Introduction to RIEMANNIAN GEOMETRY

Gert Heckman
Radboud University Nijmegen
G.Heckman@math.ru.nl

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Preface

These are lecture notes for an introductory course on Riemannian geometry. The readers are supposed to be familiar with the basic notions of the theory of smooth manifolds, such as vector fields and their flows, differential forms, integration of volume forms and the theorem of Stokes. This material has been covered in the first semester of the third year by Ioan Mărcut [15]. In addition, it will be helpful if the readers are familiar with the classical differential geometry of curves (the formulas of Frenet) and surfaces (various curvatures and the Theorema Egregium on the intrinsic nature of the Gauss curvature) in $\mathbb{R}^3$.

Such a course is offered at our university in the second semester of the second year, based on a book by Andrew Pressley [19] and lecture notes by the author [11]. While teaching that course I got so excited about the Theorema Egregium that I took the trouble to present six different proofs of it in my lecture notes: Theorem 4.6 as corollary of Theorem 4.5 (a very interesting proof, as will become clear in our section on sectional curvature), Theorem 4.9 (a neat formula, but the proof is just tricky algebra), Theorem 5.9 (a very nice proof, which was a starting point for Riemann), two proofs in Theorem 5.10 (both formulas give a nice geometric interpretation) and finally a maybe not too rigorous geometric (but very beautiful) argument right above Theorem 4.10. As Michael Atiyah tells in his interview [1] with Iz Singer on the occasion of their joint Abel prize in 2004: "I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better." I took this advice of Sir Michael literally.

However, in order to avoid too many constraints on the audience, we start in the first chapter with a short discussion of classical differential geometry of submanifolds M in a Euclidean space $\mathbb{R}^n$. We discuss the extrinsic definition of geodesics on M, and explain the intrinsic nature of the geodesic equations. The principal curvatures for hypersurfaces are introduced, and we end with a formulation of the Theorema Egregium.

In the second chapter we discuss vector bundles with connections on a manifold M. This material is the basic language to be spoken for modern differential geometry. The curvature of a connection, parallel transport and holonomy, flat connections and monodromy are the main concepts to be discussed. Since our third year students (for whom this course is meant) are not yet familiar with the concept of fundamental group and universal covering space these notions will be discussed to some extent.

In the third chapter we discuss with the standard approach to Riemannian geometry, via the Levi-Civita connection ∇ and its Riemannian curvature ∇^2. Subsequently, we discuss various curvatures derived from the Riemannian curvature, namely the sectional curvature, the Ricci curvature and the scalar curvature. Geodesics are characterized by the property that their tangent vectors are parallel along the curve with respect to the Levi-Civita connection. In geodesic normal coordinates around a given point the Riemann curvatures are just the second order coefficient in the expansion of the Riemannian metric, which in turn gives geometric meaning to the various curvatures. We discuss the second fundamental form for submanifolds of a Riemannian manifold. Finally, we discuss in two sections some in my opinion important theorems without proofs, in the spirit of Berger's book to give a little panoramic overview [3] and end with a little bit of history. This third chapter is the central subject of these lectures.

1 Classical Differential Geometry

1.1 Smooth Curves in Euclidean Space

Let $\gamma : (a,b) \to \mathbb{R}^m$ be a smooth motion in $\mathbb{R}^m$. The parameter $t \in (a,b)$ can be thought of as time, and (following Newton) the first and second derivative at time t are denoted by

$$\dot{\gamma}(t)\ ,\ \ddot{\gamma}(t)$$

and represent the *velocity* and *acceleration* of the motion. The length of the velocity is called the *speed* of the motion. The motion is called regular if $\dot{\gamma} \neq 0$ on (a,b), in which case $\gamma(t) + \mathbb{R}\dot{\gamma}(t)$ is just the tangent line of the smooth curve γ at time t. Let us assume that the motion is regular. If $(a,b) \ni t \leftrightarrow s \in (c,d)$ is a reparametrization and we write $\gamma(t) = \delta(s(t)) \Leftrightarrow \delta(s) = \gamma(t(s))$ then we have

$$\dot{\gamma} = \delta'\dot{s}\ ,\ \ddot{\gamma} = \delta''\dot{s}^2 + \delta'\ddot{s}$$

with the prime for the derivative with respect to the new parameter s.

We denote by $x \cdot y$ the standard scalar product of the vectors $x, y \in \mathbb{R}^m$. A direct calculation shows that

$$\frac{(\dot{\gamma}\cdot\dot{\gamma})\ddot{\gamma} - (\ddot{\gamma}\cdot\dot{\gamma})\dot{\gamma}}{(\dot{\gamma}\cdot\dot{\gamma})^2} = \frac{(\delta'\cdot\delta')\delta'' - (\delta''\cdot\delta')\delta'}{(\delta'\cdot\delta')^2}$$

and so the length of this vector has geometric meaning. It is called the *curvature* of the regular curve.

A reparametrization is called proper is $\dot{s} > 0$. There is an essentially unique proper reparametrization for which $|\delta'| \equiv 1$, called the *arclength*, commonly denoted by s and given by $s = \int |\dot{\gamma}| dt$. Note that for arclength parametrization the curvature is just the length of δ''. If the curvature of a regular curve vanishes identically then the curve is just a straight line segment.

Theorem 1.1. *A planar regular curve $\gamma : (a,b) \to \mathbb{R}^2$ with curvature $\kappa(t) > 0$ for all t is up to reparametrization and up to a Euclidean motion uniquely determined by its curvature function $\kappa : (a,b) \to \mathbb{R}_+$.*

Proof. Let $s \in (c,d)$ be the arclength parameter of γ and put $\delta(s) = \gamma(s(t))$ as before. Since $\delta'' \cdot \delta' = 0$ we have

$$\delta''(s) = \kappa(s)J\delta'(s)$$

with $J \in \mathrm{SO}(\mathbb{R}^2)$ such that $J^2 = -1$. The equation $J^2 = -1$ in $\mathrm{SO}(\mathbb{R}^2)$ has just two solutions, and since κ is positive by assumption J is independent of s. For a given function $\kappa : (c, d) \to \mathbb{R}_+$ the above second order differential equation has a unique solution $\delta : (c, d) \to \mathbb{R}^2$ with freely prescribed initial conditions $\delta(s_0) \in \mathbb{R}^2$ and $\delta'(s_0) \in \mathbb{R}^2$ of length one at some initial point $s_0 \in (c, d)$. These initial conditions can be eliminated by a translation and a rotation respectively, so by a proper motion of $\mathbb{R}^2$. The two choices for J correspond to two choices for $\delta''(s_0)$ under the condition $\delta''(s_0) \cdot \delta'(s_0) = 0$, and these give mirror images of the curve. Altogether the initial conditions can be eliminated by a motion of the plane. $\square$

Exercise 1.2. *Show that the above theorem is false if the condition $\kappa(s) > 0$ for all s is deleted.*

Exercise 1.3. *Fix a complex structure $J \in \mathrm{SO}(\mathbb{R}^2)$ such that $J^2 = -1$. For an arclength parametrized planar curve $\gamma : (a, b) \to \mathbb{R}^2$ define the* signed curvature *$k : (a, b) \to \mathbb{R}$ by the equation $\delta''(s) = k(s)J\delta'(s)$, and so $\kappa(s) = |k(s)|$. Show that the above theorem, with curvature replaced by signed curvature and Euclidean motion replaced by proper Euclidean motion, remains valid also if $k(s)$ vanishes for some s.*

Exercise 1.4. *Suppose the smooth curve $\gamma : (a, b) \to \mathbb{R}^m$ is regular and parametrized by arclength, that is $|\dot{\gamma}| \equiv 1$. In addition assume that the vectors*

$$\dot{\gamma}, \ddot{\gamma}, \cdots, \gamma^{m\cdot}$$

are linearly independent, with $\gamma^{j\cdot}$ the order j derivative of γ with respect to t. The orthonormal basis $t_1, \cdots, t_m$ obtained from the above basis by the Gram–Schmidt process is called the moving frame along the given curve. Show that

$$\dot{t}_j = -\kappa_{j-1}t_{j-1} + \kappa_j t_{j+1}$$

for all $j = 1, \cdots, m$ with $t_0 = t_{m+1} = 0$. The scalars κ_j for $j = 1, \cdots, m-1$ are called the curvatures of the given curve. Hint: Show that $\dot{t}_j$ lies in the span of $t_1, \cdots, t_{j+1}$. This forces the definition of the scalars κ_j for $j = 1, \cdots, m-1$, and the Frenet formulae follow easily. Prove the fundamental theorem that for arbitrary smooth functions $\kappa_j > 0$ for $j = 1, \cdots, m-1$ there exists locally a curve in $\mathbb{R}^m$ with these curvatures, and such curve is unique up to a Euclidean motion.

1.2 Submanifolds of Euclidean Space

Let M be a smooth submanifold in $\mathbb{R}^n$ of dimension m. Assume M is given by local coordinates, that is locally given as the image of a smooth injective map

$$x = (x^1, \cdots, x^m) \mapsto y(x) = (y^1(x), \cdots, y^n(x)) \in \mathbb{R}^n$$

defined on an open subset U of $\mathbb{R}^m$, such that the vectors

$$\partial_i y(x)$$

for $i = 1, \cdots, m$ are independent for all $x \in U$. Here ∂_i stands for $\partial/\partial x^i$. These m vectors span the tangent space $T_y M$ at the given point y of M.

Definition 1.5. *In the coordinate map $x \mapsto y(x)$ on M the symmetric expression*

$$ds^2 = \sum g_{ij}(x) dx^i dx^j$$

with coefficients $g_{ij}(x) = \partial_i y(x) \cdot \partial_j y(x)$ is called the first fundamental form. *Its square root ds is also called the* length element *on M. The summation is always taken over those indices which appear both as upper and lower index.*

The first fundamental form on M enables one to compute the length of piecewise smooth curves on M. Indeed, let $[a, b] \ni t \mapsto x(t) \in U$ be a piecewise smooth curve in U. The arclength

$$s = \int ds = \int \sqrt{\sum g_{ij}(x) \dot{x}^i \dot{x}^j} dt$$

of the piecewise smooth curve $t \mapsto y(x(t))$ on M gives a new parametrization, for which the curve on M is traversed with unit speed. Equivalently the length L of this curve on M is given by

$$L = \int_a^b \sqrt{\sum g_{ij}(x) \dot{x}^i \dot{x}^j} dt$$

since the length is just the total arclength.

If $B \subset U$ is a compact box then the volume V of the image $y(B) \subset M$ can be computed using the first fundamental form by integration of the smooth density

$$dV = (\det g_{ij}(x))^{1/2} dx^1 \cdots dx^m$$

over the region B. The proof follows from the Jacobi substitution theorem, and the fact that the volume of the parallelotope spanned by the m vectors $\partial_i y(x)$ is equal to the square root of the determinant of the Gram matrix $g_{ij}(x)$ of these m vectors in $\mathbb{R}^n$. The smooth density dV is called it the *volume element* on M.

Decomposing vectors in $\mathbb{R}^n$ according to $\mathbb{R}^n = T_yM \oplus N_yM$ with the normal space N_yM the orthogonal complement of the tangent space T_yM at $y \in M$ we can write

$$\partial_i \partial_j y(x) = \sum \Gamma_{ij}^k(x) \partial_k y(x) + n_{ij}(x)$$

with $\Gamma_{ij}^k : U \to \mathbb{R}$ the so called *Christoffel symbols.*

Theorem 1.6. *The Christoffel symbols Γ_{ij}^k are given by*

$$2\Gamma_{ij}^k = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}$$

and so are rational expressions in the coefficients of the first fundamental form and their first order partial derivatives. Here g^{ij} denotes the inverse matrix of g_{ij}.

Proof. Taking the scalar product of $\partial_i \partial_j y(x)$ with $\partial_l y(x)$ yields

$$\partial_i \partial_j y(x) \cdot \partial_l y(x) = \sum \Gamma_{ij}^k(x) g_{kl}(x)$$

with the usual convention of differential geometry that sums run over all those indices which appear both as lower and upper index. Hence we get

$$\Gamma_{ij}^k(x) = \sum (\partial_i \partial_j y(x) \cdot \partial_l y(x)) g^{lk}(x)$$

and in addition

$$\partial_i g_{jl}(x) + \partial_j g_{il}(x) - \partial_l g_{ij}(x) = 2\partial_i \partial_j y(x) \cdot \partial_l y(x)$$

by direct verification. $\square$

The Riemann curvature coefficients are defined by

$$R_{kij}^l = \partial_i \Gamma_{kj}^l - \partial_j \Gamma_{ki}^l + \sum \{\Gamma_{ni}^l \Gamma_{kj}^n - \Gamma_{nj}^l \Gamma_{ki}^n\}$$

and at this point these are just functions with four indices, more precisely three lower indices and one upper index, just like the Christoffel symbols have two lower indices and one upper index. If the submanifold $M \subset \mathbb{R}^n$ with local coordinates $(x^1, \cdots, x^m) \in U$ is deformed in a rigid way, meaning that the first fundamental form remains constant, then the Christoffel symbols Γ_{ij}^k and the Riemann curvature coefficients R_{kij}^l do not change likewise.

However, different local coordinates give different Γ_{ij}^k and R_{kij}^l. But at this point it is unclear what the geometric meaning is of these curvature coeficients. The key point is that there exist in an intrinsic way a first order differential operator ∇ mapping vector fields to vector fields with one forms as coefficients and an element $R \in \Omega^2(\mathrm{End}(TM))$ such that in local coordinates

$$\nabla(\partial_i) = \sum_k (\sum_j \Gamma_{ij}^k dx^j) \otimes \partial_k \quad , \quad R(\partial_k) = \sum_l (\sum_{i<j} R_{kij}^l dx^i \wedge dx^j) \otimes \partial_l$$

with ∇ the Levi-Civita connection and R the Riemann curvature. So in local coordinates ∇ is a matrix with entries $\sum_j \Gamma_{ij}^k dx^j$ from $\Omega^1(M)$ and R is a matrix with entries $\sum_{i<j} R_{kij}^l dx^i \wedge dx^j$ from $\Omega^2(M)$. But this probably does not make any sense right now.

Hence our goals will be an introduction of the various concepts, both in an *intrinsic* and *coordinate free* way. For this we shall develop the language of vector bundles and connections on a manifold, and subsequently specialize this to the case of the tangent bundle with its canonical Levi-Civita connection on a Riemannian manifold.

1.3 Geodesics on Submanifolds of Euclidean Space

We keep the notation of the previous section. A smooth curve $(a, b) \to U, t \mapsto x(t)$ in U yields a smooth curve

$$\gamma : (a, b) \to M, t \mapsto \gamma(t) = y(x(t))$$

in $M \subset \mathbb{R}^n$ and any smooth curve in $y(U) \subset M$ can be obtained this way. The velocity $\dot{\gamma}$ and the acceleration $\ddot{\gamma}$ are given by

$$\dot{\gamma}(t) = \sum \dot{x}^i(t)\partial_i y(x(t))$$
$$\ddot{\gamma}(t) = \sum \ddot{x}^k(t)\partial_k y(x(t)) + \sum \dot{x}^i(t)\dot{x}^j(t)\partial_i\partial_j y(x(t))$$

with $\dot{\gamma}(t) \in T_{y(x(t))}M$ while $\ddot{\gamma}(t) \in \mathbb{R}^n$.

Definition 1.7. *The curve γ is called a* geodesic *on M if its acceleration $\ddot{\gamma}(t)$ lies in the normal space $N_{y(x(t))}M$ for all $t \in (a,b)$.*

Exercise 1.8. *Geodesics are traversed on M with constant speed.*

Taking the scalar product of $\ddot{\gamma}$ with $\partial_l y(x)$ shows that the curve γ is a geodesic on M if and only if the curve $t \mapsto u(t)$ satisfies the *geodesic equations*

$$\ddot{x}^k + \sum \Gamma_{ij}^k(x)\dot{x}^i\dot{x}^j = 0$$

for all k. This is a system of nonlinear second order ordinary differential equations. The local existence and uniqueness theorem for such differential equations says that for a given initial point $y \in y(U) \subset M$ and a given initial tangent vector $v \in T_yM$ there exists an $\epsilon > 0$ and a unique solution $(\epsilon, \epsilon) \to U, t \mapsto x(t)$ such that $\gamma(t) = y(x(t))$ is a geodesic on M with $\gamma(0) = y$ and $\dot{\gamma}(0) = v$.

Our definition of geodesic on M is extrinsic in the sense that we need to go out from the submanifold M to the ambient vector space $\mathbb{R}^n$. However using Theorem 1.6 it follows that the geodesic equations (and hence also the geodesics on M as their solutions) can be derived only using the first fundamental form. Concepts on M that can be defined just using the first fundamental form are so called intrinsic notion. However the fact that the concept of geodesic is intrinsic may not come as a surprise, because it turns out that the distance between two nearby points on a geodesic is the length of that geodesic segment, and this property characterizes constant speed curves on M as geodesics.

Exercise 1.9. *Suppose in local coordinates $(x^1, \cdots, x^{m-1}, x^m) \in V \times (-\epsilon, \epsilon)$ on M that the curves $x^1, \cdots, x^{m-1}$ constant and $x^m = t$ are unit speed geodesics and are orthogonal to the hypersurface $x^m = 0$. In other words we have $g_{im}(x^1, \cdots, x^{m-1}, 0) \equiv \delta_{im}$ (Kronecker delta) for $i = 1, \cdots, m$ and the geodesic equations hold for the curves $x^1, \cdots, x^{m-1}$ constant and $x^m = t$. Show that $\Gamma_{mm}^k \equiv 0$ for $k = 1, \cdots, m$ and conclude that $g_{lm} \equiv \delta_{lm}$ for $l = 1, \cdots, m$. Conclude that geodesics are locally length minimizing curves.*

1.4 Hypersurfaces in Euclidean Space

Suppose M is a oriented hypersurface of dimension m in $\mathbb{R}^{m+1}$ and so we have given a global smooth choice of unit normal $n(y)$ orthogonal to T_yM for all $y \in M$.

Definition 1.10. *If* $x = (x^1, \cdots, x^m) \mapsto y(x) = (y^1(x), \cdots, y^{m+1}(x)) \in \mathbb{R}^{m+1}$ *are local coordinates and we write*

$$\partial_i \partial_j y(x) = \sum \Gamma_{ij}^k(x) \partial_k y(x) + h_{ij}(x) n(y(x))$$

then the symmetric expression

$$\sum h_{ij}(x) dx^i dx^j$$

is called the second fundamental form *of the oriented hypersurface M in the given local coordinates.*

Fix a point $y = y(x) \in M$ and a nonzero tangent vector $t(y) \in T_y M$. The plane $y + \mathbb{R}t(y) + \mathbb{N}n(y)$ cuts M locally near y in a planar regular curve. Choose a smooth parametrization $(-\epsilon, \epsilon) \ni t \mapsto \gamma(t) = y(x(t))$ of this planar curve with $\gamma(0) = y$ and $\dot{\gamma}(0) = t(y)$. The expression

$$k(\dot{x}^1, \cdots, \dot{x}^m) = \frac{\sum h_{ij} \dot{x}^i \dot{x}^j}{\sum g_{ij} \dot{x}^i \dot{x}^j}$$

(with $g_{ij} = g_{ij}(x)$, $h_{ij} = h_{ij}(x)$ the coefficients at $x = x(0)$ and $\dot{x}^i = \dot{x}^i(0)$ the derivatives at $t = 0$) is the signed curvature of this plane curve at y relative to the orientation $Jt(y) = n(y)$.

If we vary $v = (v^1, \cdots, v^m)$ over nonzero vectors in $\mathbb{R}^m$ then we get a family of planar curves $t \mapsto \gamma(t)$ whose tangent vector $\dot{\gamma}(0) = \sum v^i \partial_i y(x)$ varies over nonzero vectors in $T_y(M)$. The nonzero critical points of the signed curvature function $(v^1, \cdots, v^m) \mapsto k(v^1, \cdots, v^m)$ are solutions of the equations

$$\frac{\partial k(v^1, \cdots, v^m)}{\partial v^i} = \frac{2 \sum (h_{ij} - k(v^1, \cdots, v^m) g_{ij}) v^j}{\sum g_{ij} v^i v^j} = 0$$

for $i = 1, \cdots, m$. Hence the critical values of k as function on $\mathbb{R}^m - \{0\}$ are the solutions of the equation

$$\det(h_{ij} - k g_{ij}) = 0$$

and called the *principal curvatures* $k_1, \cdots, k_m$ of the hypersurface $M \subset \mathbb{R}^{m+1}$ at the point $x \in M$. The corresponding nonzero linear subspaces $\ker(h - kg)$ of $\mathbb{R}^m$ are called the *principal directions*.

Exercise 1.11. *Show that the principal directions corresponding to distinct principal curvatures are orthogonal with respect to the first fundamental form.*

In the case $m = 2$ of surfaces in $\mathbb{R}^3$ the principal curvatures k_1, k_2 were introduced by Monge. They are defined extrinsically, and the example of a vertical cylinder over a plane curve shows that they are not intrinsic quantities. The symmetric expressions

$$H = k_1 + k_2 \ , \ K = k_1 k_2$$

are called the mean curvature and the Gauss curvature respectively. It was an utmost remarkable insight of Gauss from 1827 that the the Gauss curvature K is an intrinsic quantity: it depends only on the first fundamental form [10]. Gauss was so excited about this fact that he baptized his result the Theorema Egregium, which is Latin for the splendid theorem. Around that time Latin was still the common language for writing mathematics papers. Riemann in his famous Habilitationsvortrag from 1854, entitled Über die Hypothesen, welche der Geometrie zu Grunde liegen, gave an outline of an intrinsic road towards the notion of curvatures in spaces of dimension $m \geq 2$ just in terms of its length element. The ideas of Riemann were taken up by the Italian geometers Enrico Betti and Francesco Brioschi, and subsequently by Eugenio Beltrami (student of Brioschi), Luigi Bianchi and Gregorio Ricci-Curbasto (students of Betti) and Tullio Levi-Civita (student of Ricci-Curbasto).

For the moment this ends our discussion of classical differential geometry. From the next section on we will follow the modern approach to differential geometry, using the language of manifolds, vector bundles and connections. From the start we shall work intrinsically, and if possible we shall work in a coordinate free manner in order that the geometric ideas are not obscured by extensive manipulation with formulas.

2 Vector Bundles and Connections

2.1 Tensor Products

In this section all vector spaces will be real vector spaces, although everything extends verbatim to vector spaces over arbitrary fields.

Definition 2.1. *Let U and V be vector spaces. The* tensor product *of U and V is a vector space $U \otimes V$, equipped with a bilinear map*

$$U \times V \to U \otimes V,\ (u, v) \mapsto u \otimes v$$

such that any bilinear map $b : U \times V \to W$ in a third vector space W defines a unique linear map $B : U \otimes V \to W$ with

$$B(u \otimes v) = b(u, v)$$

for all $(u, v) \in U \times V$.

The vector $u \otimes v \in U \otimes V$ is called a pure tensor, and the uniqueness of the linear map B (given the bilinear map b) implies that $U \otimes V$ is spanned by pure tensors. If $U \otimes V$ and $U \circledast V$ are two tensor products with corresponding bilinear maps $(u, v) \mapsto u \otimes v$ and $(u, v) \mapsto u \circledast v$ respectively, then the well defined linear map $U \otimes V \to U \circledast V, u \otimes v \mapsto u \circledast v$ has an inverse, and so is a linear isomorphism. Hence, if the tensor product exists then it is unique up to isomorphism.

For the existence, we take $U \otimes V$ the quotient of the free vector space on the set $U \times V$ modulo the linear subspace R generated by the elements

$$(u_1 + u_2, v) - (u_1, v) - (u_2, v)\ ,\ (\lambda u, v) - \lambda(u, v)$$
$$(u, v_1 + v_2) - (u, v_1) - (u, v_2)\ ,\ (u, \lambda v) - \lambda(u, v)$$

for $u, u_1, u_2 \in U$ and $v, v_1 v_2 \in V$ and $\lambda \in \mathbb{R}$. Moreover put $u \otimes v = (u, v) + R$.

We have the following natural isomorphism

$$\mathbb{R} \otimes U \cong U\ ,\ U \otimes V \cong V \otimes U$$
$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$
$$(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$$

for all vector spaces U, V, W. If $A \in \mathrm{End}(U)$ and $B \in \mathrm{End}(V)$ then $A \otimes B \in \mathrm{End}(U \otimes V)$ is defined by $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$ for $u \in U, v \in V$.

Indeed, the bilinear map $U \times V \to U \otimes V, (u, v) \mapsto (Au) \otimes (Bv)$ lifts by definition to a linear $A \otimes B \in \mathrm{End}(U \otimes V)$.

If U and V are subspaces of the real function spaces on sets X and Y respectively then $U \otimes V$ becomes a subspace of the real function space on the set $X \times Y$ by $(u \otimes v)(x, y) = u(x)v(y)$ for $u \in U$, $v \in V$ and $x \in X$, $y \in Y$. In fact $U \otimes V$ is the subspace of the real function space on $X \times Y$ generated by all functions $u \otimes v$ for $u \in U, v \in V$. In case X and Y are finite sets and U and V are equal to the full function spaces on X and Y respectively then $U \otimes V$ is equal to the full function space on the finite set $X \times Y$. Hence we conclude that $\dim(U \otimes V) = \dim(U)\dim(V)$.

If U is a finite dimensional vector space with dual vector space U^* then

$$U^* \otimes V \cong \mathrm{Hom}(U, V)$$

with $\mathrm{Hom}(U, V)$ the space of linear maps from U to V, and the isomorphism is given by $(u^* \otimes v)(u) = u^*(u)v$ for $u \in U, u^* \in U^*, v \in V$.

If both U and V are finite dimensional vector spaces then the vector space $U^* \otimes V^* \cong (U \otimes V)^*$ can be identified with the vector space of bilinear maps $U \times V \to \mathbb{R}$, and more generally for any vector space W the vector space $\mathrm{Hom}(U \otimes V, W)$ can be identified with the vector space of bilinear maps $U \times V \to W$.

Definition 2.2. *If U is a finite dimensional vector space then we denote inductively $T^0(U) = \mathbb{R}$ and $T^{p+1}(U) = T^p(U) \otimes U$ for all $p \in \mathbb{N}$. Since $\otimes : T^p(U) \otimes T^q(U) \to T^{p+q}(U)$ is a linear isomorphism for all $p, q \in \mathbb{N}$ the* tensor algebra

$$T(U) = \oplus_{p \geq 0} T^p(U)$$

becomes a graded associative algebra with respect to tensor multiplication $\otimes$.

Elements of $T^p(U^*)$ are multilinear functions of p arguments from U by putting

$$(u_1^* \otimes \cdots \otimes u_p^*)(u_1, \cdots, u_p) = u_1^*(u_1) \cdots u_p^*(u_p)$$

and the tensor product $\otimes$ turns $T(U^*) = \oplus_{p \geq 0} T^p(U^*)$ into an associative algebra, in fact the universal associative algebra generated by the vector space U^* of linear functionals on U.

Let $I_{\pm}(U)$ be the two sided homogeneous ideal of $T(U)$ generated by the quadratic tensors $(u \otimes v) \pm (v \otimes u)$ for $u, v \in U$ respectively. The quotient algebras

$$S(U) = \oplus_{p \geq 0} S^p(U) = T(U)/I_-(U) \;,\; \wedge(U) = \oplus_{p \geq 0} \wedge^p (U) = T(U)/I_+(U)$$

are graded associative algebras, called the *symmetric algebra* and *exterior algebra* on the vector space U respectively.

The commutative associative algebra $P(U) = \oplus_{p\geq 0} P^p(U)$ of polynomial functions on U is generated by U^*, and so is a quotient algebra of $T(U^*)$. In fact $P(U)$ is the universal commutative associative algebra generated by U^*, from which it easily follows that $P(U) \cong S(U^*)$ with $u_1^* u_2^* \equiv u_1^* \otimes u_2^*$ for $u_1^*, u_2^* \in U^*$ under this identification.

Let $A^p(U)$ be the vector space of alternating multilinear functions of p arguments from U in the sense that

$$\alpha(u_{\sigma(1)}, \cdots, u_{\sigma(p)}) = \varepsilon(\sigma)\alpha(u_1, \cdots, u_p)$$

for all $\sigma \in \mathfrak{S}_p$ and $u_1, \cdots, u_p \in U$. For $\alpha \in A^p(U)$ and $\beta \in A^q(U)$ the wedge product $\alpha \wedge \beta \in A^{p+q}(U)$ is defined by

$$(\alpha \wedge \beta)(u_1, \cdots, u_{p+q}) = \sum_{\sigma \in \mathfrak{S}(p,q)} \varepsilon(\sigma)\alpha(u_{\sigma(1)}, \cdots, u_{\sigma(p)})\beta(u_{\sigma(p+1)}, \cdots, u_{\sigma(p+q)})$$

with the sum over all shuffles $\sigma \in \mathfrak{S}_{p+q}$ with $\sigma(i) < \sigma(j)$ for all $i < j$ taken from either $\{1, \cdots, p\}$ or $\{p+1, \cdots, p+q\}$ and $u_1, \cdots, u_{p+q} \in U$. For example, for $p = q = 1$ this amounts to

$$u_1^* \wedge u_2^* = u_1^* \otimes u_2^* - u_2^* \otimes u_1^*$$

for $u_1^*, u_2^* \in U^*$. It is easy to check that

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$$

for $\alpha \in A^p(U), \beta \in A^q(U), \gamma \in A^r(U)$, and so $A(U) = \oplus_{p\geq 0} A^p(U)$ becomes an associative algebra with respect to the wedge product generated by U^*. It follows that $A(U) \cong \wedge(U^*)$ with $u_1^* \wedge u_2^* \equiv 2u_1^* \otimes u_2^*$ under this identification. For $u_1^*, \cdots, u_p^* \in U^*$ and $u_1, \cdots, u_p \in U$ we have $(u_1^* \wedge \cdots \wedge u_p^*)(u_1, \cdots, u_p) = \det(u_i^*(u_j))$.

Exercise 2.3. *Show that* $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ *and* $\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$ *for all* $\alpha \in A^p(U), \beta \in A^q(U), \gamma \in A^r(U)$.

Exercise 2.4. *For* $u \in U$ *let* $\mathrm{i}_u : A^p(U) \to A^{p-1}(U)$ *be defined by*

$$\mathrm{i}_u(\alpha)(u_1, \cdots, u_{p-1}) = \alpha(u, u_1, \cdots, u_{p-1})$$

for $u_1, \cdots, u_{p-1} \in U$. *The element* $\mathrm{i}_u(\alpha)$ *is called the* contraction *of the vector* $u \in U$ *with the form* $\alpha \in A^p(U)$. *Show that*

$$\mathrm{i}_u(\alpha \wedge \beta) = \mathrm{i}_u(\alpha) \wedge \beta + (-1)^p \alpha \wedge \mathrm{i}_u(\beta)$$

for $\alpha \in A^p(U)$ *and* $\beta \in A^q(U)$.

2.2 Vector Bundles

Let M be a smooth manifold of dimension m.

Definition 2.5. *A smooth real* vector bundle *of rank n on a smooth manifold M of dimension m is a smooth map $p : E \to M$ of manifolds, such that*

1. *$\forall\, x \in M$, the* fiber *$E_x := p^{-1}(x)$ is a real vector space of dimension n,*

2. *each $x \in M$ has an open neighborhood U in M and a diffeomorphism φ_U from $E_U := p^{-1}(U)$ onto $U \times \mathbb{R}^n$, called a local trivialization over U, such that the following diagram*

$$\begin{tikzcd} E_U \arrow[rr, "\varphi_U"] \arrow[dr, "p"'] & & U \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

   commutes with pr_U the projection on the first factor,

3. *the induced map $\varphi_x : E_x \to \mathbb{R}^n$ is a linear isomorphism.*

The manifold E is called the total space *of the vector bundle, the manifold M the* base space*, and p the* projection map*.*

We shall write vector bundle for smooth real vector bundle. The trivial vector bundle $\mathrm{pr}_M : M \times \mathbb{R}^n \to M$ on M is the simplest example of a vector bundle on M of rank n. Vector bundles of rank one are called line bundles. Any construction of linear algebra on vector spaces, such as the direct sum $\oplus$, the tensor product $\otimes$, the dual vector space and symmetric power S^p or exterior powers $\wedge^p$, or their duals P^p or A^p, all can be performed likewise with vector bundles on M.

A smooth map $s : U \to E_U$ with $ps = \mathrm{id}_U$ is called a section on U, and we denote by $\Gamma(E_U)$ the real vector space of all sections on U under pointwise addition and scalar multiplication. In case $U = M$ we simply write $\Gamma(E_M) = \Gamma(E)$. The algebra of smooth functions on M is denoted by $\mathcal{F}(M)$. Pointwise multiplication gives $\Gamma(E)$ the structure of a module over $\mathcal{F}(M)$. The zero section, defined by $x \mapsto 0_x$ with 0_x the zero vector in E_x, is the zero vector in $\Gamma(E)$. The zero section gives an embedding of the manifold M inside a vector bundle E on M.

If $\phi_U : E_U \to U \times \mathbb{R}^n$ and $\phi_V : E_V \to V \times \mathbb{R}^n$ are local trivializations of a vector bundle $p : E \to M$ over open neighborhoods U and V of M then we denote for $x \in U \cap V$ and $v \in \mathbb{R}^n$

$$h_{VU} : U \cap V \to \mathrm{GL}_n(\mathbb{R}),\ \phi_V \phi_U^{-1}(x, v) = (x, h_{VU}(x)v)$$

for the *transition function* from E_U to E_V. It is clear that $h_{WV} h_{VU} = h_{WU}$ on the intersection $U \cap V \cap W$ and $h_{UU} = 1$, which are called the *cocycle conditions*. Up to isomorphism the vector bundle E can be recovered from its transition functions for an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M by forming

$$\{\sqcup_{\alpha \in I}(U_\alpha \times \mathbb{R}^n)\}/_\sim$$

with $U_\alpha \times \mathbb{R}^n \ni (x, v) \sim (x, h_{\beta\alpha}(x)v) \in U_\beta \times \mathbb{R}^n$ for $x \in U_\alpha \cap U_\beta$ and $v \in \mathbb{R}^n$. The cocycle conditions $h_{\gamma\beta} h_{\beta\alpha} = h_{\gamma\alpha}, h_{\alpha\alpha} = 1$ turn the relation $\sim$ into an equivalence relation.

The space of sections $\Gamma(M \times \mathbb{R}^n)$ for the trivial bundle on M of rank n is equal to $\mathcal{F}(M)^n$. The tangent bundle TM of M has rank $m = \dim M$ and its sections are just the smooth vector fields on M. We denote $\mathfrak{X}(M) = \Gamma(TM)$ for the space of all smooth vector fields on M. The dual cotangent bundle T^*M has again rank m and its sections are the smooth differentials. The p^{th} exterior power $\wedge^p T^*M$ has rank $\binom{m}{p}$ and its sections are the smooth p-forms on M. We denote $\Omega^p(M) = \Gamma(\wedge^p T^*M)$ for the space of smooth p-forms on M. In case $p = m$ the vector bundle $\wedge^m T^*M$ is a line bundle, and its sections are the smooth volume forms on M. The manifold M is orientable if there exists a nowhere vanishing volume form in $\Omega^m(M)$. The elements of $\Omega^p(E) := \Gamma(\wedge^p T^*M \otimes E)$ are called smooth p-forms with values in E. A section for the vector bundle $T^p(T(M)) \otimes T^q(T^*(M)) = T(M)^{\otimes p} \otimes T^*(M)^{\otimes q}$ on M is called a tensor field of type (p, q).

We now come to the important concept of pull back of vector bundles. Let $f : M \to N$ be a smooth map of manifolds and let $p : E \to N$ be a vector bundle on N. Then the pull back $f^*p : f^*E \to M$ will be a vector bundle on M, defined as follows.

As a set, f^*E is the collection of those points $(x, e) \in M \times E$ for which $f(x) = p(e)$, or equivalently e lies in the fiber $E_{f(p)}$. The projection map $f^*p : f^*E \to M$ is defined by $f^*p(x, e) = x$, so just the restriction to $f^*E \subset M \times E$ of the projection on the first factor. A local trivialization of f^*E around $x \in M$ is obtained as follows. Let V be a trivializing neighborhood of $f(x)$ for the vector bundle $E \to N$, with $\psi_V : E_V \to V \times \mathbb{R}^n$ the local trivialization.

Then $U := f^{-1}(V)$ becomes a trivializing neighborhood of x for the vector bundle $f^*E \to M$. Indeed, the map $\mathrm{id}_U \times \psi_V$ is a diffeomorphism from $(f^*E)_U = (U \times E_V) \cap f^*E$ onto $\mathrm{graph}(f|_U) \times \mathbb{R}^n \subset U \times V \times \mathbb{R}^n$. Here we have used that

$$\mathrm{graph}(f|_U) = \{(y, f(y)); y \in U\} \subset U \times V$$

is a smooth submanifold and the projection $\mathrm{pr} : \mathrm{graph}(f|_U) \to U$ on the first factor is a diffeomorphism. The composition $\phi_U := (\mathrm{pr} \times \mathrm{id}_{\mathbb{R}^n})(\mathrm{id}_U \times \psi_V)$ gives the desired local trivialization from $(f^*E)_U$ onto $U \times \mathbb{R}^n$. If $s \in \Gamma(E)$ is a section, that is $s : N \to E$ with $p(s(y)) = y$ for all $y \in N$, then $f^*s \in \Gamma(f^*E)$ is defined by $f^*s(x) = (x, s(f(x)))$ for all $x \in M$. Indeed $f(x) = p(s(f(x)))$ and so $f^*s(x) \in f^*E$, and also $f^*p(f^*s(x)) = f^*p(x, s(f(x))) = x$ for all $x \in M$. This identifies $f^*\Gamma(E)$ with a linear subspace of $\Gamma(f^*E)$. The former space is module over $f^*\mathcal{F}(N)$ while the latter is a module over $\mathcal{F}(M)$. In fact $\Gamma(f^*E) = \mathcal{F}(M) \otimes f^*\Gamma(E)$ with the tensor product taken as modules over $f^*(\mathcal{F}(N))$.

Exercise 2.6. *A Euclidean structure g on a vector bundle $p : E \to M$ is a function which assigns to each $x \in M$ a Euclidean form g_x on the fiber E_x at x varying smoothly with x, in the sense that $x \mapsto g_x(e_x, f_x)$ is a smooth function on M for any two smooth sections $e, f \in \Gamma(E)$. Using a partition of unity argument show that vector bundles always admit a Euclidean structure.*

Exercise 2.7. *If $f : M \to N$ is a smooth map of manifolds and $p : E \to N$ a vector bundle on N with a Euclidean structure g then the pull back bundle $f^*p : f^*E \to M$ inherits a natural Euclidean structure f^*g.*

2.3 Connections

A smooth vector field $X \in \mathfrak{X}(M)$ has a time t flow $\phi_t : D_t \to M$ defined on its maximal domain $D_t \subset M$. For each $x \in M$ there exists $\epsilon > 0$ such that $x \in D_t$ for all $|t| < \epsilon$, and the curve $t \mapsto \gamma(t) = \phi_t(x)$ is an integral curve of X in the sense that $\dot{\gamma}(t) = X_{\gamma(t)}$. Under proper domain restrictions we have $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ and we speak of a one parameter group of diffeomorphisms.

The *Lie derivative* $\mathcal{L}_X$ is a linear operator defined on the vector spaces of p-forms $\Omega^p(M)$ and vector fields $\mathfrak{X}(M)$ by

$$\mathcal{L}_X(\alpha) = \frac{d}{dt}\{\phi_t^*\alpha\}_{t=0}\ ,\ \mathcal{L}_X(Y) = \frac{d}{dt}\{\phi_t^*Y\}_{t=0}$$

for $\alpha \in \Omega^p(M)$ and $X, Y \in \mathfrak{X}(M)$. On differential forms the Lie derivative is just given by the Cartan formula

$$\mathcal{L}_X = \mathrm{d}\mathrm{i}_X + \mathrm{i}_X\mathrm{d}$$

with $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ the exterior derivative and $\mathrm{i}_X : \Omega^p(M) \to \Omega^{p-1}(M)$ the contraction with the vector field X. For $X, Y \in \mathfrak{X}(M)$ the Lie bracket $[X, Y] \in \mathfrak{X}(M)$ is defined by $[X, Y] = \mathcal{L}_X(Y)$. It turns out that $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ as linear operators on $\Omega^p(M)$ or on $\mathfrak{X}(M)$ with the latter bracket $[\cdot, \cdot]$ the commutator bracket of linear operators. Written out for the action on $\mathfrak{X}(M)$ this amounts to

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

which is called the Jacobi identity. Hence the vector space $\mathfrak{X}(M)$ becomes a Lie algebra, and in fact the Lie algebra of the diffeomorphism group $\mathrm{Diff}(M)$ of M. The linear maps $X \mapsto \mathcal{L}_X$ become Lie algebra representations on $\Omega^p(M)$ and $\mathfrak{X}(M)$.

The name "derivative" in Lie derivative is justified because

$$\mathcal{L}_X(f\alpha) = \mathcal{L}_X(f)\alpha + f\mathcal{L}_X(\alpha)\ , \mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X(Y)$$

for $f \in \mathcal{F}(M), \alpha \in \Omega^p(M)$ and $X, Y \in \mathfrak{X}(M)$. For all $f, g \in \mathcal{F}(M)$ we have

$$\mathcal{L}_{fX}(g) = f\mathcal{L}_X(g)$$

and in fact the value of $\mathcal{L}_X(f)$ at some point $x \in M$ depends only on the value of $X_x \in T_xM$. This need no longer be true for the action of $\mathcal{L}_X$ on $\Omega^p(M)$ for $p \geq 1$ or on $\mathfrak{X}(M)$ where the outcome at some point $x \in M$ will also depend on the values of X in a small neighborhood of x. For example, we have

$$\mathcal{L}_{fX}(Y) = f\mathcal{L}_X(Y) - \mathcal{L}_Y(f)X$$

for $f \in \mathcal{F}(M)$ and $X, Y \in \mathfrak{X}(M)$. Hence the value of of $\mathcal{L}_X(Y)$ at $x \in M$ depends only on $X_x \in T_xM$ for all $Y \in \mathfrak{X}(M)$ if and only if $X_x = 0$.

Definition 2.8. *A* connection ∇ *on a vector bundle* $p : E \to M$ *is a linear map*

$$\nabla : \Gamma(E) \to \Omega^1(E)$$

with the property (Leibniz rule)

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s)$$

for all $f \in \mathcal{F}(M)$ and $s \in \Gamma(E)$. Moreover we require the connection to be local in nature, which means that for each open subset U of M we also have compatible connections

$$\nabla_U : \Gamma(E_U) \to \Omega^1(E_U)$$

in the sense that $\nabla(s)|U = \nabla_U(s|U)$ for all $s \in \Gamma(E)$.

For a given connection ∇ and $X \in \mathfrak{X}(M)$ the covariant derivative ∇_X *along X is the linear map*

$$\nabla_X : \Gamma(E) \to \Gamma(E)\ ,\ \nabla_X(s) = \mathrm{i}_X(\nabla(s))$$

which has the properties

$$\nabla_X(fs) = \mathcal{L}_X(f)s + f\nabla_X(s),\ \nabla_{fX}(s) = f\nabla_X(s),\ \nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s)$$

for all $f \in \mathcal{F}(M), s \in \Gamma(E)$ and $X, Y \in \mathfrak{X}(M)$.

An important remark is that the value of the covariant derivative $\nabla_X(s)$ of a section $s \in \Gamma(E)$ along $X \in \mathfrak{X}(M)$ at some point $x \in M$ only depends on the value $X_x \in T_xM$. So for $X \in \mathfrak{X}(M)$ nonzero there does not exist a connection ∇ on the tangent bundle TM whose covariant derivative along X is equal to the Lie derivative $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$.

If $\nabla_1, \nabla_2 : \Gamma(E) \to \Omega^1(E)$ are two connections on the same vector bundle then for any $f \in \mathcal{F}(M)$ the affine combination

$$f\nabla_1 + (1-f)\nabla_2 : \Gamma(E) \to \Omega^1(E)$$

is again a connection. Hence the connections form an affine subspace of the vector space $\mathrm{Hom}(\Gamma(E), \Omega^1(E))$. The difference

$$\nabla_1 - \nabla_2 : \Gamma(E) \to \Omega^1(E)$$

of two connections is in fact a linear map of modules over $\mathcal{F}(M)$, and so is a global section of the vector bundle $\mathrm{Hom}(E, T^*M \otimes E)$ over M, that is an element of $\Omega^1(\mathrm{End}(E))$.

Exercise 2.9. *Let $(E_1 \to M, \nabla_1)$ and $(E_2 \to M, \nabla_2)$ be two vector bundles with connections on M. Show that the direct sum bundle $E_1 \oplus E_2 \to M$ and the tensor product bundle $E_1 \otimes E_2 \to M$ have natural connections*

$$\nabla(s_1 \oplus s_2) = \nabla_1(s_1) \oplus \nabla_2(s_2)\ ,\ \nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$$

for $s_1 \in \Gamma(E_1)$ *and* $s_2 \in \Gamma(E_2)$*. In addition, we have a natural connection* ∇ *on the vector bundle* $\mathrm{Hom}(E_1, E_2)$ *defined by*

$$\nabla(S)(s) = \nabla_2(S(s)) - S(\nabla_1(s))$$

for $S \in \Gamma(\mathrm{Hom}(E_1, E_2))$ *and* $s \in \Gamma(E_1)$*, so that* $S(s) \in \Gamma(E_2)$

Exercise 2.10. *Show that any vector bundle* $E \to M$ *on a manifold* M *has a connection* ∇*.*

Exercise 2.11. *Suppose* (E, g) *is a vector bundle on* M *with a Euclidean stucture* g*. A connection* ∇ *on* E *is called compatible with* g *if*

$$d(g(s_1, s_2)) = g(\nabla(s_1), s_2) + g(s_1, \nabla(s_2))$$

for all sections $s_1, s_2 \in \Gamma(E)$*. Show that for any connection* ∇ *on* E *there exists a connectionn* ∇^g *on* E *with*

$$d(g(s_1, s_2)) = g(\nabla^g s_1, s_2) + g(s_1, \nabla s_2)$$

for all $s_1, s_2 \in \Gamma(E)$*, and conclude that any Euclidean vector bundle* (E, g) *on* M *has a connection, which is compatible with the* g*.*

A simple example of a connection on the rank one trivial bundle $\mathrm{pr}_M : M \times \mathbb{R} \to M$ with is given by the exterior derivative $\mathrm{d} : \mathcal{F}(M) \to \Omega^1(M)$ with covariant derivative along X the Lie derivative $\mathcal{L}_X$. More generally, any $\omega \in \Omega^1(M)$ determines a connection $\nabla(f) = \mathrm{d}f + f\omega$ with covariant derivative $\nabla_X(f) = \mathcal{L}_X(f) + f\omega(X)$, and any connection on the trivial line bundle on M is of this form.

More generally, consider the rank n trivial bundle $\mathrm{pr}_M : M \times \mathbb{R}^n \to M$ with $\Gamma(M \times \mathbb{R}^n) = \mathcal{F}(M) \otimes \mathbb{R}^n$ and $\Omega^1(M \times \mathbb{R}^n) = \Omega^1(M) \otimes \mathbb{R}^n$. The standard basis of $\mathbb{R}^n$ is denoted $\{e_p; p = 1, \cdots, n\}$. Given a connection ∇ on this bundle the differentials $\omega^p_q \in \Omega^1(M)$ are defined by

$$\nabla(e_q) = \sum \omega^p_q \otimes e_p$$

and called the connection forms of ∇ in the basis e_p. A section $s \in \Gamma(M \times \mathbb{R}^n)$ can be uniquely written as

$$s = \sum s^p e_p$$

for certain functions $s^p \in \mathcal{F}(U)$ for $p = 1, \cdots, n$. Using the Leibniz rule the connection ∇ is given by

$$\nabla(s) = \sum(\mathrm{d}s^p + \sum s^q \omega_q^p) \otimes e_p$$

and the covariant derivative ∇_X along $X \in \mathfrak{X}(M)$ becomes

$$\nabla_X(s) = \sum\{\mathcal{L}_X(s^p) + \sum s^q \omega_q^p(X)\}e_p$$

and we can think of a connection in a local frame as given by a "matrix of one forms".

2.4 Curvature of a Connection

Let $(E \to M, \nabla)$ be a vector bundle with connection on M. The connection $\nabla : \Gamma(E) \to \Omega^1(E)$ can be extended to a linear map

$$\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$$

for all $p \geq 0$ by the Leibniz rule

$$\nabla(\alpha \otimes s) = \mathrm{d}\alpha \otimes s + (-1)^p \alpha \wedge \nabla(s)$$

for all $\alpha \in \Omega^p(M)$ and $s \in \Gamma(E)$.

Exercise 2.12. *Show that $\nabla((f\alpha) \otimes s) = \nabla(\alpha \otimes (fs))$ as should and hence $\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$ is well defined.*

Proposition 2.13. *Let $(E_1 \to M, \nabla_1)$ and $(E_2 \to M, \nabla_2)$ be two vector bundles with connections on M, and let ∇ be the corresponding connection on the tensor product bundle $E = E_1 \otimes E_2 \to M$. Define natural maps*

$$\wedge : \Omega^p(E_1) \otimes \Omega^q(E_2) \to \Omega^{p+q}(E_1 \otimes E_2)$$

by $(\alpha \otimes s_1) \wedge (\beta \otimes s_2) = (\alpha \wedge \beta) \otimes (s_1 \otimes s_2)$. Then we have

$$\nabla(\varepsilon_1 \wedge \varepsilon_2) = (\nabla_1 \varepsilon_1) \wedge \varepsilon_2 + (-1)^p \varepsilon_1 \wedge (\nabla_2 \varepsilon_2)$$

for all $\varepsilon_1 \in \Omega^p(E_1)$ and $\varepsilon_2 \in \Omega^q(E_2)$.

Proof. For $\varepsilon_1 = \alpha \otimes s_1 \in \Omega^p(E_1)$ and $\varepsilon_2 = \beta \otimes s_2 \in \Omega^q(E_2)$ we have

$$\nabla(\varepsilon_1 \wedge \varepsilon_2) = \nabla((\alpha \wedge \beta) \otimes (s_1 \otimes s_2)) = \mathrm{d}(\alpha \wedge \beta) \otimes (s_1 \otimes s_2) +$$
$$(-1)^{p+q}(\alpha \wedge \beta) \wedge \nabla(s_1 \otimes s_2) = (\mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta) \otimes (s_1 \otimes s_2) +$$
$$(-1)^{p+q}(\alpha \wedge \beta) \wedge (\nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2) = (\mathrm{d}\alpha \otimes s_1) \wedge (\beta \otimes s_2) +$$
$$(-1)^p (\alpha \otimes s_1) \wedge (\mathrm{d}\beta \otimes s_2) + (-1)^p (\alpha \otimes \nabla_1 s_1) \wedge (\beta \otimes s_2) +$$
$$(-1)^{p+q}(\alpha \otimes s_1) \wedge (\beta \otimes \nabla_2 s_2) = \nabla_1(\alpha \otimes s_1) \wedge (\beta \otimes s_2) +$$
$$(-1)^p (\alpha \otimes s_1) \wedge \nabla_2(\beta \otimes s_2) = (\nabla_1 \varepsilon_1) \wedge \varepsilon_2 + (-1)^p \varepsilon_1 \wedge (\nabla_2 \varepsilon_2)$$

and the desired formula follows. □

Theorem 2.14. *If $(E \to M, \nabla)$ is a vector bundle with connection then*

$$\nabla\nabla(\alpha \wedge \varepsilon) = \alpha \wedge R(\varepsilon)$$

with $\alpha \in \Omega^p(M), \varepsilon \in \Omega^q(E)$ and $R \in \Omega^2(\mathrm{End}(E))$ the so called curvature *of the connection.*

Proof. We apply the above proposition with $(E_1 = M \times \mathbb{R}, \nabla_1 = \mathrm{d})$ the trivial rank one bundle with the trivial connection and $(E_2 = E, \nabla_2 = \nabla$ the given vector bundle with connection on M, and hence for $\alpha \in \Omega^p(M)$ and $\varepsilon \in \Omega^q(E)$ we get

$$\nabla(\alpha \wedge \varepsilon) = (\mathrm{d}\alpha) \wedge \otimes \varepsilon + (-1)^p \alpha \wedge (\nabla \varepsilon)$$

by the above theorem. In turn we get

$$\nabla\nabla(\alpha \wedge \varepsilon) = (\mathrm{dd}\alpha) \wedge \varepsilon + (-1)^{p+1} \mathrm{d}\alpha \wedge \nabla \varepsilon + (-1)^p \mathrm{d}\alpha \wedge \nabla \varepsilon + \alpha \wedge \nabla\nabla\varepsilon$$

which implies that $\nabla\nabla(\alpha \wedge \varepsilon) = \alpha \wedge \nabla\nabla\varepsilon$ and so $\nabla\nabla : \Omega^p(E) \to \Omega^{p+2}(E)$ is a morphism of modules over $\Omega(M)$. □

Theorem 2.15. *Let $(E \to M, \nabla)$ be a vector bundle with connection on M. Equip the vector bundle $\mathrm{End}(E) \to M$ with its natural connection ∇ (by abuse of notation), given by $\nabla(A)s = \nabla(As) - A\nabla s$ for $A \in \Gamma(\mathrm{End}(E))$ and $s \in \Gamma(E)$. Then the curvature $R \in \Omega^2(\mathrm{End}(E))$ satisfies*

$$\nabla R = 0$$

which is called the second Bianchi identity.

Proof. For all $\varepsilon = (\alpha \otimes s) \in \Omega^p(E)$ we have the *Ricci identity*

$$\nabla\nabla(\varepsilon) = R \wedge \varepsilon$$

and since $\nabla(\nabla\nabla) = (\nabla\nabla)\nabla$ we find

$$\nabla(R \wedge \varepsilon) = \nabla(\nabla\nabla\varepsilon) = \nabla\nabla(\nabla\varepsilon) = R \wedge \nabla\varepsilon$$

which in turn implies $\nabla(R) \wedge \varepsilon = 0$ for all $\varepsilon \in \Omega^p(E)$. Hence $\nabla(R) = 0$. □

Theorem 2.16. *As equality in* $\Gamma(\mathrm{End}(E))$ *we have*

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Recall that for $\alpha \in \Omega^p(M)$ we have (using the Cartan formula and induction on p) the identity

$$\begin{aligned} \mathrm{d}\alpha(X_0, \cdots, X_p) &= \sum_i (-1)^i \mathcal{L}_{X_i}(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p)) \end{aligned}$$

for $X_0, \cdots, X_p \in \mathfrak{X}(M)$. Using the Leibniz rule

$$\nabla(\alpha \otimes s) = \mathrm{d}\alpha \otimes s + (-1)^p \alpha \wedge \nabla(s)$$

it follows that for $\varepsilon = \alpha \otimes s \in \Omega^p(E)$ we get likewise

$$\begin{aligned} (\nabla\varepsilon)(X_0, \cdots, X_p) &= \sum_i (-1)^i \nabla_{X_i}(\varepsilon(X_0, \cdots, \hat{X}_i, \cdots, X_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \varepsilon([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p)) \end{aligned}$$

as an identity in $\Gamma(E)$. In particular, for $X, Y \in \mathfrak{X}(M)$ one has

$$(\nabla\nabla s)(X, Y) = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s$$

which in turn implies

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

as identity in $\Gamma(\mathrm{End}(E))$. □

Definition 2.17. *A connection $\nabla : \Gamma(E) \to \Omega^1(E)$ on a vector bundle $p : E \to M$ with curvature $R \in \Omega^2(\mathrm{End}(E))$ is called* flat *or* integrable *if $R \equiv 0$.*

Clearly connections on vector bundles on a manifold M of dimension *one* are always flat. The exterior derivative d as the trivial connection on the trivial line bundle $M \times \mathbb{R} \to M$ (or trivial vector bundle $M \times \mathbb{R}^n \to M$) is flat.

Exercise 2.18. *Verify the identity*

$$(\nabla\varepsilon)(X_0, \cdots, X_p) = \sum_i (-1)^i \nabla_{X_i}(\varepsilon(X_0, \cdots, \hat{X}_i, \cdots, X_p))$$
$$+ \sum_{i<j} (-1)^{i+j} \varepsilon([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p))$$

in the above proof using the analogous (and familiar) identity for exterior derivation $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ instead of $\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$.

Exercise 2.19. *For the trivial vector bundle $\mathrm{pr}_M : M \times \mathbb{R}^n \to M$ with sections $s = \sum s^p e_p \in \Gamma(M \times \mathbb{R}^n)$ and a connection ∇ given by*

$$\nabla(s) = \sum(\mathrm{d}s^p + \sum s^q \omega^p_q) \otimes e_p$$

with matrix of one forms ω^p_q show that the curvature is given by

$$\nabla\nabla(s) = \sum s^q(\mathrm{d}\omega^p_q + \sum \omega^p_r \wedge \omega^r_q) \otimes e_p \ .$$

These formulas are abbreviated by $\nabla = \mathrm{d} + \omega$ and $R = \mathrm{d}\omega + \omega \wedge \omega$.

Exercise 2.20. *Keep the notation of the previous exercise and choose local coordinates $(x^1, \cdots, x^m)$ on M. Suppose the matrix of connection one forms has the form $\omega^p_q = \sum \Gamma^p_{qi}\mathrm{d}x^i$ for smooth functions Γ^p_{qi} of these local coordinates. Show that the matrix of curvature two forms R^p_q is given by*

$$R^p_q = \sum_{i<j} R^p_{qij}\mathrm{d}x^i \wedge \mathrm{d}x^j, \ R^p_{qij} = \partial_i\Gamma^p_{qj} - \partial_j\Gamma^p_{qi} + \sum\{\Gamma^p_{ri}\Gamma^r_{qj} - \Gamma^p_{rj}\Gamma^r_{qi}\}$$

with as usual $\partial_i = \partial/\partial x^i$ for $i = 1, \cdots, m$. Here the indices $i, j = 1, \cdots, m$ while $p, q = 1, \cdots, n$ with m the dimension of the base manifold M and n the rank of vector bundle E.

2.5 Parallel Transport and Holonomy

Let $f : M \to N$ be a smooth map between manifolds. If $p : E \to N$ is a smooth vector bundle on N then the pull back vector bundle f^*E along f was defined as the set of pairs $(x, e) \in M \times E$ with $f(x) = p(e)$. This set f^*E is a submanifold on $M \times E$, and the projection $f^*p : f^*E \to M$ on the first factor makes it a vector bundle.

A connection ∇ on E determines a connection $f^*\nabla$ on f^*E, called the pull back of ∇ along f and defined as follows. Let $V \subset N$ be a trivializing open set for E with a local frame $e_p \in \Gamma(E_V)$ for $p = 1, \cdots, n$. Then $f^*e_p = e_p f$ for $p = 1, \cdots, n$ is a local frame for $(f^*E)_U$ with $U = f^{-1}(V) \subset M$. If ∇ on V has connection one forms $\omega^p_q \in \Omega^1(V)$ in the frame $\{e_p\}$ then the connection one forms of $f^*\nabla$ on U in the frame f^*e_p are given by $f^*\omega^p_q \in \Omega^1(U)$. In other words, the covariant derivative $(f^*\nabla)_X$ along $X \in \mathfrak{X}(M)$ acts on a local section $s = \sum s^p f^*e_p$ on U as

$$(f^*\nabla)_X(s) = \sum \{\mathcal{L}_X(s^p) + \sum s^q \mathrm{i}_X(f^*\omega^p_q)\} f^*e_p$$

with $\mathrm{i}_X(f^*\omega^p_q) = \mathrm{i}_{f_*(X)}\omega^p_q \in \mathcal{F}(U)$.

For the rest of this section let (E, ∇) be a vector bundle with connection on M. Let $\gamma : [a, b] \to M$ be a smooth arc in M from $x = \gamma(a)$ to $y = \gamma(b)$, by which we mean that γ has a smooth extension to $(a - \epsilon, b + \epsilon)$ for some $\epsilon > 0$. A section s for $(\gamma^*E, \gamma^*\nabla)$ is called *horizontal* if $(\gamma^*\nabla)s = 0$. If $U \subset M$ is a trivializing open set for E and $\gamma([a, b]) \subset U$ the condition $(\gamma^*\nabla)s = 0$ can be written as

$$\dot{s}^p(t) + \sum s^q(t) A^p_q(t) = 0$$

for all $p = 1, \cdots, n$ with $A^p_q(t) = \gamma^*(\omega^p_q)(\partial_t)$ smooth functions of $t \in [a, b]$. This is a linear first order system of ordinary differential equations. By the existence and uniqueness theorem for first order ordinary differential equations there exists for each vector $s(a) \in (\gamma^*E)_a$ a unique horizontal section s for γ^*E with the prescribed initial value for $t = a$. By linearity of the system of differential equations the map $(\gamma^*E)_a \to (\gamma^*E)_b, s(a) \mapsto s(b)$ is a linear map. If $[a, b] \ni t \leftrightarrow \tilde{t} \in [\tilde{a}, \tilde{b}]$ is a smooth reparametrization and we denote $\tilde{\gamma}(\tilde{t}) := \gamma(t(\tilde{t}))$ then $t \mapsto s(t)$ is a horizontal section for $(\gamma^*E, \gamma^*\nabla)$ if and only if $\tilde{t} \mapsto \tilde{s}(\tilde{t}) := s(t(\tilde{t}))$ is a horizontal section for $(\tilde{\gamma}^*E, \tilde{\gamma}^*\nabla)$. Indeed, we have

$$\partial_t s^p(t) + \sum s^q(t) A^p_q(t) = \{\partial_{\tilde{t}} s^p(\tilde{t}) + \sum \tilde{s}^q(\tilde{t}) \tilde{A}^p_q(\tilde{t})\} \frac{d\tilde{t}}{dt}$$

since $\tilde{A}_q^p(\tilde{t})\mathrm{d}\tilde{t} = \tilde{\gamma}^*(\omega_q^p) \leftrightarrow \gamma^*(\omega_q^p) = A_q^p(t)\mathrm{d}t$ under the substitution $\tilde{t} \leftrightarrow t$. The corresponding linear map

$$\mathrm{Par}(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}$$

is called *parallel transport* in (E, ∇) along the arc γ. Parallel transport is invariant under smooth reparametrizations of the arc γ.

For γ a piecewise smooth arc in M parallel transport is defined as the composition of the parallel transports along the smooth pieces of γ. More precisely, if $\gamma : [a, b] \to M$ and $\delta : [b, c] \to M$ are two piecewise smooth arcs in M with the end point $\gamma(b)$ of γ equal to the begin point $\delta(b)$ of δ then the composition $\delta\gamma : [a, c] \to M$ is the piecewise smooth arc

$$\delta\gamma(t) = \begin{cases} \gamma(t) & \forall t \in [a, b] \\ \delta(t) & \forall t \in [b, c] \end{cases}$$

taken in this order: traverse δ after γ is traversed. Hence

$$\mathrm{Par}(\delta\gamma) = \mathrm{Par}(\delta)\mathrm{Par}(\gamma)$$

just by definition.

Exercise 2.21. *Show that parallel transport* $\mathrm{Par}(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}$ *in* (E, ∇) *along an arc* γ *in* M *is an invertible linear map.*

If $\gamma : [a, b] \to M$ is a piecewise smooth loop based at $x = \gamma(a) = \gamma(b) \in M$ then parallel transport along the loop γ based at $x \in M$ is also called the *holonomy* along γ. The set of all holonomies of piecewise smooth loops based at $x \in M$ is a subgroup of the general linear group $\mathrm{GL}(E_x)$. It is called the *holonomy group* of (E, ∇) at $x \in M$ and is denoted $\mathrm{Hol}_x(E, \nabla)$.

Exercise 2.22. *Let* (E, ∇) *be a vector bundle with connection on a connected manifold* M. *Show that parallel transport along an arc* γ *from* x *to* y *induces an isomorphism from* $\mathrm{Hol}_x(E, \nabla)$ *onto* $\mathrm{Hol}_y(E, \nabla)$.

Exercise 2.23. *A Euclidean structure* g *on a vector bundle* $E \to M$ *is called horizontal for a connection* ∇ *on* E *if* $\nabla(g) = 0$, *or equivalently if*

$$\mathcal{L}_X(g(s_1, s_2)) = g(\nabla_X(s_1), s_2) + g(s_1, \nabla_X(s_2))$$

for all $X \in \mathfrak{X}(M)$ and all $s_1, s_2 \in \Gamma(E)$. Suppose g is a horizontal Euclidean structure on a vector bundle with connection (E, ∇) on M. Show that parallel transport

$$\mathrm{Par}(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}$$

along a piecewise smooth arc $\gamma : [a, b] \to M$ preserves the corresponding Euclidean structures, in the sense that

$$g_{\gamma(a)}(e, f) = g_{\gamma(b)}(\mathrm{Par}(\gamma)e, \mathrm{Par}(\gamma)f)$$

for any $e, f \in E_{\gamma(a)}$. Conclude that the holonomy group $\mathrm{Hol}_x(E, \nabla)$ based at $x \in M$ is a subgroup of the orthogonal group $\mathrm{O}(E_x, g_x)$.

2.6 Fundamental Group

Let Z be an arcwise connected topological space. A good example to have in mind is a domain Z in $\mathbb{C}$.

Definition 2.24. *A path in Z is a continuous map $\gamma : [0, 1] \to Z, t \mapsto \gamma(t)$. The point $\gamma(0)$ is called the begin point and the point $\gamma(1)$ the end point of γ. If begin and end point of γ coincide then γ is called a loop with base point $\gamma(0) = \gamma(1)$.*

Definition 2.25. *Let γ_1 and γ_2 be two paths in Z with equal begin points $\gamma_1(0) = \gamma_2(0)$ and equal end points $\gamma_1(1) = \gamma_2(1)$. The paths γ_1 and γ_2 are called homotopic if there exists a continuous map $h : [0, 1] \times [0, 1] \to Z, (s, t) \mapsto h(s, t)$ such that*

$$h(0, t) = \gamma_1(t)\ ,\ h(1, t) = \gamma_2(t)\ \forall t \in [0, 1]\ ,$$

$$h(s, 0) = \gamma_1(0) = \gamma_2(0)\ ,\ h(s, 1) = \gamma_1(1) = \gamma_2(2)\ \forall s \in [0, 1]\ .$$

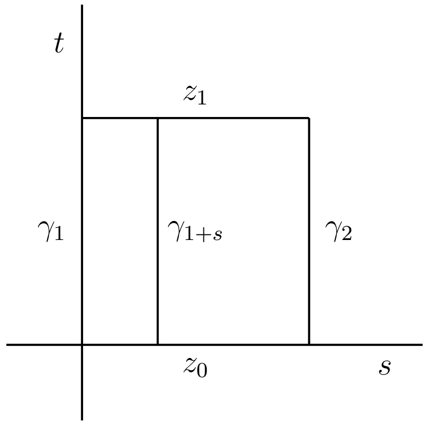
The map h is called the homotopy between the paths γ_1 and γ_2.

In other words the two paths γ_1 and γ_2 are homotopic if there exists a one parameter continuous family (with parameter $s \in [0, 1]$) of paths

$$\gamma_{1+s} : [0, 1] \to Z$$

$$\gamma_{1+s}(0) = \gamma_1(0) = \gamma_2(0)\ ,\ \gamma_{1+s}(1) = \gamma_1(1) = \gamma_2(1)\ \forall s \in [0, 1]\ .$$

The link with our previous notation is $\gamma_{1+s}(t) = h(s, t)$. If $\gamma_1(0) = \gamma_2(0) = z_0$ and $\gamma_1(1) = \gamma_2(1) = z_1$ then we draw the following schematic picture.



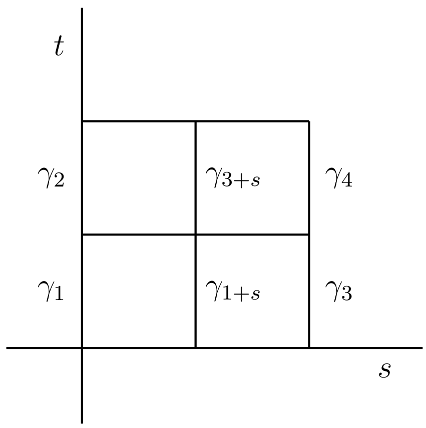
We shall write $\gamma_1 \sim \gamma_2$ if the paths γ_1 and γ_2 in Z with equal begin points and equal end points are homotopic. It is easy to show that being homotopic is an equivalence relation. The equivalence class of a path $\gamma : [0,1] \to Z$ is denoted by $[\gamma]$.

Definition 2.26. *Let $\gamma_1, \gamma_2 : [0,1] \to Z$ be two paths in Z with $\gamma_1(1) = \gamma_2(0)$. We define a new path $\gamma_2\gamma_1 : [0,1] \to Z$ by*

$$\gamma_2\gamma_1(t) = \begin{cases} \gamma_1(2t) & \forall t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \forall t \in [\frac{1}{2}, 1] \end{cases}$$

The path $\gamma_2\gamma_1$ is called the product of γ_2 and γ_1, and is always taken in this order, start with γ_1 and then follow with γ_2.

It is easy to show that if $\gamma_1 \sim \gamma_3$ and $\gamma_2 \sim \gamma_4$ and the end point z_1 of γ_1, γ_3 coincides with the begin point z_1 of γ_2, γ_4 then $\gamma_2\gamma_1 \sim \gamma_4\gamma_3$. Here is a schematic picture of the homotopy.



Hence the product $[\gamma_2][\gamma_1]$ of the homotopy classes of paths γ_2 and γ_1 as in Definition 2.26 is well defined. We leave it as an exercise to show that the product of paths is associative on homotopy classes of paths.

Theorem 2.27. *For $z_0 \in Z$ a fixed point let $\Pi_1(Z, z_0)$ denote the collection of homotopy classes of loops in Z with base point z_0. The product rule on paths in Z according to Definition 2.26 defines a group structure on $\Pi_1(Z, z_0)$. The unit element is represented by the constant path*

$$\epsilon(t) = z_0 \ \ \forall t \in [0, 1]$$

based at z_0. The inverse $[\gamma]^{-1}$ of $[\gamma] \in \Pi_1(Z, z_0)$ is represented by the loop

$$\gamma^{-1}(t) = \gamma(1 - t) \ \ \forall t \in [0, 1]$$

which is just the original loop γ but traversed in opposite direction.

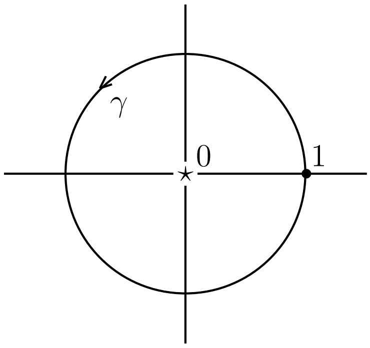
Definition 2.28. *The group $\Pi_1(Z, z_0)$ is called the* fundamental group *of the arcwise connected topological space Z with base point z_0.*

Elements of $\Pi_1(Z, z_0)$ are homotopy classes of loops, but sometimes one refers to the elements of $\Pi_1(Z, z_0)$ simply as loops (based at z_0). Even worse, in the notation one simply writes $\gamma \in \Pi_1(Z, z_0)$ rather than $[\gamma] \in \Pi_1(Z, z_0)$. If $\delta : [0, 1] \to M$ is a path with begin point $z_0 = \delta(0)$ and end point $z_1 = \delta(1)$ then the map

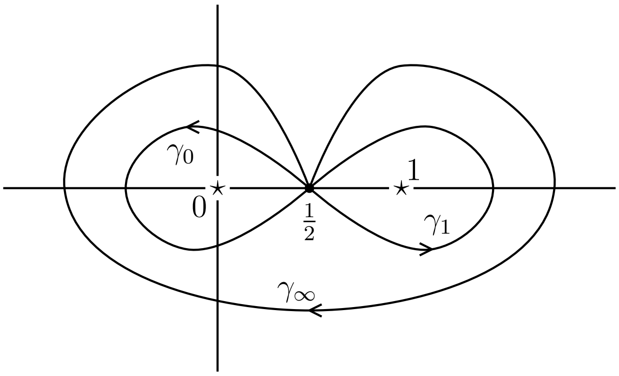
$$\Pi_1(Z, z_0) \to \Pi_1(Z, z_1), \ \gamma \mapsto \delta\gamma\delta^{-1}$$

is an isomorphism of groups. This isomorphism depends only on the homotopy class of δ and is unique (that is independent of the choice of some δ connecting the two base points) up to inner automorphisms. The inner isomorphism class of the fundamental group of Z at some base point is denoted $\Pi_1(Z)$. The arcwise connected space Z is called *simply connected* if its fundamental group is trivial. The space Z is called *contractible* if the identity map and the constant map, sending all of Z to some base point in Z, are homotopic. Clearly contractible spaces are simply connected. Examples of contractible spaces are star shaped regions in $\mathbb{R}^m$, like the open unit ball $\mathbb{B}^m$ or the hypercube $(-1, 1)^m$ or $\mathbb{R}^m$ itself.

Example 2.29. *Let $Z = \mathbb{C}^\times = \mathbb{C} - \{0\}$ and $z_0 = 1$. If $\gamma(t) = exp(2\pi it)$ for $t \in [0, 1]$ then $\Pi_1(Z, z_0)$ is a cyclic group with generator γ.*



Example 2.30. *Let $Z = \mathbb{P} - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$ with $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ the complex projective line and take $z_0 = \frac{1}{2}$. Choose loops $\gamma_0, \gamma_1, \gamma_\infty$ around the points $0, 1, \infty$ respectively as in the picture.*



It is easy to see that $\gamma_\infty \gamma_1 \gamma_0 = 1$ in $\Pi_1(Z, \frac{1}{2})$. It can be shown that $\Pi_1(Z, \frac{1}{2})$ is isomorphic to the group on three generators $\gamma_0, \gamma_1, \gamma_\infty$ with the single relation $\gamma_\infty \gamma_1 \gamma_0 = 1$.

Remark 2.31. *Because the fundamental group $\Pi_1(Z)$ is unique up to inner automorphisms the Abelianized fundamental group*

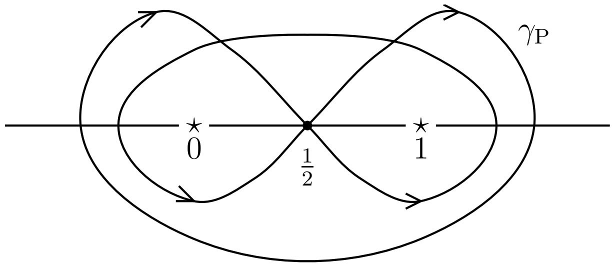
$$\Pi_1(Z)^{\text{Abel}} = \Pi_1(Z)/[\Pi_1(Z), \Pi_1(Z)]$$

is a canonically defined Abelian group, and is called the first homology group of the space Z, denoted $H_1(Z)$. By abuse of notation the class $[\gamma] \in H_1(Z)$ is called the cycle of the loop $[\gamma] \in \Pi_1(Z, z_0)$.

Example 2.32. *The Pochhammer contour $\gamma_{\mathrm{P}} \in \Pi_1(\mathbb{P} - \{0, 1, \infty\}, \frac{1}{2})$ is defined by*

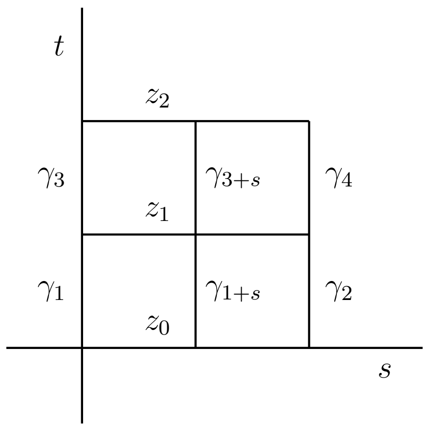
$$\gamma_{\mathrm{P}} = [\gamma_0, \gamma_1] = \gamma_0 \gamma_1 \gamma_\infty$$

with $[\cdot,\cdot]$ *for the commutator in the fundamental group* $\Pi_1(\mathbb{P}-\{0,1,\infty\},\frac{1}{2})$. *The second equality follows from the topological relation* $\gamma_\infty\gamma_1\gamma_0 = 1$.

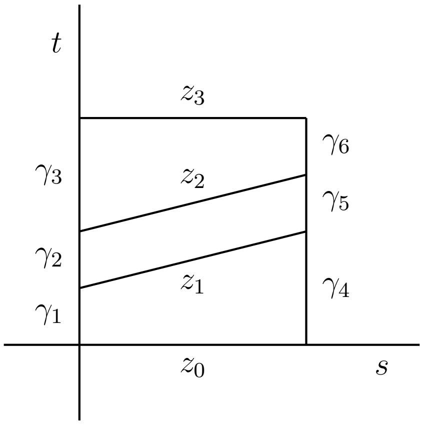


The Pochhammer contour was introduced independently by Jordan in 1887 *and Pochhammer in* 1890. *The Pochammer contour is nontrivial in homotopy, but the associated cycle is trivial in homology.*

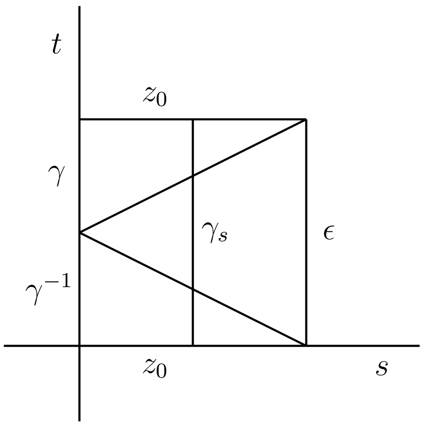
Exercise 2.33. *Show that if the end points of* $\gamma_1 \sim \gamma_2$ *coincide with the begin points of* $\gamma_3 \sim \gamma_4$ *then* $\gamma_3\gamma_1 \sim \gamma_4\gamma_2$.



Exercise 2.34. *Suppose that* $\gamma_1, \cdots, \gamma_6$ *are paths in* Z *such that* $\gamma_1 \sim \gamma_4$, $\gamma_2 \sim \gamma_5$, $\gamma_3 \sim \gamma_6$ *and the products* $\gamma_3(\gamma_2\gamma_1)$ *and* $(\gamma_6\gamma_5)\gamma_4$ *are well defined. In other words we assume that the begin points of* γ_1, γ_4 *equal* z_0, *the end points of* γ_1, γ_4 *and the begin points of* γ_2, γ_5 *equal* z_1, *the end points of* γ_2, γ_5 *and the begin points of* γ_3, γ_6 *equal* z_2, *and finally the end points of* γ_3, γ_6 *equal* z_3. *Show that* $\gamma_5(\gamma_3\gamma_1) \sim (\gamma_6\gamma_4)\gamma_2$. *In turn this implies that the group law on the fundamental group is associative. A picture of the homotopy is given by the picture below.*



Exercise 2.35. *Show that in the notation of Theorem 2.27 we have* $\epsilon\gamma \sim \gamma\epsilon \sim \gamma$ *and* $\gamma\gamma^{-1} \sim \gamma^{-1}\gamma \sim \epsilon$.



Here $\gamma_s(t)$ *is equal to* $\gamma^{-1}(2t)$ *for* $t \in [0,(1-s)/2]$, *is constant equal to* $\gamma^{-1}(1-s) = \gamma(s)$ *for* $t \in [(1-s)/2,(1+s)/2]$, *and is equal to* $\gamma(2t-1)$ *for* $t \in [(1+s)/2, 1]$

2.7 Flat Connections and Monodromy

Let M be a connected manifold, and (E, ∇) a vector bundle with connection on M. If $\gamma : [0, 1] \to$ is a piecewise smooth path with begin point $\gamma(0)$ and end point $\gamma(1)$ then parallel transport

$$\mathrm{Par}(\gamma) : E_{\gamma(0)} \to E_{\gamma(1)}$$

in (E,∇) along γ is a linear isomorphism. The product $\gamma_2\gamma_1$ of two piecewise smooth paths with the end point $\gamma_1(1)$ of γ_1 equal to the begin point $\gamma_2(0)$ of γ_2 was defined by first traversing γ_1 and subsequently γ_2 both at double speed, which in turn implies

$$\mathrm{Par}(\gamma_2\gamma_1) = \mathrm{Par}(\gamma_2)\mathrm{Par}(\gamma_1)$$

as equality in $\mathrm{Hom}(E_{\gamma_1(0)}, E_{\gamma_2(1)})$ with $\gamma_1(0)$ the begin point of γ_1 and $\gamma_2(1)$ the end point of γ_2.

Definition 2.36. *The connection ∇ on a vector bundle $E \to M$ is called flat or integrable if its curvature $R = \nabla\nabla$ vanishes identically.*

Theorem 2.37. *Suppose (E,∇) is a vector bundle with a flat connection on a connected manifold M. Each $x \in M$ has an open neighborhood U such that for each vector $e \in E_x$ there exists a unique horizontal section $s \in \Gamma(E_U)$ with the prescribed initial value $s(x) = e$.*

Proof. Since the theorem is local in nature we can and will take $M = (-1,1)^m$ an open hypercube and $E = M \times \mathbb{R}^n$ the trivial vector bundle on M. Hence a section $s \in \Gamma(E)$ can be identified with a smooth function $s : M \to \mathbb{R}^n$. Writing $s = \sum s^p e_p$ with $\{e_p\}$ the standard basis of $\mathbb{R}^n$ we get

$$\nabla_i(s) = \sum\{\partial_i(s^p) + \sum \Gamma^p_{iq} s^q\} e_p$$

with $\omega^p_q = \sum \Gamma^p_{iq} \mathrm{d}x^i$ as in Exercise 2.20. The flatness of ∇ amounts to

$$[\nabla_i, \nabla_j] = 0$$

as operators on $\Gamma(E)$ for all $i, j = 1, \cdots, m$.

Pick a vector $e \in \mathbb{R}^n$. Integration along the first coordinate axis of the equation $\nabla_1(s) = 0$ yields a unique function

$$(-1,1) \ni x^1 \mapsto s(x^1, 0, \cdots, 0)$$

with values in $\mathbb{R}^n$ and initial value $s(0) = e$. For each $x^1 \in (0,1)$ we can integrate the equation $\nabla_2(s) = 0$ along the second coordinate and obtain a unique function

$$(-1,1)^2 \ni (x^1, x^2) \mapsto s(x^1, x^2, 0, \cdots, 0)$$

with values in $\mathbb{R}^n$ and equal to the previous function for $x^2 = 0$.

We claim that $\nabla_1 s(x^1, x^2, 0 \cdots, 0) = 0$ for all $x^1, x^2 \in (-1, 1)$. Indeed, we know by construction that

$$\nabla_1 s(x^1, 0, 0, \cdots, 0) = 0, \ \nabla_2 s(x^1, x^2, 0, \cdots, 0) = 0$$

for all $x^1, x^2 \in (-1, 1)$. Since $\nabla_2 \nabla_1 = \nabla_1 \nabla_2$ we get

$$\nabla_2 \nabla_1 s(x^1, x^2, 0, \cdots, 0) = \nabla_1 \nabla_2 s(x^1, x^2, 0, \cdots, 0) = 0$$

and so by the existence and uniqueness theorem the claim follows. This proves the theorem in case $m = 2$.

Proceed by induction on j. Say we have a function $s(x^1, \cdots, x^j, 0, \cdots, 0)$ satisfying

$$\nabla_i s(x^1, \cdots, x^j, 0, \cdots, 0) = 0$$

for $i = 1, \cdots, j$ and all $x^1, \cdots, x^j \in (-1, 1)$. Hence we get a function $s(x^1, \cdots, x^{j+1}, 0 \cdots, 0)$ for all $x^1, \cdots, x^{j+1} \in (-1, 1)$ by integration of the equation

$$\nabla_{j+1} s(x^1, \cdots, x^{j+1}, 0, \cdots, 0) = 0$$

along the coordinate x^{j+1} with the already found initial value for $x^{j+1} = 0$. By the same argument as above using $\nabla_{j+1} \nabla_i = \nabla_i \nabla_{j+1}$ we find

$$\nabla_i s(x^1, \cdots, x^{j+1}, 0 \cdots, 0) = 0$$

for all $x^1, \cdots, x^{j+1} \in (-1, 1)$ and $i = 1, \cdots, m$. This gives the desired horizontal section s on $M = (-1, 1)^m$ with initial value $s(0) = e$ $\square$

In the setting of the above theorem let $\gamma : [0, 1] \to U$ be a piecewise smooth arc in U met begin point $x = \gamma(0)$ and end point $y = \gamma(1)$ some point in U. The parallel transport operator

$$\mathrm{Par}(\gamma) : E_x \to E_y$$

sends a vector $e \in E_x$ to the vector $s(y) \in E_y$ with $s \in \Gamma(E_U)$ solution of $\nabla s = 0$ and $s(x) = e$. Indeed $\gamma^* s$ is solution of $(\gamma^* \nabla) \gamma^* s = \gamma^*(\nabla s) = 0$ and $\gamma^* s(0) = s(x)$ is sent by parallel transport to $\gamma^* s(1) = s(y)$. Using the Lemma of Lebesgue it follows that for homotopic piecewise smooth arcs γ_1, γ_2 in M both with begin point x and end point y we get $\mathrm{Par}(\gamma_1) = \mathrm{Par}(\gamma_2)$ in $\mathrm{Hom}(E_x, E_y)$.

Definition 2.38. *Let (E, ∇) be a vector bundle with flat connection on M. For $[\gamma] \in \Pi_1(M, x)$ the holonomy along a representing loop γ is called the monodromy of $[\gamma]$ and is denoted $\mathrm{Mon}_x([\gamma]) \in \mathrm{GL}(E_x)$. The map*

$$\mathrm{Mon}_x : \Pi_1(M, x) \to \mathrm{GL}(E_x)$$

is a homomorphism, called the monodromy representation of (E, ∇) with base point $x \in M$.

Let M be a connected manifold and $x \in M$ a base point. The monodromy map

$$(E, \nabla) \mapsto (E_x, \mathrm{Mon}_x)$$

is a functor from the category of real vector bundles (E, ∇) with a flat connection on M to the category of finite dimensional real representations of the fundamental group $\Pi_1(M, x)$. Being a functor means that the assignment $(E, \nabla) \mapsto (E_x, \mathrm{Mon}_x)$ is compatible with constructions of linear algebra (like $\oplus, \otimes, \mathrm{Hom}, S^p, \wedge^p, \cdots$) in the two categories.

A natural question at this point is to describe the image of this functor: Which representations of $\Pi_1(M, x)$ can occur as monodromy representations on the fiber E_x of some vector bundle (E, ∇) with flat connection on M? The remarkable answer is that any finite dimensional real representation of $\Pi_1(M, x)$ can occur in this way!

It is even easy to understand why this is true, once you understand the concept of the universal covering space. For M a connected manifold with a fixed base point $x \in M$ we denote by $\tilde{M}$ the set of homotopy classes of paths with begin point x. The natural map

$$p : \tilde{M} \to M, \; [\delta] \mapsto \delta(1)$$

is a local homeomorphism, which in turn provides $\tilde{M}$ with a unique manifold structure for which $p : \tilde{M} \to M$ becomes a local diffeomorphism. The fiber $p^{-1}(x)$ is naturally identified as a set with the fundamental group $\Pi_1(M, x)$. Moreover the map

$$\tilde{M} \times \Pi_1(M, x) \to \tilde{M}, \; (\delta, \gamma) \mapsto \delta\gamma$$

defines a right action of $\Pi_1(M, x)$ on $\tilde{M}$ with quotient space M. Restriction of this action to $p^{-1}(x)$ is just right multiplication in $\Pi_1(M, x)$. The map $p : \tilde{M} \to M$ is just the quotient map for this action of $\Pi_1(M, x)$ on $\tilde{M}$ by so called deck transformations.

Now let $\Pi_1(M,x) \to \mathrm{GL}(\mathbb{R}^n)$ be a real representation of dimension n. Consider the trivial vector bundle $(\tilde{M} \times \mathbb{R}^n, \tilde{\nabla} = \mathrm{d})$ with connection on $\tilde{M}$. Let $\Pi_1(M,x)$ act on $\tilde{M} \times \mathbb{R}^n$ from the right by $(\delta, v)\gamma = (\delta\gamma, \gamma^{-1}v)$. This action maps horizontal local sections to horizontal local sections, and so defines a vector bundle with flat connection

$$(E = (\tilde{M} \times \mathbb{R}^n)/\Pi_1(M,x), \nabla)$$

on M with $p^*(E,\nabla) = (\tilde{M} \times \mathbb{R}^n, \mathrm{d})$ and with monodromy equal to the original representation.

Remark 2.39. *The same result is valid in the setting of a complex manifold M with E a holomorphic vector bundle on M and ∇ a flat holomorphic connection on $E \to M$. Again all representations of the fundamental group $\Pi_1(M,x)$ do occur as monodromy representations of such a pair (E,∇). In case M is the complement in a smooth projective manifold $\overline{M} = M \sqcup \partial M$ of a normal crossings subvariety $\partial M \subset \overline{M}$ it was shown by Pierre Deligne that E has an extension over ∂M to an algebraic vector bundle $\overline{E}$ on $\overline{M}$ and the connection ∇ has an extension $\overline{\nabla}$ as a rational connection on $\overline{E} \to \overline{M}$ with simple poles along ∂M. Such a connection $\overline{\nabla}$ on $\overline{E} \to \overline{M}$ is called regular singular along ∂M. In case M is the complement of a finite set S in the projective line $\mathbb{P}$ the existence of such a regular singular connection was conjectured in 1857 by Riemann [22]. The conjecture was formulated by Hilbert as Problem 21 in his famous list of mathematical problems from 1900 [12]. The Riemann–Hilbert problem was solved by Plemelj in 1908 in the original case that M is the complement in the projective line $\mathbb{P}$ of a finte set. In the general univariable setting it was solved by Röhrl in 1957 [8]. The Riemann–Hilbert problem in the general multivariable setting was solved by Deligne in 1970 [5].*

3 Riemannian Geometry

3.1 Riemannian Manifolds

Let M be a smooth manifold of dimension m.

Definition 3.1. *A Riemannian metric g on M assigns to each point $x \in M$ a positive definite scalar product g_x on the tangent space T_xM at x of M, which depends smoothly on x in the sense that $x \mapsto g_x(X_x, Y_x)$ is a smooth function on M for any two smooth vector fields $X, Y \in \mathfrak{X}(M)$. The pair (M, g) of a smooth manifold M with a Riemannian metric g on M is called a Riemannian manifold.*

It follows from Exercise 2.6 that smooth manifolds admit Riemann metrics, in fact in abundance. The study of smooth manifolds is the domain of differential *topology*, while the additional ingredient of a Riemannian metric is the starting point for differential *geometry*. For example, on a Riemannian manifold (M, g) one can measure the length L of a piecewise smooth arc $\gamma : [a, b] \to M$ on M by the formula

$$L = \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}dt$$

where $\dot\gamma(t) = (T\gamma)(d/dt)$ is the tangent vector along the arc in M. In local coordinates $x = (x^1, \cdots, x^m)$ on M a Riemannian metric just takes the form

$$ds^2 = \sum g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$$

with $g_{ij}(x) = ds^2(\partial_i, \partial_j)$ a positive definite symmetric matrix, depending smoothly on x. The volume element in local coordinates

$$dV = (\det(g_{ij}(x))^{1/2}dx^1 \cdots dx^m$$

defines a smooth density on M, and enables one to compute the Riemannian volume of compact domains in M.

Clearly the restriction of a Riemannian metric on M to a submanifold N of M is a Riemannian metric on N. The restriction of the Euclidean metric $\sum(\mathrm{d}y^i)^2$ on $\mathbb{R}^n$ to a submanifold M is a Riemannian metric, which in local coordinates is just the first fundamental form. However the classical terminology first fundamental form is somewhat outdated, and Riemannian metric has become the standard terminology these days.

A smooth map f from a Riemannian manifold (M, g) to a Riemannian manifold (N, h) is called an isometric immersion if

$$h_{f(x)}(T_x f(u), T_x f(v)) = g_x(u, v)$$

for all $x \in M$ and all $u, v \in T_x M$. Since $T_x f : T_x M \to T_{f(x)} N$ is then an injective linear map for all $x \in M$ isometric immersions are indeed immersions. An isometric immersion, which is also an embedding, is called an *isometric embedding*. The Whitney embedding theorem says that each manifold M of dimension m can be embedded in $\mathbb{R}^n$ for n sufficiently large, and in fact $n = 2m + 1$ is always possible. The proof in case M has a finite atlas is in fact quite easy, and is given in the notes of Marcut. However a much more difficult and spectacular result is the Nash *isometric* embedding theorem, obtained by John Nash in 1956.

Theorem 3.2. *Every smooth Riemannian manifold (M, g) of dimension m can be isometrically embedded in Euclidean space $\mathbb{R}^n$ for n sufficiently large, and in fact $n = (m + 2)(m + 3)/2$ is always possible.*

The dramatic life of John Nash has been recorded by Sylvia Nasar [18]. Nash graduated from Princeton University in 1950 with a PhD (of just 28 pages) in game theory, in which he introduced what are now called Nash equilibria for noncooperative games. After that he went to MIT, where he obtained the isometric embedding theorem in 1956. In 1994 Nash was awarded the Nobel Prize for economy for his work in game theory. In 2015 he was awarded the Abel Prize for his work on partial differential equations leading to the isometric embedding theorem. On the way home from the latter ceremony he and his wife died in a car traffic accident.

In the *real analytic* setting of smooth functions with locally convergent power series expansions a *local* isometric embedding theorem in $\mathbb{R}^n$ with $n = m(m + 1)/2$ was conjectured by Ludwig Schläfli in 1873, and proved independently by Élie Cartan and Maurice Janet in 1926.

An isometric immersion, which is also a diffeomeorphism, is simply called an *isometry*. The isometries of a Riemannian manifold (M, g) onto itself form a subgroup $\mathrm{Isom}(M, g)$ of the diffeomorphism group $\mathrm{Diff}(M)$ of the underlying manifold M, which is called the isomety group of (M, g).

Both translations and orthogonal linear transformations are isometries of the Euclidean space $\mathbb{E}^m$, which by definition is the Cartesian space $\mathbb{R}^n$ with the standard translation invariant Riemannian metric. Hence $\mathbb{R}^m \rtimes \mathrm{O}(\mathbb{R}^m)$ is a subgroup of $\mathrm{Isom}(\mathbb{E}^m)$.

Exercise 3.3. *Show that a distance preserving transformation A of $\mathbb{E}^m$ with $A(0) = 0$ satisfies $(Au, Av) = (u, v)$ for all $u, v \in \mathbb{R}^m$. Conclude that each $A \in \mathrm{Isom}(\mathbb{E}^m)$ with $A(0) = 0$ is a linear transformation of $\mathbb{R}^m$, and so $A \in \mathrm{O}(\mathbb{R}^m)$. In turn deduce that* $\mathrm{Isom}(\mathbb{E}^m) = \mathbb{R}^m \rtimes \mathrm{O}(\mathbb{R}^m)$.

Let $\mathbb{S}^m$ be the unit sphere in $\mathbb{E}^{m+1}$ with the induced Riemannian metric. The orthogonal group $\mathrm{O}(\mathbb{R}^{m+1})$ acts on the *round sphere* $\mathbb{S}^m$ by isometries, and so $\mathrm{O}(\mathbb{R}^{m+1})$ becomes a subgroup of $\mathrm{Isom}(\mathbb{S}^m)$. It will be shown later on that $\mathrm{Isom}(\mathbb{S}^m) = \mathrm{O}(\mathbb{R}^{m+1})$.

Let $\mathbb{R}^{m,1}$ be the vector space $\mathbb{R}^{m+1}$ with Lorentzian scalar product

$$(x, y) = x^1 y^1 + \cdots + x^m y^m - x^{m+1} y^{m+1}$$

and let $\mathrm{O}(\mathbb{R}^{m,1})$ be the associated Lorentz group. The associated Lorentzian space with translation invariant Lorentzian metric

$$ds^2 = (dy^1)^2 + \cdots + (dy^m)^2 - (dy^{m+1})^2$$

is denoted by $\mathbb{L}^{m,1}$. The hypersurface

$$\mathbb{H}^m = \{y \in \mathbb{L}^{m,1}; (y, y) = -1, y^{m+1} > 0\}$$

is a connected component of the two sheeted hyperboloid and is acted upon by the forward Lorentz group $\mathrm{O}_+(\mathbb{R}^{m,1})$, which is an index two subgroup of of the full Lorentz group $\mathrm{O}(\mathbb{R}^{m,1})$. It has still two connected components (for $m \geq 1$) which are separated by the determinant. The Lorenztian metric on $\mathbb{L}^{m,1}$ restricts to a Riemannian metric on $\mathbb{H}^m$. The Riemannian manifold $\mathbb{H}^m$ is called *hyperbolic space* of dimension m, and the forward Lorentz group $\mathrm{O}_+(\mathbb{R}^{m,1})$ becomes a subgroup of $\mathrm{Isom}(\mathbb{H}^m)$. It will be shown later on that $\mathrm{Isom}(\mathbb{H}^m) = \mathrm{O}_+(\mathbb{R}^{m,1})$.

The above three spaces $\mathbb{E}^m, \mathbb{S}^m, \mathbb{H}^m$ have transitive isometry groups, for which the stabilizer group of a point acts on the tangent space at that point as the full orthogonal group. It can be shown that a connected Riemannian manifold with such a highly transitive isometry group is one of these three spaces. For a general Riemannian manifold the isometry group will usually be trivial.

Exercise 3.4. *In the ball model for hyperbolic geometry one considers the stereographic projection*

$$y : \mathbb{B}^m = \{x \in \mathbb{R}^m; |x| < 1\} \to \mathbb{H}^m \ , \ y(x) = (2x, 1 + |x|^2)/(1 - |x|^2)$$

with center the south pole $s = (0, \cdots, 0, -1) \in \mathbb{R}^{m,1}$. *Show that under this projection the Riemannian metric in the hyperbolic ball model becomes*

$$ds^2 = 4\frac{(dx^1)^2 + \cdots + (dx^m)^2}{(1 - |x|^2)^2}$$

with

$$dV = \frac{2^m dx^1 \cdots dx^m}{(1 - |x|^2)^m}$$

the associated volume element.

The hyperbolic metric on $\mathbb{B}^m$ is conformal with the Euclidean metric. The four Circle Limit woodcuts by M.C. Escher are nice illustrations of isometric tesselations of the hyperbolic disc $\mathbb{D} = \mathbb{B}^2$.

3.2 Levi-Civita Connection and Riemann Curvature

Let (M, g) be a Riemannian manifold. The next key result is called the *Fundamental Theorem of Riemannian Geometry* and is due to Levi-Civita.

Theorem 3.5. *There is a unique connection* ∇ *on the tangent bundle* TM *such that*

- $\mathcal{L}_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
- $\nabla_X Y - \nabla_Y X - [X, Y] = 0$

for all $X, Y, Z \in \mathfrak{X}(M)$. *This connection is called the Levi-Civita connection of* (M, g).

The first condition expresses that the Riemannian metric g is horizontal (or flat) for the Levi-Civita connection ∇, see Exercise 2.23, and is called the *flatness of metric condition.* Hence parallel transport preserves the Riemannian metric, and so the holonomy group $\mathrm{Hol}_x(M, g)$ with respect to the Levi-Civita connection will be a subgroup of the orthogonal group $\mathrm{O}(T_xM, g_x)$ for all $x \in M$. The second condition is called the *symmetry condition* of the Levi-Civita connection.

For $f \in \mathcal{F}(M)$ the *gradient* vector field $\nabla f \in \mathfrak{X}(M)$ on a Riemannian manifold (M, g) is defined by

$$g(\nabla f, X) = \mathrm{d}f(X) = \mathcal{L}_X(f)$$

for all $X \in \mathfrak{X}(M)$, so the Riemannian metric g is used to dualize $\mathrm{d}f \in \Omega^1(M)$ to $\nabla f \in \mathfrak{X}(M)$. Subsequently the element $\nabla^2 f = \nabla(\nabla f) \in \Omega^1(TM) = \Omega^1(M) \otimes \mathfrak{X}(M)$ can be dualized using the Riemannian metric g to an element $\mathrm{Hess} f \in \Omega^1(M) \otimes \Omega^1(M)$. Explicitly, since $\nabla g = 0$ we have

$$\mathrm{Hess} f(X, Y) := g(\nabla_X \nabla f, Y) = \mathcal{L}_X(g(\nabla f, Y)) - g(\nabla f, \nabla_X Y)$$

and so

$$\mathrm{Hess} f(X, Y) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathrm{d}f(\nabla_X Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Hence

$$\mathrm{Hess} f(X, Y) - \mathrm{Hess} f(Y, X) = -\mathrm{d}f(\nabla_X Y - \nabla_Y X - [X, Y])$$

and so the second condition in the theorem can be rewritten as

$$\mathrm{Hess} f(X, Y) = \mathrm{Hess} f(Y, X)$$

for all $X, Y \in \mathfrak{X}(M)$, which for this reason is called the *symmetry condition.* The symmetric bilinear form $\mathrm{Hess} f(X, Y) = g(\nabla^2 f(X), Y)$ on $\mathfrak{X}(M)$ is called the *Hessian* of the function $f \in \mathcal{F}(M)$.

After these initial remarks we now come to the (rather straightforward) proof of the Fundamental Theorem of Riemannian Geometry.

Proof. The flatness of metric and symmetry conditions in the above theorem amount to

$$\mathcal{L}_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

and

$$g(\nabla_X Y, Z) - g(\nabla_Y X, Z) - g([X, Y], Z) = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The trick for the proof is to consider the expression

$$\mathcal{L}_X g(Y, Z) + \mathcal{L}_Y g(X, Z) - \mathcal{L}_Z g(X, Y)$$

for $X, Y, Z \in \mathfrak{X}(M)$. Using the flat metric condition this can be rewritten as

$$g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X)$$

and becomes

$$2g(\nabla_X Y, Z) - g([X,Y], Z) + g([X,Z], Y) + g([Y,Z], X)$$

by the symmetry condition. Hence we get

$$\begin{aligned} 2g(\nabla_X Y, Z) = \mathcal{L}_X g(Y,Z) + \mathcal{L}_Y g(X,Z) - \mathcal{L}_Z g(X,Y) + \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, which gives a well defined formula for ∇. $\square$

Let us spell out the above formulas in local coordinates $(x^1, \cdots, x^m)$ on M with ∂_i the derivative and $\mathrm{d}x^i$ the differential for the coordinate x^i. The connection one forms ω_j^k and the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection in these local coordinates are defined by

$$\nabla(\partial_j) = \sum \omega_j^k \otimes \partial_k,\ \omega_j^k = \sum \Gamma_{ij}^k \mathrm{d}x^i,\ \nabla_{\partial_i}\partial_j = \sum \Gamma_{ij}^k \partial_k$$

and so with Riemannian metric $ds^2 = \sum g_{ij} dx^i dx^j$ we get

$$\sum \Gamma_{ij}^k g_{kl} = g(\nabla_{\partial_i}\partial_j, \partial_l) = \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\}/2$$

using the last formula in the above proof. Hence the Christoffel symbols are given by

$$\Gamma_{ij}^k = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}/2$$

with g^{ij} the inverse matrix of g_{ij}. This is just the formula already found in Theorem 1.6 in the setting of submanifolds of Euclidean space. In this setting the covariant derivative $\nabla_{\partial_i}\partial_j$ is just the orthogonal projection of the Euclidean derivative along ∂_i of the tangent field ∂_j onto the tangent bundle TM of the submanifold $M \subset \mathbb{R}^n$, as defined just before Theorem 1.6. Note that Γ_{ij}^k is symmetric under $i \leftrightarrow j$, but this is of course an immediate consequence of the symmetry condition for the Levi-Civita connection.

Definition 3.6. *For a Riemannian manifold (M, g) the Riemann curvature $R \in \Omega^2(\mathrm{End}(TM))$ is defined as the square of the Levi-Civita connection*

∇, just like in the case of a general vector bundle with connection, and so $R(X,Y) \in \Gamma(\mathrm{End}(TM))$ for all $X, Y \in \mathfrak{X}(M)$. By Theorem 2.16 this then amounts to the formula

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

It will be our goal to come grips with this admittedly abstract definition of Riemann curvature in more geometric terms. The next theorem gives some simple algebraic properties of the Riemann curvature, and is proved by just calculations.

Theorem 3.7. *The Riemann curvature tensor satisfies*

- $g(R(Y,X)Z,W) = -g(R(X,Y)Z,W)$
- $g(R(X,Y)W,Z) = -g(R(X,Y)Z,W)$
- $g(R(X,Y)Z,W) + g(R(Y,Z)X,W) + g(R(Z,X)Y,W) = 0$
- $g(R(Z,W)X,Y) = g(R(X,Y)Z,W)$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof. The first item is trivial, because by definition $R \in \Omega^2(\mathrm{End}(TM))$ is antisymmetric under interchange of first and second argument.

The second item that $R(X,Y) \in \Gamma(\mathrm{End}(TM))$ is antisymmetric with respect to g follows from the flatness of metric condition of the Levi-Civita coonection ∇. Indeed, we have by definition

$$g(R(X,Y)W,Z) = g(\nabla_X \nabla_Y W, Z) - g(\nabla_Y \nabla_X W, Z) - g(\nabla_{[X,Y]} W, Z)$$

and by flatness of metric we get

$$\begin{gathered} g(R(X,Y)W,Z) = \mathcal{L}_X g(\nabla_Y W, Z) - g(\nabla_Y W, \nabla_X Z) + \\ -\mathcal{L}_Y g(\nabla_X W, Z) + g(\nabla_X W, \nabla_Y Z) - g(\nabla_{[X,Y]} W, Z) \end{gathered}$$

and adding $g(R(X,Y)Z,W)$ to it yields

$$(\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{[X,Y]}) g(W,Z) = 0$$

using again the flatness of metric.

The third item, which is called the (first, as opposed to the second) Bianchi identity, is a consequence of the symmetry condition of the Levi-Civita connection. Indeed, addition of the three equations

$$\begin{aligned}R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\\ R(Y,Z)X &= \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X\\ R(Z,X)Y &= \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y\end{aligned}$$

and using the symmetry condition gives on the right hand side

$$\begin{gathered}\nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y\\ = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\end{gathered}$$

by again using the symmetry condition.

The fourth item is just a formal consequence of the other three items. Indeed, addition of the four Bianchi identities

$$\begin{aligned}g(R(X,Y)Z,W) + g(R(Y,Z)X,W) + g(R(Z,X)Y,W) &= 0\\ -g(R(X,Y)W,Z) - g(R(Y,W)X,Z) - g(R(W,X)Y,Z) &= 0\\ -g(R(Z,W)X,Y) - g(R(W,X)Z,Y) - g(R(X,Z)W,Y) &= 0\\ g(R(Z,W)Y,X) + g(R(W,Y)Z,X) + g(R(Y,Z)W,X) &= 0\end{aligned}$$

using the antisymmetry in the first and last two arguments gives that all terms on the left hand side cancel, except the terms $1 + 4 + 7 + 10$, which implies

$$2g(R(X,Y)Z,W) + 2g(R(Z,W)Y,X) = 0$$

and the desired formula follows. Note that this proof can recovered from this last equation, by remembering that the four Bianchi identities needed have all four W, Z, Y, X as their last argument. The first term of these Bianchi identities with an appropriate sign then follow, and by cyclic symmmetry in the first three arguments the complete set of Bianchi identities can be written down. $\square$

In local coordinates $(x^1, \cdots, x^m)$ we shall denote

$$R(\partial_i, \partial_j)\partial_k = \sum R^l_{ijk}\partial_l$$

and

$$R_{ijkh} = g(R(\partial_i, \partial_j)\partial_k, \partial_h) = \sum R_{ijk}^l g_{lh}$$

for the coefficients of the Riemann curvature. The Christoffel symbols were found to be

$$\Gamma_{ij}^k = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}/2$$

and since by Exercise 2.20 we have

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum \{\Gamma_{ir}^l \Gamma_{jk}^r - \Gamma_{jr}^l \Gamma_{ik}^r\}$$

we see that the Riemann curvature coefficients are explicit expressions in the Riemannian metric coefficients and their first and second order derivatives. In local coordinates the above theorem reads

$$R_{ijkh} = -R_{jikh},\ R_{ijkh} = -R_{ijhk},\ R_{ijkh} = R_{khij}$$
$$R_{ijkh} + R_{kijh} + R_{jkih} = 0$$

and these are the symmetry relations and the Bianchi identities.

Theorem 3.8. *If the Riemann curvature R of a Riemannian manifold (M, g) vanishes identically then there exist local coordinates $x^1, \cdots, x^m$ in which the Riemannian metric becomes locally Euclidean in the sense that $g(\partial_i, \partial_j) = \delta_{ij}$ or equivalently $ds^2 = \sum (dx^i)^2$.*

Proof. Let $x \in M$ be a base point, and pick an orthonormal basis e_i of $(T_x M, g_x)$. By Theorem 2.37 there exist flat vector fields $X_i \in \mathfrak{X}(U)$ on a suitable open neighborhood U of x in M and equal to e_i at x. By the flatness of metric condition for ∇ we get $g(X_i, X_j) = \delta_{ij}$ on U. By the symmetry condition for ∇ and the flatness of the X_i we get

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$$

for $i, j = 1, \cdots, m$. Since the vector fields X_i commute with each other the corresponding flows $t \mapsto \phi_{i,t}$ commute likewise. For $\epsilon > 0$ sufficiently small and $|x^i| < \epsilon$ the smooth map

$$(x^1, x^2 \cdots, x^m) \mapsto \phi_{1,x^1}(\phi_{2,x^2}(\cdots \phi_{m,x^m}(x) \cdots))$$

defines local coordinates around x. In these coordinates the vector fields X_i correspond to ∂_i and so in these coordinates the Riemannian metric becomes $ds^2 = \sum (dx^i)^2$. Hence (M, g) is a locally Euclidean manifold. $\square$

Exercise 3.9. *If $f \in \mathcal{F}(M)$ is a smooth function on a manifold M then $x \in M$ is called a critical point of f if the differential* $\mathrm{d}f \in \Omega^1(M)$ *vanishes at x, or equivalently if $\mathcal{L}_X(f)$ vanishes at x for all $X \in \mathfrak{X}(M)$. In this case the* Hessian $\mathrm{Hess}f$ *of f at the critical point x is the bilinear form on T_xM defined by taking for* $\mathrm{Hess}f(X,Y)$ *the value of $\mathcal{L}_X\mathcal{L}_Y(f)$ at the point x. Show that* $\mathrm{Hess}f$ *defines a symmetric bilinear form on T_xM which coincides with the previously defined Hessian $\nabla^2 f$ in case of a Riemannian manifold (M,g).*

Exercise 3.10. *Check from the definition of the Levi-Civita connection that the familiar formulas $\Gamma_{ij}^k = \sum\{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\}g^{lk}/2$ for the Christoffel symbols hold.*

It will be convenient and natural to denote

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

and so the Riemann curvature tensor R will be considered both an element of $\Omega^2(\mathrm{End}(TM))$, so a trilinear form on $\mathfrak{X}(M)$ with values in $\mathfrak{X}(M)$, or a quadrilinear form on $\mathfrak{X}(M)$ with values in $\mathcal{F}(M)$.

3.3 Sectional Curvature

We have seen that he Riemann curvature tensor of a Riemannian manifold (M,g) is a quadrilinear form

$$\mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathbb{R}, (X,Y,Z,W) \mapsto g(R(X,Y)Z,W)$$

which is antisymmetric under both $X \leftrightarrow Y$ and $Z \leftrightarrow W$ and symmetric under $(X,Y) \leftrightarrow (Z,W)$. Hence it can also be considered as a symmetric bilinear form on $\wedge^2\mathfrak{X}(M)$. We know from linear algebra that a symmetric bilinear form is uniquely determined by its associated quadratic form. This leads to the following definition.

For two vector fields $X,Y \in \mathfrak{X}(M)$ the expression

$$K(X,Y) = \frac{-R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

is well defined at those points $x \in M$ where X_x, Y_x are linearly independent, and is called the *sectional curvature* of the pair (X,Y). If U is the maximal open subset of M on which X and Y are pointwise linearly independent, then $K(X,Y) \in \mathcal{F}(U)$.

The denominator in the above formula at $x \in M$ is just the square of the Riemannian area of the parallelogram in T_xM spanned by X_x and Y_x. If we denote by $P = \mathbb{R}X + \mathbb{R}Y$ the plane (that is rank two) subbundle of TU spanned by X and Y then it follows that

$$K(P) = K(X, Y) \in \mathcal{F}(U)$$

is independent of the chosen frame X, Y for P.

Definition 3.11. *Let V be a finite dimensional real vector space. A form of curvature type on V is a quadrilinear form*

$$R : V^4 \to \mathbb{R}, (x, y, z, w) \mapsto R(x, y, z, w)$$

with the properties

$$R(y, x, z, w) = -R(x, y, z, w), \; R(x, y, w, z) = -R(x, y, z, w)$$
$$R(z, w, x, y) = R(x, y, z, w)$$
$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$$

for all $x, y, z, w \in V$. We call these relations the antisymmetry, the symmetry and the Bianchi identity respectively.

If $\dim V = m$ then the forms of curvature type on V form a vector space of dimension $m^2(m^2 - 1)/12$ (see the proof of Riemann's formula).

Theorem 3.12. *A form of curvature type R on V is completely determined by the knowledge of the values $R(x, y, x, y)$ for all $x, y \in V$.*

Proof. Suppose we know the values $R(x, y, x, y)$ for all $x, y \in V$. Using the symmetry identity we have

$$R(x + z, y, x + z, y) = R(x, y, x, y) + 2R(x, y, z, y) + R(z, y, z, y)$$

and so we also know the values $R(x, y, z, y)$ for all $x, y, z \in V$. Likewise the expression

$$R(x, y + w, z, y + w) - R(x, y, z, y) - R(x, w, z, w)$$

is known for all $x, y, z, w \in V$. This expression can be rewritten as

$$R(x, y, z, w) + R(x, w, z, y) = R(x, y, z, w) - R(y, z, x, w)$$

using the symmetry and antisymmetry identities.

Suppose that R' is yet another form of curvature type on V such that $R(x,y,x,y) = R'(x,y,x,y)$ for all $x, y \in V$. Then we conclude from the above that

$$R(x,y,z,w) - R(y,z,x,w) = R'(x,y,z,w) - R'(y,z,x,w)$$

for all $x, y, z, w \in V$, and so the expression $R(x,y,z,w) - R'(x,y,z,w)$ is invariant under cyclic permutations of x, y, z. In turn this implies that

$$3\{R(x,y,z,w) - R'(x,y,z,w)\} = 0$$

by the Bianchi identity. Hence $R = R'$ as forms of curvature type. $\square$

Corollary 3.13. *The Riemann curvature of a Riemannian manifold is completely determined by its sectional curvature.*

In the definition of the sectional curvature $K(X,Y)$ the minus sign in the numerator is chosen in order that the sectional curvature coincides with the Gauss curvature in the case of surfaces, as will be shown in the next theorem.

Theorem 3.14. *Let M be a smooth surface in $\mathbb{R}^3$ with local coordinates*

$$\mathbb{R}^2 \supset U \to M \subset \mathbb{R}^3, x = (x^1, x^2) \mapsto y(x) = (y^1(x), y^2(x), y^3(x))$$

as described in [11] and [19]. Then the Gauss curvature K of M is equal to the sectional curvature $K(\partial_1, \partial_2)$.

Proof. Let K denote the Gauss curvature. The Gauss equations

$$\begin{aligned}
EK &= (\Gamma^2_{11})_v - (\Gamma^2_{21})_u + \Gamma^2_{21}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^2_{21} \\
FK &= (\Gamma^2_{12})_v - (\Gamma^2_{22})_v + \Gamma^2_{21}\Gamma^1_{12} + \Gamma^2_{22}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^1_{22} - \Gamma^2_{12}\Gamma^2_{22} \\
FK &= (\Gamma^1_{21})_u - (\Gamma^1_{11})_v + \Gamma^1_{11}\Gamma^1_{21} + \Gamma^1_{12}\Gamma^2_{21} - \Gamma^1_{21}\Gamma^1_{11} - \Gamma^1_{22}\Gamma^2_{11} \\
GK &= (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^2_{12}
\end{aligned}$$

were derived as Theorem 4.5 of [11]. Note that in the second and the third equations two terms on the right hand side cancel. The classical notation E, F, G for the coefficients of the first fundamental form with coordinates (u, v) as used in [11] is related to our present notation by $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$ and $u = x^1, v = x^2$. We have slightly rewritten the Gauss equations from [11] using that $\Gamma^k_{ij} = \Gamma^k_{ji}$. The coefficients of the Riemann curvature form are given by

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \sum \{\Gamma^l_{ir}\Gamma^r_{jk} - \Gamma^l_{jr}\Gamma^r_{ik}\}$$

with indices i, j, k, l taken from $1, 2$. Hence the Gauss equations become

$$EK = R^2_{211},\ FK = R^2_{212},\ FK = R^1_{121},\ GK = R^1_{122}$$

which in turn implies that

$$R_{1212} = R^1_{121}F + R^2_{121}G = (F^2 - EG)K$$

and so the sectional curvature $K(\partial_1, \partial_2) := -R_{1212}/(EG - F^2)$ equals the Gauss curvature K. This legitimates the use of the same letter K for both sectional curvature and Gauss curvature. □

3.4 Ricci Curvature

Let R_x denote the Riemann curvature form on the tangent space T_xM at $x \in M$, so that $R_x(u, v) \in \mathrm{End}(T_xM)$ for all $u, v \in T_xM$. For $u, v \in T_xM$ the map

$$T_xM \to T_xM,\ e \mapsto R_x(e, u)v$$

is linear over $\mathbb{R}$, and its trace is denoted $\mathrm{Ric}_x(u, v)$ and is called the *Ricci form* at x of $u, v \in T_xM$. In other words, if $e_1, \cdots, e_m$ is an orthonormal basis of T_xM with respect to g_x then

$$\mathrm{Ric}_x(u, v) = \sum_{i=1}^m g_x(R_x(e_i, u)v, e_i) = -\sum_{i=1}^m R_x(u, e_i, v, e_i)$$

and so the symmetry relation for the first two and last two arguments of the Riemann curvature form $R_x(\cdot, \cdot, \cdot, \cdot) = g_x(R_x(\cdot, \cdot), \cdot, \cdot)$ implies that the Ricci form Ric_x is a symmetric bilinear form on T_xM. As a tensor the Ricci curvature Ric has the same type as the Riemannian metric g, namely both assign to a pair of vector fields a function.

In local coordinates $(x^1, \cdots, x^m)$ with local vector fields $\partial_i = \partial/\partial x^i$ and Riemann curvature coefficients R^l_{ijk} defined by $R(\partial_i, \partial_j)\partial_k = \sum R^l_{ijk}\partial_l$ the Ricci curvature coefficients become $\mathrm{Ric}_{ij} = \mathrm{Ric}(\partial_i, \partial_j) = \sum R^k_{kij}$.

By abuse of notation, the Ricci form Ric_x on T_xM defines a symmetric linear operator Ric_x on (T_xM, g_x) by

$$\mathrm{Ric}_x(u, v) = g_x(\mathrm{Ric}_x u, v)$$

for $u, v \in T_xM$. Its trace is a real number and is denoted

$$S(x) = \operatorname{tr}\mathrm{Ric}_x = \sum_{j=1}^{m} \mathrm{Ric}_x(e_j, e_j)$$

and called the *scalar curvature* of M.

Exercise 3.15. *Show that for a surface the Ricci form is equal to the Gauss curvature times the first fundamental form. Hence the Ricci operator is equal to the multiplication operator by the Gauss curvature and the scalar curvature is equal to twice the Gauss curvature. Hint: Use formulas from the proof of Theorem 3.14.*

Exercise 3.16. *Let $e_1, \cdots, e_m$ be an orthonormal basis of the Euclidean space (T_xM, g_x) and let $P_{ij} = \mathbb{R}e_i + \mathbb{R}e_j$ be the plane spanned by e_i and e_j for $i < j$. Show that $S(x) = \sum_{i<j} 2K_x(P_{ij})$ with K the sectional curvature and S the scalar curvature.*

Exercise 3.17. *Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields on M. How do the Riemann curvature $R(X, Y) \in \Gamma(\mathrm{End}(TM))$, the sectional curvature $K(X, Y) \in \mathcal{F}(M)$ and the Ricci curvature $\mathrm{Ric}(X, Y) \in \mathcal{F}(M)$ change under a constant scaling $g \mapsto \lambda g$ for $\lambda > 0$ of the Riemannian metric g.*

Exercise 3.18. *For $m = 3$ we have $m^2(m^2-1)/12 = m(m+1)/2 = 6$ and so the number of parameters of Riemann curvature forms and Ricci curvature forms coincides. Show that for a Riemannian manifold of dimension $m = 3$ the Riemann curvature is determined by the Ricci curvature.*

Exercise 3.19. *A connected Riemannian manifold (M, g) is called Einstein manifold if $\mathrm{Ric} = \lambda g$ for some scalar λ. Show that for $m \geq 3$ the condition $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathcal{F}(M)$ implies that λ is in fact a constant. Hint: Use the Bianchi identity.*

Remark 3.20. *The Ricci form was introduced by the Italian mathematician Gregorio Ricci-Curbasto in 1887 [20]. Ricci introduced the curvature form* *named after him with the hope that the integral curves of the directions given by the diagonalization of the Ricci curvature form with respect to the Riemann metric form would yield nice curves in the manifold, analogous to the lines of curvature for the second fundamental form of a smooth surface in* $\mathbb{R}^3$. *But for a general Riemannian manifold* (M, g) *Ricci was quite disappointed to find no good geometric interpretation of the Ricci curvature lines. However, in the twentieth century the Ricci form turned out to become an utmost interesting concept.*

At the beginning of this century the Ricci form became a crucial tool in the theory of general relativity of Einstein from 1915. The manifolds are four dimensional, representing spacetime, and the Riemannian metric g *gets replaced by a Lorenztian metric* g *of signature* $(+---)$ *for time and space respectively. If we denote in local coordinates the Einstein tensor by*

$$G_{ij} = \mathrm{Ric}_{ij} - \tfrac{1}{2} S g_{ij}$$

then the Einstein field equations are given by

$$G_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij}$$

with Λ *the cosmological constant,* T_{ij} *the stress-energy tensor,* G *Newton's gravitational constant and* c *the speed of light in vacuum. The Einstein field equations can be interpreted as a set of equations, which describe how the matter-energy distribution over spacetime (the right hand side) determines the curvature of spacetime (the left hand side). For the development of his theory Einstein came to grips with the intricacies of Riemannian geometry with the help of his friend and classmate Marcel Grossmann, who was a mathematician with keen knowledge of geometry.*

At the end of this century Ricci flow became a central object of study. The Ricci flow equation for a connected Riemannian manifold (M, g) *is given by the geometric evolution equation*

$$\partial_t g = -2\mathrm{Ric}(g)$$

(the factor 2 *is irrelevant and just for historic reasons and we write* $\mathrm{Ric}(g)$ *just to emphasize the dependence of* Ric *on* g*) and in case* M *is compact also the normalized geometric evolution equation*

$$\partial_t g = \frac{2}{m} S_0 g - 2\mathrm{Ric}(g)$$

for a time dependent Riemannian metric g (starting at time $t = 0$ with the given metric g on M). They were introduced by Richard Hamiton in 1981. Here S_0 is the mean (average) of the scalar curvature over M. For example, for a compact connected surface the normalized Ricci flow equation becomes

$$\partial_t g = (S_0 - S)g = 2(K_0 - K)g$$

with mean Gauss curvature $K_0 = 1, 0$ or -1 if the genus of the surface is $0, 1$ or ≥ 2 respectively, using Exercise 3.15. This equation has the property of smearing out the Gauss curvature, and makes g converge to the constant curvature metric on the surface. In dimension three the Ricci flow was deeply studied by Grigori Perelman in his proof of the Poincaré Conjecture from 2003, which says that a connected and simply connected smooth threefold is homemorphic to the standard round sphere $\mathbb{S}^3$.

3.5 Geodesics

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and Riemann curvature $R = \nabla^2$.

Definition 3.21. *A smooth curve $(a, b) \ni t \mapsto \gamma(t) \in M$ is called a* geodesic *if the tangent field $\dot\gamma(t) = d\gamma(d/dt) \in T_{\gamma(t)M}$ is parallel along γ, that is $\nabla_{\dot\gamma(t)}(\dot\gamma(t)) = 0$ for all $t \in (a, b)$.*

Geodesics are traversed wth constant speed, because

$$\frac{d}{dt}\{g(\dot\gamma(t), \dot\gamma(t)\} = 2g(\nabla_{\dot\gamma(t)}(\dot\gamma(t)), \dot\gamma(t)) = 0$$

by flatness of metric for ∇. In local coordinates $x^1, \cdots, x^m$ the Levi-Civita connection becomes

$$\nabla_{\partial_i}(\partial_j) = \sum \Gamma_{ij}^k \partial_k$$

with Christoffel symbols by Exercise 3.10 given by

$$\Gamma_{ij}^k = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}/2$$

the geodesic equations take the form

$$\ddot x^k + \sum \Gamma_{ij}^k \dot x^i \dot x^j = 0$$

just like we found for submanifolds of $\mathbb{R}^n$ in the first chapter. Indeed we have $\dot\gamma = \sum \dot x^i \partial_i$ and so $\nabla_{\dot\gamma}(\dot\gamma) = \sum\{\ddot x^k + \sum \Gamma_{ij}^k \dot x^i \dot x^j\}\partial_k$.

Exercise 3.22. *Check that in local coordinates $x^1, \cdots, x^m$ for geodesics $t \mapsto x^i(t)$ the expression $\sum g_{ij}\dot{x}^i\dot{x}^j$ is constant, and so geodesics are traversed with constant speed.*

The geodesic equations are second order nonlinear differential equations. The existence and uniqueness theorem for such equations gives that for fixed initial point $x \in M$ there exists $\delta, \epsilon > 0$ such that for all $v \in T_xM$ with $g_x(v,v) < \delta^2$ there exist a geodesic

$$(-\epsilon, \epsilon) \ni t \mapsto \gamma_v(t) \in M$$

with initial position $\gamma_v(0) = x$ and initial velocity $\dot{\gamma}_v(0) = v$.

The geodesic equations are invariant under rescaling of time $t \mapsto ct$ with $0 < c \leq 1$ a constant. Therefore it follows from the chain rule that for all $0 \leq c \leq 1$ we have

$$\gamma_{cv}(t) = \gamma_v(ct)$$

whenever both sides are defined. If we take $\rho = \delta\epsilon/2$ and $g_x(v,v) < \rho^2$ then $\gamma_v(t) = \gamma_{2v/\epsilon}(\epsilon t/2)$ is defined for $|t| < 2$. Hence the *exponential map*

$$\exp_x : \{v \in T_xM; g_x(v,v) < \rho^2\} \to M, \ \exp_x(v) = \gamma_v(1)$$

is well defined and smooth.

The derivative $d(\exp_x)_0$ of the exponential map at the origin of T_xM is equal to the identity, as linear map from T_xM to T_xM. Indeed, we have

$$d(\exp_x)_0(v) = \frac{d}{dt}\{\exp_x(tv)\}|_{t=0} = \frac{d}{dt}\{\gamma_{tv}(1)\}|_{t=0} = \frac{d}{dt}\{\gamma_v(t)\}|_{t=0} = v$$

which proves the claim. Hence it follows from the inverse function theorem that after a possibly shrinking of $\rho > 0$ the exponential map $\exp_x$ becomes a diffeomorphism from $\{v \in T_xM; g_x(v,v) < \rho^2\}$ onto an open neighborhood of x in M. The image under $\exp_x$ of orthonormal coordinates $x^1, \cdots, x^m$ on the Euclidean space (T_xM, g_x) gives *geodesic normal coordinates* around the point x in the Riemannian manifold M. The geodesics that begin at x are called *radial geodesics*.

Theorem 3.23 (Gauss's lemma)**.** *Given $x \in M$ a fixed base point, let S be the unit sphere in (T_xM, g_x) and let*

$$F : [0, \rho) \times S \to U, \ F(r, v) = \exp_x(rv)$$

be so called geodesic polar coordinates on a suitable neighborhood U of x in M. Then the pull back F^*g of the Riemannian metric g under F has the form

$$F^*g = dr^2 + r^2 h(r)$$

with $h(r)$ a Riemannian metric on S depending on $r \in [0, \rho)$ and with $h(0)$ the standard Euclidean metric on S, or equivalently with

$$dr^2 + r^2 h(0) = (dx^1)^2 + \cdots + (dx^m)^2$$

equal to the standard Euclidean metric on T_xM in orthonormal coordinates $x^1, \cdots, x^m$. In particular, in geodesic normal coordinates around x the radial geodesics intersect the spheres $S_r = \{v \in T_xM; g_x(v, v) = r^2\}$ orthogonally for all $r \in (0, \rho)$.

Proof. Take $y^1 = r = \sqrt{(x^1)^2 + \cdots + (x^m)^2}$ and local coordinates $y^2, \cdots, y^m$ on S. The fact that $y^1 = t$ while $y^2, \cdots, y^m$ remain constant give unit speed geodesics implies (using the geodesic equations $\ddot{y}^k + \sum \Gamma^k_{ij} \dot{y}^i \dot{y}^j = 0$) that $\Gamma^k_{11} \equiv 0$ in these coordinates for $k = 1, \cdots, m$. Using the familiar expression for the Christoffel symbols $\Gamma^k_{ij} = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}/2$ of the metric $F^*g = \sum g_{ij} dy^i dy^j$ this implies that

$$\partial_1 \{g_{1k}(y^1, \cdots, y^m)\} \equiv 0$$

for $k = 1, \cdots, m$. Note that this argument already occurred in Exercise 1.9.

Since $d(\exp_x)_0$ is the identity operator we find that

$$g_{1k}(0, y^2, \cdots, y^m) \equiv 0$$

for $k = 1, \cdots, m$ and hence we conclude

$$g_{1k}(y^1, y^2, \cdots, y^m) \equiv \delta_{1k}$$

for $k = 1, \cdots, m$. But this says that in the Riemannian metric $\sum g_{ij} dy^i dy^j$ the term $(dy^1)^2 = dr^2$ appears with coefficient 1 while all terms $dy^1 dy^k$ have coefficient 0 for $k = 2, \cdots, m$.

Finally the coefficients of $dy^k dy^l$ remain smooth after division by r^2 for $k, l = 2, \cdots, m$ because we are working with polar coordinates. This proves the formula $F^*g = dr^2 + r^2 h(r)$ with $dr^2 + r^2 h(0)$ equal to the Euclidean metric on T_xM. $\square$

In case M is a surface, we can carry the calculation a little further, which presumably was done by Riemann in order to come to the next theorem.

Lemma 3.24. *In the notation of Gauss's lemma consider the case of a surface M with geodesic normal coordinates u, v and geodesic polar coordinates r, θ related by $u = r\cos\theta, v = r\sin\theta$. Then the Riemannian metric in these coordinates has the form*

$$ds^2 = dr^2 + r^2 h(r,\theta)d\theta^2 = du^2 + dv^2 + H(u,v)(udv - vdu)^2$$

with $h(0,\theta) \equiv 1$ and $H(u,v) = (h(r,\theta) - 1)/r^2$ a smooth function around the origin, with $3H(0,0)$ equal to minus the Gauss curvature at the given point.

Proof. Since $r = \sqrt{u^2 + v^2}$ and $\theta = \arctan(v/u)$ we get

$$dr = \frac{udu + vdv}{\sqrt{u^2 + v^2}},\ d\theta = \frac{d(v/u)}{1 + (v/u)^2} = \frac{udv - vdu}{u^2 + v^2}$$

and hence

$$dr^2 = \frac{u^2du^2 + 2uvdudv + v^2dv^2}{u^2 + v^2} = du^2 + dv^2 - \frac{(udv - vdu)^2}{u^2 + v^2}$$

which in turn implies that $H(u,v) = (h(r,\theta) - 1)/r^2$. In the classical notation $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ this gives

$$E = 1 + H(u,v)v^2,\ F = -H(u,v)uv,\ G = 1 + H(u,v)u^2$$

and since E, F, G are smooth functions of u, v we conclude that $H(u,v)$ is smooth as well. See however the remark after this lemma.

Using the familiar formulas (with $x^1 = u, x^2 = v$)

$$\Gamma_{ij}^k = \sum\{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\}g^{lk}/2,\ R_{ijkh} = \sum R_{ijk}^l g_{lh}$$
$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum\{\Gamma_{ir}^l \Gamma_{jk}^r - \Gamma_{jr}^l \Gamma_{ik}^r\}$$

it follows that

$$R_{1212}(0,0) = 3H(0,0)$$

by straightforward calculation. Hence the lemma follows from Theorem 3.14 where it is shown that $R_{1212}(0,0)$ is equal to the value of $-K(EG - F^2)$ at the origin. $\square$

Remark 3.25. *In the proof we have claimed that around the origin H is a smooth function of u, v because E, F, G are smooth functions of u, v. This is obviously true outside the origin but I do not quite understand how the smoothness of $H(u, v)$ at the origin follows. However, it is clearly true in the analytic context by the theorem of Hartogs, and so we might have to work in the more restricted analytic context.*

Theorem 3.26 (Riemann's formula). *If $x^1, \cdots, x^m$ are geodesic normal coordinates around a point $x \in M$ of an analytic Riemannian manifold (M, g) then the Riemannian metric $(\exp_x)^* g = \sum g_{ij} dx^i dx^j$ has the form*

$$\sum (dx^i)^2 + \frac{1}{12} \sum R_{ijkh}(0)(x^i dx^j - x^j dx^i)(x^k dx^h - x^h dx^k) + O(r^3)$$

for $r^2 = \sum (x^i)^2 \downarrow 0$. Here the value of $g(R(\partial_i, \partial_j)\partial_k, \partial_h)$ at the origin 0 of $T_x M$ is denoted by $R_{ijkh}(0)$.

Proof. Let P^d be the vector space of degree d homogenous polynomials in $\mathbb{R}[x^1, \cdots, x^m]$, and so $\dim P^d = m(m+1)\cdots(m+d-1)/d!$ for all $d \in \mathbb{N}$. Let D^d be the vector space of differentials on $\mathbb{R}^m$ with coefficients from P^d, and so $\dim D^d = m^2(m+1)\cdots(m+d-1)/d!$ for all $d \in \mathbb{N}$. Let $E = \sum x^i \partial_i$ be the Euler operator. The contraction operator

$$\mathrm{i}_E : D^d \twoheadrightarrow P^{d+1}$$

with the Euler field is clearly surjective, and so the kernel K^d has dimension equal to $d(m-1)m\cdots(m+d-1)/(d+1)!$ for all $d \in \mathbb{N}$.

In particular, the vector space K^1 has dimension $m(m-1)/2$ with basis $(x^i dx^j - x^j dx^i)$ for $1 \le i < j \le m$ and $\dim K^2 = (m-1)m(m+1)/3$. The multiplication map

$$P^1 \otimes K^1 \twoheadrightarrow K^2$$

is a linear surjection with kernel of dimension $m(m-1)(m-2)/6$ spanned by the independent vectors

$$x^k \otimes (x^i dx^j - x^j dx^i) + x^i \otimes (x^j dx^k - x^k dx^j) + x^j \otimes (x^k dx^i - x^i dx^k)$$

for $1 \le i < j < k \le m$. It follows that any differential in K^2 can be uniquely written in the form

$$\sum c_{ijk} x^k (x^i dx^j - x^j dx^i)$$

(sum over *all* indices $1 \leq i, j, k \leq m$) under the restrictions

$$c_{ijk} + c_{jik} = 0,\ c_{ijk} + c_{kij} + c_{jki} = 0$$

for all i, j, k.

The vector space Q^2 of (symmetric) quadratic differentials on $\mathbb{R}^m$ with degree 2 homogeneous coefficients has basis $x^i x^j dx^k dx^h$ for $1 \leq i \leq j \leq m$ and $1 \leq k \leq h \leq m$. Let C^2 be the linear subspace of Q^2 spanned by the products

$$(x^i dx^j - x^j dx^i)(x^k dx^h - x^h dx^k)$$

for all $1 \leq i, j, k, h \leq m$. Any quadratic differential in C^2 can be uniquely written in the form

$$\sum C_{ijkh}(x^i dx^j - x^j dx^i)(x^k dx^h - x^h dx^k)$$

under the restrictions

$$C_{ijkh} = -C_{jikh},\ C_{ijkh} = -C_{ijhk},\ C_{ijkh} = C_{khij}$$
$$C_{ijkh} + C_{kijh} + C_{jkih} = 0$$

for all $1 \leq i, j, k, h \leq m$. Indeed the antisymmetry condition $C_{ijkh} = -C_{ijhk}$ allows to rewrite the above element from C^2 as

$$\sum 2C_{ijkh} x^k (x^i dx^j - x^j dx^i) \otimes dx^h$$

and so the C_{ijkh} are unique under the given restrictions by the argument from the previous paragraph. Since $\dim K^2 = (m-1)m(m+1)/3$ this implies that

$$\dim C^2 = (m-1)m^2(m+1)/12$$

because of the two symmetry conditions $C_{ijkh} = -C_{ijhk}$ and $C_{ijkh} = C_{khij}$.

By Gauss's lemma we can expand the Riemannian metric $(\exp_x)^* g$ in geodesic normal coordinates as

$$\sum (dx^i)^2 + \sum C_{ijkh}(x^i dx^j - x^j dx^i)(x^k dx^h - x^h dx^k) + O(r^3)$$

for unique real constants C_{ijkh} with the above constraints. Restriction to the linear plane $\mathbb{R}e_i + \mathbb{R}e_j$ for $i < j$ gives $12C_{ijij} = R_{ijij}(0)$ by the previous lemma, and since this is valid in any system of geodesic normal coordinates around the given point x Riemann's formula follows from Theorem 3.12 $\square$

While in geodesic normal coordinates around the point $x \in M$ the first order deviation of the Riemannian metric $(\exp_x)^* g$ from the flat Euclidean metric $(dx^1)^2 + \cdots + (dx^m)^2$ is equal to zero the coefficients $R_{ijkh}(0)$ of the Riemann curvature tensor describe the next second order deviation. This is how Riemann in his famous Habilitation lecture from 1854 introduced and motivated his curvature concept for a space of *arbitrary* dimension $m \geq 2$ with an *arbitrary* Riemannian metric as a natural generalization of the Gauss curvature for surfaces in $\mathbb{R}^3$: The curvature of space describes the deviation from flat space, and in order to control this deviation it is necessary and sufficient to know in each point the curvature for any pair of $m(m-1)/2$ planar directions $\mathbb{R}e_i + \mathbb{R}e_j$ for $1 \leq i < j \leq m$.

The Theorema Egregium on the intrinsic nature of the Gauss curvature came in Riemann's approach for free. Indeed the very concept of geodesics is intrinsic and hence also geodesic normal coordinates have intrinsic meaning. For a more extensive discussion on Riemann' s original work with an English translation of his Habilitation lecture we refer to Spivak [23].

The next result is a direct consequence of Riemann's formula.

Corollary 3.27. *In geodesic normal coordinates around $x \in M$ the Riemannian volume element dV has an expansion*

$$dV = \{1 - \frac{1}{6} \sum \mathrm{Ric}_{ij}(0) x^i x^j + O(r^3)\} dx^1 \cdots dx^m$$

for $r \downarrow 0$ with $\mathrm{Ric}_{ij}(0) = \sum R^k_{kij}(0) = \sum R_{kijh}(0)\delta^{hk}$ the Ricci form at the origin.

Proof. Recall that in local coordinates $x^1, \cdots, x^m$ the volume element has the form $dV = \det(g_{ij})^{1/2} dx^1 \cdots dx^m$. In geodesic normal coordinates around $x \in M$ Riemann's formula can be rewritten as

$$g_{kh} = \delta_{kh} + \frac{1}{3} \sum R_{ikjh}(0) x^i x^j + O(r^3)$$

which in turn implies that

$$\det(g_{kh})^{1/2} = \{1 - \frac{1}{3} \sum R_{kijh}(0) \delta^{hk} x^i x^j + O(r^3)\}^{1/2}$$

and the desired formula follows. $\square$

In the case of surfaces with $m = 2$ the Ricci form is just the Gauss curvature times the first fundamental form, and indeed for positive Gauss curvature the area element on the surface is locally decreasing relative to the Euclidean area form. In the case of surfaces the next corollary was found by Diquet in 1848.

Corollary 3.28 (Diquet's formula)**.** *The volume $V(x, r)$ of a small radial geodesic ball around $x \in M$ with radius $r > 0$ is given by*

$$V(x, r) = \omega_m r^m \{1 - \frac{1}{6(m+2)} S(x) r^2 + O(r^3)\}$$

for $r \downarrow 0$ with $\omega_m = \pi^{m/2}/\Gamma(m/2+1)$ the volume of the unit ball in $\mathbb{R}^m$ and $S(x) = \mathrm{tr}(\mathrm{Ric}_x)$ the scalar curvature at x.

For example, for the unit sphere $\mathbb{S}^2$ with $K \equiv 1$ a geodesic disc with radius $r \in [0, \pi]$ has area $A(r) = 2\pi(1 - \cos r)$ as shown by Archimedes. Hence $A(r) = \pi r^2\{1 - r^2/12 + O(r^4)\}$ for $r \downarrow 0$ in accordance with Diquet's formula.

Exercise 3.29. *Prove Diquet's formula using the previous corollary with geodesic normal coordinates around $x \in M$ in which the Ricci operator is diagonalized.*

Exercise 3.30. *Let (M, g) be a Riemannian manifold with isometry group $\mathrm{Isom}(M, g)$ and stabilizer subgroup $\mathrm{Isom}_x(M, g)$ for some $x \in M$. Show that the homomorphism*

$$\mathrm{Isom}_x(M, g) \to \mathrm{O}(T_x M, g_x),\ f \mapsto T_x f$$

is a monomorphism. Hint: Use geodesic normal coordinates.

Exercise 3.31. *Show that $\mathbb{E}^m, \mathbb{S}^m, \mathbb{H}^m$ are model spaces for geometries of constant sectional curvature equal to $0, +1, -1$ respectively, by showing that the isometry groups have point stabilizer equal to the full orthogonal group of the tangent space at the given point.*

3.6 The Second Fundamental Form

Suppose M is a smooth submanifold of dimension m of a smooth manifold $\tilde{M}$ of dimension n, and so the codimension of M in $\tilde{M}$ is equal to $n - m$.

Suppose $\tilde{g}$ is a Riemannian metric on $\tilde{M}$. By restriction we get a Riemannian metric g on M, and so $(M,g) \hookrightarrow (\tilde{M},\tilde{g})$ is an isometric embedding. The goal of this section will be to understand for $x \in M$ and P a plane in T_xM the relation between on the one hand the sectional curvature $K(P)$ measured inside the submanifold (M,g) and on the other hand the sectional curvature $\tilde{K}(P)$ measured inside the ambient space $(\tilde{M},\tilde{g})$. The motivation comes from the classical example of a surface S in flat Euclidean space, which is the situation dealt with by Gauss.

For each point $x \in M$ the inner product $\tilde{g}_x$ on $T_x\tilde{M}$ splits $T_x\tilde{M}$ into an orthogonal direct sum

$$T_x\tilde{M} = T_xM \oplus N_xM$$

with the normal space N_xM the orthogonal complement of T_xM in $T_x\tilde{M}$. Varying x along M gives a direct sum decomposition of vector bundles

$$T\tilde{M}|_M = TM \oplus NM$$

with $NM = \sqcup_x N_xM$ the normal bundle of M in $(\tilde{M},\tilde{g})$. The tangent bundle TM and the normal bundle NM are vector bundles on M of rank m and n respectively. Let $p : T\tilde{M}|_M \to TM$ be the orthogonal projection morphism. Any vector field $X \in \mathfrak{X}(M)$ extends, at least locally, to a vector field $\tilde{X} \in \mathfrak{X}(\tilde{M})$. The Levi-Civita connection on TM will be denoted by ∇. The Levi-Civita connection on both $T\tilde{M}$ and its restriction $T\tilde{M}|_M$ to M will be denoted by $\tilde{\nabla}$. The latter connection is the pullback of $\tilde{\nabla}$ on $T\tilde{M}$ under the inclusion $M \hookrightarrow \tilde{M}$. Hence the restriction of $\tilde{\nabla}_{\tilde{X}}(\tilde{Y})$ to M is equal to $\tilde{\nabla}_X(Y)$ as section of $T\tilde{M}|_M$.

Proposition 3.32. *For $X,Y \in \mathfrak{X}(M)$ we have $\nabla_X(Y) = p(\tilde{\nabla}_X(Y))$.*

Proof. We check that $p(\tilde{\nabla}_X(Y))$ satisfies the two defining properties of the covariant derivative $\nabla_X(Y)$. For $X,Y,Z \in \mathfrak{X}(M)$ we in

$$\begin{gathered}\mathcal{L}_X g(Y,Z) = (\mathcal{L}_{\tilde{X}}\tilde{g}(\tilde{Y},\tilde{Z}))|_M = \tilde{g}(\tilde{\nabla}_{\tilde{X}}(\tilde{Y}),\tilde{Z})|_M + \tilde{g}(\tilde{Y},\tilde{\nabla}_{\tilde{X}}(\tilde{Z}))|_M = \\ \tilde{g}(\tilde{\nabla}_X(Y),Z) + \tilde{g}(Y,\tilde{\nabla}_X(Z)) = g(p(\tilde{\nabla}_X(Y)),Z) + g(Y,p(\tilde{\nabla}_X(Z)))\end{gathered}$$

and so the flatness of metric condition of $\nabla_X(Y)$ follows. Likewise the symmetry condition

$$p(\tilde{\nabla}_X(Y)) - p(\tilde{\nabla}_Y(X)) = p((\tilde{\nabla}_{\tilde{X}}(\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}}(\tilde{X}))|_M) = p([\tilde{X},\tilde{Y}]|_M) = [X,Y]$$

of $\nabla_X(Y)$ follows. $\square$

For $X, Y \in \mathfrak{X}(M)$ let $H(X,Y) \in \Gamma(NM)$ denote the normal component of $\tilde{\nabla}_X Y$ and so

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y)$$

in the above notation.

Proposition 3.33. *We have $H(X,Y) = H(Y,X)$ for all $X, Y \in \mathfrak{X}(M)$. Moreover, the value $H(X,Y)_x \in N_xM$ of $H(X,Y)$ at $x \in M$ depends only on the values $X_x, Y_x \in T_xM$ of the vector fields X, Y at $x \in M$. In other words*

$$H : TM \otimes TM \to NM$$

is a symmetric vector bundle morphism, called the second fundamental form *of M in $\tilde{M}$.*

Proof. The symmetry is obvious since

$$H(X,Y) - H(Y,X) = (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) - (\nabla_X Y - \nabla_Y X) = [X,Y] - [X,Y] = 0$$

for all $X, Y \in \mathfrak{X}(M)$. For fixed $Y \in \mathfrak{X}(M)$ the values of $\tilde{\nabla}_X Y$ and $\nabla_X Y$ at $x \in M$ depend only on the value X_x of X at x. Hence the value $H(X,Y)_x$ at $x \in M$ depends for fixed Y only on X_x. By symmetry we conclude that the value $H(X,Y)_x \in N_xM$ depends only on the values $X_x, Y_x \in T_xM$. $\square$

The Riemannian submanifold $(M, g) \hookrightarrow (\tilde{M}, \tilde{g})$ is called *totally geodesic* if each geodesic in M remains geodesic in $\tilde{M}$. The second fundamental form measures the failure of M to be totally geodesic in $\tilde{M}$.

Proposition 3.34. *The submanifold M is totally geodesic in $\tilde{M}$ if and only if the second fundamental form H vanishes identically.*

Proof. If $t \mapsto \gamma(t)$ is a smooth path in M then $\tilde{\nabla}_{\dot{\gamma}}(\dot{\gamma}) = \nabla_{\dot{\gamma}}(\dot{\gamma}) + H(\dot{\gamma}, \dot{\gamma})$. Suppose $t \mapsto \gamma(t)$ is a geodesic in M, that is $\nabla_{\dot{\gamma}}(\dot{\gamma}) = 0$. Then $t \mapsto \gamma(t)$ remains a geodesic in $\tilde{M}$ if and only if $\tilde{\nabla}_{\dot{\gamma}}(\dot{\gamma}) = 0$ or equivalently if and only if $H(\dot{\gamma}, \dot{\gamma}) = 0$. Hence all geodesics in M remain geodesics in $\tilde{M}$ if and only if $H \equiv 0$. $\square$

The curvature R of M can be expressed in terms of the curvature $\tilde{R}$ of $\tilde{M}$ with help of the second fundamental form.

Theorem 3.35. *We have the so called* Gauss equation

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \\ \tilde{g}(H(X,Z),H(Y,W)) - \tilde{g}(H(X,W),H(Y,Z))$$

for all $X,Y,Z,W \in \mathfrak{X}(M)$.

Proof. Since $\tilde{\nabla}_Y Z = \nabla_Y Z + H(Y,Z)$ we get

$$\tilde{\nabla}_X \tilde{\nabla}_Y Z = \nabla_X \nabla_Y Z + H(X, \nabla_Y Z) + \tilde{\nabla}_X H(Y,Z)$$

and hence also

$$\tilde{g}(\tilde{\nabla}_X \tilde{\nabla}_Y Z, W) = g(\nabla_X \nabla_Y Z, W) + \tilde{g}(\tilde{\nabla}_X H(Y,Z), W)$$

for all $X,Y,Z,W \in \mathfrak{X}(M)$. Differentiation of the identity $\tilde{g}(H(Y,Z),W) = 0$ in the direction of X gives

$$\tilde{g}(\tilde{\nabla}_X H(Y,Z), W) + \tilde{g}(H(Y,Z), \tilde{\nabla}_X W) = 0$$

and since $\tilde{\nabla}_X W = \nabla_X W + H(X,W)$ the second term in this expression can also be rewritten as $\tilde{g}(H(Y,Z),H(X,W))$. Hence we find

$$\tilde{g}(\tilde{\nabla}_X \tilde{\nabla}_Y Z, W) = g(\nabla_X \nabla_Y Z, W) - \tilde{g}(H(Y,Z),H(X,W))$$

for all $X,Y,Z,W \in \mathfrak{X}(M)$. If we subtract this equality with X and Y interchanged and use that $\tilde{g}(\tilde{\nabla}_{[X,Y]} Z, W) = g(\nabla_{[X,Y]} Z, W)$ then the Gauss equation follows. □

For $X,Y \in \mathfrak{X}(M)$ two independent vector fields the sectional curvature $K(P)$ in M of the plane subbundle $P = \mathbb{R}X + \mathbb{R}Y$ of TM was defined by

$$K(P) = \frac{-R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

and likewise for $\tilde{K}$ in $\tilde{M}$. Hence the Gauss equation can be rewritten in the form

$$K(P) = \tilde{K}(P) + \frac{\tilde{g}(H(X,X),H(Y,Y)) - \tilde{g}(H(X,Y),H(X,Y))}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

with $P = \mathbb{R}X + \mathbb{R}Y$. In particular, in case $(\tilde{M}, \tilde{g})$ is the flat Euclidean space $\mathbb{R}^n$ we get

$$K(P) = \frac{(H(X,X), H(Y,Y)) - (H(X,Y), H(X,Y))}{(X,X)(Y,Y) - (X,Y)^2}$$

with $(\cdot,\cdot)$ the scalar product on $\mathbb{R}^n$.

In the special case that M is an oriented hypersurface in $\mathbb{R}^{m+1}$ the unit normal map $n : M \to \mathbb{S}^m$ is a section in the normal bundle $NM \to M$ and classically called the Gauss map. If we write $H(X,Y) = h(X,Y)n$ then the second fundamental form

$$h : TM \otimes TM \to \mathbb{R}$$

becomes scalar valued, and so h is a scalar tensor field on M of the same type as the Riemannian metric g. The m solutions $k_1, \cdots, k_m$ of the equation

$$\det(h - kg) = 0$$

at a given point are called the principal curvatures of the oriented hypersurface $M \hookrightarrow \mathbb{R}^{m+1}$ at the given point, in accordance with the discussion in Section 1.4. In local coordinates

$$x = (x^1, \cdots, x^m) \mapsto y(x) = (y^1(x), \cdots, y^{m+1}(x))$$

for the hypersurface M of $\mathbb{R}^{m+1}$ we have

$$\partial_i \partial_j y(x) = \sum \Gamma_{ij}^k(x) \partial_k + h_{ij}(x) n(x)$$

with $\Gamma_{ij}^k = \sum \{\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\} g^{lk}/2$ the Christoffel symbols and h_{ij} the coefficients of the second fundamental form. Hence

$$h_{ij}(x) = (n(x), \partial_i \partial_j y(x)) = -(\partial_i n(x), \partial_j y(x))$$

since $(n(x), \partial_j y(x)) = 0$ for all j. Since $(n(x), n(x)) = 1$ we obtain

$$\partial_i n(x) = -\sum h_{ij}(x) g^{jk}(x) \partial_k y(x)$$

which are the so called *Weingarten equations*.

Exercise 3.36. *Consider a smooth hypersurface M in $\mathbb{R}^{m+1}$ around the origin $0 \in \mathbb{R}^{m+1}$ given as the graph $x^{m+1} = f(x^1, \cdots, x^m)$ of a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ with $f(0) = 0$ and $\partial_i f(0) = 0$ for $i = 1, \cdots, m$. Let $n(0) = (0, \cdots, 0, 1)$ be the unit normal to M at the origin. Show that the second fundamental form h of M at the origin is equal to the Hessian of f at the origin in the sense that $h(\partial_i, \partial_j) = \partial_i \partial_j f(0)$.*

Exercise 3.37. *Let M be a smooth hypersurface in $\mathbb{R}^{m+1}$ and for $x \in M$ let V_x be a three dimensional linear subspace of $\mathbb{R}^{m+1}$ containing the normal line $N_x M$. Show that locally near x the intersection*

$$S = M \cap \{x + V_x\}$$

is a smooth surface with Gauss curvature at x equal to the sectional curvature $K_x(P_x)$ with $P_x = T_x M \cap V_x$.

Exercise 3.38. *Show that for a hypersurface M in Euclidean space $\mathbb{R}^{m+1}$ the principal curvatures k_i need not be intrinsic quantities, whereas their pairwise products $k_i k_j$ for $i \neq j$ are intrinsic quantities.*

3.7 The Laplace–Beltrami operator

Let (M, g) be a connected oriented Riemannian manifold with Riemannian volume form

$$d\mathrm{vol} = (\det g_{ij}(x))^{1/2} dx^1 \wedge \cdots \wedge dx^m$$

in local coordinates $x = (x^1, \cdots, x^m)$. For $X \in \mathfrak{X}(M)$ a smooth vector field on M the divergence $\mathrm{div}(X) \in \mathcal{F}(M)$ is the unique smooth function characterized by

$$\mathcal{L}_X(d\mathrm{vol}) = \mathrm{div}(X)\, d\mathrm{vol}$$

with $\mathcal{L}_X(\omega) = d/dt\{\phi_t^* \omega\}_{t=0}$ for $\omega \in \Omega^m(M)$ and ϕ_t the (local) flow of the vector field X. For $f \in \mathcal{F}(M)$ a smooth function the gradient vector field $\nabla f = \mathrm{grad} f \in \mathfrak{X}(M)$ was defined as the dual (via the Riemannian metric) of the exterior derivative $\mathrm{d}f \in \Omega^1(M)$ of the function f.

Definition 3.39. *The Laplace–Beltrami operator Δ is the linear second order differential operator on $\mathcal{F}(M)$ defined by $\Delta f = \mathrm{div}(\mathrm{grad} f)$.*

The vector space $\mathcal{F}_c(M)$ of smooth real valued functions with compact support on M carries a natural inner product given by

$$(f_1, f_2) = \int_M f_1 f_2 \, d\mathrm{vol}$$

for $f_1, f_2 \in \mathcal{F}_c(M)$. It is easy to check that

$$\int_M g(\mathrm{grad} f, X) \, d\mathrm{vol} + \int_M f \mathrm{div}(X) \, d\mathrm{vol} = 0$$

by Stokes' theorem. In turn this implies that

$$(f_1, \Delta f_2) = -\int_M g(\mathrm{grad} f_1, \mathrm{grad} f_2) \, d\mathrm{vol} = (\Delta f_1, f_2)$$

for all $f_1, f_2 \in \mathcal{F}_c(M)$, and so Δ is a symmetric linear operator on $\mathcal{F}_c(M)$. The Laplace–Beltrami operator is a nonpositive symmetric operator in the sense that $(\Delta f, f) \leq 0$ for all $f \in \mathcal{F}_c(M)$. For this reason some people call the nonnegative operator $-\Delta \geq 0$ the Laplace–Beltrami operator of (M, g).

Exercise 3.40. *Show that in local coordinates $x = (x^1, \cdots, x^m)$ the Laplace–Beltrami operator takes the form $\Delta f = \sum |g(x)|^{-1/2} \partial_i (g^{ij}(x) |g(x)|^{1/2} \partial_j f)$ with $|g(x)| = \det g_{ij}(x)$.*

Exercise 3.41. *Show that the Laplace–Beltrami operator is also defined in case (M, g) is unoriented.*

Exercise 3.42. *Show that the Laplace–Beltrami operator Δ is the unique second order linear differential operator on $\mathcal{F}(M)$ with leading symbol the Riemannian metric (that is in local coordinates $\Delta = \sum g^{ij}(x) \partial_i \partial_j + \cdots$ with $\cdots$ a first order linear differential operator), which is symmetric on $\mathcal{F}_c(M)$ (that is $(\Delta f_1, f_2) = (f_1, \Delta f_2)$ for all $f_1, f_2 \in \mathcal{F}_c(M)$) and with $\Delta 1 = 0$. In the possibly unoriented case we write*

$$(f_1, f_2) = \int_M f_1 f_2 \, dV$$

with $dV = |d\mathrm{vol}|$ the associated smooth positive Riemannian density.

Suppose now that (M, g) is a compact connected Riemannian manifold, and consider the eigenvalue problem

$$\Delta f + \lambda f = 0$$

for $f \in \mathcal{F}(M)$ a nonzero eigenfunction and $\lambda \in \mathbb{R}$ the corresponding eigenvalue. It is known that the sequence of eigenvalues

$$\Lambda = \{\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\}$$

of $-\Delta$, counted with multiplicity and called the *spectrum* of the Riemannian manifold, increases towards ∞. In particular, all eigenspaces of $-\Delta$ are finite dimensional. The spectral counting function $N(\lambda)$ is defined for $\lambda > 0$ as the number of eigenvalues less than λ.

Theorem 3.43 (Weyl's law)**.** *The growth for $\lambda \to \infty$ of the spectral counting function $N(\lambda)$ is given by*

$$N(\lambda) = (2\pi)^{-m}\omega_m \mathrm{Vol}(M, g)\lambda^{m/2} + O(\lambda^{(m-1)/2})$$

with $\omega_m = \pi^{m/2}/\Gamma(1 + m/2)$ the volume of the unit ball in $\mathbb{R}^m$.

This result was obtained by Hermann Weyl in 1911 in the case of a bounded domain in $\mathbb{R}^m$ (for $m = 2, 3$), and had been conjectured by Arnold Sommerfeld and Hendrik Lorentz in 1910. In the case of compact connected Riemannian manifolds the result is due to Minakshisundaram and Pleijel in 1948 [16], with the error term due to Hörmander in 1968 [13]. Minakshisundaram and Pleijel obtained their result from the (so called heat trace) asymptotic expansion

$$\sum_{k\geq 0} e^{-\lambda_k t} \;\sim\; (4\pi t)^{-m/2} \sum_{i\geq 0} a_i t^i \quad (t > 0)$$

for $t \downarrow 0$ [17],[2]. The scalars a_i are of the form $\int_M u_i dV$ with $u_i : M \to \mathbb{R}$ certain Riemannian invariants. For example, we have $u_0 = 1$ – and so $a_0 = \mathrm{vol}(M, g)$ – and $u_1 = S/6$ with S the scalar curvature. But for large i the functions u_i become rather obscure algebraic expressions in terms of Riemann curvatures. The next beautiful result was obtained independently by Chazarain [4] and Duistermaat–Guillemin [7].

Theorem 3.44. *For (M, g) a compact connected Riemannian manifold the (so called wave trace) series*

$$\sum_{k \geq 0} \cos(\sqrt{\lambda_k} t)$$

defines a tempered distribution on $\mathbb{R}$, which for $t \geq 0$ is smooth outside the set of the lengths of the closed geodesics on (M, g).

The set of the lengths of all closed geodesics on (M, g) is sometimes called the *length spectrum*. Closed geodesics need not necessarily be simple, so if l is contained in the length spectrum $\mathcal{L}$ then $\mathbb{N}l$ is also contained in $\mathcal{L}$.

For special compact connected Riemannian manifolds with a sufficiently transitive group of isometries the spectrum can be compute explicitly and tends to be highly degenerate. The most typical example is the round sphere $\mathbb{S}^m$ with spectrum equal to $\{k(k + m - 1); k = 0, 1, 2, \cdots\}$ and multiplicities $m_k = n_k - n_{k-2}$ and $n_k = (k + m)(k + m - 1) \cdots (k + 1)/m!$ the dimension of the vector space of degree k homogeneous polynomials in $m + 1$ variables. In particular $\mathbb{S}^2$ has spectrum $\{k(k + 1); k = 0, 1, 2, \cdots\}$ with multiplicities $2k + 1$, while $\mathbb{S}^3$ has spectrum $\{k(k + 2); k = 0, 1, 2, \cdots\}$ with multiplicities $(k + 1)^2$. However, for a general compact connected Riemannian manifold both the spectrum and the length spectrum are transcendental concepts, with no hope of explicit computation. Nevertheless, the above truly remarkable theorem gives an intimate relation between the spectrum and the length spectrum for any compact connected Riemannian manifold (M, g).

3.8 Historical remarks

The founders of the classical differential geometry of surfaces in $\mathbb{R}^3$ are Gaspard Monge (1746-1818), who introduced the principal curvatures k_1, k_2 of such a surface, and more importantly Carl Friedrich Gauss (1777-1855), whose main contribution was the insight that the product $K = k_1 k_2$ of the principal curvatures, named Gauss curvature, remains invariant under rigid bending of the surface. Gauss was so excited by this result, that he named it the Theorema Egregium, the splendid theorem. In the year 1827 Gauss wrote his papers in Latin.

Bernard Riemann (1826-1866) was a student at Göttingen, and might even have learned the Theorema Egregium from the master himself? On June 10, 1854 he delivered his Habilitationsvortrag with the title "Über die

Hypothesen welche der Geometrie zu Grunde liegen". Gauss had chosen this topic out of three possible subjects proposed by Riemann. It seems that Riemann prepared his lecture in just a couple of months. There was some time pressure, because of the declining health of Gauss. Riemann studied a space of *arbitrary* dimension $m \geq 2$ as being given locally by coordinates $x^1, \cdots, x^m$ with a metric $\sum g_{ij} dx^i dx^j$ with $m(m+1)/2$ *arbitrary* smooth functions $g_{ij} = g_{ji}$ but with the matrix g_{ij} positive definite. So one should view the space intrinsically, and not embedded locally in $\mathbb{R}^n$ and performing calculations extrinsically in the ambient Euclidean space. In such a space length of curves can be measured and so geodesics can be defined, and subsequently Riemann introduced the sectional curvatures at each point by working in geodesic normal coordinates. The lecture of Riemann exceeded all expectations of Gauss and greatly surprised him. Returning to the faculty meeting, he spoke with the greatest praise and rare enthusiasm to Wilhelm Weber (1804-1891) about the depth of the thoughts that Riemann had presented.

Riemann was appointed professor in Göttingen in 1857, and two years later he succeeded Peter Gustav Lejeune Dirichlet (1805-1859) on the chair of mathematics, formerly held by Gauss until 1855.

In 1858 the Italian mathematician Enrico Betti (1823-1892) and Francesco Brioschi (1824-1897) visited Riemann in Göttingen. These contacts were renewed when Riemann visited Betti in Pisa from 1863 till 1865. At that time Riemann suffered from tuberculosis and sought relief in the gentle mediterranean climate. Brioschi with his student Eugenio Beltrami (1835-1900) and more notably Betti with his students Luigi Bianchi (1856-1928) and Gregorio Ricci-Curbasto (1853-1925) developed the ideas of Riemann on differential geometry further.

The final ideas on parallel transport coming from a compatible connection were developed by Tullio Levi-Civita (1873-1941), who was a student of Ricci. Levi-Civita was appointed on the chair of Rational Mechanics in 1898 in Padua until 1918, when he moved to Rome on the chair of Higher Analysis. Einstein had an extensive correspondence with Levi-Civita. Regarding Levi-Civita's new work, Einstein wrote "I admire the elegance of your method of computation. It must be nice to ride through these fields upon the horse of true mathematics, while the like of us have to make our way laboriously on foot". Being of Jewish descent Levi-Civita was deprived in 1938 due to the racial laws from the fascist Italian government of his professorship and his membership of all scientific societies. Isolated from the scientific community he died in his apartment in Rome in 1941. Later on, when asked what he liked best about Italy, Einstein said "spaghetti and Levi-Civita".

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