TOPOLOGICAL K-THEORY, EXERCISE SHEET 9, 16.04.2015

Exercise 1. Let $\mathbf{C}$ be a (small) category. Associated to $\mathbf{C}$ is a simplicial set (the *nerve* of $\mathbf{C}$)

$$N\mathbf{C}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set};\ \ [n] \longmapsto \mathrm{Fun}([n], \mathbf{C})$$

sending each $[n] = [0 < ... < n]$ to the set of functors from $[n]$ to $\mathbf{C}$.

(1) Show that the set $N\mathbf{C}_n$ can be identified with the set of n-tuples of composable arrows $c_0 \to c_1 \to ... \to c_n$ in $\mathbf{C}$. Using this, identify the face and degeneracy maps

$$d_i\colon N\mathbf{C}_n \longrightarrow N\mathbf{C}_{n-1} \qquad s_i\colon N\mathbf{C}_n \longrightarrow N\mathbf{C}_{n+1}$$

for all $0 \leq i \leq n$.

(2) A functor $f\colon \mathbf{C} \to \mathbf{D}$ induces a map of simplicial sets $N(f)\colon N\mathbf{C} \to N\mathbf{D}$, by sending a tuple

$$N\mathbf{C}_n \ni \Big(c_0 \to c_1 \to ... \to c_n\Big) \mapsto \Big(f(c_0) \to f(c_1) \to ... \to f(c_n)\Big).$$

Show that every map of simplicial sets $N\mathbf{C} \to N\mathbf{D}$ is given by $N(f)$ for some functor $f\colon \mathbf{C} \to \mathbf{D}$.

Conversely, show that for any two functors $f, g\colon \mathbf{C} \to \mathbf{D}$ such that $N(f) = N(g)$, we have that $f = g$.

(3) Show that $N(\mathbf{C} \times \mathbf{D}) \simeq N(\mathbf{C}) \times N(\mathbf{D})$.

(4) Recall that a category $\mathbf{C}$ is a groupoid if each arrow in $\mathbf{C}$ is an isomorphism. Show that a category $\mathbf{C}$ is a groupoid precisely if $N\mathbf{C}$ satisfies the following '*horn-filling condition*': given two elements $x, y \in N\mathbf{C}_1$ such that $d_1(x) = d_1(y)$, there exists an element $z \in N\mathbf{C}_2$ such that

$$x = d_2(z) \qquad y = d_1(z).$$

(5) The classifying space $B\mathbf{C}$ of a category $\mathbf{C}$ is the geometric realization of the simplicial set $N\mathbf{C}$. Show that for any groupoid $\mathbf{C}$, the set of path-components $\pi_0(B\mathbf{C})$ is isomorphic to the set of isomorphism classes of objects in $\mathbf{C}$.

Exercise 2. Let $f_0, f_1\colon A \to X$ be two maps between simplicial sets. A homotopy from f_0 to f_1 is a map of simplicial sets

$$H\colon A \times \Delta[1] \to X$$

such that $H \circ (\mathrm{id}_A \times d^1) = f_0$ and $H \circ (\mathrm{id}_A \times d^0) = f_1$.

(1) Suppose that $A = N\mathbf{C}$ and $X = N\mathbf{D}$. Show that a homotopy from $N(f_0)$ to $N(f_1)$ is the same as a natural transformation from f_0 to f_1.

(2) Suppose $\mathbf{C}$ is a category with a terminal object c. Show that there is a homotopy from the identity map $N\mathbf{C} \to N\mathbf{C}$ to the map $N\mathbf{C} \to N\mathbf{C}$ sending

$$N\mathbf{C} \ni \Big(c_0 \to ... \to c_n\Big) \longmapsto \Big(c = c = ... = c\Big)$$

(3) If A is a simplicial set, then there is a natural homeomorphism $|A \times \Delta[1]| \simeq |A| \times |\Delta[1]|$. Use this to show that the classifying space of a category with a terminal object is contractible.

Exercise 3. If P is a set with a left action of a group G, let $P//G$ denote the category whose objects are elements $p \in P$ and where a morphism $g\colon p \to q$ is an element $g \in G$ such that $g \cdot p = q$.

(1) Show that there is a bijection $N(P//G)_n \simeq P \times G^{\times n}$ and identify the face and degeneracy maps in these terms.

(2) Consider the action of G on itself by *left* multiplication. For each $g \in G$, show that *right* multiplication induces a map of simplicial sets $\rho_g\colon N(G//G) \to N(G//G)$. Prove that this defines a right G-action on $N(G//G)$, such that the G-action on each $N(G//G)_n$ is free.

(3) Consider the trivial action of G on the one-element set $*$. Show that there is a map $N(G//G) \to N(*//G)$ that realizes each $N(*//G)_n$ as the quotient of $N(G//G)_n$ by the right action of G.

(4) Show that the classifying space $B(G//G)$ is a contractible space and that the right G-action on $N(G//G)$ induces a free right G-action on $B(G//G)$. Also show that the map $B(G//G) \to B(*//G)$ realizes $B(*//G)$ as the quotient of $N(G//G)$ by this free G-action.

The map $B(G//G) \to B(*//G)$ is called the 'universal G-bundle' and is usually denoted by $EG \to BG$.