TOPOLOGICAL K-THEORY, EXERCISE SHEET 5, 05.03.2015

Exercise 1. For any pointed compact Hausdorff space (X,x), realize the (unreduced) suspension ΣX as the quotient of $X \times [-1,1]$ by collapsing each of $X \times \{-1\}$ and $X \times \{1\}$ to a point. View the result as a pointed space with basepoint $(x,0)$. For sufficiently nice X (e.g. a CW-complex, whose basepoint is a 0-cell) this is homotopy equivalent to the reduced suspension.

(1) View $\mathrm{GL}(n,\mathbb{C})$ as a pointed space with basepoint the identity matrix. Construct a natural map

$$\mathsf{Top}_*\Big(X \times (-1/2,1/2), \mathrm{GL}(n,\mathbb{C})\Big) \longrightarrow \mathrm{Vect}^n(\Sigma X)$$

from the set of *pointed* $\mathrm{GL}(n,\mathbb{C})$-valued functions on $X \times (-1/2,1/2)$ to the set of isomorphism classes of vector bundles on ΣX.
Hint: the function $g\colon X \times (-1/2,1/2) \to \mathrm{GL}(n,\mathbb{C})$ determines the transition function for a bundle which is trivial on $X \times (-1/2,1]/_\sim$ and on $X \times [-1,1/2)/_\sim$.

(2) Show that the above map is surjective.

(3) Show that pointed homotopic maps $X \times (-1/2,1/2) \to \mathrm{GL}(n,\mathbb{C})$ determine isomorphic vector bundles on X.
Hint: from a pointed homotopy you can construct a vector bundle on $\Sigma(X \times I)$, which is a quotient of $\big(\Sigma X\big) \times I$.

(4) Show that the map

$$[X, \mathrm{GL}(n,\mathbb{C})]_* \simeq [X \times (-1/2,1/2), \mathrm{GL}(n,\mathbb{C})]_* \longrightarrow \mathrm{Vect}^n(\Sigma X)$$

is a natural isomorphism.
Hint: if two (pointed) maps $g_1, g_2\colon X \times (-1/2,1/2) \to \mathrm{GL}(n,\mathbb{C})$ determine isomorphic bundles, show that there are maps

$$\begin{aligned} f_0\colon X \times [-1,1/2)/X \times \{-1\} &\longrightarrow \mathrm{GL}(n,\mathbb{C}) \\ f_1\colon \quad X \times (-1/2,1]/X \times \{1\} &\longrightarrow \mathrm{GL}(n,\mathbb{C}) \end{aligned}$$

such that $f_0 \cdot g_0 = g_1 \cdot f_1$ on $X \times (-1/2,1/2)$.

(5) Convince yourself that the same thing works when we replace $\mathrm{GL}(n,\mathbb{C})$ by $\mathrm{GL}(n,\mathbb{R})$ and complex vector bundles by real vector bundles.

Exercise 2. There are natural inclusions

$$\mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(n+1,\mathbb{C});\;\; A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

preserving the basepoint. Let $\mathrm{colim}_n\, \mathrm{GL}(n,\mathbb{C})$ be the union of all general linear groups, *with the weak topology.*

(1) For $f \in [X, \mathrm{GL}(n,\mathbb{C})]_*$, let $[E_f] \in \mathrm{Vect}^n(\Sigma X)$ be the vector bundle associated to it by the previous exercise. Show that the map

$$[X, \mathrm{colim}_n \mathrm{GL}(n,\mathbb{C})]_* \longrightarrow \widetilde{K}(\Sigma X);\ \ f \longmapsto [E_f] - [\mathrm{rank}(E_f)]$$

is a bijection (X is pointed compact Hausdorff as in the previous exercise).

(2) The space $\mathrm{colim}_n \mathrm{GL}(n,\mathbb{C})$ comes equipped with a continuous binary operation induced by the maps

$$\oplus\colon \mathrm{GL}(m,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(m+n,\mathbb{C});\ \ A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Show that the induced binary operation on $[X, \mathrm{colim}_n \mathrm{GL}(n,\mathbb{C})]_*$ agrees with the addition in $\tilde{K}(\Sigma X)$.

(3) The space $[X, \mathrm{colim}_n \mathrm{GL}(n,\mathbb{C})]_*$ also comes with another group structure, induced by matrix multiplication. Show that this group structure agrees with the addition in $\tilde{K}(\Sigma X)$ by constructing a homotopy between matrix multiplication and the map $\oplus$.

This shows that matrix multiplication becomes commutative up to homotopy, at least when the dimension n tends to ∞. On the other hand, it shows that the block sum $\oplus$ turns $\mathrm{colim}_n \mathrm{GL}(n,\mathbb{C})$ into a commutative H-group.

Exercise 3. Let X be a pointed compact Hausdorff space and let $T\colon \Sigma X \to \Sigma X$ be the map sending $[X,t]$ to $[X,-t]$.

(1) Let $g\colon X \times (-1/2, 1/2) \to \mathrm{GL}(n,\mathbb{C})$ be a (pointed) map and let E_g be the associated vector bundle. Show that $T^* E_g$ is the bundle associated to the map $\overline{g}\colon X \times (-1/2, 1/2) \to \mathrm{GL}(n,\mathbb{C})$ given by

$$\overline{g}(x,t) = g(x,-t)^{-1}$$

(2) Conclude with the help of the previous exercises that the map $T\colon \Sigma X \to \Sigma X$ induces the map $T^*\colon \tilde{K}(\Sigma X) \to \tilde{K}(\Sigma X)$ sending an element to its (additive) inverse.