ALGEBRAIC TOPOLOGY, EXERCISE SHEET 9, 27.11.2015

Exercise 1 (More on the degree)**.**

(1) Show that for any permutation $\sigma \in \Sigma_{n+1}$, the map
$$S^n \to S^n; (x_1, ..., x_{n+1}) \mapsto (x_{\sigma(1)}, ..., x_{\sigma(n)})$$
has degree sign(σ) (the sign of σ).

(2) Consider two maps $f, g\colon X \to S^n$ with the property that $f(x) \neq -g(x)$ for all x. Show that f is homotopic to g.

(3) Let $f\colon S^n \to S^n$ be any map. Show that f has degree 1 if there are no points $x \in S^n$ such that $f(x) = -x$.

(4) Let $f\colon S^n \to S^n$ be any map. Show that f has degree $(-1)^{n+1}$ whenever f has no fixed point.

Exercise 2. A vector field in $\mathbb{R}^n$ is a function that associates to each point of $\mathbb{R}^n$ a n-dimensional real vector in a continuous way; it can be regarded as a continuous function $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$. Let $D^n \subset \mathbb{R}^n$ be the n-disc of radius one and let ∂D^n be its boundary. Suppose that Φ is a vector field in $\mathbb{R}^n$ such that for every $x \in \partial D^n$ the vector $\Phi(x)$ is pointing outside D^n (i.e. $\langle x, \Phi x\rangle \geq 0$). Show that there is a point s inside D^n such that $\Phi(s) = 0$.

Similarly, a vector field on $S^n \subseteq \mathbb{R}^{n+1}$ is a continuous function $\Phi\colon S^n \to \mathbb{R}^{n+1}$ with the property that $\big\langle \Phi(x), x\big\rangle = 0$ for all $x \in S^n$ (i.e. $\Phi(x)$ lies in the hyperplane orthogonal to x).

Exercise 3 (Hairy ball theorem)**.**

(1) Show that all spheres S^{2n+1} admit a vector field which is nowhere zero (i.e. $\Phi(x) \in \mathbb{R}^{2n+2}$ is not the zero vector).

(2) Show that any vector field on a sphere S^{2n} has a zero.

(3) Show that there are three tangent vector fields α, β, γ on S^3 that are everywhere linearly independent, i.e. for every $x \in S^3$ the vectors $\alpha(x), \beta(x), \gamma(x)$ are linearly independent.
Hint: identify $S^3 \subseteq \mathbb{R}^4$ with the space of real quaternions of unitary norm.

Exercise 4. Let (X, Y) be a CW pair. Then the quotient space X/Y can be turned into a CW complex such that the quotient map $X \to X/Y$ is cellular.

Exercise 5. Let W be a finite CW complex whose cells are parametrized by sets J_n. We define the **Euler characteristic** $\chi(W)$ of W to be the integer
$$\chi(W) = \sum_{n\in\mathbb{N}} (-1)^n |J_n|.$$
Note that the sum is finite by our assumptions on W. Show that the Euler characteristic has the following properties (all spaces are finite CW complexes):

(1) For subcomplexes A and B of X such that $A \cup B = X$ we have:
$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B) \qquad \text{(additivity)}$$

(2) For CW complexes X and Y we have:

$$\chi(X \times Y) = \chi(X) \times \chi(Y) \qquad \text{(multiplicativity)}$$

(3) A combination of some of the constructions introduced so far allows us to deduce that the cone CX of a finite CW complex X is again a finite CW complex (show this using, for example, the CW structure on the interval consisting of two 0-cells and one 1-cell). Use this explicit cell structure on CX to show that we have

$$\chi(CX) = 1.$$

Exercise 6. Let X be a CW complex and for every $n > 0$ let $\{e_\sigma\}_{\sigma \in J_n}$ be the set of n-cells of X. Let us set $J = \bigcup_{n>0} J_n$ and let $A \subseteq X$ be a subspace.

(1) Given a tuple $\varepsilon = \{\varepsilon_\sigma\}_{\sigma \in J}$ of numbers $\varepsilon_\sigma \in (0, 1]$ we want to inductively construct an 'ε-small' neighborhood $N_\varepsilon(A)$ of A in X. Begin by setting $N^0_\varepsilon(A) = A \cap X^{(0)}$. For the induction step, show that from $N^{n-1}_\varepsilon(A)$ you can build a neighborhood $N(A)^n_\varepsilon$ of $A \cap X^n$ in X^n which has the following two properties:
   - For the intersection with $X^{(n-1)}$ we obtain $N(A)^n_\varepsilon \cap X^{(n-1)} = N(A)^{n-1}_\varepsilon$.
   - For every n-cell $\sigma \in J_n$ and every $x \in \chi^{-1}_\sigma(N(A)^n_\varepsilon)$ the distance from x to the preimage $\chi^{-1}_\sigma(A \cup N(A)^{n-1}_\epsilon)$ is less than ε_σ.

   Now observe that the union $N_\varepsilon(A) = \bigcup_{n \in \mathbb{N}} N^n_\varepsilon(A)$ is a neighborhood of A in X. (Hint: it might be useful to identify use "polar coordinates", i.e., identify D^n with the cone over S^{n-1}).

(2) Show that every CW complex is normal. Thus show that disjoint closed subsets have disjoint open neighborhoods, and that points are closed.