TOPOLOGICAL K-THEORY, EXERCISE SHEET 7, 19.03.2015

Exercise 1. Let γ be the canonical line bundle on $\mathbb{C}P^1$ and recall that there is a homeomorphism $\mathbb{C}P^1 \simeq S^2$. Show that γ is the line bundle associated to the clutching function

$$S^1 \longrightarrow \mathbb{C} \setminus \{0\}; \ \ z \longmapsto z$$

Hint: one can realize the northpole of S^2 as the point $[0,1] \in \mathbb{C}P^1$, the southpole as the point $[1,0] \in \mathbb{C}P^1$ and the equator as the set of points $[z,1] \in \mathbb{C}P^1$ with $|z| = 1$.

Exercise 2. Let $f\colon S^1 \to \mathrm{GL}(k, \mathbb{C})$ be a clutching function of the form

$$f(z) = A_n \cdot z^n + ... + A_1 \cdot z + A_0$$

where $A_n, ..., A_0$ are $k \times k$ matrices. Let E_f be the rank k vector bundle over S^2 associated to f. Show that the direct sum $\mathbb{C}^{n \cdot k} \oplus E_f$ is isomorphic to E_g, where $g\colon S^1 \to \mathrm{GL}\big((n+1) \cdot k, \mathbb{C}\big)$ is of the form

$$g(z) = B_1 \cdot z + B_0$$

for matrices B_1 and B_0 with $(n+1)k$ rows and columns.

Hint: the bundle $\mathbb{C}^{n \cdot k} \oplus E_f$ is presented by the clutching function $S^1 \to \mathrm{GL}\big((n+1)k, \mathbb{C}\big)$ given by

$$z \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(z) \end{pmatrix} \tag{1}$$

Now consider the linear clutching function $S^1 \to \mathrm{GL}\big((n+1)k, \mathbb{C}\big)$ given by

$$z \mapsto \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 & A_0 \end{pmatrix} \tag{2}$$

Show that you can obtain clutching function (1) from clutching function (2) by elementary matrix operations (adding a multiple of a row to another row, etc...). Note that elementary matrix operations preserve the determinant of a matrix, and are all these operations are homotopic to the identity. Conclude from this that the two clutching functions (1) and (2) are homotopic.