TOPOLOGICAL K-THEORY, EXERCISE SHEET 1, 02.02.2015

Exercise 1. Show that the Möbius line bundle over S^1 is nontrivial.
Hint: any nowhere vanishing section $s\colon [0,1]/\sim \longrightarrow [0,1] \times \mathbb{R}^\times/\sim$ determines a continuous section $\overline{s}\colon [0,1]/\sim \longrightarrow [0,1] \times \{\pm 1\}/\sim$ of the twofold cover of the circle.

Exercise 2. Recall that the tangent bundle to $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is given by the space $TS^n = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : v \perp x\}$, equipped with the subspace topology.

(1) Show that the natural projection $p\colon TS^n \to S^n$ determines a vector bundle over the n-sphere.

(2) Prove that the direct sum of TS^n with a trivial (real) line bundle is a trivial (real) rank $(n+1)$-bundle over S^n.

(3) Prove that the tangent bundles to S^1 and S^3 (and S^7) are trivial vector bundles.
Hint: realize S^1 as the complex numbers of unit length and realize S^3 as the quaternions of unit length.

(4) Prove that for any $n \geq 0$, the tangent bundle to S^{2n+1} admits a nowhere vanishing section.

(5) For any $n \geq 1$, the antipodal map

$$S^{2n} \longrightarrow S^{2n}; \qquad x \longmapsto -x$$

is not homotopic to the identity map - if you know something about (de Rham) cohomology, prove this by showing that the antipodal map induces multiplication by -1 on $H^{2n}(S^{2n})$.

Use this to prove that any section of TS^{2n} must be zero at some point of S^{2n}. In particular, the tangent bundles to even-dimensional spheres are nontrivial.
Hint: assuming that there is a nowhere vanishing vector field on S^{2n}, construct a homotopy between the identity and the antipodal map on S^{2n}.

Exercise 3. The (real) Stiefel manifolds $V_k(\mathbb{R}^n) \subseteq (\mathbb{R}^n)^{\times k}$ are the subspaces consisting of k-tuples of vectors in $\mathbb{R}^n$ which are linearly independent (for $k \leq n$). Note that there is an action of $GL_k(\mathbb{R})$ on $V_k(\mathbb{R}^n)$, given by

$$A \cdot (v_1, \cdots, v_k) = \big(\sum_i A_{1i} v_i, \cdots, \sum_i A_{ki} v_i\big).$$

(1) The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is defined to be the quotient of $V_k(\mathbb{R}^n)$ by this action (with the quotient topology). Show that (as a set) $\mathrm{Gr}_k(\mathbb{R}^n)$ can be identified with the set of k-dimensional linear subspaces of $\mathbb{R}^n$.

(2) Let $E = \{(V, v) \in \mathrm{Gr}_k(\mathbb{R}^n) \times \mathbb{R}^n : v \in V\}$ be equipped with the subspace topology. Show that the natural projection

$$p\colon E \longrightarrow \mathrm{Gr}_k(\mathbb{R}^n); \qquad (V, v) \longmapsto V$$

determines a rank k vector bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$, which we call the tautological k-plane bundle.

Hint: for any subspace $V \subseteq \mathbb{R}^n$, let $\pi\colon \mathbb{R}^n \to V$ be the othogonal projection onto V. Show that

$$\mathcal{U} := \Big\{ W \in \mathrm{Gr}_k(\mathbb{R}^n) : \pi(W) = V \Big\}$$

is an open neighbourhood of V in $\mathrm{Gr}_k(\mathbb{R}^n)$. Show that the map

$$p^{-1}\big(\mathcal{U}\big) \longrightarrow \mathcal{U} \times V; \qquad \big(W, w\big) \longmapsto \big(W, \pi(w)\big)$$

provides a trivialization of E over the open subset $\mathcal{U} \subseteq \mathrm{Gr}_k(\mathbb{R}^n)$.