Exercise 1. Calculate the homology groups of the wedge of two spheres $S^n \vee S^m$.

Exercise 2. Consider the circle $S^1 \subset \mathbb{R}^2$. Let $\sigma_1: \Delta^1 \rightarrow S^1$ and $\sigma_2: \Delta^1 \rightarrow S^1$ be paths from $(-1,0)$ to $(1,0)$ parameterizing the lower and the upper semicircle, respectively. Show that $\sigma_2 - \sigma_1$ represents the generator $\omega_1 \in H_1(S^1)$. (Hint: Hurewicz isomorphism)

Exercise 3 (Reduced homology). Show that given a pointed space $(X,x_0)$ there is an isomorphism $\tilde{H}_i(X) \cong H_i(X,x_0)$ which is natural with respect to pointed maps, i.e., maps sending base points to base points.

Exercise 4 (Suspension). Define the “unreduced suspension” $\Sigma X$ of a space $X$ to be the quotient space of $I \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. Show that there is a natural isomorphism $\tilde{H}_i(X) \rightarrow \tilde{H}_{i+1}(\Sigma X)$ for every $i \in \mathbb{N}$.

Exercise 5 (Mayer-Vietoris for pairs).

(1) Let $X = X_1 \cup X_2$ for two subspaces $X_1, X_2 \subset X$ with non-empty intersection and let $A \subset X_1 \cap X_2$ be a subspace then there is a long exact sequence in homology, called the relative Mayer-Vietoris sequence (Hint: Exercise 27):

$$\ldots \rightarrow H_n(X_1 \cap X_2, A) \rightarrow H_n(X_1, A) \oplus H_n(X_2, A) \rightarrow H_n(X, A) \rightarrow \ldots$$

(2) Deduce the following reduced version of Mayer-Vietoris: if $X_1, X_2$ are subspaces of $X$ with $X = X_1 \cup X_2$ and $X_1 \cap X_2 \neq \emptyset$ then there is an exact sequence:

$$\ldots \rightarrow \tilde{H}_n(X_1 \cap X_2) \rightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \rightarrow \tilde{H}_n(X) \rightarrow \ldots$$

Exercise 6 (Homology of the torus and the Klein bottle).

(1) Let us recall that the torus is obtained from the following rectangle by identifying edges marked with the same letter in a way that the orientations of the arrows match:

![Diagram of a rectangle with identified edges](image)

Calculate the homology groups of the torus $X$. (Hint: try to apply Mayer-Vietoris with the interior of the rectangle as $X_1$ and the complement of a point in the interior of the rectangle as $X_2$).

(2) The Klein bottle is obtained in a similar way by identifying edges of a rectangle as indicated in the following picture:
Calculate the homology of the Klein bottle.

**Exercise 7.** Show that the following two statements are equivalent:

1. Let $U \subseteq A \subseteq X$ be subspaces of $X$ such that $U \subseteq A^\circ$. Then the inclusion $(X \setminus U, A \setminus U)(X, A)$ induces isomorphisms on relative homology groups.

2. Let $X_1, X_2 \subseteq X$ be subspaces of $X$ such that $X_1^\circ \cup X_2^\circ = X$. Then the inclusion $(X_1, X_1 \cap X_2)(X, X_2)$ induces isomorphisms on relative homology groups.

**Exercise 8 (Algebraic Mayer-Vietoris sequence.)** Conclude the proof of Lemma 4, Lecture 6. In more detail, establish the exactness of the algebraic Mayer-Vietoris sequence in the remaining two cases.

**Exercise 9.** Let $(X, d)$ be a compact metric space and let $(U_i)_{i \in I}$ be an open cover of $X$. Then there is a positive real number $\lambda$, called a *Lebesgue number of the cover*, such that every subset of $X$ of diameter less than $\lambda$ is entirely contained in $U_i$ for some $i$.

(Hint: we can suppose $I$ to be finite (why?), consider the following function, check that it is continuous and admits a minimum:

$$m: X \to \mathbb{R}_{\geq 0}: \quad x \mapsto \max_{i \in I} \{d(x, X - U_i)\}.$$