DERIVATIONS, THE LAWRENCE-SULLIVAN INTERVAL AND THE FIORENZA-MANETTI MAPPING CONE

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Abstract. We describe the rational homotopy type of any component of the based mapping space $\operatorname{map}^*(X,Y)$ as an explicit L_∞ algebra defined on the (desuspended and positive) derivations between Quillen models of X and Y. When considering the Lawrence-Sullivan model of the interval, we obtain an L_∞ model of the contractible path space of Y. We then relate this, in a geometrical and natural manner, to the L_∞ structure on the Fiorenza-Manetti mapping cone of any differential graded Lie algebra morphism, two in principal different algebraic objects in which Bernoulli numbers appear.

1. Introduction

The theory of L_∞ *algebras* (or *strongly homotopy Lie algebras*) was introduced in [27] as a generalization of differential graded Lie algebras in the context of deformation theory of algebraic structures. Recently, and among other geometrical applications [8, 16], this theory has been used to sharpen the Quillen approach to rational homotopy [26] by describing the rational homotopical behavior of nilpotent spaces in terms of L_∞ algebras [12, 13]. We begin by following this approach for the class of based mapping spaces.

Indeed, in classical rational homotopy theory, geometrical properties of mapping spaces are often described in terms of derivations of certain differential graded Lie algebras. A general and recent result in this context reads as follows (see Section 2 for a precise definition of the objects involved):

Let $f\colon X \to Y$ be a map between nilpotent CW-complexes of finite type, and denote by $\operatorname{map}^*_f(X,Y)$ the path component of the space of based continuous maps from X to Y containing f. Let C be a finite type graded differential coalgebra model of X and let M be a Lie model of Y. Denote by $\mathcal{L}(C)$ the Quillen functor on C and let $\gamma'\colon \mathcal{L}(C) \to M$ be a model for f. Then [7, Theorem 3], whenever X is finite, the Lie bracket in M and the coalgebra structure on C induce a Lie bracket on the graded vector space

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$s^{-1}\mathcal{D}er_{\gamma'}(\mathcal{L}(C),M)$ for which it becomes a Lie model of $\mathrm{map}^*_f(X,Y)$. A similar result also holds when X is of finite type, non necessarily finite [7, Theorem 7].

On the other hand, see [2], starting with the Quillen minimal model $\mathcal{Q}$ of X and any Lie model $\gamma\colon \mathcal{Q}\to M$ of f, there is a natural and explicit L_∞ structure on $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathcal{Q},M)$ encoding the higher Whitehead products on the rational homotopy groups of $\mathrm{map}^*_f(X,Y)$. Then we prove:

Theorem 1.1. *As L_∞ algebras, $s^{-1}\mathcal{D}er_{\gamma'}(\mathcal{L}(C),M)$ and $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathcal{Q},M)$ are quasi-isomorphic. In particular, the latter is an L_∞ model of $map^*_f(X,Y)$.*

Again, a similar result holds when X is of finite type but non necessarily finite (see Theorem 3.1).

Then, we consider, on the one hand, the Lawrence-Sullivan completed Lie model of the interval $\mathcal{I}$, see [22]. On the other hand, we choose as C the dual of the standard acyclic free algebra $\Lambda(t,dt)$. Even though $\mathcal{I}$ is clearly not the Quillen minimal model of the interval, we may follow the approach in the proof of the previous result to show (see Theorem 4.1 for a precise statement):

Theorem 1.2. *If M is any differential graded Lie algebra, then $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)$ and $s^{-1}\mathcal{D}er(\mathcal{L}(C),M)$ are quasi-isomorphic as L_∞ algebras.*

In particular, we obtain:

Corollary 1.3. *If M is any Lie model of a nilpotent complex Y, then $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)$ is an L_∞ model of the based path space $PY = map^*(I,Y)$.*

We finish by identifying in Theorem 4.3 the homotopy fiber of any differential graded Lie algebra morphism $\chi\colon L\to M$ via the acyclic L_∞ algebra $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)$. Let C_χ be the *suspended mapping cone* of χ, which is known to posses an L_∞ structure lifting the Lie bracket on L and preserving the original differential [10]. On the other hand, consider the natural L_∞ algebra $L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)$ obtained as the pullback of χ and the evaluation morphism $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)\to M$. Then, we prove:

Theorem 1.4. *The L_∞ algebras C_χ and $L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{I},M)$ are strongly isomorphic.*

In particular, this establishes an explicit and close relation between the Lawrence-Sullivan model for the interval and the Fiorenza-Manetti mapping cone, which explain why in these apparently different objects, Bernoulli numbers appear.

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2. Preliminaries

We will rely on known results from rational homotopy theory, in particular from the Quillen approach [26], for which [9] and [28] are standard references. We also assume the reader is aware of the principles on L_∞ algebras being [19, 20] original references. For a more geometrical or rational homotopical approach we refer to [10, 13, 16] or [5, 12] respectively. With the only purpose of fixing notation and certain conventions we give some definitions and briefly sketch basic facts together with some results of general nature we will use.

In general, every considered algebraic object (unless explicitly stated otherwise) will be $\mathbb{Z}$-graded. However, in the framework of rational homotopy theory, most considered algebraic models of homotopy types are concentrated in non negative degrees.

Let CDGC (resp. DGL) denote the category of (augmented and coaugmented) *cocommutative differential graded coalgebras*, CDGC's henceforth (resp. *differential graded Lie algebras*, DGL's henceforth). We denote by $\overline{C}$ the kernel of the augmentation morphism of a given CDGC C, and by $\overline{\Delta}\colon \overline{C} \to \overline{C}\otimes\overline{C}$ the reduced comultiplication whose kernel is the graded subspace of primitive elements. In the free commutative algebra ΛV generated by the graded vector space V, we consider the canonical structure of cofree cocommutative graded coalgebra whose comultiplication is the unique morphism of graded algebras such that $\Delta(v) = v\otimes 1 + 1\otimes v$, $v\in V$. On the other hand, we denote by $\mathbb{L}(V)$ the *free differential graded Lie algebra* generated by V, that is, the Lie subalgebra of the tensor Lie algebra $T(V)$, where the bracket is given by commutators.

These two categories are related by the *Quillen Functor* $\mathcal{L}\colon \mathsf{CDGC}\to\mathsf{DGL}$. If (C,δ), or simply C, is any CDGC, then $\mathcal{L}(C)$ is the DGL $(\mathbb{L}(s^{-1}\overline{C}),\partial)$ where s^{-1} denotes *desuspension* and $\partial = \partial_1 + \partial_2$, in which $\partial_1(s^{-1}c) = -s^{-1}\delta c$ and

$$\partial_2(s^{-1}c) = \frac{1}{2}\sum_i (-1)^{|a_i|}[s^{-1}a_i, s^{-1}b_i],$$

being $c\in\overline{C}$ and $\overline{\Delta}c = \sum_i a_i\otimes b_i$.

Another graded object of special interest along the paper is the space of positive derivations of DGL's and its universal cover. Let $\gamma\colon L\to M$ be any morphism in DGL and consider the differential graded vector space of γ*-derivations* $(\mathrm{Der}_\gamma(L,M),\delta)$ in which $\mathrm{Der}_\gamma(L,M)_n$ is the space of linear maps $\theta\colon L_*\to M_{*+n}$ such that

$$\theta[x,y] = [\theta(x),\gamma(y)] + (-1)^{n|x|}[\gamma(x),\theta(y)],$$

and $\delta\theta = \partial\circ\theta + (-1)^{n+1}\theta\circ\partial$. The complex of *positive* γ*-derivations* $\mathcal{D}\mathrm{er}_\gamma(L,M)$ is defined as

$$\mathcal{D}\mathrm{er}_\gamma(L,M)_i = \begin{cases} \mathrm{Der}_\gamma(L,M)_i & \text{for } i>1,\\ Z\,\mathrm{Der}_\gamma(L,M)_1 & \text{for } i=1,\end{cases}$$

where Z denotes the space of cycles. Its *universal cover* $\widetilde{\mathcal{D}\mathrm{er}}_\gamma(L,M)$ is defined as

$$\widetilde{\mathcal{D}\mathrm{er}}_\gamma(L,M)_i = \begin{cases} \mathrm{Der}_\gamma(L,M)_i & \text{for } i>2, \\ Z\,\mathrm{Der}_\gamma(L,M)_2 & \text{for } i=2. \end{cases}$$

An L_∞ *algebra* is a graded vector space L together with a CDGC structure on the coalgebra ΛsL, generated by the suspension on L. The existence of such a structure is equivalent to the existence of degree $k-2$ linear maps $\ell_k\colon L^{\otimes k}\to L$, for $k\geq 1$, satisfying the following two conditions:

(i) For any permutation σ of k elements,

$$\ell_k(x_{\sigma(1)},\dots,x_{\sigma(k)}) = \epsilon_\sigma\epsilon'\ell_k(x_1,\dots,x_k),$$

where ϵ_σ is the signature of the permutation and ϵ' is the sign given by the Koszul convention.

(ii) The *generalized Jacobi identity* holds, that is

$$\sum_{i+j=n+1}\ \sum_{\sigma\in S(i,n-i)} \epsilon_\sigma\epsilon'(-1)^{i(j-1)}\ell_{n-i}(\ell_i(x_{\sigma(1)},\dots,x_{\sigma(i)}),x_{\sigma(i+1)},\dots,x_{\sigma(n)})=0,$$

where $S(i,n-1)$ denotes the set of $(i,n-i)$-shuffles, i.e., permutations σ of n-elements such that $\sigma(1)<\cdots<\sigma(i)$ and $\sigma(i+1)<\cdots<\sigma(n)$.

Every differential graded Lie algebra (L,∂) is therefore an L_∞ algebra by setting $\ell_1=\partial$, $\ell_2=[-,-]$ and $\ell_k=0$ for $k>2$.

Observe that, if L is a finite type graded vector space, an L_∞ structure on L induces a commutative differential graded algebra (CDGA) structure on $\mathcal{C}^\infty(L)=\Lambda(sL)^\sharp$ where $\sharp$ denotes the dual vector space. This can be seen as a generalization of the cochain algebra on a classical DGL. Moreover $\mathcal{C}^\infty(L)$ is a minimal Sullivan algebra [9, §12] if and only if L is a non negatively graded minimal L_∞ algebra (that is, $\ell_1=0$) in which L_0 acts nilpotently.

Given two L_∞ algebras L and L', a *morphism of L_∞ algebras or L_∞ morphism* is a morphism of differential graded coalgebras $f\colon(\Lambda sL,\delta)\to(\Lambda sL',\delta')$. Abusing notation, and whenever there is no ambiguity, we shall often write simply $f\colon L\to L'$. Note that such a morphism f is determined by the projection $\pi f\colon\Lambda sL\to sL'$ which is the sum of a system of skew-symmetric maps $f^{(k)}\colon\otimes^k L\longrightarrow L'$ of degree $1-k$ satisfying an infinite sequence of equations involving the brackets ℓ_k and ℓ'_k, $k\geq 1$. In particular, these equations are satisfied if $f\colon L\to L'$ satisfies $\ell_k f^{\otimes k}=f\ell_k$ for all k. In this case we say that f is a *strong L_∞ morphism.*

An L_∞ morphism is a *quasi-isomorphism* if $f^{(1)}$ is a quasi-isomorphism of complexes. Being quasi-isomorphic is an equivalence relation among L_∞ algebras. Indeed [16], if $f\colon\Lambda sL\to\Lambda sL'$ is a quasi-isomorphism, then there exist an L_∞ morphism $g\colon\Lambda sL'\to\Lambda sL$ such that $H_*(g^{(1)})$ is the inverse of $H_*(f^{(1)})$.

Given a finite type complex X, an L_∞ algebra of finite type L is an L_∞ *model* of X if $\mathcal{C}^\infty(L)$ is a Sullivan model of X. On the other hand, given a 1-connected complex (not necessarily of finite type) X, an L_∞ algebra is an L_∞ *model* of X if it is quasi-isomorphic to $\lambda(X)$. Here λ denotes the functor described in [26] that associates to any 1-connected space X a DGL (free as Lie algebra) $\lambda(X)$, which determines an equivalence between the homotopy categories of rational 1-connected complexes and that of non negatively graded DGL's over $\mathbb{Q}$.

Observe that, if X is 1-connected and of finite type then $\mathcal{C}^\infty\lambda(X)$ is quasi-isomorphic to the Sullivan minimal model of X [23], and both definitions above coincide in the intersection class of 1-connected and finite type complexes.

It is known [16, 4.9] that any L_∞ algebra is isomorphic to the direct sum of a minimal L_∞ algebra, i.e. with $\ell_1 = 0$, and a linear contractible one. We present now an explicit and detailed decomposition of this kind in the classical case of free and positively graded DGL's [9, Theorem 22.13]:

Let $\mathbb{L}(V)$ be any free positively graded DGL and write $V = W \oplus U \oplus \partial_1 U$ in which ∂_1 denotes the indecomposable part of the differential ∂ on $\mathbb{L}(V)$. Consider in $\mathbb{L}(V)$ the ideal I generated by U and ∂U and observe that the inclusions in the tensor algebra $W, U, \partial U \subset T(V)$ induce an algebra isomorphism $T(W \oplus U \oplus \partial U) \xrightarrow{\cong} T(V)$ which restricts to an isomorphism of Lie algebras

$$\mathbb{L}(W \oplus U \oplus \partial U) \xrightarrow{\cong} \mathbb{L}(V).$$

Thus, we endow $\mathbb{L}(W)$ with a (decomposable) differential which makes the induced isomorphism $\mathbb{L}(W) \cong \mathbb{L}(V)/I$ a DGL one. Observe also that the composition of the quotient map $\mathbb{L}(V) \twoheadrightarrow \mathbb{L}(V)/I$ with the inverse isomorphism yields a surjective quasi-isomorphism

$$\varphi\colon \mathbb{L}(V) \stackrel{\simeq}{\twoheadrightarrow} \mathbb{L}(W)$$

which, by the lifting lemma, has a section

$$\psi\colon \mathbb{L}(W) \xrightarrow{\simeq} \mathbb{L}(V).$$

Explicitly, given $w \in W, u \in U$, $\varphi(w) = w$, $\varphi(u) = 0$ and $\varphi(\partial_1 u) = -\varphi(\Phi)$ where $\partial u = \partial_1 u + \Phi$. Finally, consider the coproduct $\mathbb{L}(W) * \mathbb{L}(A \oplus \partial A) = \mathbb{L}(W \oplus A \oplus \partial A)$ of the minimal model $\mathbb{L}(W)$ and the acyclic DGL $\mathbb{L}(A \oplus \partial A)$, with A a copy of U, and define the DGL isomorphism

$$\Gamma\colon \mathbb{L}(W) * \mathbb{L}(A \oplus \partial A) \xrightarrow{\cong} \mathbb{L}(V), \quad \Gamma(a) = u, \quad \Gamma(\partial a) = \partial u, \quad \Gamma(w) = \psi(w).$$

Consider now the inclusion and the projection

$$\mathbb{L}(W) \underset{q}{\overset{j}{\rightleftarrows}} \mathbb{L}(W) * \mathbb{L}(A \oplus \partial A),$$

which are obviously quasi-isomorphisms, and observe that $\Gamma j = \psi$ and $q\Gamma^{-1} = \varphi$. Moreover, $qj = 1_{\mathbb{L}(W)}$ and $jq \sim 1 = 1_{\mathbb{L}(W \oplus A \oplus \partial A)}$, not just as DGL's but also as chain complexes, since $\mathrm{Im}(jq - 1) \subset \mathbb{L}(W) * \mathbb{L}^+(A \oplus \partial A)$ which is an acyclic ideal.

It would be ideal to have such a chain homotopy, not just as a linear map, but also as a derivation. However, and this is why the DGL cylinder in [1] does not work properly, the formula $1-jq = k\partial+\partial k$ prevents any such k to be a derivation as $1-jq$ is not in general. For this, and for computational purposes in next sections, we fix now a particular chain homotopy between 1 and jq. Consider the (non differential!) Lie algebra morphism

$$\alpha\colon \mathbb{L}(W)*\mathbb{L}(A\oplus\partial A)\to\mathbb{L}(W)*\mathbb{L}(A\oplus\partial A)$$

which is equal to $(1+jq)/2$ on the generators. That is, $\alpha(w)=w$, $\alpha(a)=a/2$, $\alpha(\partial a)=\partial a/2$, for $w\in W$ and $a\in A$ (note that $(1+jq)/2$ is not a Lie algebra morphism). Finally, define

$$k\colon \mathbb{L}(W)*\mathbb{L}(A\oplus\partial A)\longrightarrow\mathbb{L}(W)*\mathbb{L}(A\oplus\partial A)$$

as the degree one α-derivation satisfying $k(w)=k(a)=0$ and $k(\partial a)=a$. An easy induction on the bracket length shows that, in fact,

$$1-jq=k\partial+\partial k,$$

and moreover, $k_{|_{\mathbb{L}(W)}}=0$. In the same way, $\varphi\psi=1_{\mathbb{L}(W)}$ and $\psi\varphi\sim 1_{\mathbb{L}(V)}$ via the chain homotopy

$$K=\Gamma k\Gamma^{-1}\colon\mathbb{L}(V)\to\mathbb{L}(V).$$

Note that K is then a β-derivation where $\beta=\Gamma\alpha\Gamma^{-1}\colon\mathbb{L}(V)\to\mathbb{L}(V)$ is the Lie algebra morphism defined as $(1_{\mathbb{L}(V)}+\psi\varphi)/2$ on V. One checks that, on generators, if $w\in W$ and $u\in U$, then $K(u)=0$, $K(\partial_1 u)=u-K(\Phi)$ and $K(w)=-K(\Psi)$, where $\partial u=\partial_1 u+\Phi$ and $\psi(w)=w+\Psi$. As the image of k is contained in $\mathbb{L}(W)*\mathbb{L}^+(A\oplus\partial A)$, the image of K lives also in the acyclic ideal I. Then, a straightforward computation shows that $\varphi\beta=\varphi$ while $\varphi K=0$. Finally note that one may change basis by redefining w to be $w-\Psi$ so that all the properties above still hold and K is zero on $\mathbb{L}(W)$.

3. L_∞ models of mapping spaces in terms of derivations

Here we prove Theorem 1.1. We begin by briefly explaining why we make different finiteness assumptions in what follows. When X is finite and nilpotent, and Y is a finite type nilpotent complex, $\operatorname{map}^*(X,Y)$ is again a finite type nilpotent complex whose rational homotopy type is fully modeled by algebraic objects. However, if X is only considered to be of finite type, then $\operatorname{map}^*(X,Y)$ is again a nilpotent CW-complex but no longer of finite type. In this case, only its universal cover admits a DGL model [7, Lemma 5]. In any case, finite type assumptions are needed on both X and Y so that $\operatorname{map}^*(X,Y)$ is naturally a CW-complex.

Let $f\colon X\to Y$ be a based map between nilpotent CW-complexes of finite type, and let $\gamma\colon\mathbb{L}(W)\to M$ be a Lie model of f in which $\mathbb{L}(W)$ is the minimal model of X and M is any Lie model of Y. Choose also C, a CDGC model of X and consider its Quillen construction $\mathcal{L}(C)$ which is again a free

Lie model of X. With the same notation as in the section above, we obtain quasi-isomorphisms

$$\mathbb{L}(W) \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \mathcal{L}(C)$$

which satisfy: $\varphi\psi = 1_{\mathbb{L}(W)}$ and $\psi\varphi \sim 1_{\mathcal{L}(C)}$.

On the one hand, $\gamma\varphi\colon \mathcal{L}(C) \to M$ is a DGL model of f for which we may apply [7, Theorem 3] to deduce that, when X is finite, the DGL $s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M)$, equipped with the canonical bracket defined, up to suspension-desuspension, by

$$[g,h](c) = \sum_i (-1)^{1+|a_i|\,|g|}[g(a_i), h(b_i)]\,, \quad \text{where} \quad \overline{\Delta}(c) = \sum_i a_i \otimes b_i\,,$$

is a Lie model of $\mathrm{map}^*_f(X,Y)$. In the same way [7, Theorem 7(2)], if X is of finite type, the universal cover $s^{-1}\widetilde{\mathcal{D}er}_{\gamma\varphi}(\mathcal{L}(C), M)$ is a Lie model of $\widetilde{\mathrm{map}}^*_f(X,Y)$.

On the other hand, see [2], there is an L_∞ structure on $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathcal{Q}, M)$, whose brackets $\{\ell_j\}_{j\geq 1}$ are defined as the Eckmann-Hilton dual of certain operations introduced in [3, §4], which provide the higher Whitehead products on the rational homotopy groups of $\mathrm{map}_f(X,Y)$. Let us recall these brackets: given $\theta_1,\dots,\theta_n$ in $\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$, we define, in a recursive way, a linear map of degree $|\theta_1| + \cdots + |\theta_n|$ denoted by

$$\{\theta_1,\dots,\theta_n\}\colon \mathbb{L}(W) \longrightarrow M,$$

as follows. If $n = 1$, then $\{\theta_1\} = \theta_1$. For $n \geq 2$

(i) If $\Psi \in \mathbb{L}^{<n}(W)$, then $\{\theta_1,\dots,\theta_n\}(\Psi) = 0$.

(ii) If $\Psi \in \mathbb{L}^{\geq n}(W)$ and $\Psi = [\alpha,\beta]$, where $\alpha,\beta \in \mathbb{L}^{<n}(W)$ then

$$\begin{aligned}
\{\theta_1,\dots,\theta_n\}(\Psi) &= \{\theta_1,\dots,\theta_n\}([\alpha,\beta]) \\
&= [\{\theta_1,\dots,\theta_n\}\alpha, \gamma\beta] + (-1)^{|\alpha|(|\theta_1|+\cdots+|\theta_n|)}\,[\gamma\alpha, \{\theta_1,\dots,\theta_n\}\beta] \\
&+ \sum_{k=1,\sigma\in S(k,n-k)}^{n-1} \Big(\epsilon'\,\big[\{\theta_{\sigma(1)},\dots,\theta_{\sigma(k)}\}\alpha, \{\theta_{\sigma(k+1)},\dots,\theta_{\sigma(n)}\}\beta\big] \\
&+ \epsilon''\,\big[\{\theta_{\sigma(k+1)},\dots,\theta_{\sigma(n)}\}\alpha, \{\theta_{\sigma(1)},\dots,\theta_{\sigma(k)}\}\beta\big]\Big),
\end{aligned}$$

where ϵ' and ϵ'' are given by the Koszul sign convention. Then, define brackets $\{\ell_i\}_{i\geq 1}$ on $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$ as follows. The first bracket is the differential of the complex $\ell_1(s^{-1}\theta) = -s^{-1}\delta\theta$, and for $n > 1$,

$$s\ell_n(s^{-1}\theta_1,\dots,s^{-1}\theta_n)(v) = (-1)^\epsilon\{\theta_1,\dots,\theta_n\}(\partial v), \tag{3.1}$$

where $\epsilon = n + 1 + \sum_{j=1}^n (n+1-j)|\theta_j|$.

With this notation, Theorem 1.1 of the Introduction is contained in:

Theorem 3.1. *$s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M)$ and $s^{-1}\widetilde{\mathcal{D}er}_{\gamma\varphi}(\mathcal{L}(C), M)$ are isomorphic as L_∞ algebras to $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$ and $s^{-1}\widetilde{\mathcal{D}\mathrm{er}}_\gamma(\mathbb{L}(W), M)$ respectively. In particular,*

(i) *If X is finite, $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W),M)$ is an L_∞ model of $map^*_f(X,Y)$.*

(ii) *If X is of finite type, non necessarily finite, $s^{-1}\widetilde{\mathcal{D}\mathrm{er}_\gamma}(\mathbb{L}(W),M)$ is an L_∞ model of the universal cover $\widetilde{map}^*_f(X,Y)$.*

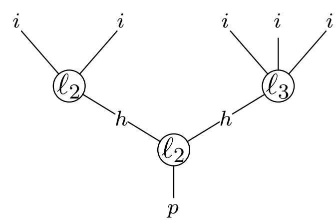
The proof lies on the following particular instance of the classical *Homotopy Transfer Theorem* [11, 17, 18, 21, 24], which goes back to [14, 15] for the case of A_∞ structures.

Theorem 3.2. *Consider the following diagram*

$$h \circlearrowright (L,\ell_1) \underset{i}{\overset{p}{\rightleftarrows}} (V,d)$$

in which $(L,\{\ell_k\}_{k\geq 1})$ is an L_∞ algebra; (V,d) is a differential graded vector space; i and p are morphisms of chain complexes with i quasi-isomorphism; and h is a chain homotopy between Id_L and ip, i.e., it is a degree one linear map h such that $\mathrm{Id}_L - ip = \ell_1 h + h\ell_1$. Then, there exists an L_∞ structure $\{\ell'_k\}_{k\geq 1}$ on V and an L_∞ quasi-isomorphism $i\colon (V,\{\ell'_k\}_{k\geq 1}) \xrightarrow{\simeq} (L,\{\ell_k\}_{k\geq 1})$ such that $\ell'_1 = d$ and $i^{(1)} = i$.

Moreover, the transferred higher brackets ℓ'_k can be explicitly described in terms of rooted trees as follows. Let $\mathcal{T}_k$ be the groupoid of directed rooted trees with internal vertices of valence at least two and exactly k leaves. The leaves of the trees in $\mathcal{T}_k$ are labeled by i, each internal edge is labeled by h and the root edge is labeled by p. Every internal vertex v is labeled by the operation ℓ_r, where r is the number of input edges of v. For example, the following picture represents a (planar embedding of a) tree in $\mathcal{T}_5$:



We denote by T_k the set of isomorphism classes of $\mathcal{T}_k$. Each planar embedding $\widetilde{\Gamma}$ of a tree Γ in $\mathcal{T}_k$ gives rise to a linear map

$$T_{\widetilde{\Gamma}}(i,p,h,\ell)\colon V^{\otimes k} \longrightarrow V$$

according to the usual operadic rules. For the previous tree this linear map corresponds to

$$T_{\widetilde{\Gamma}}(i,p,h,\ell)(v_1\otimes\cdots\otimes v_5) = p(\ell_2(h\ell_2(i(v_1)\otimes i(v_2))\otimes h\ell_3(i(v_3)\otimes i(v_4)\otimes i(v_5)))).$$

Then, we define $T_\Gamma(i,p,h,\ell)$ as the composition of $T_{\widetilde{\Gamma}}(i,p,h,\ell)$ with the symmetrization map $\Lambda^k V \to V^{\otimes k}$ given by

$$v_1\wedge\cdots\wedge v_k \mapsto \sum_{\sigma\in S_k} \epsilon_\sigma \epsilon\, v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)},$$

where S_k denotes the symmetric group on k letters, ϵ_σ denotes the signature of the permutation and ϵ stands for the sign given by the Koszul convention. Then, for $k \geq 2$, each ℓ'_k is given by the formula

$$\ell'_k = \sum_{\Gamma \in T_k} \frac{T_\Gamma(i,p,h,\ell)}{|\operatorname{Aut}\Gamma|}, \tag{3.2}$$

where $\operatorname{Aut}\Gamma$ is the automorphism group of the tree Γ.

Proof of Theorem 3.1. As the restriction of the quasi-isomorphism we will construct restricts to universal covers, we only show that $s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M)$ and $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$ are quasi-isomorphic as L_∞ algebras.

Applying directly the decomposition process stated in Section 2, and with the same notation, the quasi-isomorphisms

$$\mathbb{L}(W) \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \mathcal{L}(C)$$

satisfy: $\varphi\psi = 1_{\mathbb{L}(W)}$ and $\psi\varphi \sim 1_{\mathcal{L}(C)}$ via the chain homotopy

$$K = \Gamma^{-1} k \Gamma\colon \mathcal{L}(C) \to \mathcal{L}(C),$$

with $\Gamma\colon \mathbb{L}(W) * \mathbb{L}(A \oplus \partial A) \xrightarrow{\cong} \mathcal{L}(C)$. In this particular case, due to the explicit differential in $\mathcal{L}(C) = \mathbb{L}(s^{-1}\overline{C})$, if we write $s^{-1}\overline{C} = s^{-1}B \oplus \partial_1 s^{-1}B \oplus s^{-1}H$, then we get

$$K(s^{-1}b) = 0, \quad b \in B,$$

and

$$K(\partial_1 s^{-1}b) = s^{-1}b - \frac{1}{2}\sum_i (-1)^{|z'_i|} k[s^{-1}z'_i, s^{-1}z''_i],$$

with $\overline{\Delta} b = \sum_i z'_i \otimes z''_i$.

We now transfer this DGL structure to an L_∞ algebra structure on $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$. To this end, we prove that applying $\operatorname{Hom}(-, M)$ to the diagram

$$K \circlearrowright \mathcal{L}(C) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbb{L}(W)$$

it yields another diagram

$$K^* \circlearrowright s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M) \underset{\varphi^*}{\overset{\psi^*}{\rightleftarrows}} s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M)$$

suitable to use Theorem 3.2.

First, a simple inspection proves that $\varphi^* = s^{-1}\operatorname{Hom}(\varphi, M)$ restricts to

$$s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M) \xrightarrow{\varphi^*} s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M).$$

Another computation, in which the fact $\varphi\psi = 1_{\mathbb{L}(W)}$ is strongly needed, shows that $\psi^* = s^{-1}\operatorname{Hom}(\psi, M)$ restricts to

$$s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C), M) \xrightarrow{\psi^*} s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W), M).$$

Next we prove that K^* also restricts to a map on $s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C),M)$. For it, and just for clearness and cleanness in the exposition, we suspend everything in the following computation avoiding in this way the s^{-1} operation and the signs it gives rise to. Also, recall that K is defined as a β-derivation where $\beta\colon \mathcal{L}(C)\to\mathcal{L}(C)$ is the Lie algebra morphism defined as $(1_{\mathcal{L}(C)}+\psi\varphi)/2$ on $s^{-1}\overline{C}$. Then, given $\theta\in\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C),M)$:

$$\begin{aligned}\theta K[x,y] &= \theta([Kx,\beta y]+(-1)^{|x|}[\beta x,Ky])\\ &= [\theta Kx,\gamma\varphi\beta y]+(-1)^{(|x|+1)|\theta|}[\gamma\varphi Kx,\theta\beta y]\\ &\quad +(-1)^{|x|}[\theta\beta x,\gamma\varphi Ky]+(-1)^{|x|(|\theta|+1)}[\gamma\varphi\beta x,\theta Ky].\end{aligned}$$

However, since $\varphi\beta=\varphi$ and $\varphi K=0$, we conclude that

$$\theta K[x,y]=[\theta Kx,\gamma\varphi y]+(-1)^{|x||\theta K|}[\gamma\varphi x,\theta Ky],$$

that is, $K^*(\theta)$ is also a $\gamma\varphi$ derivation.

Now, the equality $\psi^*\varphi^*=1_{s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W),M)}$ is obvious while a straightforward computation shows that $\varphi^*\psi^*\sim 1_{s^{-1}\mathcal{D}er_{\gamma\varphi}(\mathcal{L}(C),M)}$ via K^*. Finally, Lemma 6 of [7] proves that φ^* is a quasi-isomorphism and we have all the required data in Theorem 3.2 to transfer an L_∞ algebra structure to $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W),M)$.

The proof of the theorem ends by checking that the transferred bracket ℓ_k' on $s^{-1}\,\mathcal{D}\mathrm{er}_\gamma(\mathbb{L}(W),M)$ coincides with ℓ_k for $k\geq 1$. Since this is only a (long and tedious) inductive computation, requiring all the properties of the objects involved, we just sketch it here for the case $n=2$. Also, for the sake of clearness, we avoid any suspension and desuspension operation, and we write every sign arising from the Koszul convention as $\pm$.

On the one hand:

$$\ell_2(\theta,\eta)(w)=\pm\{\theta,\eta\}\big(\varphi\partial\psi(w)\big)=\pm\{\theta,\eta\}\big(\varphi\partial(w)\big)$$

since $\psi(w)=w+\Phi$ where Φ belongs to a differential ideal annihilated by φ. Now, as w is decomposed by the differential, $\partial w=\frac{1}{2}\sum_i\pm[z_i',z_i'']$. Thus:

$$\begin{aligned}\ell_2(\theta,\eta)(w)=&\frac{1}{2}\sum_i\pm\{\theta,\eta\}[\varphi(z_i'),\varphi(z_i'')]=\\ =&\frac{1}{2}\sum_i\pm[\{\theta,\eta\}\varphi(z_i'),\gamma\varphi(z_i'')]\pm[\gamma\varphi(z_i'),\{\theta,\eta\}\varphi(z_i'')]+ \qquad (3.3)\\ &\frac{1}{2}\sum_i\pm[\theta\varphi(z_i'),\eta\varphi(z_i'')]\pm[\eta\varphi(z_i'),\theta\varphi(z_i'')]. \qquad (3.4)\end{aligned}$$

On the other hand:

$$\begin{aligned}\ell_2'(\theta,\eta)(w)&=\pm\frac{1}{2}\psi_*\ell_2(\varphi_*\theta,\varphi_*\eta)(w)\pm\frac{1}{2}\psi_*\ell_2(\varphi_*\eta,\varphi_*\theta)(w)=\\ &=\pm\frac{1}{2}\ell_2(\varphi_*\theta,\varphi_*\eta)\big(\psi(w)\big)\pm\frac{1}{2}\big(\ell_2(\varphi_*\eta,\varphi_*\theta)\big(\psi(w)\big).\end{aligned}$$

Now observe that $\psi(w) = \psi\varphi(w) = w - K\partial w - \partial K w = w - K\partial w$ since K is zero in $\mathbb{L}(W)$. Hence:

$$\ell_2'(\theta,\eta)(w) = \pm\frac{1}{2}\ell_2(\varphi_*\theta,\varphi_*\eta)(w) \pm \frac{1}{2}\ell_2(\varphi_*\eta,\varphi_*\theta)(w) \tag{3.5}$$

$$\pm\frac{1}{2}\ell_2(\varphi_*\theta,\varphi_*\eta)(K\partial w) \pm \frac{1}{2}\ell_2(\varphi_*\eta,\varphi_*\theta)(K\partial w). \tag{3.6}$$

Observe that (3.4)=(3.5) by definition, while (3.6) can be expressed as

$$\begin{aligned}&\pm\frac{1}{2}K_*\ell_2(\varphi_*\theta,\varphi_*\eta)(\partial w) \pm \frac{1}{2}K_*\ell_2(\varphi_*\eta,\varphi_*\theta)(\partial w)\\ =&\frac{1}{2}\sum_i \pm K_*\ell_2(\varphi_*\theta,\varphi_*\eta)[z_i',z_i''] + \frac{1}{2}\sum_i \pm K_*\ell_2(\varphi_*\eta,\varphi_*\theta)[z_i',z_i''].\end{aligned}$$

Since $\ell_2(\varphi_*\theta,\varphi_*\eta)$ and $\ell_2(\varphi_*\eta,\varphi_*\theta)$ are $\gamma\varphi$-derivations, so is their composition with K_*. Moreover, it is easy to see that, only on generators,

$$\frac{1}{2}\big(\pm K_*\ell_2(\varphi_*\theta,\varphi_*\eta) \pm K_*\ell_2(\varphi_*\eta,\varphi_*\theta)\big) = \varphi_*\{\theta,\eta\}.$$

This readily implies that the above expression of (3.6) is equal to (3.3) and the theorem is proved.

□

4. The Lawrence-Sullivan interval and the Fiorenza-Manetti mapping cone

In what follows, given a graded vector space V, the completed free Lie algebra $\widehat{\mathbb{L}}(V)$ is the subalgebra of the completed tensor algebra $\widehat{T}(V) = \Pi_{n\geq 0}T^n(V)$ generated by $T^1(V) = V$ and with the Lie bracket given by commutators.

As in [22, 25] we shall consider the *Lawrence-Sullivan model* $(\widehat{\mathbb{L}}(a,b,x),\partial)$ in which a,b are *flat* or *Maurer-Cartan* elements of degree -1 and x is a degree zero element giving a gauge equivalence between a and b. These conditions define the differential ∂ on the generators a,b,x to be $\partial a = -\frac{1}{2}[a,a]$, $\partial b = -\frac{1}{2}[b,b]$, and

$$\partial x = (ad_x)b + \sum_{i=0}^{\infty}\frac{B_i}{i!}(ad_x)^i(b-a),$$

where B_i denotes the ith Bernoulli number. Recall that the Bernoulli numbers can be defined recursively as follows

$$B_0 = 1, \quad -\frac{B_n}{n!} = \sum_{k=0}^{n-1}\frac{B_{n-1-k}}{(n-1-k)!(k+2)!} \quad \text{for } n\geq 1.$$

Geometrically, one can think of this construction as the model, in the based homotopy category, of the disjoint union $I\dot\cup *$ of the interval I and a point [4]. Thus, a good candidate for a completed model of the interval would be then a model of the cofibre of the map $S^0 \to I\dot\cup *$ which sends -1

to 0 or 1 and it sends 1 to $*$. Since S^0 is modeled by $\mathbb{L}(b)$, in which b is a Maurer-Cartan element, the cofibre of this map is then modeled by

$$(\widehat{\mathbb{L}}(a,x),\partial) \quad \text{in which} \quad \partial(a) = -\frac{1}{2}[a,a] \quad \text{and} \quad \partial(x) = -\sum_{i\geq 0}\frac{B_i}{i!}\mathrm{ad}_x^i(a),$$

which we also call the *Lawrence-Sullivan construction*. In the same way, given M a Lie model of the CW-complex Y, the zero map $\phi\colon (\widehat{\mathbb{L}}(a,x),\partial) \to M$ could also be seen as a model of the constant map in the based path space $PY = \mathrm{map}^*(I,Y)$. In what follows we omit the subindex ϕ in $s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$, and endow this chain complex with the higher brackets given in formula (3.1) which are now drastically simplified since $\phi = 0$:

$$s\ell_k(s^{-1}\theta_1,\dots,s^{-1}\theta_k)(x) = (-1)^{\epsilon}\{\theta_1,\dots,\theta_k\}\left(-\sum_{i\geq 0}\frac{B_i}{i!}\mathrm{ad}_x^i(a)\right) \tag{4.1}$$

$$= (-1)^{\epsilon}\left(-\frac{B_{k-1}}{(k-1)!}\sum_{\sigma\in S_k}\epsilon'[\theta_{\sigma(1)}x,[\theta_{\sigma(2)}x,\dots[\theta_{\sigma(k-1)}x,\theta_{\sigma(k)}a]]\dots]\right).$$

On the other hand, consider $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{L}(C),M)$ (equipped with the usual DGL structure explicitly recalled at the beginning of Section 3), and where C is the dual of the standard acyclic CDGA $A = \Lambda(t,dt)$ with $|t| = 0$. Note that basis for A^0 and A^1 are given by $\{t^j\}_{j\geq 0}$ and $\{t^j dt\}_{j\geq 0}$ respectively. Thus, C is a differential vector space also concentrated in (subscript) degrees 0 and 1. Observe that any vector of C_0 can be written as a formal series $\sum_{j\geq 0}\lambda_j\alpha_j$ representing the function $\{t^j\}_{j\geq 0}\to\mathbb{Q}$ which assigns to each t^j the scalar λ_j. Respectively, any element in C_1 can be expressed as $\sum_{j\geq 0}\mu_j\beta_j$ representing the map $\{t^j dt\}_{j\geq 0}\to\mathbb{Q}$ sending $t^j dt$ to μ_j. Observe that, with this notation, the differential δ on C is the only linear and formal extension to the above series for which $\delta(\beta_j) = (j+1)\alpha_{j+1}$ and $\delta(\alpha_j) = 0$, for all $j\geq 0$. As A is not of finite type, C does not inherits directly a coalgebra structure. However, for each $j\geq 0$ we may set

$$\Delta(\alpha_j) = \sum_{k=0}^{j}\alpha_k\otimes\alpha_{j-k}, \quad \Delta(\beta_j) = \sum_{k=0}^{j}\beta_k\otimes\alpha_{j-k} + \sum_{k=0}^{j}\alpha_k\otimes\beta_{j-k},$$

and again, extend Δ in a linear and formal way to C so that it becomes a CDGC.

Then, we prove:

Theorem 4.1. $(s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M),\{\ell_k\}_{k\geq 1})$ *is an* L_∞ *algebra quasi-isomorphic to* $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{L}(C),M)$*. In particular, it is an* L_∞ *model of the based path space.*

Remark 4.2. Note that, since the completed Lawrence-Sullivan construction is not a minimal Quillen model in its classical sense, neither the results of [2] nor Theorem 3.1 can be readily applied here. However, we follow the same approach, transferring directly the DGL structure of $s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{L}(C), M)$, a natural DGL model of the based path space.

Proof. Consider the following diagram

$$K \circlearrowright s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{L}(C), M) \underset{Q}{\overset{J}{\rightleftarrows}} s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x), M).$$

in which:
(i) Q is defined by

$$\begin{aligned} &sQ(s^{-1}\eta)(s^{-1}\alpha_1) = \eta(a),\\ &sQ(s^{-1}\eta)(s^{-1}\beta_0) = \eta(x),\\ &sQ(s^{-1}\eta)(s^{-1}\alpha_j) = sQ(s^{-1}\eta)(s^{-1}\beta_{j-1}) = 0, \quad j \geq 2. \end{aligned}$$

(iii) J is defined by

$$\begin{aligned} sJ(s^{-1}\theta)(a) &= \sum_{j\geq 1} \theta(s^{-1}\alpha_j),\\ sJ(s^{-1}\theta)(x) &= \sum_{j\geq 0} \frac{1}{j+1}\theta(s^{-1}\beta_j). \end{aligned}$$

(iv) And finally, K is defined by

$$\begin{aligned} &sK(s^{-1}\theta)(s^{-1}\beta_j) = 0 \quad j \geq 0,\\ &sK(s^{-1}\theta)(s^{-1}\alpha_1) = (-1)^{|\theta|+1}\sum_{j\geq 1}\frac{1}{j+1}\theta(s^{-1}\beta_j)),\\ &sK(s^{-1}\theta)(s^{-1}\alpha_j) = (-1)^{|\theta|}\frac{1}{j}\theta(s^{-1}\beta_{j-1}) \quad j \geq 2. \end{aligned}$$

Note also that Q, J are quasi-isomorphism, as their domain and codomain are both contractible, while JQ is trivially the identity on $s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x), M)$. We now show that

$$\mathrm{Id}_L - QJ = s^{-1}\delta K + K s^{-1}\delta$$

with $(L, \ell_1) = (s^{-1}\,\mathcal{D}\mathrm{er}(\mathcal{L}(C), M), s^{-1}\delta)$. The left part of the formula reads:

$$s(\mathrm{Id}_L - QJ)(s^{-1}\theta)(s^{-1}\alpha_j) = \begin{cases} -\sum_{i\geq 2}\theta(s^{-1}\alpha_i) & \text{if } j = 1,\\ \theta(s^{-1}\alpha_j) & \text{if } j \geq 2, \end{cases}$$

$$s(\mathrm{Id}_L - QJ)(s^{-1}\theta)(s^{-1}\beta_j) = \begin{cases} -\sum_{i\geq 1}\frac{1}{i+1}\theta(s^{-1}\beta_i)) & \text{if } j = 0,\\ \theta(s^{-1}\beta_j) & \text{if } j \geq 1. \end{cases}$$

For the right part, a direct computation shows that

$$sK(s^{-1}\delta\theta)(s^{-1}\beta_j) = 0 \text{ if } j \geq 0,$$

$$sK(s^{-1}\delta\theta)(s^{-1}\alpha_j) = \begin{cases} (-1)^{|\theta|+1}\sum_{i\geq 1}\left(\frac{1}{i+1}\partial_M\theta(s^{-1}\beta_i) + (-1)^{|\theta|}\theta(s^{-1}\alpha_{i+1})\right) & \text{if } j=1,\\ (-1)^{|\theta|}\frac{1}{j}\partial_M\theta(s^{-1}\beta_{j-1}) + \theta(s^{-1}\alpha_j) & \text{if } j\geq 2,\end{cases}$$

$$s^{-1}\delta K(s^{-1}\theta)(s^{-1}\beta_j) = \begin{cases} -\sum_{i\geq 1}\frac{1}{i+1}\theta(s^{-1}\beta_i) & \text{if } j=0,\\ \theta(s^{-1}\beta_j) & \text{if } j\geq 1,\end{cases}$$

$$s^{-1}\delta K(s^{-1}\theta)(s^{-1}\alpha_j) = \begin{cases} (-1)^{|\theta|}\sum_{i\geq 1}\frac{1}{i+1}\partial_M\theta(s^{-1}\alpha_i) & \text{if } j=1,\\ (-1)^{|\theta|+1}\frac{1}{j}\partial_M\theta(s^{-1}\beta_{j-1}) & \text{if } j\geq 2.\end{cases}$$

Finally, comparing these expressions, one gets the required equality.

Thus, we have all the data needed to apply Theorem 3.2 and obtain an explicit L_∞ structure on $s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$ with brackets $\{\ell'_k\}_{k\geq 1}$ given by equation (3.2).

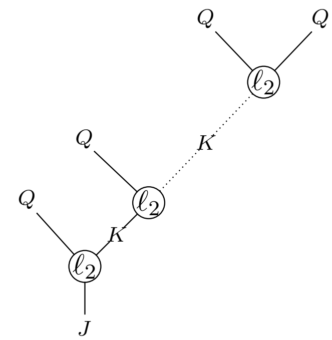
We finish by showing that these brackets agree with the ones in (4.1). In the case $k=2$, the formula (3.2) produces the following ℓ'_2,

$$\begin{aligned} s\ell'_2(s^{-1}\eta_1,s^{-1}\eta_2)(x) &= \Big(sJ\ell_2(Q(s^{-1}\eta_1),Q(s^{-1}\eta_2))(x)\\ &\quad -(-1)^{|s^{-1}\eta_1||s^{-1}\eta_2|}sJ\ell_2(Q(s^{-1}\eta_2),Q(s^{-1}\eta_1))(x)\Big)/2\\ &= \frac{1}{2}\Big(s\ell_2(Q(s^{-1}\eta_1),Q(s^{-1}\eta_2))(s^{-1}\beta_1)\\ &\quad -(-1)^{|s^{-1}\eta_1||s^{-1}\eta_2|}s\ell_2(Q(s^{-1}\eta_2),Q(s^{-1}\eta_1))(s^{-1}\beta_1)\Big)/2\\ &= \frac{1}{2}\Big([\eta_1(a),\eta_2(x)] - (-1)^{|\eta_2|}[\eta_2(a),\eta_1(x)]\Big)\\ &= (-1)^{\epsilon}\left(-\frac{B_1}{1!}\sum_{\sigma\in S_2}\epsilon'[\eta_{\sigma(1)}(x),\eta_{\sigma(2)}(a)]\right), \end{aligned}$$

which is precisely ℓ_2 of (4.1).

Now, we proceed by induction on k. In order to make formulas clearer we will omit signs replacing them with $\pm$ and we will remove all suspensions and desuspensions. Note that, by definition of the map K, for $k\geq 2$ the only trees providing a non-trivial contribution to ℓ'_k are the trees with k

leaves of the form



Now, the induction hypotheses for $l < k$ reads

$$\begin{aligned}
& (-1)^{\epsilon}\left(-\frac{B_{l-1}}{(l-1)!}\sum_{\sigma\in S_l}\epsilon'[\eta_{\sigma(1)}x,[\eta_{\sigma(2)}x,\ldots[\eta_{\sigma(l-1)}x,\eta_{\sigma(l)}a]]\ldots]\right)\\
= \;& \ell'_l(\eta_1,\ldots,\eta_l)(x)\\
= \;& \left(\sum_{\sigma\in S_l}\pm J\ell_2(Q\eta_{\sigma(1)},K\ell_2(\ldots,K\ell_2(Q\eta_{\sigma(l-1)},Q\eta_{\sigma(l)})\ldots)(x)\right)/2\\
= \;& \sum_{\substack{\sigma\in S_l\\ \sigma(l-1)<\sigma(l)}}\pm J\ell_2(Q\eta_{\sigma(1)},K\ell_2(\ldots,K\ell_2(Q\eta_{\sigma(l-1)},Q\eta_{\sigma(l)}\ldots)(x)\\
= \;& \sum_{\substack{\sigma\in S_l\\ \sigma(l-1)<\sigma(l)}}\sum_{i\geq 0}\frac{\pm 1}{i+1}\ell_2(Q\eta_{\sigma(1)},K\ell_2(\ldots,K\ell_2(Q\eta_{\sigma(l-1)},Q\eta_{\sigma(l)})\ldots)(\beta_i).
\end{aligned}$$

By a similar computation we have

$$\begin{aligned}
& \ell'_k(\eta_1,\ldots,\eta_k)(x)\\
= \;& \sum_{\substack{\sigma\in S_k\\ \sigma(k-1)<\sigma(k)}}\sum_{i\geq 0}\frac{\pm 1}{i+1}\ell_2(Q\eta_{\sigma(1)},K\ell_2(\ldots,K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\ldots)(\beta_i)\\
= \;& \sum_{\substack{\sigma\in S_k\\ \sigma(k-1)<\sigma(k)}}\sum_{i\geq 1}\frac{\pm 1}{i+1}\Big[\eta_{\sigma(1)}(x),K\ell_2(\ldots,K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\ldots)(\alpha_i)\Big]\\
= \;& \sum_{\substack{\sigma\in S_k\\ \sigma(k-1)<\sigma(k)}}\Big(\pm\frac{1}{2}\Big[\eta_{\sigma(1)}(x),\sum_{j\geq 1}\frac{1}{j+1}\ell_2(Q\eta_{\sigma(2)},K\ell_2(\ldots\\
& \qquad\qquad\ldots,K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\ldots)(\alpha_j)\Big]
\end{aligned}$$

$$
\begin{aligned}
\pm\ & \frac{1}{3}\Big[\eta_{\sigma(1)}x, K\ell_2(Q\eta_{\sigma(2)},\dots K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\dots)(\alpha_2)\Big]\\
\pm\ & \frac{1}{4}\Big[\eta_{\sigma(1)}x, K\ell_2(Q\eta_{\sigma(2)},\dots K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\dots)(\alpha_3)\Big]\pm\dots\Big)\\
=\ & \sum_{\sigma\in S_k}\pm\frac{1}{2}\frac{B_{k-2}}{(k-2)!}[\eta_{\sigma(1)}x,[\eta_{\sigma(2)}x,\dots[\eta_{\sigma(k-1)}x,\eta_{\sigma(k)}a]]\dots]\\
+\ & \sum_{\substack{\sigma\in S_k\\ \sigma(k-1)<\sigma(k)}}\Big(\pm\frac{1}{3}\frac{1}{2}\Big[\eta_{\sigma(1)}x,\ell_2(Q\eta_{\sigma(2)},K\ell_2(\dots K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\dots)(\beta_1)\Big]\\
+\ & \sum_{\substack{\sigma\in S_k\\ \sigma(k-1)<\sigma(k)}}\Big(\pm\frac{1}{4}\frac{1}{3}\Big[\eta_{\sigma(1)}x,\ell_2(Q\eta_{\sigma(2)},K\ell_2(\dots K\ell_2(Q\eta_{\sigma(k-1)},Q\eta_{\sigma(k)})\dots)(\beta_2)\Big]\\
\pm\ & \dots\Big)
\end{aligned}
$$

Repeating the above argument recursively we finally obtain that

$$
\begin{aligned}
&\ell'_k(\eta_1,\dots,\eta_k)(x) =\\
&= \pm\sum_{\sigma\in S_k}\left(\sum_{i=0}^{k-2}\frac{B_{k-2-i}}{(k-2-i)!(i+2)!}\right)[\eta_{\sigma(1)}x,[\eta_{\sigma(2)}x,\dots[\eta_{\sigma(k-1)}x,\eta_{\sigma(k)}a]]\dots]\\
&\qquad = \pm\left(-\frac{B_{k-1}}{(k-1)!}\sum_{\sigma\in S_k}\epsilon'[\eta_{\sigma(1)}x,[\eta_{\sigma(2)}x,\dots[\eta_{\sigma(k-1)}x,\eta_{\sigma(k)}a]]\dots]\right),
\end{aligned}
$$

where in the last step we use the recurrence formula for the Bernoulli numbers. □

We now see how the homotopy fiber of a given DGL morphism $\chi\colon L\to M$ is the common link of all of the above with the Fiorenza-Manetti *mapping cone* of χ, which we now recall. In general, the complex

$$
C_\chi=(C_\chi,\delta),\qquad (C_\chi)_i=L_i\oplus M_{i-1},\qquad \delta(l,m)=(\partial_L l,\chi(l)-\partial_M m),
$$

does not have a DGL structure. However, it can be canonically endowed with an L_∞ structure lifting the Lie algebra structure on L and the differential of the mapping cone [10, Theorem 1]. Explicitly, this structure is given by

$$
\ell_2(l_1,l_2)=[l_1,l_2]_L,\quad \ell_2(m,l)=\frac{1}{2}[m,\chi(l)]_M,\quad \ell_2(m_1,m_2)=0,
$$

and, for $k>2$, the only nonzero higher order brackets are

$$
\begin{aligned}
&\ell_k(m_1,\dots,m_{k-1},l)\\
&\qquad = (-1)^\epsilon\left(-\frac{B_{k-1}}{(k-1)!}\sum_{\sigma\in S_{k-1}}\epsilon'[m_{\sigma(1)},[m_{\sigma(2)},\dots[m_{\sigma(k-1)},\chi(l)]]\dots]\right),
\end{aligned}
$$

where B_i is again the ith Bernoulli number.

On the other hand, recall that the homotopy fiber of a given DGL morphism $\chi\colon L\to M$ is the homotopy pullback of

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ L & \xrightarrow{\chi} & M. \end{array}$$

To effectively compute this, one has to find a fibrant replacement $P\to M$ of $0\to M$, i.e., one has to replace 0 by a based path object P for M, and then compute the ordinary pullback. In [10, §3], this replacement is chosen to be the DGL morphism $M\otimes\Lambda^+(t,dt)\to M$ induced by the evaluation at $t=1$. Then, as shown in [10, §3], C_χ and $L\times_M M\otimes\Lambda^+(t,dt)$ are quasi-isomorphic as L_∞ algebras, and this exhibits C_χ as an L_∞ model of the homotopy fibre of χ.

Here we choose a different fibrant replacement, namely, the L_∞ morphism $ev\colon s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)\to M$, given by $ev(s^{-1}\theta)=(-1)^{|\theta|+1}\theta(a)$. At the sight of Theorem 4.1, this is a good path object which geometrically models the evaluation at the endpoint on based paths.

Next, observe that ev is a strong L_∞ morphism and therefore, the componentwise L_∞ structure on the product $L\times s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$ restricts to an L_∞ structure on the pullback

$$L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M).$$

Note that this is not true in general if χ or ev are regular L_∞ morphisms.

Thus, both C_χ and $L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$ are L_∞ models (and therefore quasi-isomorphic) of the homotopy fibre of χ. Furthermore, we obtain:

Theorem 4.3. *The morphism $\Psi\colon L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)\to C_\chi$ defined as $\Psi(l,s^{-1}\theta)=(l,\theta(x))$ is a strong isomorphism of L_∞ algebras.*

Proof. Given an element $(l,s^{-1}\theta)\in L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$, note that θ is determined by $\theta(x)$ since $\theta(a)=\chi(l)$. It is then clear that Ψ is an isomorphism of graded vector spaces. It remains to show that Ψ is compatible with the brackets defining the L_∞ structures. We will denote by $\{\ell_k\}$ and $\{\widetilde{\ell}_k\}$ the higher brackets on C_χ and $L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$, respectively.

Let $(l,s^{-1}\theta)\in L\times_M s^{-1}\,\mathcal{D}\mathrm{er}(\widehat{\mathbb{L}}(a,x),M)$. Recall that $\ell_1=\delta$ and $\widetilde{\ell}_1=d$. It is easy to check that Ψ is compatible with the differentials, i.e., $\delta\Psi(l,s^{-1}\theta)=\Psi d(l,s^{-1}\theta)$. Indeed,

$$\begin{aligned}
\Psi d(l,s^{-1}\theta) &= \Psi(\partial_L l,-s^{-1}\delta\theta)=(\partial_L l,-\delta\theta(x))\\
&= (\partial_L l,-\partial_M\theta(x)-(-1)^{|\theta|}\theta\Big(\sum_{k\geq 0}\frac{B_k}{k!}\mathrm{ad}_x^k(a)\Big))\\
&= (\partial_L l,-\partial_M\theta(x)-(-1)^{|\theta|}\theta(a))\\
&= (\partial_L l,-\partial_M\theta(x)+\gamma(l))=\delta\Psi(l,s^{-1}\theta).
\end{aligned}$$

Now we need to show that Ψ is also compatible with the higher brackets. We will denote by $\theta = \theta_{l,m}$ the derivation defined by $\theta(a) = \chi(l)$ and $\theta(x) = m$. For $k = 2$,

$$\begin{aligned}
\Psi\widetilde{\ell}_2((l_1, s^{-1}\theta_{l_1,0}), (l_2, s^{-1}\theta_{l_2,0})) &= \Psi([l_1, l_2]_L, \ell_2'(s^{-1}\theta_{l_1,0}, s^{-1}\theta_{l_2,0})) \\
&= ([l_1, l_2]_L, s\ell_2'(s^{-1}\theta_{l_1,0}, s^{-1}\theta_{l_2,0})(x)) \\
&= ([l_1, l_2]_L, 0) = \ell_2((l_1, 0), (l_2, 0)) \\
&= \ell_2(\Psi(l_1, s^{-1}\theta_{l_1,0}), \Psi(l_2, s^{-1}\theta_{l_2,0})),
\end{aligned}$$

$$\begin{aligned}
\Psi\widetilde{\ell}_2((0, s^{-1}\theta_{0,m_1}), (0, s^{-1}\theta_{0,m_2})) &= \Psi(0, \ell_2'(s^{-1}\theta_{0,m_1}, s^{-1}\theta_{0,m_2})) \\
&= (0, s\ell_2'(s^{-1}\theta_{0,m_1}, s^{-1}\theta_{0,m_2})(x)) = (0, 0) \\
&= \ell_2(\Psi(0, s^{-1}\theta_{0,m_1}), \Psi(0, s^{-1}\theta_{0,m_2})),
\end{aligned}$$

$$\begin{aligned}
\Psi\widetilde{\ell}_2((0, s^{-1}\theta_{0,m}), (l, s^{-1}\theta_{l,0})) &= \Psi(0, \ell_2'(s^{-1}\theta_{0,m}, s^{-1}\theta_{l,0})) \\
&= (0, s\ell_2'(s^{-1}\theta_{0,m}, s^{-1}\theta_{l,0})(x)) = (0, \frac{1}{2}[m, \chi(l)]_M) \\
&= \ell_2(\Psi(0, s^{-1}\theta_{0,m}), \Psi(l, s^{-1}\theta_{l,0})).
\end{aligned}$$

The case $k > 2$ is obtained by a similar computation using (4.1). □

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