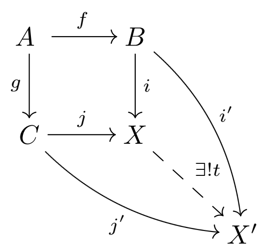
ALGEBRAIC TOPOLOGY, EXERCISE SHEET 8, 20.11.2015

Exercise 1 (Pushout of topological spaces)**.** Let $\mathcal{C}$ be a category and let $f\colon A \to B$ and $g\colon A \to C$ be morphisms in $\mathcal{C}$ with a common domain. A pair of maps $i\colon B \to X$ and $j\colon C \to X$ is called a *pushout* of f and g if

- $i \circ f = j \circ g$ and
- for every other pair $i'\colon B \to X', j'\colon C \to X'$ such that $i' \circ f = j' \circ g$, there exists a unique morphism $t\colon X \to X'$ such that $t \circ i = i'$ and $t \circ j = j'$.

More diagrammatically:



(1) Let $(i\colon B \to X, j\colon C \to X)$ and $(i'\colon B \to Y, j'\colon C \to Y)$ both be pushouts of morphisms f and g. Show that there is a unique isomorphism $p\colon X \to Y$ such that $i' = p \circ i, j' = p \circ j$. This justifies that we talk about *the* pushout of f and g.

(2) Show that for every two continuous maps of topological spaces with a common domain there is a pushout of these maps.
Hint: use disjoint unions and quotients of spaces.

(3) Let $(i\colon B \to X, j\colon C \to X)$ be a pushout of f and g in $\mathcal{T}\mathrm{op}$. Show that U is open in X if and only if $i^{-1}(U)$ and $j^{-1}(U)$ are open in B and C respectively.

(4) Let $(i\colon B \to X, j\colon C \to X)$ be a pushout of f and g in $\mathcal{T}\mathrm{op}$. Show that if g is a closed embedding then this is also the case for i.

Exercise 2.

(1) The torus can be obtained from the square by identifying opposite sides. Use an adapted CW decomposition of the square to also turn the torus into a CW complex.

(2) Similarly we can obtain the Klein bottle from the unit square by identifying $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$. Show that there is a similar CW decomposition of the Klein bottle.

(3) Can you come up with CW decompositions of the torus and the Klein bottle which have the same number of cells in each dimension? In particular this shows the obvious fact that the number of cells does *not* determine the space.

(4) Give a CW-structure on the real line, the real line without a finite number of points and the plane without one point.

Exercise 3 (Hawaiian earrings)**.** Let $E_n \subset \mathbb{R}^2$ be the circle with radius $1/n$ centered at $(1/n, 0)$ and radius $1/n$. Moreover, let $E = \bigcup_{n\in\mathbb{N}} E_n$ be topologized as subspace of $\mathbb{R}^2$. Show that there is no CW structure on E.

Recall from Lecture 6 that the reduced homology of the spheres is given by

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

Any map $f\colon S^n \to S^n$ induces a group homomorphism $f_*\colon \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$. If we choose a generator ω of $\tilde{H}_n(S^n)$, then we can write $f_*(\omega) = p \cdot \omega$ for a unique integer p. We will call this integer p the *degree of f* and denote it by $\deg(f)$. Note that this is independent of the chosen generator ω.

Exercise 4 (Degrees of maps between spheres)**.**

(1) Show that two homotopic maps between S^n have the same degree. Use this to prove that for any line l in $\mathbb{R}^{n+1}$ through the origin, the map $f_{l,\theta}\colon S^n \to S^n$ obtained by rotating the sphere by θ degrees around the line l has degree 1.
(2) Let $f, g\colon S^n \to S^n$. Show that $\deg(fg) = \deg(f) \cdot \deg(g)$ and that $\deg(\mathrm{id}_{S^n}) = 1$.
(3) If $f\colon S^n \to S^n$ is not surjective, show that $\deg(f) = 0$.
(4) Recall that the unreduced suspension $\Sigma(S^n)$ of S^n is the quotient of $S^n \times [-1,1]$ by identifying $S^n \times \{1\}$ and $S^n \times \{-1\}$ with points. Show that there is a well-defined homeomorphism

$$\Sigma(S^n) \longrightarrow S^{n+1};\ [v,t] \longmapsto (\sqrt{1-t^2} \cdot v, t)$$

where we think of S^n as the unit sphere in $\mathbb{R}^{n+1}$.
(5) It follows from (4) that any map $f\colon S^n \to S^n$ induces a map $\Sigma(f)\colon S^{n+1} \to S^{n+1}$. Show that $\deg(f) = \deg(\Sigma(f))$.
Hint: use Exercise 4 of sheet 6.
(6) Show that for any $p \in \mathbb{Z}$, there exists a map $f\colon S^n \to S^n$ which has degree p.
(7) Let $H \subseteq \mathbb{R}^{n+1}$ be a hyperplane through the origin. Show that reflection at H induces a map $S^n \to S^n$ of degree -1.
Hint: use (1) to prove that all reflections at hyperplanes have the same degree. Now use (5) to prove that the map $S^n \to S^n; (x_0, x_1, \cdots, x_n) \mapsto (-x_0, x_1, \cdots, x_n)$ has degree -1.

Exercise 5 (Fundamental theorem of algebra)**.** Realize S^2 as the one point compactification of $\mathbb{C}$, where we will call the extra point ∞.

(1) Show that every non-constant polynomial map $f\colon \mathbb{C} \to \mathbb{C}$ can be continuously extended to a map $\hat{f}\colon S^2 \to S^2$ that sends ∞ to ∞.
Hint: use that the open neighbourhoods of ∞ correspond to complements of compact subspaces of $\mathbb{C}$.
(2) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of degree k with leading coefficient 1 and let $g\colon \mathbb{C} \to \mathbb{C}$ be given by $g(z) = z^k$. Show that $\hat{f}$ and $\hat{g}$ are homotopic and conclude that they have the same degree.
(3) Let $\hat{g}\colon S^2 \to S^2$ be the map induced by $g(z) = z^k$. Notice that $\hat{g}$ preserves the subspaces $S^2 \backslash \{\infty\}$ and $S^2 \backslash \{0\}$. Use this, together with the naturality of the Mayer-Vietoris sequence, to prove that $\hat{g}$ has degree k.
(4) Conclude that every non-constant polynomial $f\colon \mathbb{C} \to \mathbb{C}$ has a zero.

Remark 6. Recall that $S^2 \cong \mathbb{C}P^1$ and that the inclusion $\mathbb{C} \to \mathbb{C}P^1; z \mapsto [z : 1]$ realizes $\mathbb{C}P^1$ at the one point compactification of $\mathbb{C}$. Using this, the map $\hat{f}$ associated to a polynomial $f(z) = a_k z^k + \cdots + a_0$ is simply the holomorphic map

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1;\ \big[z_0 : z_1\big] \longmapsto \big[a_k z_0^k + a_{k-1} z_0^{k-1} z_1 + \cdots + a_0 z_1^k : z_1^k\big].$$