TOPOLOGICAL K-THEORY, EXERCISE SHEET 4, 26.02.2015

Exercise 1. Show that (complex/real) K-theory defines a functor $K\colon \mathsf{CTop}^{\mathrm{op}} \to \mathsf{Ring}$.

Exercise 2. Recall that we can identify $\mathbb{R}P^n$ with the quotient of S^n by the relation $x \sim -x$. Define the (total space of the) tangent bundle $T\mathbb{R}P^n$ of $\mathbb{R}P^n$ as the quotient of the space

$$TS^n = \big\{(x,v) \in S^n \times \mathbb{R}^{n+1} : v \perp x\big\}$$

by the relation $(x,v) \sim (-x,-v)$.

(1) Show that the canonical projection $\tau\colon T\mathbb{R}P^n \to \mathbb{R}P^n; [(x,v)] \mapsto [x]$ gives a vector bundle over $\mathbb{R}P^n$.

(2) Let γ be the canonical line bundle over $\mathbb{R}P^n$ and let $\gamma^\perp$ be the bundle

$$\Big\{(l,v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \perp l\Big\}$$

Show that there is an isomorphism of vector bundles over $\mathbb{R}P^n$

$$\tau \xrightarrow{\simeq} \mathrm{Hom}(\gamma, \gamma^\perp).$$

Hint: by the 'calculus of vector bundles', giving a map $\tau \to \mathrm{Hom}(\gamma,\gamma^\perp)$ is equivalent to giving a map $\phi\colon \tau \otimes \gamma \to \gamma^\perp$. Let $[x]$ be the line through the unit vector x. Given a vector $\lambda \cdot x$ in $[x]$ and a vector v orthogonal to x, define

$$\phi(v \otimes \lambda \cdot x) = \lambda \cdot v \in [x]^\perp$$

Check that this is a well defined map of vector bundles, which induces an isomorphism between τ and $\mathrm{Hom}(\gamma,\gamma^\perp)$.

(3) Use (2) to prove that $\tau \oplus \mathbb{R} \simeq \mathbb{R}^{n+1}$ is a trivial vector bundle of rank $n+1$.

Exercise 3. Let X be a paracompact Hausdorff space and let $E \to X$ be a vector bundle over X.

(1) Let $A \subseteq X$ be a closed subset of X over which E admits a section s. Shows that this section extends to a section of E over the whole space X.

Hint: let $U_i \subseteq X$ be opens such that the $A \cap U_i$ cover A and such that E is trivial over U_i. Over each $U_i \cap A$, we can identify the section s with a $\mathbb{R}^n$-valued function. But on a paracompact Hausdorff space (such as U_i) we can always extend real-valued functions from a closed subset to the whole space (Tietze's extension theorem).

Use this to extend the section $s\big|_{U_i \cap A}$ to a section of E over U_i. Finally, use a partition of unity to construct from these local extensions a global extension of s to the whole of X.

(2) Let $F \to X$ be another vector bundle over X. Suppose that there is a closed subset $A \subseteq X$ over which there exists an isomorphism of vector bundles $\phi\colon F\big|_A \to E\big|_A$. Show that there is an open $U \subseteq X$ containing A, over which there exists an isomorphism of vector bundles

$$\tilde{\phi}\colon F\big|_U \longrightarrow E\big|_U$$

which extends the isomorphism ϕ we already had on A.

Hint: ϕ determines a section of $\mathrm{Hom}(F, E)$ over A, which takes values in linear isomorphisms.