category. Since $\lambda \circ m \circ h \simeq \mathrm{id}_X$ it then follows that h is a homotopy equivalence as well. ◻