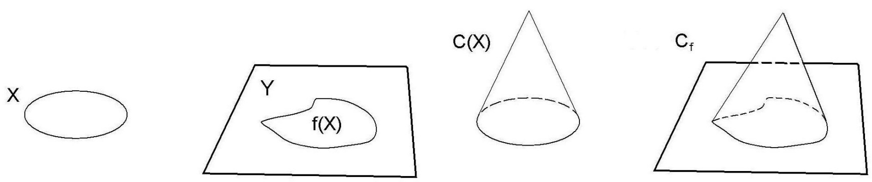
ALGEBRAIC TOPOLOGY, EXERCISE SHEET 4, 16.10.2015

Exercise 1. Let $f\colon X \to Y$ be a continuous map and let $i\colon X \to CX$ be the inclusion of X into its cone as defined in Lecture 4 . Let $\sim_f$ be the equivalence relation on the disjoint union $Y \sqcup CX$ generated by $f(x) \sim_f i(x)$ for all $x \in X$. We define the *mapping cone* of the map f as the quotient space $C_f := Y \sqcup CX/_{\sim_f}$. Let $q\colon Y \sqcup CX \to C_f$ be the canonical quotient map. Let $p\colon Y \to C_f$



be the composition of q with the inclusion $Y \to Y \sqcup CX$. Show that for every continuous map $g\colon Y \to Z$ such that $g \circ f$ is homotopic to a constant map there is a map $h\colon C_f \to Z$ such that $h \circ p = g$. More diagrammatically:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle i}\downarrow & & {\scriptstyle p}\downarrow & \nearrow{\scriptstyle \exists h} & \\ CX & \longrightarrow & C_f & & \end{array}$$

Exercise 2 (The category of pairs of topological spaces)**.**

(1) Define the category Top^2 of pairs of topological spaces.
(2) Show that the formation of relative singular chain complexes and relative singular homology groups define functors $C\colon \mathsf{Top}^2 \to \mathsf{Ch}(\mathbb{Z})$ and $H_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$ respectively.

Exercise 3 (Alternative description of relative singular homology)**.** Let (X, A) be a pair of spaces. Recall that the relative homology groups $H_n(X, A)$ are defined as the homology groups of the chain complex with components $C_n(X)/C_n(A)$. For each n, we define subsets of $C_n(X)$ by

$$Z_n(X, A) = \{\sigma \in C_n(X) \mid \partial(\sigma) \in C_{n-1}(A)\},$$
$$B_n(X, A) = \{\sigma \in C_n(X) \mid \exists \sigma' \in C_n(A)\colon \sigma - \sigma' \in B_n(X)\}.$$

Show that

(1) $Z_n(X, A)$ is a subgroup of $C_n(X)$ and $B_n(X, A)$ is a subgroup of $Z_n(X, A)$.
(2) $H_n(X, A) \cong Z_n(X, A)/B_n(X, A)$ for each $n \geq 0$.

Exercise 4 (Chain homotopy)**.** Let C, D and E be chain complexes.

(1) Show that chain homotopy defines an equivalence relation on the set $\mathsf{Ch}(C, D)$ of chain maps from C to D. Denote by $[C, D] := \mathsf{Ch}(C, D)/\sim$ be the set of chain homotopy classes of maps from C to D.

(2) Let $f_1, f_2, g_1, g_2 \colon C \to D$ be chain maps such that f_1 is chain homotopic to f_2 and g_1 is chain homotopic to g_2. Show that $f_1 + g_1$ is chain homotopic to $f_2 + g_2$ and that $-f_1$ is chain homotopic to $-f_2$. Conclude from this that $[C, D]$ has the natural structure of an abelian group.
(3) Let $C = \mathbb{Z}[n]$ be the chain complex with $C_n = \mathbb{Z}$ and $C_m = 0$ for all $m \neq n$ (with zero differentials). Try to identify the abelian group $[\mathbb{Z}[n], D]$.
(4) Let $f_1, f_2 \colon C \to D$ be chain homotopic maps and let $g \colon D \to E$ be a chain map. Show that $h \circ f_1$ is chain homotopic to $h \circ f_2$.

Dually, if $f \colon C \to D$ is a chain map and $g_1, g_2 \colon D \to E$ are chain homotopic, show that $g_1 \circ f$ is chain homotopic to $g_2 \circ f$. Conclude from this that there is a well-defined composition map

$$\circ : [D, E] \times [C, D] \longrightarrow [C, E]$$

We recall the definition of *split* exact sequence. Given two abelian groups A' and A'', there is always the short exact sequence

$$0 \longrightarrow A' \xrightarrow{i} A' \oplus A'' \xrightarrow{p} A'' \longrightarrow 0$$

where i is the inclusion $a' \mapsto (a', 0)$ and p the projection $(a'; a'') \mapsto a''$. A short exact sequence $0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} A'' \longrightarrow 0$ is *split* if there is an isomorphism $f \colon A \mapsto A' \oplus A''$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{j} & A & \xrightarrow{q} & A'' & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow f & & \downarrow = & & \\ 0 & \longrightarrow & A' & \xrightarrow[i]{} & A' \oplus A'' & \xrightarrow[p]{} & A'' & \longrightarrow & 0 \end{array}$$

Exercise 5 (Splitting exact sequences)**.**

(1) Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$ be a short exact sequence of abelian groups. Show that the following statements are equivalent:
 (a) The map q admits a section, i.e., there is a homomorphism $s \colon A'' \to A$ such that $q\, s = \mathrm{id} \colon A'' \to A''$;
 (b) The map j admits a retraction, i.e., there is a homomorphism $r \colon A \to A'$ such that $r\, j = \mathrm{id} \colon A' \to A'$;
 (c) The short exact sequence is split.
(2) Let $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$ be a short exact sequence of abelian groups with $A'' = \mathbb{Z}$ (or more generally: any free abelian group). Show that this short exact sequence is split.
(3) Try to find an example of a short exact sequence of abelian groups that is not split.
(4) State and prove a statement analogous to (1) for chain complexes.
(5) Show that if B is a retract of space X then $H_n(X) \simeq H_n(A) \oplus H_n(X, A)$ for every $n \in \mathbb{N}$.

Warning: (2) is not true when A'' is a *chain complex* consisting of free abelian groups.

Exercise 6. Use the previous exercise and the long exact sequence of pairs to show that $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and specify a basis for it.