On solid and rigid monoids in monoidal categories

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Introduction

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- Solid rings are completely classified and all of them are commutative and countable. The only torsion-free solid rings are the subrings of the rationals.
- [Bousfield–Kan] $H_*(f; R)$ is an isomorphism if and only if $H_*(f; cR)$ is an isomorphism. $R_\infty X \simeq (cR)_\infty X$.

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- The products $\prod_{p\in P}\mathbb{Z}/p$ and $\prod_{p\in P}\widehat{\mathbb{Z}}_p$, where P is any set of primes are rigid. However, the Prüfer group $\mathbb{Z}/p^\infty$ or the p-adic field $\widehat{\mathbb{Q}}_p$ do not admit a rigid ring structure.

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- There exist rigid rings of arbitrarily large cardinality.
- [Casacuberta–Rodríguez–Tai] Rigid rings appear naturally as localizations of $\mathbb{Z}$ and as homotopical localizations of S^1.

Localization and colocalization functors

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- The objects in the image of L are called L-**local objects** and the objects in the image of C are called C-**colocal objects**.
- A morphism f is called an L-**local equivalence** if $L(f)$ is an isomorphism, and it is called a C-**colocal equivalence** if $C(f)$ is an isomorphism.

Closed (co)localizations

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A **closed localization** is a localization functor (L, l) in $\mathcal{E}$ such that for every L-local equivalence $f\colon X \to Y$ and every L-local object Z, the induced map

$$f^*\colon \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$$

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A **closed colocalization** is a colocalization functor (C, c) in $\mathcal{E}$ such that for every C-colocal equivalence $f\colon X \to Y$ and every C-colocal object Z in the induced map

$$f_*\colon \mathrm{Hom}(Z, X) \longrightarrow \mathrm{Hom}(Z, Y)$$

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Smashing and mapping

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- If L is smashing then $A \cong LI$ and $X \otimes LY \cong L(X \otimes Y)$ for all X and Y.
- If C is mapping, then $C(\mathrm{Hom}(X, Y)) \cong \mathrm{Hom}(X, CY)$ for all X and Y.

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- We can define a functor $F\colon \mathcal{E} \to \mathrm{Fun}(\mathcal{E}, \mathcal{E})$ by setting $F(X)(-) = - \otimes X$ and another functor $G\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E}, \mathcal{E})$ by setting $G(X)(-) = \mathrm{Hom}(X, -)$.

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- Moreover, F preserves solid monoids and the functor G sends solid monoids to solid comonoids.

Solid monoids

Theorem

Let $\mathcal{E}$ be a closed symmetric monoidal category. Then, there is a one to one correspondence between the following classes:

(i) *Solid monoids.*

(ii) *Smashing localization functors.*

(iii) *Mapping colocalization functors.*

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Theorem

Let (R, μ, η) be a solid monoid, and let $L_R = - \otimes R$ and $C_R = \mathrm{Hom}(R, -)$. Then the following categories are equivalent:

(i) *L_R-loc the full subcategory of L_R-local objects*
(ii) *C_R-coloc the full subcategory of C_R-colocal objects*
(iii) *R-mod the category of R-modules.*

Rigid monoids

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Theorem

Let (L, l) be a closed localization in $\mathcal{E}$.

(i) *LI is rigid and all rigid monoids appear this way.*

(ii) *Every rigid monoid is commutative.*

Stable homotopy theory

Let $\mathcal{S}p$ be the stable homotopy category of spectra. This is a triangulated category equipped with a compatible closed symmetric monoidal structure, where the unit is given by the sphere spectrum S.

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Theorem

Let L be any localization functor in $\mathcal{S}p$.

(i) *If L is smashing, then LS is a solid ring spectrum, and all solid ring spectra appear as smashing localizations of the sphere spectrum.*

(ii) *If L is closed, then the spectrum LS is a rigid ring spectrum and all rigid ring spectra appear as closed localizations of the sphere spectrum.*

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Theorem

If R is a solid ring spectrum, e.g., $R = H\mathbb{Q}$, $L_K S$ or $L_{E(n)}S$, then there is an equivalence of categories $L_R\mathcal{S}p \cong R\text{-}mod \cong Cell_R\mathcal{S}p$.

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Let L be any localization functor in $\mathcal{S}p$.

(i) *$LH\mathbb{Z} \cong HA$ for some rigid ring A and all (algebraic) rigid rings appear this way.*

(ii) *If L is smashing, then A is a subring of the rationals.*

(iii) *If LS is connective, then LS is a solid ring spectrum if and only if $LS \cong MA$, where A is a subring of the rationals.*

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Corollary

If R is a connective solid ring spectrum, then $R \cong MA$ for some subring of the rationals A.