LECTURE 6: K-THEORY AS A COHOMOLOGY THEORY

In this lecture we will prove that K-theory is a generalized cohomology theory. For this we will need to use the Bott periodicity theorem in order to define the *positive* K-groups. A detailed proof of Bott periodicity will be given in the third part of the course.

6.1. Eilenberg–Steenrod axioms for cohomology

Eilenberg and Steenrod introduced in 1945 an axiomatic approach to cohomology (and homology) theory by abstracting the fundamental properties that any cohomology theory should satisfy.

6.1.1. A *cohomology theory* h^* on Top^2 (or any nice subcategory like compact pairs, pairs of CW-complexes, ...) is a collection of *contravariant* functors

$$h^n\colon \mathsf{Top}^2 \longrightarrow \mathsf{Ab},\ n \in \mathbb{Z},$$

where Ab denotes the category of abelian groups, and natural transformations

$$\delta^n\colon h^n \circ R \longrightarrow h^{n+1},$$

where $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ is the functor that sends (X, A) to $(A, \emptyset)$ and f to $f|A$, satisfying the following axioms:

(i) *Homotopy invariance.* If $f \simeq g$, then $h^n(f) = h^n(g)$ for every n in $\mathbb{Z}$.

(ii) *Excision.* For every pair (X, A) and $U \subseteq A$ such that the closure $\overline{U}$ is contained in the interior A^o, the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces and isomorphism

$$h^n(X \setminus U, A \setminus U) \cong h^n(X, A),\ \text{for every } n \in \mathbb{Z}.$$

(iii) *Exactness.* For every par (X, A), consider the inclusions $i\colon A \to X$ and $j\colon (X, \emptyset) \to (X, A)$. Then there is a long exact sequence

$$\cdots \longrightarrow h^{n-1}(A) \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*}$$
$$\xrightarrow{i^*} h^n(A) \xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} h^{n+1}(X) \xrightarrow{i^*} h^{n+1}(A) \longrightarrow \cdots$$

If moreover h^* satisfies the *dimension axiom*, that is, $h^n(*) = 0$ for every n in $\mathbb{Z}$, then h^* is called an *ordinary cohomology theory*; otherwise it is called an *generalized* or *extraordinary cohomology theory.*

For example, singular, cellular, de Rham and Čech cohomology are all ordinary cohomology theories. They all coincide on finite CW-pairs. However, K-theory will be a generalized cohomology theory.

6.1.2. A *reduced cohomology theory* $\widetilde{h}^*$ on Top_* is a collection of *contravariant* functors

$$\widetilde{h}^n\colon \mathsf{Top}_* \longrightarrow \mathsf{Ab},\ n \in \mathbb{Z}$$

and natural equivalences $\widetilde{h}^n \circ \Sigma \overset{\cong}{\to} \widetilde{h}^{n+1}$ satisfying the following axioms:

(i) *Homotopy invariance.* If $f \simeq g$, then $\widetilde{h}^n(f) = \widetilde{h}^n(g)$ for every n in $\mathbb{Z}$.

(ii) *Exactness*. For every pair (X, A) in Top^2 and A in Top_* there is an exact sequence

$$\widetilde{h}^n(X \cup CA) \xrightarrow{j^*} \widetilde{h}^n(X) \xrightarrow{i^*} \widetilde{h}^n(A), \text{ for every } n \in \mathbb{Z},$$

where $i\colon A \to X$ and $j\colon X \to X \cup CA$ denote the inclusions.

Theorem 6.1.3. *K-theory and reduced K-theory are a generalized cohomology theory and a reduced cohomology theory, respectively.*

Proof. We have already defined the negative K-groups (see Definition 5.2.1). To define the positive ones, we have to use the Bott periodicity theorem. This theorem states that there is an isomorphism

$$\beta\colon \widetilde{K}^{-n}(X) \xrightarrow{\cong} \widetilde{K}^{-n-2}(X), \text{ for all } n \geq 0.$$

Since, for a space X in Top we have that $K^{-n}(X) = \widetilde{K}^{-n}(X_+)$ there is also an isomorphism $K^{-n}(X) \cong K^{-n-2}(X)$ in the unreduced case.

Thus, for a space X in Top_*, we can define

$$\widetilde{K}^{2n}(X) = \widetilde{K}^0(X) \text{ and } \widetilde{K}^{2n+1}(X) = \widetilde{K}^{-1}(X) \text{ for every } n \text{ in } \mathbb{Z}.$$

And similarly, for a space X in Top, we define

$$K^{2n}(X) = K^0(X) \text{ and } K^{2n+1}(X) = K^{-1}(X) \text{ for every } n \text{ in } \mathbb{Z}.$$

This allows to extend all the results about exact sequences from the previous lectures to all the integers. In particular we can extend the long exact sequence of Corollary 5.3.6 to an infinite long exact sequence also on the right.

Homotopy invariance for K^* and $\widetilde{K}^*$ follows from the fact that if we have a vector bundle and we pullback along homotopic maps, then we get isomorphic bundles, and is left as an exercise.

Exactness for K^* is precisely Corollary 5.3.6 and for $\widetilde{K}^*$ it follows from Corollary 5.3.3.

The excision axiom is essentially the fact that $K^n(X, A) = \widetilde{K}(X/A)$. To prove it, let $X = X_1 \cup X_2$ and also $X = X_1^o \cup X_2^o$. Then $X_1/X_1 \cap X_2 \cong X/X_2$ and thus

$$K^n(X_1, X_1 \cap X_2) = \widetilde{K}^n(X_1/X_1 \cap X_2) \cong \widetilde{K}^n(X/X_2) = K^n(X, X_2).$$

Now, let $X_1 = X \setminus U$ and $X_2 = A$ and observe that

$$(X \setminus U)^o \cup A^o = (X \setminus \overline{U}) \cup A^o \supseteq (X \setminus A^o) \cup A^o = X.$$

So, we can apply the previous result to get excision. □

Remark 6.1.4. Due to Bott periodicity we have only two different K-groups K^0 and K^1. So, alternatively, the long exact sequence of Corollary 5.3.6 can be written as an exact sequence of six terms

$$\begin{array}{ccccc} K^0(X,A) & \longrightarrow & K^0(X) & \longrightarrow & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,A). \end{array}$$

Corollary 6.1.5. *Let X and Y in Top_*. Then $\widetilde{K}^{-n}(X \vee Y) \cong \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$ for every $n \geq 0$.*

Proof. We have (pointed) inclusions $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ and surjections $r_1\colon X \vee Y \to X$ and $r_2\colon Y \to X \vee Y$. They satisfy that $r_1 i_1 = \mathrm{id}_X$ and $r_2 i_2 = \mathrm{id}_Y$. So taking $\widetilde{K}^{-n}$ we have maps

$$\widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y) \xrightarrow{r_1^* + r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{(i_1^*, i_2^*)} \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$$

and $(i_1^*, i_2^*) \circ (r_1^* + r_2^*) = \mathrm{id}$. This means that (i_1^*, i_2^*) is surjective.

To prove that it is also injective, let $\xi \in \ker(i_1^*, i_2^*)$. Then $i_1^*(\xi) = 0$ and $i_2^*(\xi) = 0$. Consider the pair $(X \vee Y, X)$ and apply Corollary 5.3.5 to get an exact sequence

$$K^{-n}(X \vee Y, X) = \widetilde{K}^{-n}(X \vee Y/X) \cong \widetilde{K}^{-n}(Y) \xrightarrow{r_2^*} \widetilde{K}^{-n}(X \vee Y) \xrightarrow{i_1^*} \widetilde{K}^{-n}(X).$$

Since $\xi \in \ker i_1^*$, there exists an element $\eta \in \widetilde{K}^{-n}(Y)$ such that $r_2^*(\eta) = \xi$. But $\eta = i_2^* r_2^*(\eta) = i_2^*(\xi) = 0$. Hence $\eta = 0$ and so $\xi = 0$ too. □

Corollary 6.1.6. *Let (X, A) in Top^2 and A in Top_*. If A is contractible, then $\widetilde{K}^{-n}(X/A) \cong \widetilde{K}^{-n}(X)$ for every $n \geq 0$.*

Proof. Use the long exact sequence of Corollary 5.3.5 and the fact that if A is contractible, then $\Sigma^n A$ is also contractible, hence $\widetilde{K}^{-n}(A) = 0$ for every $n \geq 0$. □

Corollary 6.1.7. *Let X and Y in Top_* and Y be a retract of X. Then*

$$\widetilde{K}^{-n}(X) \cong K^{-n}(X, Y) \oplus \widetilde{K}^{-n}(Y)$$

for every $n \geq 0$.

Proof. Since Y is a retract of X, there exists a map $r\colon X \to Y$ such that $ri = \mathrm{id}_Y$, where i denotes the inclusion. This means that $i^* r^* = \mathrm{id}$ and hence that i^* in injective. Thus, in the long exact sequence of Corollary 5.3.5 the map δ factors through the zero map and therefore we have *split* short exact sequences

$$0 \longrightarrow K^{-n}(X, Y) \xrightarrow{j^*} \widetilde{K}^{-n}(X) \xrightarrow{i^*} \widetilde{K}^{-n}(Y) \longrightarrow 0,$$

since r^* is a section. So $\widetilde{K}^{-n}(X) \cong K^{-n}(X, Y) \oplus \widetilde{K}^{-n}(Y)$. □

Remark 6.1.8. The same result is true in the unreduced case for X and Y in Top and Y a retract of X. Namely,

$$K^{-n}(X) \cong K^{-n}(X, Y) \oplus K^{-n}(Y)$$

for every $n \geq 0$. It can be deduced from the previous case by replacing X and Y by X_+ and Y_+, respectively, and using that $\widetilde{K}^{-n}(X_+) = K^{-n}(X)$.

Corollary 6.1.9. *Let X and Y in Top_*. Then the projection maps $\pi_1\colon X \times Y \to X$, $\pi_2\colon X \times Y \to Y$ and the quotient map $q\colon X \times Y \to X \times Y / X \vee Y = X \wedge Y$ induce an isomorphism*

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y)$$

for every $n \geq 0$.

Proof. Using the map $X \to X \times Y$ that sends x to (x, y_0) and the projection π_1 we can see that X is a retract of $X \times Y$. By Corollary 6.1.7 we have that

$$\widetilde{K}^{-n}(X \times Y) \cong K^{-n}(X \times Y, X) \oplus \widetilde{K}^{-n}(X).$$

Now, $K^{-n}(X \times Y, X) = \widetilde{K}^{-n}(X \times Y/X)$. But Y is a retract of $X \times Y/X$, so applying Corollary 6.1.7 again, we obtain

$$\widetilde{K}^{-n}(X \times Y/X) \cong K^{-n}(X \times Y/X, Y) \oplus \widetilde{K}^{-n}(Y).$$

But $K^{-n}(X \times Y/X, Y) = \widetilde{K}^{-n}(X \vee Y)$ yielding the result. $\square$

Remark 6.1.10. If we assume Bott periodicity, then all of the previous corollaries hold for K^n and $\widetilde{K}^n$ for $n \in \mathbb{Z}$.

6.2. The external product for reduced K-theory

6.2.1. Let X and Y in Top_* and consider the *external product*

$$\begin{array}{ccc} K^0(X) \otimes K^0(X) & \longrightarrow & K^0(X \times Y) \\ \xi \otimes \xi' & \longmapsto & \xi * \xi' = \pi_1^*(\xi) \cdot \pi_2^*(\xi'), \end{array}$$

where π_1 and π_2 denote the projections. This external product is induced by the tensor product of vector bundles

$$\begin{array}{ccc} \mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(Y) & \longrightarrow & \mathrm{Vect}_{\mathbb{C}}(X \times Y) \\ E \otimes F & \longmapsto & \pi_1^*(E) \otimes \pi_2^*(F). \end{array}$$

6.2.2. Now, let us see what happens if we restrict this external product to elements in $\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y)$. Recall that $\widetilde{K}^0(X) = \ker(K^0(X) \to K^0(x_0))$. Let $\xi \in \widetilde{K}^0(X)$ and $\xi' \in \widetilde{K}^0(Y)$. Then from the following commutative diagram

$$\begin{array}{ccccc} K^0(\{x_0\} \times Y) & \longleftarrow & K^0(X \times Y) & \xleftarrow{\pi_1^*} & K^0(X) \\ \uparrow & & & \swarrow & \\ K^0(x_0) & & & & \end{array}$$

it follows that $\pi_1^*(\xi)$ that lies in $K^0(X \times Y)$ restricts to zero in $K^0(\{x_0\} \times Y)$. Similarly $\pi_2^*(\xi')$ restricts to zero in $K^0(X \times \{y_0\})$. So, $\pi_1^*(\xi) \cdot \pi_2^*(\xi')$ restrict to zero in $K^0(X \vee Y)$ and hence, it lies in the kernel of $K^0(X \times Y) \to K^0(*)$ which is $\widetilde{K}^0(X \times Y)$. By Corollary 6.1.9 there is a split short exact sequence

$$0 \longrightarrow \widetilde{K}^0(X \wedge Y) \longrightarrow \widetilde{K}^0(X \times Y) \longrightarrow \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \cong \widetilde{K}^0(X \vee Y) \longrightarrow 0.$$

Since $\pi_1^*(\xi) \cdot \pi_2^*(\xi')$ lies in $\widetilde{K}^0(X \times Y$ and is zero in $\widetilde{K}^0(X \times Y)$ it lies in the kernel of the third map in the above sequence, which is $\widetilde{K}^0(X \wedge Y)$. So we have defined a map

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \longrightarrow \widetilde{K}^0(X \wedge Y). \tag{1}$$

This map is, in fact the restriction of the exterior product on K^0 as we can see in the following diagram

$$\begin{array}{ccc} K^0(X) \otimes K^0(Y) \cong & \widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) & \oplus\ \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \oplus \mathbb{Z} \\ \downarrow & \downarrow & \| \\ K^0(X \times Y) \quad \cong & \widetilde{K}^0(X \wedge Y) & \oplus\ \widetilde{K}^0(X) \oplus \widetilde{K}^0(Y) \oplus \mathbb{Z}, \end{array}$$

where the first isomorphism is obtained by using that $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$ and similarly for Y, and the isomorphism on the second row is obtained by using Corollary 6.1.9.

6.2.3. We can replace X by $\Sigma^n X$ and Y by $\Sigma^m Y$ in (1) to obtain a pairing

$$\widetilde{K}^{-n}(X) \otimes \widetilde{K}^{-m}(Y) \longrightarrow \widetilde{K}^{-n-m}(X \wedge Y).$$

If X and Y are in Top, we can replace X by X_+ and Y by Y_+ in the previous pairing to obtain a pairing

$$K^{-n}(X) \otimes K^{-m}(Y) \longrightarrow K^{-n-m}(X \wedge Y),$$

in the unreduced case.

6.3. Vector bundles on spheres and clutching functions

The only computation we have so far for the K-groups is $K^0(X) \cong \mathbb{Z}$ if X is a contractible space. The next natural step is to consider spheres.

6.3.1. We can decompose the sphere S^k as the union of the upper and the lower hemisphere. Since each hemisphere is contractible (it is homotopy equivalent to a disk) every fiber bundle on S^k restricts to a trivial bundle on each of the hemispheres. So a fiber bundle on S^k "should be determined" by a map from the intersection of the two hemispheres to $GL_n(\mathbb{C})$.

Definition 6.3.2. A *clutching function* for S^k is a map $f\colon S^{k-1} \to GL_n(\mathbb{C})$, where $GL_n(\mathbb{C})$ is the group of invertible $n \times n$ matrices with coefficients in $\mathbb{C}$.

6.3.3. Every clutching function $f\colon S^{k-1} \to GL_n(\mathbb{C})$ gives rise to a vector bundle E_f over S^k of rank n. We define

$$E_f = (D^- \times \mathbb{C}^n) \cup_{S^{k-1}\times\mathbb{C}^n} (D^+ \times \mathbb{C}^n),$$

where $D^- = \{(x_1, \dots, x_{k+1}) \in S^k \mid x_{k+1} \le 0\}$ is the lower hemisphere, and similarly, $D^+ = \{(x_1, \dots, x_{k+1}) \in S^k \mid x_{k+1} \ge 0\}$ is the upper hemisphere. If $x \in S^{k-1}$, then we identify (x, v) in $D^- \times \mathbb{C}^n$ with $(x, f(x)v)$ in $D^+ \times \mathbb{C}^n$. If f is homotopic to g, then $E_f \cong E_g$.

In fact, as we saw in the exercises, there is a bijection between homotopy classes of clutching functions from a space X to $GL_n(\mathbb{C})$ and $\mathrm{Vect}^n_{\mathbb{C}}(\Sigma X)$. In the case of spheres, this particularizes to the following result.

Proposition 6.3.4. *There is an isomorphism* $\mathrm{Vect}^n_{\mathbb{C}}(S^k) \cong [S^{k-1}, GL_n(\mathbb{C})]$ *for every* $n, k \ge 1$. □

Lemma 6.3.5. *The group* $GL_n(\mathbb{C})$ *is path-connected for every* $n \ge 1$.

Proof. The case $n = 1$ is trivial since $GL_1(\mathbb{C}) = \mathbb{C} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\}$, which is path-connected. Let $n \ge 2$ and let $A \in GL_n(\mathbb{C})$. We are going to show that A is connected to the identity matrix by a path. Let B be the Jordan canonical form of A, that is,

$$B = \begin{pmatrix} \boxed{J_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{J_k} \end{pmatrix}, \text{ where } J_i = \begin{pmatrix} \lambda_i & 1 & 1 \\ & \ddots & 1 \\ 0 & & \lambda_i \end{pmatrix},$$

and there exists and invertible matrix C such that $A = CBC^{-1}$.

For each $\lambda_i \in \mathbb{C}$, let $\gamma_i\colon I \to \mathbb{C}$ be a path from λ_i to 1 *not passing* through the origin. Let $B(t)$ be matrix obtained from B by replacing λ_i by γ_i and multiplying by $(1-t)$ all the elements above the diagonal.

Now define the path $\gamma\colon I \to GL_n(\mathbb{C})$ by $\gamma(t) = CB(t)C^{-1}$. This path satisfies that $\gamma(0) = CBC^{-1} = A$ and $\gamma(1) = CC^{-1} = \mathrm{Id}$. $\square$

Corollary 6.3.6. *Every complex vector bundle over S^1 is trivial. In particular, $K^0(S^1) \cong \mathbb{Z}$.*

Proof. By Proposition 6.3.4 we know that $\mathrm{Vect}^n_{\mathbb{C}}(S^1) \cong [S^0, GL_n(\mathbb{C})]$ consists of one element only, since $GL_n(\mathbb{C})$ is path-connected by Lemma 6.3.5. $\square$

Corollary 6.3.7. *Every complex line bundle over S^k for $k > 2$ is trivial.*

Proof. By Proposition 6.3.4 we know that $\mathrm{Vect}^1_{\mathbb{C}}(S^k) \cong [S^{k-1}, GL_1(\mathbb{C})] \cong [S^{k-1}, S^1]$, since $GL_1(\mathbb{C}) \simeq U(1) = S^1$, where $U(1)$ is the unitary group of order one (in fact, $GL_n(\mathbb{C}) \simeq U(n)$ for every n).

The sphere S^{k-1} is simply connected for $k > 2$, hence any map $S^{k-1} \to S^1$ factors through the universal cover $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is contractible, any map $S^{k-1} \to S^1$ is homotopic to a constant map. But any two constant maps on S^1 are homotopic because S^1 is path-connected. So $[S^{k-1}, S^1]$ has one element only. $\square$

Corollary 6.3.8. *As abelian group under the tensor product* $\mathrm{Vect}^2_{\mathbb{C}}(S^2) \cong \mathbb{Z}$.

Proof. By Proposition 6.3.4 we know that $\mathrm{Vect}^2_{\mathbb{C}}(S^2) \cong [S^1, S^1] \cong \mathbb{Z}$, since there is an isomorphism $\pi_1(S^1, x) \cong [S^1, S^1]$. $\square$