Topological K-theory, Lecture 2

Matan Prasma

March 2, 2015

Again, we assume throughout that our base space B is connected.

1 Direct sums

Recall from last time:

Given vector bundles $p_1 : E_1 \longrightarrow B$ and $p_2 : E_2 \longrightarrow B$, their **direct sum** is $E_1 \oplus E_2 := E_1 \times_B E_2$ together with the projection map $p : E_1 \oplus E_2 \longrightarrow B$. Note that $p^{-1}(b) = p_1^{-1}(b) \oplus p_2^{-1}(b)$.

Proposition 1.1. *The projection $E_1 \oplus E_2 \longrightarrow B$ is a vector bundle.*

Example 1. The canonical line bundle on $\mathbf{R}P^n$, $E \longrightarrow \mathbf{R}P^n$ has an orthogonal complement given by $E^\perp = \{(\ell, v) \in \mathbf{R}P^n \times \mathbf{R}^{n+1} | v \perp \ell\}$. The map $E^\perp \longrightarrow \mathbf{R}P^n$, $(\ell, v) \mapsto \ell$ is an n-dimensional vector bundle, whose fiber over ℓ is $\ell^\perp$.

Observe 1.2. We have an isomorphism of vector bundles $E \oplus E^\perp \xrightarrow{\cong} \mathbf{R}P^n \times \mathbf{R}^{n+1}$ given by $(\ell, v, w) \mapsto (\ell, v + w)$. When $n = 1$, $E \longrightarrow \mathbf{R}P^1 = S^1$ is the Mobius line bundle which we have shown to be non-trivial. Since in this case $E \cong E^\perp$, we see that the (direct) sum of two non-trivial bundles may be trivial. We will explore this algebraic structure more thoroughly later in the course.

2 Operations on vector bundles

Let $\mathrm{Vect}_{\Bbbk}$ be the category of finite dimensional vector spaces over $\mathbf{k}$ $(= \mathbf{R}, \mathbf{C})$. This category is **enriched** over topological spaces in that for every $V, W \in \mathrm{Vect}_{\Bbbk}$, the set of linear maps $\mathrm{Hom}(V, W)$ admits a topology for which the composition rule is continuous.

Definition 2.1. An endofunctor $T : \mathrm{Vect}_{\Bbbk} \longrightarrow \mathrm{Vect}_{\Bbbk}$ is called **topological** if for every $V, W \in \mathrm{Vect}_{\Bbbk}$, the map $T : \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(TV, TW)$ is continuous.

Our goal is now to show that such a topological functor T induces an endofunctor of vector bundles, obtained by applying T "fiberwise".

If $p : E \longrightarrow B$ is a vector bundle, we define the set TE to be the union $\bigcup_{b \in B} T(E_b)$ and if $\varphi : E \longrightarrow F$ is a map of vector bundles we define the function

$T(\varphi): TE \longrightarrow TF$ by the maps $T(\varphi_b): T(E_b) \longrightarrow T(F_b)$. We want to define a topology on TE such that $T\varphi$ will be continuous.

If $E = B \times V$, $TE = B \times TV$ already admits a topology. If, furthermore, $F = B \times W$ and $\varphi : E \longrightarrow F$ a map of vector bundles, then as we saw last time, φ corresponds to a map $\Phi : B \longrightarrow \mathrm{Hom}(V, W)$ and we obtain a map $T\Phi : B \longrightarrow \mathrm{Hom}(TV, TW)$ which then corresponds back to $T\varphi : TE \longrightarrow TF$. Thus, $T(\varphi)$ is continuous because $T\Phi$ is so. Note that, if φ is an isomorphism, then so is $T\varphi$ since in that case $T(\varphi_b)$ is an isomorphism for each $b \in B$.

Suppose E is trivial but has no preferred product structure. Choose an isomorphism $\alpha : E \longrightarrow B \times V$ and topologize TE by requiring $T(\alpha) : TE \longrightarrow B \times TV$ to be a homeomorphism (there is only one possible topology for TE that make it so). If $\beta : E \longrightarrow B \times V$ is any other isomorphism, then for $\varphi = \beta\alpha^{-1}$ we see that $T(\alpha)$ and $T(\beta)$ induce the same topology on E since $T(\beta) = T(\varphi)T(\alpha)$ is a homeomorphism as a composition of such. We thus see that the topology on TE does not depend on the choice of α.

Furthermore, it is clear that if $\varphi : E \longrightarrow F$ is a map of trivial bundles, then $T(\varphi)$ is a map of vector bundles, and that if $B \subseteq B$, $T(E)|_{B'} \cong T(E|_{B'})$ $[T(E)|_{B'} \cong B' \times TV]$.

Suppose $p : E \longrightarrow B$ is arbitrary. Then if $U \subseteq B$ is such that $E|_U$ is trivial, we topologize $T(E|_U)$ as above. We then topologize TE by declaring a set $V \subseteq T(E)$ to be open iff $V \cap T(E|_U)$ is open for every U for which $E|_U$ is trivial over. As we saw last time, continuity is a local property so that for a map of arbitrary vector bundles $\varphi : E \longrightarrow F$, $T\varphi : TE \longrightarrow TF$ is continuous. If $B' \subset B$ then again $T(E|_{B'}) \cong T(E)|_{B'}$ so that the two possible topologies agree.

Let us give few examples of the operations on vector bundles we have constructed:

i $E \otimes F$.

ii $\mathrm{Hom}(E, F)$.

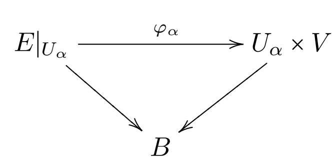
iii E^* – the dual bundle.

The identities these operations satisfy in vector spaces continue to hold for vector bundles. For example, we have an isomorphism $E \otimes (F' \oplus F'') \cong (E \otimes F') \oplus (E \otimes F'')$.

3 Transition functions

It is common to view a vector bundle is family of vector spaces, one for every point in the base, which are glued together. We now make this precise.

Definition 3.1. Let $p : E \longrightarrow B$ be a **k**-vector bundle with trivializations



that restrict to vector space isomorphisms $\varphi_\alpha|_{E_b} : E_b \xrightarrow{\cong} \{b\} \times V$. The **transition functions** are defined to be the maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(V) := \mathrm{Iso}(V)$$

given by $g_{\beta\alpha}(b) = \varphi_\beta|_{E_b} (\varphi_\alpha|_{E_b})^{-1}$. Note that $\mathrm{Iso}(V)$ is a topological space since V is a topological vector space.

Observe 3.2. The transition functions of a vector bundle satisfy the **cocycle condition**: On triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$. This can be seen by the following diagram

$$V \xrightarrow{\varphi^{-1}} E_b \xrightarrow{\varphi_\beta} V \xrightarrow{\varphi_\beta^{-1}} E_b \xrightarrow{\varphi_\gamma} V$$

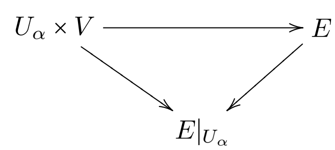
The previous observation admits a converse in the form of

Proposition 3.3. *Let $\{U_\alpha\}$ be an open cover of B and suppose we are given maps*

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(V) := \mathrm{Iso}(V)$$

satisfying the cocycle condition. Then there is a vector bundle $p : E \longrightarrow B$ with fiber V whose transition functions are $g_{\beta\alpha}$.

Proof. Define $E := \coprod_\alpha U_\alpha \times V / \simeq$ where for every $b \in U_\alpha \cap U_\beta$, (b, v) (b, w) iff $w = g_{\beta\alpha}(b)(v)$. The cocycle condition implies that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$. Thus, if (b, v) is equivalent to (b, w), $v = g_{\alpha\beta}(b)(w)$ so that is symmetric. Transitivity follows in a similar way and thus is an equivalence relation. Define $p : E \longrightarrow B$ by $p[b, v] = b$. Then the map $U_\alpha \times V \longrightarrow \coprod_\alpha U_\alpha \times V \longrightarrow E$ admits a factorization



in which the left map is a homeomorphism. We see that $p : E \longrightarrow B$ is a vector bundle with transition functions $g_{\beta\alpha}$. □

4 Paracompact spaces

Our goal for the remains of this talk and the next one is to establish a classification of vector bundles. We will need to make a mild assumption on the base space B and we review it now. The proofs of the following point-set topology assertions will be omitted. They can be found in Hatcher's book "vector bundles and K-theory".

Definition 4.1. A Hausdorff space X is **paracompact** if every open cover $\{U_\alpha\}_{\alpha \in I}$ admits a **partition of unity** with respect to it (or: subordinated to it), i.e., there are maps $\{h_\alpha : X \longrightarrow [0, 1]\}_{\alpha \in I}$ satisfying:

i For every $x \in X$, $h_\alpha(x) = 0$ for almost all α.

ii For every $x \in X$, $\Sigma_\alpha h_\alpha(x) = 1$.

iii For every α, $\overline{h_\alpha^{-1}(0,1]} \subseteq U_\alpha$.

Example 2.

i Every compact Hausdorff space.

ii Every CW-complex.

iii Every metric space.

Definition 4.2. An open cover $\{U_\alpha\}$ of X is **locally finite** if for every $x \in X$ there is an open neighbourhood V_x such that $V_x \cap U_\alpha = \varnothing$ for almost all α.

There is another equivalent efinition of paracompact spaces as follows

Theorem 4.3. *A space X is paracompact iff it is Hausdorff and every open cover has a locally finite open refinement.*

Finally, we need a technical

Lemma 4.4. *Let X be a paracompact space. If $\{U_\alpha\}$ is an open cover, there is a* ***countable*** *open cover $\{V_\beta\}$ such that each V_β is a disjoint union of opens, each contained in some U_α.*

5 Classification of vector bundles

Recall that we have defined (in the exercise) the Grassmanian $G_n = G_n(\mathbf{k}^\infty)$ to be the space of all n-dimensional subvector spaces of $\mathbf{k}^\infty$. We also defined $E_n = E_n(\mathbf{k}^\infty) = \{(V,v) \in G_n \times \mathbf{k}^\infty | v \in V\}$ and showed that the projection map $\gamma_n : G_n \longrightarrow E_n$ given by $(V,v) \mapsto V$ defines an n-dimensional vector bundle.

The following proposition asserts that every n-dimensional vector bundle can be obtained as a pullback along γ_n.

Proposition 5.1. *Let $p : E \longrightarrow B$ be a rank n vector bundle over $\mathbf{k}$ with B paracompact. Then there exists a map $f : B \longrightarrow G_n(\mathbf{k}^\infty)$ and an isomorphism of vector bundles over B, $E \cong f^* E_n$.*

Proof. We can assume that $p : E \longrightarrow B$ has trivializations $\varphi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times V$ with $\{U_\alpha\}_{\alpha \in I}$ locally finite and countable. Let $\{h_\alpha : X \longrightarrow [0,1]\}$ be a partition of unity wrt $\{U_\alpha\}$ and define $g_\alpha : E \longrightarrow V$ by $g_\alpha|_{p^{-1}(U_\alpha)} = (h_\alpha p)\cdot(\pi_2 \varphi_\alpha)$ (where $\pi_2 : U_\alpha \times V \longrightarrow V$ is the projection map) and $g_\alpha = 0$ else. Note that g_α is continuous since $\overline{h_\alpha^{-1}(0,1]} \subseteq U_\alpha$. Choose an isomorphism $\Sigma_\alpha V \cong \mathbf{k}^\infty$ (I is countable) and define $g = \Sigma_\alpha g_\alpha : E \longrightarrow \Sigma_\alpha V \cong \mathbf{k}^\infty$. Then g is well-defined since $\{U_\alpha\}$ is locally finite. We now claim that g maps each E_b isomorphically onto V. This is so since if $h_\alpha(b) \neq 0$ then for any $e \in E_b$, $g(e) = \Sigma_\alpha g_\alpha(e) = (\Sigma_\alpha h_\alpha(b)) \cdot (\pi_2 \varphi_\alpha(e)) = \pi_2(\varphi_\alpha(e)) \in V$. Define $f : B \longrightarrow G_n(\mathbf{k}^\infty)$ via $f(b) = g(E_b)$.

We consider the pullback

$$\begin{array}{ccc} f^*(E_n(\mathbf{k}^\infty)) & \longrightarrow & E_n(\mathbf{k}^\infty) \\ \downarrow & & \downarrow \\ B & \longrightarrow & G_n(\mathbf{k}^\infty). \end{array}$$

Then $f^*E_n(\mathbf{k}^\infty)$ consists of triples (b, V, v) such that g maps E_b isomorphically onto $V \subseteq \mathbf{k}^\infty$. Thus, the map $E \longrightarrow f^*(E_n(\mathbf{k}^\infty))$ given by the isomorphism $g : E_b \xrightarrow{\cong} V$ on every fiber E_b is an isomorphism of vector bundles. $\square$