ALGEBRAIC TOPOLOGY, EXERCISE SHEET 6, 06.11.2015

Exercise 1. Calculate the homology groups of the wedge of two spheres $S^n \vee S^m$.

Exercise 2. Consider the circle $S^1 \subset \mathbb{R}^2$. Let $\sigma_1\colon \Delta^1 \to S^1$ and $\sigma_2\colon \Delta^1 \to S^1$ be paths from $(-1,0)$ to $(1,0)$ parameterizing the lower and the upper semicircle, respectively. Show that $\sigma_2 - \sigma_1$ represents the generator $\omega_1 \in H_1(S^1)$. (Hint: Hurewicz isomorphism)

Exercise 3 (Reduced homology)**.** Show that given a pointed space (X, x_0) there is an isomorphism $\tilde{H}_i(X) \cong H_i(X, x_0)$ which is natural with respect to pointed maps, i.e., maps sending base points to base points.

Exercise 4 (Suspension)**.** Define the "unreduced suspension' ΣX of a space X to be the quotient space of $I \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. Show that there is a natural isomorphism $\tilde{H}_i(X) \longrightarrow \tilde{H}_{i+1}(\Sigma X)$ for every $i \in \mathbb{N}$.

Exercise 5 (Mayer-Vietoris for pairs)**.**

(1) Let $X = X_1^\circ \cup X_2^\circ$ for two subspaces X_1, $X_2 \subset X$ with non-empty intersection and let $A \subset X_1 \cap X_2$ be a subspace then there is a long exact sequence in homology, called the *relative Mayer-Vietoris sequence* (Hint: Exercise 27):

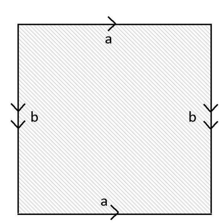
$$\ldots \to H_n(X_1 \cap X_2, A) \to H_n(X_1, A) \oplus H_n(X_2, A) \to H_n(X, A) \to \ldots$$

(2) Deduce the following *reduced version of Mayer-Vietoris*: if X_1, X_2 are subspaces of X with $X = X_1^\circ \cup X_2^\circ$ and $X_1 \cap X_2 \neq \emptyset$ then there is an exact sequence:

$$\ldots \to \tilde{H}_n(X_1 \cap X_2) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(X) \to \ldots$$

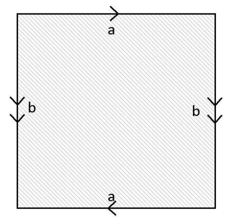
Exercise 6 (Homology of the torus and the Klein bottle)**.**

(1) Let us recall that the torus is obtained from the following rectangle by identifying edges marked with the same letter in a way that the orientations of the arrows match:



Calculate the homology groups of the torus X. (Hint: try to apply Mayer-Vietoris with the interior of the rectangle as X_1 and the complement of a point in the interior of the rectangle as X_2).

(2) The *Klein bottle* is obtained in a similar way by identifying edges of a rectangle as indicated in the following picture:



Calculate the homology of the Klein bottle.

Exercise 7. Show that the following two statements are equivalent:

(1) Let $U \subseteq A \subseteq X$ be subspaces of X such that $U \subseteq A^\circ$. Then the inclusion $(X\backslash U, A\backslash U)(X, A)$ induces isomorphisms on relative homology groups.
(2) Let $X_1, X_2 \subseteq X$ be subspaces of X such that $X_1^\circ \cup X_2^\circ = X$. Then the inclusion $(X_1, X_1 \cap X_2)(X, X_2)$ induces isomorphisms on relative homology groups.

Exercise 8 (Algebraic Mayer-Vietoris sequence.)**.** Conclude the proof of Lemma 4, Lecture 6. In more detail, establish the exactness of the algebraic Mayer-Vietoris sequence in the remaining two cases.

Exercise 9. Let (X, d) be a compact metric space and let $(U_i)_{i\in I}$ be an open cover of X. Then there is a positive real number λ, called a *Lebesgue number of the cover*, such that every subset of X of diameter less than λ is entirely contained in U_i for some i.

(Hint: we can suppose I to be finite (why?), consider the following function, check that it is continuous and admits a minimum:

$$m\colon X \to \mathbb{R}_{\geq 0}\colon \quad x \mapsto \max_{i\in I}\{d(x, X - U_i)\}\ ).$$