TOPOLOGICAL K-THEORY, EXERCISE SHEET 2, 12.02.2015

Exercise 1. Let $E \to X$ and $F \to X$ be two vector bundles over X. Show that there is a bijection between sections of the hom bundle $\mathrm{Hom}(E, F) \to X$ and maps of vector bundles from E to F (fixing the base X).

Exercise 2. A (Riemannian) metric on a real vector bundle $E \to X$ is a continuous function $E \otimes E \to \mathbb{R}$ whose restriction to each fiber $E_x \otimes E_x$ gives a positive-definite inner product on E_x.

(1) Show that specifying a metric on a vector bundle $E \to X$ is equivalent to choosing a section of the bundle $E^* \otimes E^* \to X$ whose value at each point $x \in X$ gives a positive-definite inner product on E_x.

(2) Suppose that $E \to X$ admits a metric. Show that there is an isomorphism of vector bundles $E \to E^*$ over X.

(3) Show that any (real) vector bundle over a paracompact Hausdorff space admits a metric.

Exercise 3. For X a topological space, let $\mathrm{Pic}^{\mathbb{C}}(X)$ (resp. $\mathrm{Pic}^{\mathbb{R}}(X)$) be the set of isomorphism classes of complex (real) line bundles over X.

(1) Prove that the tensor product of line bundles induces an abelian group structure on $\mathrm{Pic}^{\mathbb{C}}(X)$ (or on $\mathrm{Pic}^{\mathbb{R}}(X)$). One sometimes calls this group the Picard group of X (although the terminology arises from algebraic geometry, where one considers algebraic/holomorphic line bundles).

(2) Prove that an element in $\mathrm{Pic}^{\mathbb{R}}(X)$ has order 2 if X is paracompact Hausdorff. Does the same thing hold for $\mathrm{Pic}^{\mathbb{C}}(X)$?