Radboud University Nijmegen

Master Thesis

Hyperbolic Reflection Groups and the Leech Lattice

Date: June 2011
Author: Hester Pieters, 0537381
Supervisor: Prof. Dr. G.J. Heckman
Second Reader: Dr. B.D. Souvignier

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Introduction

The Leech lattice is the unique 24 dimensional unimodular even lattice without roots. It was discovered by Leech in 1965. In his paper from 1985 on the Leech lattice ([3]), Richard Borcherds gave new more conceptual proofs then those known before of the existence and uniqueness of the Leech lattice and of the fact that it has covering radius $\sqrt{2}$. He also gave a uniform proof of the correctness of the "holy constructions" of the Leech lattice which are described in [6], Chapter 24. An important goal of this thesis is to present these proofs. They depend on the theory of hyperbolic geometry and hyperbolic reflection groups. The first two chapters give an introduction to these and contain all that will be needed.
We will furthermore elaborate on the deep holes of the Leech lattice in relation to the classification of all 24 dimensional unimodular even lattices, the Niemeier lattices. In more detail, the thesis is organized as follows.

In Chapter 1 three models for the hyperbolic space H^n are described together with some elementary properties of this space.

In Chapter 2 we describe hyperbolic reflection groups. We show that a discrete group W generated by reflections is in fact generated by the reflections in the hyperplanes that bound a convex polyhedron $\tilde{D}$, a fundamental domain for the group W. Hence W has a Coxeter representation. We then describe criteria to determine whether the polyhedron $\tilde{D}$ is bounded or has finite volume.

Chapter 3 deals with lattices. We describe the root lattices that will later play an important role in the classification of Niemeier lattices. We also describe how a lattice determines a reflection group. Finally, we describe the theory of gluing and define the covering radius and holes of a lattice.

In Chapter 4 we describe Vinberg's algorithm. This is an algorithm to determine the fundamental domain $\tilde{D} \subset H^n$ of a discrete reflection group W. We emphasize the case where the reflection group W is the reflection group of an even hyperbolic lattice.

In Chapter 5 we turn to even unimodular lattices. First we describe what is known about the classification of such lattices. Then we describe the relation between primitive norm zero vector of a lattice L of type $\mathrm{II}_{8n+1,1}$ (so L is even unimodular with signature $(8n+1,1)$) and lattices of type II_{8n}. After that, we treat the examples $\mathrm{II}_{9,1}$ and $\mathrm{II}_{17,1}$ applying the theory from chapters 2 and 4.
We then discuss the classification of the Niemeier lattices, presenting Boris Venkov's proof of this classification. We end the chapter with some results on Niemeier lattices that we will need in the last chapter.

Finally, in Chapter 6 we discuss the Leech lattice. We present Borcherds' proofs of the existence and uniqueness of the Leech lattice and the fact that it has covering radius $\sqrt{2}$. We end the chapter with a section that discusses the deep holes of the Leech lattice.

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Chapter 1

Hyperbolic Space

In this chapter we describe some standard knowledge on hyperbolic geometry. It can all be found in [5] and [16]. We will first describe three models for the hyperbolic space. The description of these models is taken from [16]. All these models are differentiable manifolds with a Riemannian metric. Each model is defined on a different subset of $\mathbb{R}^{n,1}$ ($n \geq 2$), i.e. $\mathbb{R}^{n+1}$ with symmetric bilinear form $(x, y) = x_1y_1 + ... + x_ny_n - x_{n+1}y_{n+1}$. This subset is called the *domain* of the model. The first is the *ball model* (see [16] §4.5). Here the domain is

$$B^n = \{(x_1, ..., x_n, 0) \mid x_1^2 + ... + x_n^2 < 1\},$$

and the Riemannian metric is

$$ds^2 = 4\frac{dx_1^2 + ... + dx_n^2}{(1 - x_1^2 - ... - x_n^2)^2}.$$

The associated volume form is

$$dV = 2^n \frac{dx_1...dx_n}{(1 - |x|^2)^n}.$$

The geodesics are the circles orthogonal to the boundary sphere $\partial B^n = S^{n-1} = \{(x_1, ..., x_n, 0) \mid x_1^2 + ... + x_n^2 = 1\}$, the points at "infinity".
The second model is the *upper half-space model* (see [16] §4.6). Here the domain is

$$U^n = \{(x_1, ..., x_n, 0) \mid x_n > 0\},$$

and the Riemannian metric is

$$ds^2 = \frac{dx_1^2 + ... + dx_n^2}{x_n^2}.$$

The associated volume form is

$$dV = \frac{dx_1...dx_n}{x_n^n}.$$

The geodesics are half circles and half lines orthogonal to the boundary $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^{n,1} \mid x_n = x_{n+1} = 0\}$. The set of points at infinity is $\partial U^n = \mathbb{R}^{n-1} \cup \{\infty\}$.
The last model is the *hyperboloid model* (see [16] Chapter 3). Here the domain is

$$H^n = \{(x_1, ..., x_n, x_{n+1}) \mid x_1^2 + ... + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\},$$

and the Riemannian metric is

$$ds^2 = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2.$$

The associated volume form is

$$dV = \frac{dx_1...dx_n}{(1 + x_1^2 + ... + x_n^2)^{\frac{1}{2}}}.$$

The geodesics are intersections of two-dimensional vector subspaces of $\mathbb{R}^{n,1}$ with the hyperboloid. If we identify each point in H^n with the (unique) one-dimensional vector subspace of $\mathbb{R}^{n,1}$ that contains that point (interpret the hyperboloid model as lying in projective space) then it is clear that we can identify the boundary points with those lines that lie in the boundary of the cone $V_+ = \{x \in \mathbb{R}^{n,1} \mid (x,x) < 0 \text{ and } x_{n+1} > 0\}$.

To show that the three models are equivalent we have to describe isometries between them. The isometry $f : H^n \to B^n$ is the central projection from the point $(0, ..., 0, -1)$ (see [16] §4.5):

$$f : (x_1, ..., x_{n+1}) \mapsto (\frac{x_1}{1 + x_{n+1}}, ..., \frac{x_n}{1 + x_{n+1}}, 0).$$

The isometry $g : B^n \to U^n$ (see [16] §4.6) is given by:

$$g : x = (x_1, ..., x_n, 0) \mapsto \frac{1}{||x - e_n||^2}(2x_1, ..., 2x_{n-1}, 1 - \sum_{i=1}^{n} x_i^2, 0), \text{ (with } e_n \text{ the standard basis vector).}$$

Here $g = \rho \circ \sigma$, where σ is the reflection of $\mathbb{R}^n$ in the sphere $S(e_n, \sqrt{2})$ with center e_n and radius $\sqrt{2}$ and ρ is the reflection of $\mathbb{R}^n$ in the boundary of U^n. This isometry can be extended to the boundary. The same formula can be used except for $x = e_n$, for this set $g(e_n) = \infty$. These three models and the maps between them can also be found in [5] and [9]. The isometries do indeed induce the different metrics as they are defined above (see for example section 7 in [5]).

From now on in this chapter we use H^n as a model for the hyperbolic space. The Lorentz group is defined to be $O(n, 1) = \{A \in Gl(n + 1, \mathbb{R}) \mid (Ax, Ay) = (x, y) \text{ for all } x, y \in \mathbb{R}^{n,1}\}$. For $A \in O(n, 1)$ we have either $AH^n = H^n$ or $AH^n = -H^n$. Hence $O_+(n, 1) = \{A \in O(n, 1) \mid A(H^n) = H^n\}$ has index 2 in $O(n, 1)$. Elements of $O_+(n, 1)$ are sometimes called orthochronous transformations. The restriction of an element of $O_+(n, 1)$ to H^n is called a *linear isometry* of H^n. A *Riemannian isometry* $f : H^n \to H^n$ of H^n is a diffeomorphism of H^n that preserves the Riemannian metric. These two notions turn out to be equivalent (see [5] Theorem 10.2).
The group $O_+(n, 1)$ acts transitively on H^n. To show this consider two points $x, y \in H^n$. Then the orthogonal reflection in the linear hyperplane perpendicular to $x - y$ sends x to y and clearly is an element of $O_+(n, 1)$. The connected group $SO_+(n, 1) = \{A \in O_+(n, 1) \mid \det(A) = 1\}$ acts also transitively on H^n since if $x, y \in H^n$ and $z \in H^n$ is an arbitrary fixed point then there are orthogonal reflections $A, B \in O_+(n, 1)$ for which $Ax = z$ and $Bz = y$. Reflection matrices have determinant -1 so $BA \in SO_+(n, 1)$ and this element sends x to y.

A linear subspace of $\mathbb{R}^{n,1}$ is said to be hyperbolic (elliptic, parabolic) if the bilinear form induced on it is nondegenerate and indefinite (positive, degenerate). We remark that the orthogonal complement of a hyperbolic (elliptic, parabolic) subspace is an elliptic (hyperbolic,parabolic) subspace of complementary dimension (see [16] §3.1, note that in this book elliptic, hyperbolic and parabolic subspaces are called space-like, light-like and time-like).

Definition 1. *A hyperbolic m-plane of H^n is the nonempty intersection of H^n with a hyperbolic subspace of $\mathbb{R}^{n,1}$ of dimension $m + 1$. A hyperbolic $(n - 1)$-plane of H^n is called a hyperplane of H^n. A hyperbolic 1-plane of H^n is called a line or geodesic of H^n.*

Lemma 1. *Let $p, q \in H^n$. Then $(p, q) = -\cosh(d(p, q))$. (Where the distance $d(p, q)$ is the length of the geodesic from p to q.)*

Proof. sketch (following page 83 of [5]): $d(p, q)$ is invariant under isometries so p and q can be first translated to a standard position. In fact, they can be translated to the plane spanned by e_n and e_{n+1} as follows: Define a_n to be the unit tangent vector at p in the direction of the geodesic from p to q and define $a_{n+1} = p$. By the Gram-Schmidt process the set of orthonormal vectors $\{a_n, a_{n+1}\}$ can be extended to an orthonormal basis $\{a_1, ..., a_n, a_{n+1}\}$ for $\mathbb{R}^{n,1}$ (so:$(a_i, a_j) = (e_i, e_j)$.) The matrix A with columns $a_1, ..., a_{n+1}$ is a linear isometry of H^n. Its inverse A^{-1} takes p to e_{n+1} and the plane spanned by p and q to the plane P spanned by e_n and e_{n+1}. The intersection of P with H^n is (one branch of) the standard hyperbola and is the unique geodesic passing through $A^{-1}(p)$ and $A^{-1}(q)$ (see below). Now $A^{-1}(p) = (0, ..., 0, 1)$ and $d(p, q) = d(A^{-1}(p), A^{-1}(q))$. It can therefore be assumed that $A^{-1}(q) = (0, ..., 0, \sinh(d(p,q)), \cosh(d(p,q)))$. Hence:

$$\begin{aligned}(p,q) &= (A^{-1}(p), A^{-1}(q)) \\ &= ((0,...,0,1),(0,...,0,\sinh(d(p,q)),\cosh(d(p,q)))) \\ &= -\cosh(d(p,q))\end{aligned}$$

□

It follows from the proof above that the group of isometries acts transitively on point pairs $\{p,q\}$ in H^n with the same distance. Indeed, there is an isometry that sends $\{p,q\}$ to $\{e_{n+1}, \tilde{q}\}$ with $\tilde{q}$ a point in the plane spanned by e_n and e_{n+1} such that $d(p,q) = d(e_{n+1}, \tilde{q})$. With a rotation that leaves e_{n+1} fixed $\tilde{q}$ can be mapped to any another point in this plane at the same distance from e_{n+1}. Reversing this argument we can now send these two points to two arbitrary points $\hat{p}, \hat{q}$ in H^n with $d(p,q) = d(\hat{p}, \hat{q})$. Thus H^n is *two-point homogeneous* for the group $O_+(n,1)$ of isometries of H^n.

Geodesics

Let $p, q \in H^n$. Then the geodesic between p and q is indeed the intersection of the two-dimensional vector subspace of $\mathbb{R}^{n,1}$ containing p and q with H^n:
Since H^n is *two-point homogeneous* for its group of isometries $O_+(n,1)$ we can always assume that $p = (0, ..., 0, p_n, p_{n+1})$ and $q = (0, ..., 0, q_n, q_{n+1})$. Now let $\gamma : [a,b] \to H^n$ be a path from p to q and suppose that γ lies not in the plane spanned by the standard basis vectors e_n and e_{n+1}. Denote by λ the projection of γ onto the $x_n - x_{n+1}$ plane. Then $L(\gamma) = \int_a^b ||\gamma'(t)||_{\gamma(t)} dt = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt > \int_a^b \sqrt{g_{\lambda(t)}(\lambda'(t), \lambda'(t))} dt = L(\lambda)$, since $g = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2$ and thus $g_{\gamma(t)-\lambda(t)}(\gamma'(t) - \lambda'(t), \gamma'(t) - \lambda'(t)) \geq 0$ for all $t \in [a,b]$. Hence the geodesic between p and q lies in the $x_n - x_{n+1}$ plane and is therefore equal to the intersection of this plane with H^n. It can be shown that under the isometries f and g as defined above these geodesics are mapped to the geodesics as they were defined in the other two models of the hyperbolic space.

For more about hyperbolic geometry we refer to [5] and [16].

Chapter 2

Hyperbolic Reflection Groups

Let $V = \mathbb{R}^{n,1}$ and let H^n be the hyperboloid model of the hyperbolic space. The negative norm vectors in V consist of two connected components. Denote by V_+ the connected component that contains $H^n = \{x = (x_1, ..., x_{n+1}) \in V \mid x^2 = -1 \text{ and } x_{n+1} > 0\}$ (so V_+ is the interior of the forward light cone). If $x \in V$ has positive norm then the (linear) hyperplane perpendicular to x gives a hyperplane of H^n and reflection in this hyperplane is an isometry of H^n. If the norm of $x \in V$ is negative then, after multiplication by an element of $\mathbb{R}^\times$, x corresponds to a point in H^n. If the norm of $x \neq 0$ is zero then x corresponds to a point at infinity.
Let $\alpha \in V$ be such that $(\alpha, \alpha) > 0$. Then the reflection $r_\alpha \in O_+(n,1)$ in the hyperplane $H_\alpha := \{x \in V \mid (x, \alpha) = 0\}$ orthogonal to α is given by $r_\alpha : x \mapsto x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha$ for all $x \in V$. Let $W < O_+(n,1)$ be a discrete group generated by reflections. Because W is a discrete group the set of mirrors $\{H_\alpha\}_{r_\alpha \in W}$ is locally finite. They divide H^n into convex regions.

In general, if U is a linear hyperplane of V with nonzero intersection with V_+ then we write $\tilde{U} = U \cap H^n$ for the corresponding hyperbolic hyperplane. Denote by U^+ and U^- the two open half-spaces in V that are bounded by U and let $\tilde{U}^+$ and $\tilde{U}^-$ be the corresponding open half-spaces in H^n bounded by $\tilde{U}$. Let $\tilde{C}$ be a nonempty intersection of such half-spaces, that is

$$\tilde{C} = \bigcap_{i \in I} \tilde{U}_i^-.$$

Then the closure of $\tilde{C}$, denoted by $\tilde{D}$, is a *convex polyhedron* if $\{\tilde{U}_i\}_{i \in I}$ is locally finite. It is possible that the hyperplanes $\tilde{U}_i$ accumulate towards some point in ∂H^n. We will always assume that no $\tilde{U}_i^-$ contains the intersection of the remaining half-spaces. In this case the half-spaces $\tilde{U}_i^-$ are uniquely determined by the polyhedron $\tilde{D}$. For more background on convex sets and polyhedra see [1] and [2].

Now we return to the discrete reflection group $W < O_+(n,1)$. The locally finite set of mirrors $\tilde{H}_\alpha = \{x \in V \mid (x, \alpha) = 0\} \cap H^n$ divides H^n into convex polyhedra. It is shown below that W acts transitively on these polyhedra and each of them is a fundamental domain for the action of W.
Pick an arbitrary convex polyhedron $\tilde{D}$, i.e. $\tilde{D}$ is the closure of a connected component $\tilde{C} \subset H^n \setminus \bigcup_{r_\alpha \in W} \tilde{H}_\alpha$. Denote by D and C the connected components of $V \setminus \bigcup_{r_\alpha \in W} H_\alpha$ that contain $\tilde{D}$ and $\tilde{C}$, respectively. So D is a convex cone and $\tilde{D} = D \cap H^n$ (see Figure 2.1).

We will first show that $\tilde{D}$ is a fundamental domain for the action of the group W' generated by the reflections in the hyperplanes bounding $\tilde{D}$, i.e. $W' =< r_\alpha \mid \tilde{H}_\alpha \cap \tilde{D} \neq \emptyset >$. After that we will show that $W' = W$, thus W is generated by the reflections in the hyperplanes bounding $\tilde{D}$.

Lemma 2. *Let $p \in H^n$. Then $W'p \cap \tilde{D} \neq \emptyset$.*

Proof. Let $p_+ \in \tilde{C}$ and choose $q \in W'p$ for which $|q - p_+|^2 \leq |wq - p_+|^2$ for all $w \in W'$. Now

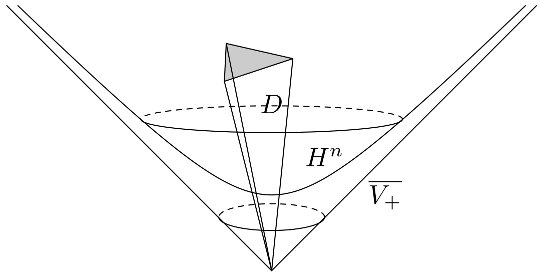

Figure 2.1: Sketch of cone D in V_+.

suppose $q \notin \tilde{D}$. Then there exists an α_i such that $q \notin \tilde{H}^-_{\alpha_i}$ and thus $(\alpha_i, q) > 0$. So

$$\begin{aligned}(r_{\alpha_i}(q), p_+) &= (q - 2(\alpha_i, q)\alpha_i, p_+) \\ &= (q, p_+) - 2(\alpha_i, q)(\alpha_i, p_+) \\ &> (q, p_+),\end{aligned}$$

since $(\alpha_i, p_+) < 0$. Hence

$$\begin{aligned}|r_{\alpha_i}(q) - p_+|^2 &= r_{\alpha_i}(q)^2 - 2(r_{\alpha_i}(q), p_+) + p_+^2 \\ &= q^2 - 2(r_{\alpha_i}(q), p_+) + p_+^2 \\ &< q^2 - 2(q, p_+) + p_+^2 \\ &= |q - p_+|^2.\end{aligned}$$

This contradicts the fact that q was chosen such that $|q - p_+|^2 \leq |wq - p_+|^2$ for all $w \in W'$. Thus $q \in \tilde{D}$ and $W'p \cap \tilde{D} \neq \emptyset$. □

So each W'-orbit meets $\tilde{D}$ in at least one point. By Theorem 5.13 in [12] it now follows that it meets each orbit in exactly one point. Hence $\tilde{D}$ is a fundamental domain for the action of W'.

Lemma 3. *Let $r_\alpha \in W$. Then there is a $w \in W'$ such that $r_{w\alpha} \in W'$.*

Proof. Let $p \in \tilde{H}_\alpha$ be a general point, i.e. $p \in \tilde{H}_\alpha \setminus \bigcup_{\{\beta \neq \alpha | r_\beta \in W\}} \tilde{H}_\beta$. It follows from Lemma 2 that there is a $w \in W'$ such that $wp \in \tilde{D}$. Then $(wp, w\alpha) = (p, \alpha) = 0$. So $wp \in \tilde{H}_{w\alpha} \cap \tilde{D}$. Thus $\tilde{H}_{w\alpha}$ is a hyperplane bounding $\tilde{D}$ and therefore $r_{w\alpha} \in W'$. □

Thus any generator $r_\alpha \in W$ can be written as a product of reflections in W', in the notation above $r_\alpha = w^{-1} r_{w\alpha} w$ with $w, r_{w\alpha} \in W'$.

The results in the remainder of this chapter are all from [19] and [20], also see [18]. Let $\alpha_i \in V, i \in I$, be vectors such that $(\alpha_i, \alpha_i) = 1$ and $\tilde{D} = \bigcap_{i \in I} \overline{\tilde{H}^-_{\alpha_i}}$ (where it is again assumed that no $\tilde{H}^-_{\alpha_i}$ contains the intersection of the other half-planes). The Gram matrix of the vector system $\{\alpha_i\}$ will be called the Gram matrix of the polyhedron $\tilde{D}$. It contains the following geometric information (see [20] §1.1.3):

1. The hyperplanes $\tilde{H}_{\alpha_i}$ and $\tilde{H}_{\alpha_j}$ intersect if and only if $|(\alpha_i, \alpha_j)| < 1$ and in this case the angle ϕ_{ij} between them is determined by the formula

$$\cos(\phi_{ij}) = -(\alpha_i, \alpha_j).$$

Since the set of hyperplanes $\{\tilde{H}_{\alpha_i}\}$ is locally finite these angles are dihedral, say $\phi_{ij} = \frac{\pi}{m_{ij}}$. In particular, in the case that $\tilde{D}$ is a fundamental domain for the action of a discrete group W generated by the reflections in the hyperplanes H_{α_i} bounding $\tilde{D}$ the group W has a Coxeter representation $< r_{\alpha_i} \mid (r_{\alpha_i} r_{\alpha_j})^{m_{ij}} = 1 >$, where $m_{ii} = 1$ and $m_{ij} = \infty$ if $\tilde{H}_{\alpha_i}$ and $\tilde{H}_{\alpha_j}$ are disjoint (in this case there is no relation between r_{α_i} and r_{α_j}).

2. The hyperplanes $\tilde{H}_{\alpha_i}$ and $\tilde{H}_{\alpha_j}$ do not intersect if and only if $|(\alpha_i, \alpha_j)| \geq 1$. Note that $(\alpha_i, \alpha_j) \geq 1$ is impossible since then $\tilde{H}^-_{\alpha_i} \subset \tilde{H}^-_{\alpha_j}$ or $\tilde{H}^-_{\alpha_j} \subset \tilde{H}^-_{\alpha_i}$ which contradicts the assumption that no half space contains the intersection of the others. So $(\alpha_i, \alpha_j) \leq -1$. If $(\alpha_i, \alpha_j) = -1$ then $\tilde{H}_{\alpha_i}$ and $\tilde{H}_{\alpha_j}$ meet at infinity and we say that the hyperplanes are *parallel*. Otherwise, they are *ultraparallel* and there is a unique geodesic from one to the other and orthogonal to both. The length of this geodesic is the distance $d(\tilde{H}_{\alpha_i}, \tilde{H}_{\alpha_j})$ between the hyperplanes and is determined by the formula
$$\cosh(d(\tilde{H}_{\alpha_i}, \tilde{H}_{\alpha_j})) = -(\alpha_i, \alpha_j).$$

The convex polyhedron $\tilde{D} = \bigcap_i \overline{\tilde{H}^-_{\alpha_i}}$ is *nondegenerate* if the hyperplanes $\tilde{H}_{\alpha_i}$ have no common point in H^n or ∂H^n and there is no hyperplane orthogonal to all the $\tilde{H}_{\alpha_i}$. Furthermore, $\tilde{D}$ is called *finite* if it is the intersection of a finite number of half-spaces $\tilde{H}^+_{\alpha_i}$. Obviously every bounded polyhedron is finite.
We will now describe the conditions to determine whether a polyhedron is bounded or has finite volume. They can be found in [19] §1. A finite convex polyhedron $\tilde{D} \subset H^n$ is bounded if and only if
$$D \subset V_+ \cup \{0\}.$$
This is clear since if $D \subset V_+ \cup \{0\}$ then there exists a ball $B \subset H^n$ with finite radius such that $\tilde{D} \subset B$.
Furthermore, a finite convex polyhedron $\tilde{D} \subset H^n$ has finite volume if and only if
$$D \subset \overline{V_+}.$$
This can easiest be seen in the upper half-space model. If $D \subset \overline{V_+}$ then the convex cone D can contain rays in the boundary of V_+. These rays correspond to vertices at infinity of $\tilde{D}$. Now let q be a vertex at infinity of $\tilde{D}$. We can assume that q is the point ∞ in the upper-half space model. Let $R > 0$ be such that the horizontal hyperplane at height R only intersects the hyperplanes that pass through ∞. So the intersection of a horizontal hyperplane at height $\geq R$ with $\tilde{D}$ has a constant volume μ in this plane. Then the volume of the part of $\tilde{D}$ with $x_n \geq R$ is:
$$\mu \int_R^\infty \frac{dx_n}{x_n^n} = \frac{\mu}{n+1} R^{-n+1} < \infty.$$
So a finite convex polyhedron $\tilde{D}$ with vertices at infinity indeed has finite volume. It is clear that if D is not contained in $\overline{V_+}$ then the volume of $\tilde{D}$ will not be finite.

The above inclusions can be translated in conditions for the Gram matrix of $\tilde{D} = \bigcap_{i\in I} \overline{\tilde{H}^-_{\alpha_i}}$. Before stating this result some further notation has to be fixed. Let A be a symmetric matrix. It will be called the *direct sum* of matrices $A_1, ..., A_k$ if by a permutation of its rows and the same permutation of its columns it can be written in the form
$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{pmatrix}$$
In this case we write $A = A_1 \oplus ... \oplus A_k$. A matrix that is not a direct sum of two or more nonempty matrices is *indecomposable*. Every symmetric matrix A can be represented uniquely as a direct sum of indecomposable matrices. These matrices will be called the *components* of A. Now define A^+ to be the direct sum of all positive definite components, A^0 to be the direct sum of all degenerate nonnegative definite components and A^- to be the direct sum of all components which are not nonnegative definite.
Finally, for $G = (g_{ij})_{i,j\in I} = ((\alpha_i, \alpha_j))_{i,j\in I}$ the Gram matrix of a nondegenerate convex polyhedron and $S \subset I$ let $G_S := (g_{ij})_{i,j\in S}$ be the principal submatrix of G obtained by deleting the kth rows and columns for $k \notin S$ and let
$$C_S = \Big(\bigcap_{i\in S} H_{\alpha_i}\Big) \bigcap \Big(\bigcap_{i\in I\setminus S} H^-_{\alpha_i}\Big) \subset D$$

be the corresponding facet of D. Clearly $C_\emptyset = C$ and $C_I = \{0\}$.

Suppose now that G is an indecomposable symmetric matrix that satisfies

$$g_{ii} = 1 \text{ and } g_{ij} \leq 0 \text{ for } i \neq j. \tag{2.1}$$

Note that in the case that G is the Gram matrix of a nondegenerate finite convex polyhedron $\tilde{D} \subset H^n$ this is equivalent to the assumption that no angle of $\tilde{D}$ exceeds $\pi/2$. The matrix G can be either nonnegative definite or indeterminate. If G is positive definite then all the elements of G^{-1} are nonnegative (see for example [7], Lemma 9.1). If G is nonnegative definite and degenerate then by the Perron-Frobenius lemma ([12], Section 2.6) its kernel is one-dimensional and spanned by a vector $\delta = \sum_{i \in I} k_i \alpha_i$ with all $k_i > 0$. Furthermore, if G is nonnegative definite then all its proper principal submatrices are positive definite.
A matrix $G = (g_{ij})$ that satisfies condition (2.1) is called *critical* if it is not positive definite, yet all its proper principal submatrices are positive definite. A critical matrix is always indecomposable.

Proposition 1. *(Theorem 1 in [19]) Let $\tilde{D} = \bigcap_{i \in I} \overline{\tilde{H}^-_{\alpha_i}} \subset H^n$ be a nondegenerate finite convex polyhedron with finite volume and with indecomposable Gram matrix G that satisfies condition 2.1. Then*

1. *$G_S = G^0_S$ is a principal submatrix of rank $n-1$ of $G \Leftrightarrow C_S \cap H^n$ is a vertex at infinity of $\tilde{D}$.*
2. *$G_S = G^+_S$ is a principal submatrix of rank m of $G \Leftrightarrow C_S \cap H^n$ is an ordinary face of $\tilde{D}$ of codimension m.*

Proof. (the proof here basically follows the proofs of Lemma 3 and Lemma 5 in [18])

1. $\Rightarrow$: Suppose $G_S = G^0_S$, where $I \supset S = \{1, ..., l\}$. Let k_i be the positive coefficients of the linear dependence between the rows of the matrix G_S. Suppose that $\delta = 0$. Then for $j > l$ we have $0 = (\delta, \alpha_j) = \sum_{i=1}^{l} k_i g_{ij}$. Because $k_i > 0$ and $g_{ij} \leq 0$ for all i it then follows that $g_{ij} = 0$ for all i and hence the Gram matrix G would be decomposable. So it follows that $\delta \neq 0$ and δ is an isotropic vector that generates the one dimensional orthogonal complement of $< \alpha_i \mid i \in S >$. Now let $j > l$. Since $(\alpha_j, \alpha_i) \leq 0$ for all $i \in S$, $(\alpha_j, \delta) \leq 0$. Suppose that $(\alpha_j, \delta) = 0$. Then $(\alpha_j, \alpha_i) = 0$ for all $i \in S$ and thus α_j is in the orthogonal complement of $U =< \alpha_i \mid i \in S >$. The subspace U is an n-dimensional parabolic subspace orthogonal to $\mathbb{R}\delta$ and hence $\alpha_j = \lambda\delta$ for a $\lambda \in \mathbb{R}$. This is impossible since $\alpha_j^2 = 1$. So $\delta^2 = 0$, $(\alpha_j, \delta) < 0$ for all $j \in I \setminus S$ and $(\alpha_i, \delta) = 0$ for all $i \in S$. Hence $C_S = \mathbb{R}\delta$ is a ray at the boundary of V_+ that corresponds to a vertex at infinity of $\tilde{D}$.
$\Leftarrow$: Suppose that the facet C_S is a ray on the boundary of V_+ corresponding to a vertex q at infinity of $\tilde{D}$. The linear plane $C_S^\perp =< \alpha_i \mid i \in S >$ orthogonal to this ray is an n-dimensional parabolic subspace of V. Thus the matrix G_S is nonnegative and hence by Perron-Frobenius has a 1-dimensional kernel. Now suppose that G_S is decomposable, say $G_S = A_1 \oplus A_2$ with $A_1 = A_1^+$. Suppose furthermore that $S = \{1, ..., k\}$ and that $\alpha_1, ..., \alpha_l$ $(l < k)$ are the vectors that participate in the formation of A_1. These vectors span an elliptic subspace M_1. Now let $x \in M_1$ be a nonzero vector that satisfies $(\alpha_i, x) \leq 0$ for $i = 1, \ldots, l$. Since $x \in M_1$ we also have $(x, \alpha_i) = 0$ for $i = l+1, \ldots, k$. If $f \in C_S \cap \overline{V}_+$ then for $\epsilon > 0$ small enough $f + \epsilon x \in D$. But also $(f + \epsilon x)^2 = \epsilon^2 x^2 > 0$ so $f + \epsilon x \notin \overline{V}_+$ which contradicts the assumption that $\tilde{D}$ has finite volume. So the matrix G_S is indecomposable and it thus follows that G_S is a principal submatrix of rank $n-1$ and $G_S = G^0_S$.

2. $\Rightarrow$: Suppose $G_S = G^+_S$ is a principal submatrix of rank m, say $S = 1, ..., m$. Then the vectors $\alpha_1, ..., \alpha_m$ form a basis of an m-dimensional elliptic subspace of $\mathbb{R}^{n,1}$. The orthogonal complement $M^\perp$ of $M =< \alpha_i \mid i \in S >$ is a hyperbolic subspace of codimension m and $\widetilde{M}^\perp = M^\perp \cap H^n$ is a hyperbolic plane of codimension m. The orthogonal projection of $x \in \mathbb{R}^{n,1}$ on $M^\perp$ is
$$x_{M^\perp} = x - \sum_{i,j \in S} h_{ij}(x, \alpha_i)\alpha_j,$$

where the $h_{ij} \geq 0$ are the entries of G_S^{-1}. Then for $x \in D$ and $k \notin S$ we see that

$$(x_{M^\perp}, \alpha_k) = (x, \alpha_i) - \sum_{i,j \in S} h_{ij}(x, \alpha_i)(\alpha_j, \alpha_k) \leq 0.$$

Since furthermore $(x_{M^\perp}, \alpha_k) = 0$ for $k \in S$ it then follows that $x_{M^\perp} \in D$. Thus also if $x \in \tilde{D}$ then $x_{M^\perp} \in \tilde{D}$. Hence the orthogonal projection of $\tilde{D}$ on $\tilde{M}^\perp$ lies in $\tilde{D}$. Since it contains a nonempty open subset of $\widetilde{M^\perp}$, $M^\perp \cap \tilde{D} = \overline{C_S} \cap H^n$ is a face of codimension m of $\tilde{D}$.
$\Leftarrow$: Now suppose that $C_S \cap H^n$ is an ordinary face of $\tilde{D}$ of codimension m and let M be the linear plane that contains the facet C_S. Since $C_S \cap H^n$ is an ordinary face this is an $m-$dimensional hyperbolic subspace. The vectors α_i with $i \in S$ generate the orthogonal complement $M^\perp$ of M that is thus an $(n-m)$-dimensional elliptic subspace. If G_S is not positive definite then there must be a linear dependence with nonnegative coefficients between the rows of G_S (G_S cannot be indeterminate since the α_i with $i \in S$ span an elliptic subspace). But if this was true there would also be a linear dependence between the α_i with $i \in S$. As this is not the case, it follows that $G_S = G_S^+$ and G_S is a principal submatrix of rank m.

□

The polyhedron $\tilde{D}$ can be shown to be bounded or to have finite volume without completely determining its combinatorial structure by verifying the following conditions (Proposition 1 in [19]):

1. $\tilde{D}$ has finite volume if and only if for any critical principal submatrix G_S of the Gram matrix G
   (a) if $G_S = G_S^0$, then there exists a subset $T \subset I$ that contains S, $G_T = G_T^0$ and rank $G_T = n-1$.
   (b) if $G_S = G_S^-$, then $C_S = \{0\}$.
2. $\tilde{D}$ is bounded if and only if the Gram matrix G contains no degenerate nonnegative definite principal submatrices and any indeterminate critical principal submatrix of G satisfies condition (b).

Proof. 1. $\Rightarrow$: Suppose that $\tilde{D}$ has finite volume and let G_S be a critical principal submatrix of G. First suppose that $G_S = G_S^0$. The kernel of G_S is spanned by a vector $\delta = \sum_{i \in S} k_i \alpha_i$ with $k_i > 0$. This δ is an isotropic vector that corresponds to a vertex at infinity q of $\tilde{D}$. Now let $T = \{i \in I \mid (\alpha_i, \delta) = 0\}$ so that $q = C_T \cap H^n$. Then $S \subset T$ and by Proposition 1, $G_T = G_T^0$ and rank $G_T = n-1$.
Now suppose that $G_S = G_S^-$. Since (by Perron-Frobenius) all principal submatrices of an indecomposable semidefinite Gram matrix satisfying condition 2.1 are positive definite, G_S cannot be contained in a semidefinite principal submatrix. It thus follows that C_S cannot be any other facet of D then $\{0\}$.
$\Leftarrow$: Now assume that conditions (a) and (b) are satisfied and suppose that C_S is an extremal ray of the convex cone D that is not in V_+. Then by Proposition 1 the matrix G_S cannot be positive definite. Hence G_S contains a critical principal submatrix G_{T_1}. Since $C_S \subset C_{T_1}$, $C_{T_1} \neq \{0\}$. Thus by condition (b) G_{T_1} is nonnegative and therefore semidefinite. Now it follows from condition (a) that there exists a $T_2 \supset T_1$ such that $G_{T_2} = G_{T_2}^0$ and rank $G_{T_2} = n-1$. By Proposition 1, $T_2 = \{i \in I \mid (\alpha_i, q) = 0\}$ where q is a vertex at infinity. Any face of $\tilde{D}$ that contains this vertex (and is not equal to this vertex) is an ordinary face and thus corresponds to a positive definite principal submatrix. G_{T_1} cannot be contained in a positive definite principal submatrix so it follows that $C_{T_1} = C_{T_2}$. Because $T_1 \subset S$ we have $C_S \subset C_{T_1}$ and C_S corresponds to the vertex q at infinity. It thus follows that $D \subset \overline{V}_+ \cup \{0\}$ and hence $\tilde{D}$ has finite volume.

2. $\Rightarrow$: If $\tilde{D}$ is bounded $G_S = G_S^0$ is impossible since by the proof of 1. above it would then follow that $\tilde{D}$ has a vertex at infinity. If $G_S = G_S^-$ the proof of 1. above shows that $C_S = \{0\}$.

$\Leftarrow$: Now we assume that G contains no semidefinite critical principal submatrices and it then follows immediately from the discussion of case 1. that $D \subset V_+$ and hence $\tilde{D}$ is bounded.

$\square$

Chapter 3

Lattices

In this chapter we describe the root lattices that will later play an important role in the classification of Niemeier lattices. We also explain how a lattice determines a reflection group. Finally, we describe the theory of gluing and define the covering radius and holes of a lattice. Most of it can be found in [6] chapters 1,2 and 4.

Definition 2. *A lattice L in a real vector space V of finite dimension n is a subgroup of V such that there exist an $\mathbb{R}$-basis $\{v_1, ..., v_n\}$ of V which is a $\mathbb{Z}$-basis of L, i.e. $L = \mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_n$. Equivalently: L is an additive subgroup that is discrete and $V \setminus L$ is compact. Furthermore, V (and thus L) is equipped with a symmetric bilinear form $(\cdot, \cdot)$.*

We write $l^2 = (l, l)$ for the norm of $l \in L$. An n-dimensional lattice $L_n = \mathbb{Z}v_1 \oplus ... \oplus \mathbb{Z}v_n$ is called *integral* if $l^2 \in \mathbb{Z}$ for all $l \in L_n$, and *even* if $l^2 \in 2\mathbb{Z}$ for all $l \in L_n$. We will say that a lattice is *positive definite*, *hyperbolic*, etc. if the bilinear form $(\cdot, \cdot)$ has this signature. A *Gram matrix of a lattice* is the Gram matrix of a basis for this lattice. The *determinant of a lattice* is the determinant of such a Gram matrix, it is denoted by $\det L$. A lattice is called unimodular if the determinant is ± 1.
The *dual lattice* of a lattice L_n is given by $L_n^* = \{x \in \mathbb{R}^n \mid (x, l) \in \mathbb{Z} \text{ for all } l \in L_n\}$. A lattice L is integral if and only if $L \subset L^*$. If G is the Gram matrix of L then L^* has Gram matrix G^{-1} for the dual basis and $\det L^* = (\det L)^{-1}$. For an integral lattice the group L^*/L of order $\det L$ is called its *dual quotient group*. So an integral lattice L is unimodular if and only if $L = L^*$.
Take a basis $\{v_1, ..., v_n\}$ of L_n and let A be the $n \times n$ matrix which has as rows the vectors v_i. A is called a *generator matrix* for the lattice L_n and L_n consists of all vectors

$$\xi A,$$

with $\xi \in \mathbb{Z}^n$. The Gram matrix of L_n is $G = AA^t$ and $\det L_n = (\det A)^2$. Furthermore, $(A^{-1})^t$ is a generator matrix for L_n^*. For any integral lattice we have:

$$L \subset L^* \subset \frac{1}{\det L} L.$$

(If $l^* \in L^*$ then $l^* = \xi(A^{-1})^t = \xi(A^{-1})^t A^{-1} A = (\det L)^{-1} \xi \mathrm{adj}(G) A = (\det L)^{-1} \eta M$, where $\xi, \eta \in \mathbb{Z}^n$).
From now on all lattices will be integral.

3.1 Root lattices and reflection groups

Let L be a lattice. A *root* $\alpha \in L$ is an element such that reflection by its orthogonal hyperplane in V is in $\mathrm{Aut}(L)$, i.e. $r_\alpha : \lambda \mapsto \lambda - 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$ is an element of $\mathrm{Aut}(L)$. Thus α is a root if $\lambda - 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}\alpha \in L$ for all $\lambda \in L$. The root system $R(L)$ of L is the set of all roots:

$$R = R(L) = \left\{ \alpha \in L \mid 2\frac{(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \lambda \in L \right\}$$

Thus for an integral lattice L the roots are the vectors of norm 1 and 2 in L, which are called the short and long roots respectively. If L is even then there are no roots of norm 1 so the root system of L in this case is $R(L) = \{\alpha \in L \mid \alpha^2 = 2\}$.

Let L be an even lattice. The subgroup $W(L) =< r_\alpha \mid \alpha \in R > \, < \mathrm{Aut}(L)$ is the reflection group of L. The hyperplanes perpendicular to the roots of L divide the real vector space $V = L \otimes \mathbb{R}$ into regions. Choose one such a connected component C, the *Weyl chamber*. Define the *positive roots* to be those roots with positive product with all vectors in C, i.e. $R_+ = \{\alpha \in R \mid (\xi, \alpha) > 0 \text{ for all } \xi \in C\}$. We will also write $\alpha > 0$ to indicate that α is a positive root. Similarly, the negative roots $\alpha < 0$ are the set of roots $R_- = \{\alpha \in R \mid (\xi, \alpha) < 0 \text{ for all } \xi \in C\}$. The simple roots are those roots that are perpendicular to the faces of C and that have product at most 0 with all elements of C. So $\alpha \in R_-$ is simple if it is not of the form $\alpha = j\beta + k\gamma$, where $j, k \geq 1$ and $\beta, \gamma \in R_-$. (Contrary to what is common, here the simple roots are negative. With this convention a root is a simple root iff it has product at most 0 with all other simple roots and it is a simple root iff it has product at most 0 with all elements of C.)
Denote by $Q = Q(R) = \mathbb{Z}R \subset L$ the root lattice with dual weight lattice $P = P(R) = Q^* = \{\lambda \in V | (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in Q\} (= \{\lambda \in V | (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in R_+\})$. Let $P_+ = \{\lambda \in P | (\lambda, \alpha) \in \mathbb{N} \text{ for all } \alpha \in R_+\}$ be the cone of dominant weights.

To each lattice L (not necessarily positive definite) with a fixed Weyl chamber C we can attach a *Coxeter-Dynkin diagram* which is determined by the Gram matrix of the simple roots. To each simple root we associate a vertex, and two vertices corresponding to distinct simple roots α_i, α_j are joined by $-(\alpha_i, \alpha_j)$ lines. If L is even and positive definite then two vertices are always joined by 0 or 1 lines, in this case we say that the diagram is simply laced. The only possible irreducible simply laced diagrams are those of the types A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7 and E_8 (for a proof see [12] section 2.7). So the diagram of an even and positive definite lattice is a union of diagrams of these types. The diagrams are shown in Figure 3.1 below. Hence if L is a positive definite even lattice then the root lattice $Q = \mathbb{Z}R(L)$ is an orthogonal sum of lattices isomorphic to one of the lattices A_n, D_n, E_6, E_7 and E_8. In the case of a hyperbolic lattice the Coxeter diagram may contain multiple bonds.

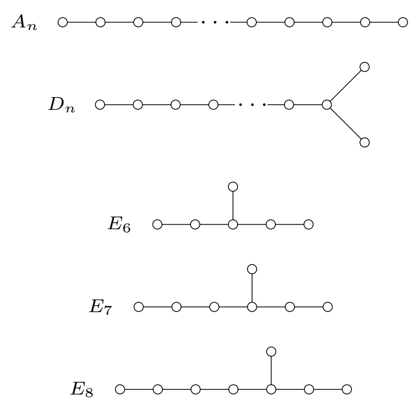

Figure 3.1: Irreducible simply laced root diagrams of finite type.

The *Coxeter number h* of an irreducible root system is the number of roots divided by the dimension. The Coxeter number of a component of a positive definite lattice L is defined as the Coxeter number of the lattice generated by the roots of that component. In the following table we list the above root lattices with their Coxeter number and the order of their dual quotient group.

For a positive definite lattice L with a chosen Weyl chamber C the Weyl vector $\rho \in L \otimes \mathbb{R}$ is the vector that has inner product -1 with all simple roots of C. It is given by $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$. This ρ does indeed satisfy $(\rho, \alpha_i) = -1$ for all simple α_i:
Let α_i be a simple root. It is well known that in this case $r_{-\alpha_i}(R_+ \setminus \{-\alpha_i\}) = R_+ \setminus \{-\alpha_i\}$ ([12]

Table 3.1: Root lattices

Root system	Coxeter number	Determinant
A_n	$n+1$	$n+1$
D_n	$2n-2$	4
E_6	12	3
E_7	18	2
E_8	30	1

Proposition 5.6). Hence

$$\begin{aligned} r_{-\alpha_i}(\rho) &= r_{-\alpha_i}\left(\frac{1}{2}\sum_{\beta\in R_+\setminus\{-\alpha_i\}}\beta - \frac{1}{2}\alpha_i\right) \\ &= \frac{1}{2}\sum_{\beta\in R_+\setminus\{-\alpha_i\}}\beta + \frac{1}{2}\alpha_i \\ &= \rho+\alpha_i \\ &= \rho+(\rho,-\alpha_i)\alpha_i. \end{aligned}$$

Clearly, the Weyl vector ρ is in the Weyl chamber C. If L is not positive definite there may or may not be a Weyl vector, that is, a vector that has product -1 with all simple roots.

Assume from now on that L is a positive definite even lattice. For a component R of the Coxeter diagram of L the orbit of roots of L under the Weyl group $W(L)$ has a unique representative α_0 that is in the Weyl chamber C. This root is called the *highest root*. If $\alpha_1, ..., \alpha_j$ are simple roots of R then $-\alpha_0 = \sum_{i=1}^{j} k_i\alpha_i$, where the k_i are positive integers called the weights of the roots α_i and $\sum_{i=1}^{j} k_i = h-1$. Equivalently, $0 = \sum_{i=0}^{j} k_i\alpha_i$, with $k_0 = 1$ and $\sum_{i=0}^{j} k_i = h$. The diagram obtained by adding the highest root is the extended Coxeter diagram of L. It is the Coxeter diagram of a semi-definite lattice $\tilde{L}$. We write $\tilde{R}_n$ for the extended Coxeter diagram corresponding to the Coxeter diagram R_n. The extended diagrams for the diagrams in Figure 3.1 are shown below in Figure 3.2 (see [12] section 4.7). The numbers next to the nodes are the coefficients k_i. Any node with coefficient 1 is called a *special vertex*, its removal would give back the diagram corresponding to (a component of) the Coxeter diagram of the positive definite lattice.

Example 1. The lattice D_n^+.
The root lattice of type D_n ($n \geq 4$) is given by

$$D_n = \{(x_1, ..., x_n) \in \mathbb{Z}^n \mid x_1 + ... + x_n \in 2\mathbb{Z}\}.$$

The determinant of the Gram matrix of D_n is 4 for all n. Coset representatives for D_n^*/D_n are:

$$\begin{aligned} [0] &= (0,0,...,0), \text{ with norm } 0, \\ [1] &= (\frac{1}{2},\frac{1}{2},...,\frac{1}{2},\frac{1}{2}), \text{ with norm } \frac{n}{4}, \\ [2] &= (0,0,...,1), \text{ with norm } 1, \\ [3] &= (\frac{1}{2},\frac{1}{2},...,\frac{1}{2},-\frac{1}{2}), \text{ with norm } \frac{n}{4}. \end{aligned}$$

Let $n \in 2\mathbb{N}$, $\omega := [1]$ and define the lattice $D_n^+ = D_n \cup (\omega + D_n)$. D_n^+ is a sublattice of the dual lattice D_n^* of index 2, since $2\omega \in D_n$ for $n \in 2\mathbb{N}$. Also, since $\omega^2 = \frac{n}{4}$, D_n^+ is even iff $n \in 8\mathbb{N}$. If $n \in 8\mathbb{N}$ then D_n^+ is the unique even unimodular lattice (up to isomorphism) that contains D_n as a sublattice of rank 2 (D_n^+ is isomorphic to $D_n \cup ([3] + D_n)$). Furthermore, $D_8^+ = E_8$. However, $D_{16}^+ \ncong 2E_8$ since the norm 2 vectors in D_{16}^+ generate the index 2 sublattice D_{16}. Indeed, D_{16} is irreducible while $2E_8$ is reducible.

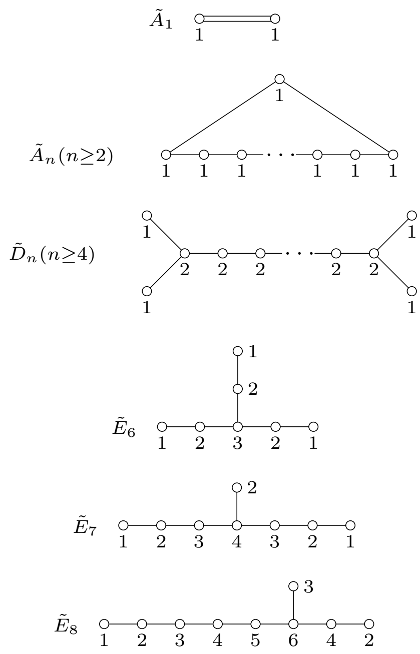

Figure 3.2: Extended root diagrams labelled with the integers k_i.

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The construction of D_n^+ is an example of gluing lattices.
In general, starting with integral lattices $L_1, ..., L_k$ (not necessarily distinct) of total dimension n one can construct an n-dimensional integral lattice L that has as a sublattice the direct sum $L_1 \oplus ... \oplus L_k$. A vector of L is of the form $l = l_1 + ... + l_k$ with $l_i \in L_i^*$ (not necessarily in L_i). Let $\{\omega_i^n \mid n = 1, \ldots, \det(L_i)\}$ be a set of standard representatives for the cosets of L_i in L_i^*, they are called the *glue vectors* for L_i. In this context the group L_i^*/L_i is also called the *glue group*. The glue vectors are usually chosen to be of minimal length in their coset. The lattice L is generated by $L_1 \oplus ... \oplus L_k$ and a set of vectors $\{g^j\}$ of the form

$$g^j = g_1^j + ... + g_k^j, \tag{3.1}$$

where $g_i^j \in \{\omega_i^n\}$ is one of the standard representatives of L_i^*/L_i for all i. These vectors are also called glue vectors. They must have integral product with each other and be closed under addition modulo $L_1 \oplus ... \oplus L_k$. The additive group formed by the glue vectors is called the *glue code*. It can happen that for a glue vector g^j only one of the g_i^j in 3.1 is nonzero. In this case the component L_i has self-glue. The lattice D_n^+ is constructed by self-gluing D_n. Since the sublattice $Q = \mathbb{Z}R(L)$ of an integral lattice L is a direct sum of root lattices, the root lattices are a particularly good choice for the lattices L_i. The classification of Niemeier lattices described in section 5.3 is a good example of the use of gluing theory.

3.2 Covering radius and holes

Let L be a positive definite lattice and let $V = L \otimes \mathbb{R}$ be the ambient Euclidean space. The *covering radius* R of L is the smallest $R > 0$ such that the spheres of radius R centered at the points of L will cover V. It is given by

$$R = \sup_{x \in V} \inf_{l \in L} \sqrt{(x - l)^2}.$$

For each point $l \in L$ its *Voronoi cell* $V(l)$ consists of those points in V that are at least as close to l as to any other $l' \in L$. Thus

$$V(l) = \{x \in V \mid (x - l)^2 \leq (x - l')^2 \text{ for all } l' \in L\}.$$

The Voronoi cells are compact convex polytopes whose union is V. Their interiors are disjoint, but they do have faces in common. All Voronoi cells of a lattice L are congruent and have volume equal to $\sqrt{\det L}$. We will sometimes call $V(0)$ the Voronoi cell of L. For any irreducible root lattice $Q = \mathbb{Z}R(L)$ the Voronoi cell is the union of the images of the fundamental simplex (or standard alcove) $A = \{\xi \in Q \otimes \mathbb{R} \mid 0 \leq (\xi, \alpha) \leq 1 \text{ for all } \alpha \in R_+\}$ under the (finite) Weyl group $W(R(L))$ (see [6], Chapter 21, Theorem 5).
The vertices of the Voronoi cells are called *holes*. The *radius* $R(\xi)$ of a hole ξ is its distance from L so it is given by

$$R(\xi)^2 = \inf\{(l - \xi)^2 \mid l \in L\}.$$

The holes with maximum radius are called the *deep holes*, their radius is equal to the covering radius of L. The other holes are called *shallow holes*. The *vertices* of a hole ξ of L are the elements of the set

$$L(\xi) = \{l \in L \mid (l - \xi)^2 = R(\xi)\},$$

i.e. the lattice points l such that ξ is a vertex of the Voronoi cell $V(l)$.
The *packing radius* of L is defined to be the largest $r > 0$ such that balls with radius r centered at the lattice points do not overlap. The covering radius is the circumradius of $V(0)$, i,e. the radius of the smallest circumscribed sphere. The packing radius is the inradius of $V(0)$, i.e. the radius of the largest inscribed sphere.

Chapter 4

The Algorithm of Vinberg

In this chapter we describe the algorithm of Vinberg. This is an algorithm to determine a fundamental domain $\tilde{D}$ for a discrete group $W < O_+(n,1)$ generated by reflections. The idea of the algorithm is as follows: Pick a point $c \in H^n$. The set of mirrors $\mathcal{H}_c = \{H_\alpha \mid r_\alpha \in W, (\alpha, c) = 0\}$ that contain c divide H^n into convex regions. Let $\tilde{D}_c$ be the closure of one of those components that contain c. It is a fundamental domain for the subgroup $W_c < W$ generated by the reflections whose mirrors contain c. Now there is a unique fundamental domain $\tilde{D}$ of W that contains c and is contained in $\tilde{D}_c$. The polyhedron $\tilde{D}$ is bounded by the mirrors in $\mathcal{H}_c$. The other mirrors that bound $\tilde{D}$ are found by moving away from the point c inside $\tilde{D}_c$ while checking which mirrors one meets. That is, the mirrors are enumerated by increasing distance from c.
The precise formulation of the algorithm is as follows: Pick a point $c \in H^n$. Let $R_c = \{\alpha \in V = \mathbb{R}^{n,1} \mid \alpha^2 > 0, (c, \alpha) \le 0 \text{ and } r_\alpha \in W\}$. Form a sequence $\alpha_1, \alpha_2, \ldots \in R_c$ according to the following rules:

1. Pick $\alpha_1, ..., \alpha_k \in R_c$ such that
$$\tilde{D}_c = \bigcap_{i=1}^{k} \overline{\tilde{H}^-_{\alpha_i}}.$$
2. for $l > k$ pick $\alpha_l \in R_c$ that satisfies
$$(\alpha_l, \alpha_i) \le 0 \text{ for all } i < l,$$
and such as to minimize the distance $d(c, \tilde{H}_{\alpha_l})$ from the point c to the hyperplane $\tilde{H}_{\alpha_l}$.

Then
$$\tilde{D} = \bigcap_i \overline{\tilde{H}^-_{\alpha_i}}$$
is a fundamental domain for W (see [19], §3).
Now suppose that Γ is a discrete subgroup of motions of H^n with $W \triangleleft \Gamma$ the subgroup generated by reflections that has fundamental domain $\tilde{D}$. Denote by $\mathrm{Sym}(\tilde{D})$ the symmetry group of $\tilde{D}$. Then the group Γ decomposes into a semidirect product
$$\Gamma = W \rtimes H,$$
where $H \subset \mathrm{Sym}(\tilde{D})$.

Below we will give a formulation of Vinberg's algorithm for the case that W corresponds to an even lattice and also give a proof for this case. This is also the formulation of Vinberg's algorithm we will use later.

In general, to obtain a hyperbolic reflection group from a lattice start with a hyperbolic lattice $L \subset V = \mathbb{R}^{n,1}$, $n \ge 1$. The automorphism group of L is the discrete subgroup $\mathrm{Aut}(L) = \{g \in O(V) \mid g(L) = L\} < GL(V)$. In the case that L is an integral lattice $\mathrm{Aut}(L) = GL(n+1, \mathbb{Z}) \cap O(V)$.

As before $V_+ = \{\lambda \in \mathbb{R}^{n,1} \mid \lambda^2 < 0 \text{ and } x_{n+1} > 0\}$ and the hyperboloid model is used for the hyperbolic space H^n. Denote the mirror of a root $\alpha \in R(L)$ by $H_\alpha = \{\xi \in V \mid (\xi, \alpha) = 0\}$. Choose a connected component $\tilde{C} = C \cap H^n \subset H^n \setminus \bigcup_{\alpha \in R} \tilde{H}_\alpha$, also called the Weyl chamber. As before, we denote by $\tilde{D}$ and D the closure of $\tilde{C}$ resp. C. Define $W'(L) =< r_\alpha \mid \alpha \text{ simple root} >$. This is the same situation as in Chapter 2 so again $W' = W$, $\tilde{D}$ is a fundamental domain for the action of W on H^n and W has a Coxeter representation.

Now let L be an even lattice. Then a fundamental domain $\tilde{D}$ and its simple roots for the action of $W = W(L)$ on H^n can be found by Vinberg's algorithm as follows:
Choose a controlling vector $c \in P_+(= P \cap \tilde{D})$. Define for $\alpha \in R_-$ the height of α with respect to c by $\mathrm{ht}(\alpha) = -(c, \alpha) \in \mathbb{N}$, it can be interpreted as a measure for the distance between $\tilde{H}_\alpha$ and c. Let $W_c \lhd W$ be the subgroup generated by the reflections whose mirrors contain c, i.e. $W_c =< r_\alpha | \alpha \in R_- \text{ and } \mathrm{ht}(\alpha) = 0 >$. This is a finitely generated group. In particular if $c^2 < 0$ then $R \cap c^\perp$ is of finite type and if $c^2 = 0$ then $R \cap c^\perp$ is of finite or affine type. Let $\tilde{D}_c$ be the closure of the connected component of $H^n \setminus \bigcup_{\{\alpha | r_\alpha \in W_c\}} \tilde{H}_\alpha$ that contains c (possibly in the closure). There is a unique fundamental domain $\tilde{D}$ of W that is contained in $\tilde{D}_c$ and its simple roots can be found as follows:

1. A positive root with $\mathrm{ht}(\alpha) = 0$ is a simple root of $\tilde{D}$ if and only if it is a simple root of $\tilde{D}_c$.
2. For $\alpha < 0$ with $\mathrm{ht}(\alpha) \geq 1$: α is a simple root if and only if $(\alpha, \beta) \leq 0$ for all simple $\beta < 0$ with $\mathrm{ht}(\beta) \leq \frac{\mathrm{ht}(\alpha)}{2}$.

Proof. (this proof follows the one in [10])

1. Clear
2. Since $(\alpha, \beta) \leq 0$ holds for all simple roots α, β it is clear that the implication $\Rightarrow$ holds. Conversely, it is also true that α is a simple root if $(\alpha, \beta) \leq 0$ for all simple roots β. We need to prove that it suffices to only check this condition for all simple β such that $\mathrm{ht}(\beta) \leq \frac{\mathrm{ht}(\alpha)}{2}$. So let $\alpha < 0$ be a root with $\mathrm{ht}(\alpha) \geq 1$ that satisfies the condition that $(\alpha, \beta) \leq 0$ for all simple $\beta < 0$ with $\mathrm{ht}(\beta) \leq \frac{\mathrm{ht}(\alpha)}{2}$ and suppose that α is not a simple root. So $\tilde{H}_\alpha$ does not bound $\tilde{D}$ and there exists a simple root γ such that $(\alpha, \gamma) \nleq 0$ and thus $(\alpha, \gamma) \geq 1$. Since $(\alpha, \beta) \leq 0$ for all simple $\beta < 0$ with $\mathrm{ht}(\beta) \leq \frac{\mathrm{ht}(\alpha)}{2}$ it follows that $\mathrm{ht}(\gamma) > \frac{\mathrm{ht}(\alpha)}{2}$. Since furthermore $r_\gamma(\alpha) \in R_-$ and thus $\mathrm{ht}(r_\gamma(\alpha)) = \mathrm{ht}(\alpha) - (\alpha, \gamma)\mathrm{ht}(\gamma) \in \mathbb{N}$ it follows that $(\gamma, \alpha) = 1$. Then $(\alpha, r_\gamma(\alpha)) = (\alpha, \alpha - \gamma) = (\alpha, -r_\alpha(\gamma)) = (\alpha, \gamma) = 1$. So also $(\alpha, r_\gamma(\alpha)) = 1$.
   On the other hand, since α is not a simple root it can be written as a linear combination $\alpha = \sum_{i=1}^m n_i \alpha_i$, where $m \geq 2$, $n_i \geq 1$ and the α_i are simple roots. Now suppose $\gamma \neq \alpha_i$ for all $i \in \{1, ..., m\}$. Then $(\alpha, \gamma) = \sum n_i (\alpha_i, \gamma) \leq 0$ since $(\alpha_i, \gamma) \leq 0$ for all simple roots α_i and $n_i \geq 1$. Thus $\gamma = \alpha_i$ for an $i \in \{1, ..., m\}$, after possible renumeration we can suppose that $\gamma = \alpha_1$. Then
   $$\mathrm{ht}(\alpha) = n_1 \mathrm{ht}(\gamma) + \sum_{i=2}^m n_i \mathrm{ht}(\alpha_i) > n_1 \frac{\mathrm{ht}(\alpha)}{2} + \sum_{i=1}^m n_i \mathrm{ht}(\alpha_i).$$
   Thus $n_1 = 1$ and $\mathrm{ht}(\alpha_i) < \frac{\mathrm{ht}(\alpha)}{2}$ for all $i \geq 2$ since otherwise the term on the right would exceed $\mathrm{ht}(\alpha)$. Since α satisfies the condition in 2. above it then follows that $(\alpha, \alpha_i) \leq 0$ for all $i \geq 2$. Hence $(\alpha, r_\gamma(\alpha)) = (\alpha, \alpha - \gamma) = \sum_{i \geq 2} n_i (\alpha, \alpha_i) \leq 0$. This is a contradiction and thus α has to be a simple root.

□

Now the group of orthochronous automorphisms of L, $\mathrm{Aut}_+(L) = \mathrm{Aut}(L) \cap O_+(n, 1)$, is given by

$$\mathrm{Aut}_+(L) = W \rtimes \mathrm{Sym}(\tilde{D}),$$

where the symmetry group of the fundamental domain $\tilde{D}$, $\mathrm{Sym}(\tilde{D})$, is the same as the symmetry group of the Coxeter diagram of L determined by the simple roots of $\tilde{D}$.

Note that if a finite set of simple roots $\{\alpha_1, \ldots, \alpha_k\}$ determined by Vinberg's algorithm is such that the volume of $D_1 = \bigcap_{i=1}^k \overline{\tilde{H}_{\alpha_i}^-}$ is finite then this set consists of all simple roots and hence D_1 is a fundamental domain of W. Indeed, suppose that α_{k+1} is a simple root that is not equal to one of the $\alpha_i, i \leq k$ and set $D_2 = \bigcap_{i=1}^{k+1} \overline{\tilde{H}_{\alpha_i}^-}$. Then the Gram matrix of D_1 is a principal submatrix of the Gram matrix of D_2. So any principal submatrix of D_1 that corresponds to a vertex (including those at infinity) is also a principal submatrix of D_2 corresponding to the same vertex. Hence all vertices of D_1 are also vertices of D_2. Since D_1 has finite volume this can only be the case if $D_1 = D_2$ which is impossible.

Chapter 5

Even Unimodular Lattices

This chapter deals with even unimodular lattices. We say that a lattice is of type $\mathrm{II}_{p,q}$ if it is an even unimodular lattice with signature (p,q). If q is zero we write II_p. First we describe what is known about the classification of such lattices. Then we describe the relation between primitive norm zero vectors of a lattice L of type $\mathrm{II}_{8n+1,1}$ and lattices of type II_{8n}. After that, we treat the examples $\mathrm{II}_{9,1}$ and $\mathrm{II}_{17,1}$ applying the theory of the chapters 2 and 4. We then turn to the classification of the Niemeier lattices, presenting Boris Venkov's proof of this classification. We end this chapter with some results on Niemeier lattices that we will need in the last chapter.

A *modular form of weight* $2k$ is a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \text{ (with } q = e^{2\pi i z})$$

which converges for $|q| < 1$ (and hence for $\mathrm{Im}(z) > 0$) and which satisfies the identity

$$f(-1/z) = z^{2k} f(z).$$

A modular form is called a *cusp form* if it furthermore satisfies $a_0 = 0$. Note that if f_1 and f_2 are modular forms of weight $2k_1$ and $2k_2$ then $f_1 f_2$ is a modular form of weight $2k_1 + 2k_2$. Chapter VII of [17] provides a good introduction into modular forms.

From now on assume that L is an even unimodular lattice. For a positive definite lattice L the *theta series* of L is

$$\theta_L(z) = \sum_{\lambda \in L} q^{\lambda^2/2} = \sum_{m=0}^{\infty} N_m \cdot q^{m/2},$$

where $q = e^{2\pi i z}$ and N_m is the number of lattice vectors $\lambda \in L$ that satisfy $\lambda^2 = m$.
The following theorem is a result of Hecke ([17], Chapter VII, Theorem 8).

Theorem 1. *If L is a lattice of type II_n then*

1. *n is divisible by 8,*
2. *θ_L is a modular form of weight $n/2$.*

The algebra of modular forms coincides with $\mathbb{C}[E_2, E_3]$ where the *Eisenstein series* E_k are modular forms of weight $2k$ given by

$$E_k(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m,$$

with $\sigma_k(m) = \sum_{d|m} d^k$ and the Bernoulli numbers B_k are defined by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_k x^{2k}}{(2k)!}$$

In particular

$$E_2(z) = 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m, \text{ and}$$
$$E_3(z) = 1 - 504\sum_{m=1}^{\infty}\sigma_5(m)q^m.$$

Let L be a lattice of type II_n with $n = 8m$. Since $\theta_L(0) = 1$ it follows from Theorem 1 that $\theta_L - E_{2m}$ is a cusp form of weight $2m$. Using this the theta series for $n = 8, 16$ and 24 can be determined. Following example 6.6 in [17] we get:

1. if $n = 8$ then $\theta_L = E_2$ since the only cusp form of weight $2m = 4$ is zero. The number of roots of L is 240,
2. if $n = 16$ then $\theta_L = E_2^2 = E_4$ since the only cusp form of weight $2m = 8$ is again zero. The the number of roots of L is 480.
3. if $n = 24$ then θ_L is a modular form of weight 12. The space of modular forms of this weight is two dimensional. A basis is formed by the functions:

$$E_6(z) = 1 + \frac{65520}{691}\sum_{m=1}^{\infty}\sigma_{11}(m)q^m = 1 + \frac{65520}{691}(q + 2049q^2 + 177148q^3 + 4196353q^4 + \ldots), \text{ and}$$
$$\Delta(z) = q\prod_{m=1}^{\infty}(1-q^m)^{24} = \sum_{m=1}^{\infty}\tau(m)q^m = q - 24q^2 + 252q^3 - 1472q^4 + \ldots,$$

where τ is called Ramanujan's function. We have $\theta_L = E_6 + c\Delta = 1 + N_2q^2 + \ldots$, with N_2 the number of roots of L. It follows that $c = N_2 - \frac{65520}{691}$ and thus

$$\theta_L = E_6 + (N_2 - \frac{65520}{691})\Delta. \tag{5.1}$$

Because the number of roots of an irreducible root system is equal to nh, where h is the Coxeter number of the root system, we can determine from Table 3.1 that the only possible root system for a lattice of type II_8 is E_8. Furthermore, for a lattice of type II_{16} the only possible root systems are $2E_8$ and D_{16}. It thus follows that up to isomorphism there is exactly one even unimodular lattice of type II_8, namely E_8, and two even unimodular lattices of type II_{16}, $2E_8$ and D_{16}^+.

The Minkowski-Siegel mass formula provides another check that these enumerations are correct. It is given by

$$\sum_L \frac{1}{|\mathrm{Aut}(L)|} = \frac{B_{2k}}{8k}\prod_{j=1}^{4k-1}\frac{B_j}{4j},$$

where $n = 8k$ and the sum is over the isomorphism classes of lattices of type II_n. In [6] (Chapter 16) the correctness of the classification of the lattices of type II_{24}, the Niemeier lattices, is verified using this formula. It also follows from the mass formula that the number of isomorphism classes of lattices of type II_n increases very rapidly as n increases. Indeed, for $n = 32$ the right hand side of the mass formula is already greater than $4 \cdot 10^7$. Since $|\mathrm{Aut}(L)| \geq 2$ for all L it then follows that there are at least $8 \cdot 10^7$ isomorphism classes of lattices of type II_{32}. Refining this argument, King [14] later showed that there are at least $116 \cdot 10^7$ isomorphism classes of such lattices. Furthermore, he shows that among these there are more than $1 \cdot 10^7$ without roots. Kervaire has shown in [13] that there are exactly 132 indecomposable even unimodular 32-dimensional lattices with a complete root system (i.e. a root system R such that $Q = \mathbb{Z}R$ is a subgroup of finite index in the corresponding lattice). Among these, several lattices happen to have the same root system. There occur 119 different root systems. A complete classification of the lattices of type II_n for $n \geq 32$ seems hopeless.

Contrary to positive or negative definite integral even unimodular lattices, the indefinite lattices of this type are classified for all dimensions. There only exists a lattice of type $\mathrm{II}_{p,q}$ with $p, q \geq 1$ if $p - q \in 8\mathbb{Z}$ and in this case it is unique (see [17], Chapter V).

5.1 Norm zero vectors in hyperbolic lattices

In this section we describe the relation between norm zero vectors in hyperbolic lattices and extended Coxeter diagrams in the Coxeter diagram of L. We follow the discussion on this in [3]. Let $n \in 8\mathbb{N}$ and L of type $\mathrm{II}_{n+1,1}$. Let U be the lattice of type $\mathrm{II}_{1,1}$ with basis $\{x, y\}$ and symmetric bilinear form $(\cdot,\cdot)$ that satisfy: $x^2 = y^2 = 0$ and $(x, y) = -1$. A vector $x \in L$ is primitive if for all $k > 1$, $\frac{x}{k} \notin L$.

Lemma 4. *If $x \in L$ is a primitive norm zero vector then there is a vector $y \in L$ with $y^2 = 0$ and $(x, y) = -1$.*

Proof. Let $x \in L$ be a primitive norm zero vector. Since L is unimodular, we have $L = L^*$ and thus there is an $y \in L$ with $(x, y) = -1$. If $y^2 = 0$ then y already satisfies the requirements in the Lemma. Now suppose $y^2 \neq 0$, say $y^2 = k \in 2\mathbb{Z}$. Then $(y + \frac{k}{2}x)^2 = y^2 + 2\frac{k}{2}(x, y) = k - k = 0$ and $(y + \frac{k}{2}x, x) = (x, y) = -1$. So $y + \frac{k}{2}x$ is an element of L that satisfies the requirements. $\square$

Thus a primitive norm zero vector $x \in L$ is contained in a rank 2 sublattice U. Since N and U are both unimodular, $N \oplus U$ is also unimodular and has signature $(25, 1)$. Hence $L = N \oplus U$ and the orthogonal complement N of U is a lattice of type II_n. It follows that there are bijective correspondences between the sets:

1. n-dimensional even unimodular lattices (up to isomorphism),
2. sublattices of L isomorphic to U (up to the action of $\mathrm{Aut}(L)$),
3. primitive norm zero vectors in L (up to the action of $\mathrm{Aut}(L)$).

Instead of the basis $\{x, y\}$ of U with symmetric bilinear form $(\cdot,\cdot)$ that satisfies: $x^2 = y^2 = 0$ and $(x, y) = -1$ we will also use the basis (x_1, x_2) for $\mathbb{R}^{1,1}$, where $x_1 = \frac{1}{2}x - y$ and $x_2 = \frac{1}{2}x + y$. So $x_1^2 = 1, x_2^2 = -1$, and $(x_1, x_2) = 0$. Note that this is not a lattice basis for U.

If N is a lattice of type II_n that corresponds to a norm zero vector $x \in L$ then $x^\perp/\langle x\rangle \cong N$ with $\langle x\rangle$ the one-dimensional singular lattice generated by x. Also, $x^\perp \cong N \oplus 0$ where we write 0 for the one-dimensional singular lattice. If N has roots then the Coxeter diagram of $N \oplus 0$ is the Coxeter diagram of N with all irreducible components R replaced by the extended diagrams $\tilde{R}$.
Now assume that N has roots and let $L = N \oplus U$ with coordinates (ν, m, n) (with respect to the basis $\{x, y\}$ of U) where $\nu \in N, m, n \in \mathbb{Z}$ and $(\nu, m, n)^2 = \nu^2 - 2mn$. Then $z = (0, 0, 1)$ is a primitive norm zero vector that corresponds to the n-dimensional even unimodular lattice N. The roots in $z^\perp \cong N \oplus 0$ are the vectors of the form $(\alpha, 0, n)$ with $\alpha \in R(N)$ and $n \in \mathbb{Z}$. Suppose that $R(N) = \sum_i R_i$ with the R_i irreducible components of $R(N)$. Let $\{\alpha_i^j \mid j = 1, \dots, \mathrm{rk}\ R_i\}$ be a set of simple roots for the components R_i and let α_i^0 be the corresponding highest roots. As a set of simple roots for $z^\perp$ we can pick the vectors

$$\begin{aligned} f_i^j &= (\alpha_i^j, 0, 0), \text{ for } j \geq 1, \text{and} \\ f_i^0 &= (\alpha_i^0, 0, 1). \end{aligned}$$

So the set of vectors $\{f_i^j \mid j = 0, \dots, \mathrm{rk}\ R_i\}$ form the extended Coxeter diagram $\tilde{R}_i$ and furthermore $\sum_j k_j f_i^j = z$, where the k_j are the weights of the vertices of $\tilde{R}_i$. If we apply Vinberg's algorithm with z as a controlling vector we find a unique fundamental domain $\tilde{D}$ of L that contains z and such that all the vectors f_i^j that form the extended Coxeter diagrams $\tilde{R}_i$ are simple roots of $\tilde{D}$.
On the other hand, suppose that $\tilde{D}$ is a fundamental domain of L and that $\tilde{R}$ is a connected extended Coxeter diagram in the Coxeter diagram of $\tilde{D}$. Then there is a primitive norm zero vector x in L that is in $\tilde{D}$ and such that $(x, \alpha_i) = 0$ for all simple roots α_i of $\tilde{R}$:
Indeed, set $x = \sum_i k_i\alpha_i$ with α_i the simple roots of $\tilde{R}$ and the k_i their weights. Then $x^2 = 0$ and $(x, \alpha_i) = 0$ for all α_i. Furthermore, if α is a simple root of $\tilde{D}$ then $(x, \alpha) \leq 0$ since all α_i have inner product at most zero with α.
It follows that there also is a bijective correspondence between the sets:

1. maximal disjoint sets of extended Coxeter diagrams in the Coxeter diagram of $\tilde{D}$ such that no two diagrams in the set are joined to each other, and
2. primitive norm zero vectors of $\tilde{D}$ that have at least one root perpendicular to them.

5.2 The lattices $\mathrm{II}_{9,1}$ and $\mathrm{II}_{17,1}$

Consider $\mathbb{R}^{n+1,1}$ with quadratic form $x^2 = \sum_{i=1}^{n+1} x_i^2 - x_{n+2}^2$. Denote by $\{\epsilon_i\}_{i=1}^{n+2}$ the standard basis of $\mathbb{R}^{n+1,1}$. Consider $\mathrm{II}_{n+1,1}$ with $n \in 8\mathbb{N}$. Let $r = (\frac{1}{2}, ..., \frac{1}{2}) \in \mathbb{R}^{n+1,1}$. Then $x \in \mathbb{R}^{n+1,1}$ is an element of $\mathrm{II}_{n+1,1}$ if:

1. $x \cdot r \in \mathbb{Z}$, i.e. $\sum_{i=1}^{n+1} x_i - x_{n+2} \in 2\mathbb{Z}$, and
2. all $x_i \in \mathbb{Z}$ or all $x_i - \frac{1}{2} \in \mathbb{Z}$.

Write $R \equiv R(\mathrm{II}_{n+1,1})$. The group of orthochronous automorphisms of $\mathrm{II}_{n+1,1}$ (i.e. the group of automorphisms that do not interchange the positive and negative time cones) is the semidirect product of the group of reflections W of $\mathrm{II}_{n+1,1}$ with the symmetry group of the Weyl chamber $\tilde{D}$ determined by W. The polyhedron $\tilde{D}$ can be determined by applying Vinberg's algorithm. Choose the controlling vector $c = (0, ..., 0, 2) \in \mathbb{R}^{n+1,1}$. Then $c^2 = -4$ and $c \in P = \{\lambda \in \mathbb{R}^{n+1,1} \mid (\lambda, \alpha) \in \mathbb{Z}$ for all $\alpha \in R\}$ since $-(c, x) = 2x_{n+2} \in \mathbb{Z}$ for all $x \in \mathrm{II}_{n+1,1}$. Now

$$\begin{aligned} R \cap c^{\perp} &= \left\{ \alpha = \sum_{i=1}^{n+2} a_i \epsilon_i \in R \;\middle|\; a_{n+2} = 0 \right\} \\ &= \left\{ \alpha \in \mathbb{R}^{n+1,1} \;\middle|\; a_{n+2} = 0, a_i \in \mathbb{Z} \text{ for all } i, \sum_i a_i \in 2\mathbb{Z} \text{ and } \sum_i a_i^2 = 2 \right\} \\ &= D_{n+1}. \end{aligned}$$

Use for D_{n+1} the basis:

$$\begin{aligned} e_i &= \epsilon_i - \epsilon_{i+1}, \text{ for } i = 1, ..., n, \\ e_{n+1} &= \epsilon_n + \epsilon_{n+1}. \end{aligned}$$

So here we have chosen the controlling vector c that corresponds with the sublattice D_{n+1} of $\mathrm{II}_{n+1,1}$. There are however many more possibilities for the controlling vector. Another 'good' choice would be the norm zero vector $(0, ..., 0, 1, -1)$ that corresponds to the sublattice $n\tilde{E}_8$. If it exists, the Weyl vector ρ is another possible controlling vector. In this case all the simple roots have height 1 (and all roots of height 1 are simple). For $n = 8, 16$ or 24 there in fact exists a Weyl vector ρ_{n+1} given by:

$$\begin{aligned} \rho_9 &= (0, 1, \ldots, 8, 38), \\ \rho_{17} &= (0, 1, \ldots, 16, 46), \\ \rho_{25} &= (0, 1, \ldots, 24, 70). \end{aligned}$$

This is shown in Chapter 27 of [6]. There it is also remarked that for $n = 32, 40, ...$ there exists no vector in $\mathbb{R}^{n+1,1}$ having constant inner product with all simple roots of the lattice of type $\mathrm{II}_{n+1,1}$.

Example 2. $\mathrm{II}_{9,1}$
Now assume $n = 8$. As above we pick the controlling vector $c = (0, \ldots, 0, 2) \in \mathbb{R}^{9,1}$. So $R \cap c^{\perp} = D_9$ and we use the basis $\{e_1, \ldots, e_9\}$ as described above. Let $\alpha = \sum_i a_i \epsilon_i$ be a root of height 1, i.e. $-(c, \alpha) = 2a_{10} = 1$. Thus $a_{10} = \frac{1}{2}$. This vector is accepted as a simple root by Vinberg's algorithm if it satisfies:

$$\begin{aligned} a_i - a_{i+1} &\leq 0, \text{ for } i = 1, ..., 8 \\ a_8 + a_9 &\leq 0 \end{aligned}$$

Since α is a root it also satisfies $\alpha^2 = \sum_{i=1}^{9} a_i^2 - \frac{1}{4} = 2$. Furthermore, since $a_{10} = \frac{1}{2}$, $a_i + \frac{1}{2} \in \mathbb{Z}$ for all i. Then the only possibility is that all $a_i \in \{\frac{1}{2}, -\frac{1}{2}\}$. It follows from the criteria above that if $a_i = \pm\frac{1}{2}$ for all i then

$$\begin{aligned} a_i &= \frac{1}{2} \Rightarrow a_{i+1} = \frac{1}{2} \text{ for } i = 1, ..., 8, \\ a_{i+1} &= -\frac{1}{2} \Rightarrow a_i = -\frac{1}{2} \text{ for } i = 1, ..., 8. \end{aligned}$$

From $a_8 + a_9 \leq 0$ it then follows that $a_1, ..., a_8$ are equal to $-\frac{1}{2}$ and from $\alpha \cdot r \in \mathbb{Z}$ it follows that $a_9 = \frac{1}{2}$. So there is just one vector e_{10} of height 1 that satisfies the criteria in Vinberg's algorithm, namely

$$e_{10} = -\frac{1}{2}\sum_{i=1}^{8} \epsilon_i + \frac{1}{2}\epsilon_9 + \frac{1}{2}\epsilon_{10}.$$

The vectors e_i are a basis for $\mathbb{R}^{n+1,1}$. The dual basis is:

$$\begin{aligned}
f_i &= \epsilon_1 + ... + \epsilon_i - i\epsilon_{10}, \text{ for } i = 1, ..., 7 \\
f_8 &= \frac{1}{2}\sum_{i=1}^{8} \epsilon_i - \frac{1}{2}\epsilon_9 - \frac{9}{2}\epsilon_{10} \\
f_9 &= \frac{1}{2}\sum_{i=1}^{9} \epsilon_i - \frac{7}{2}\epsilon_{10} \\
f_{10} &= -2\epsilon_{10}
\end{aligned}$$

Suppose that α is a root of height greater than one. Since the f_i form a basis α can be written as $\alpha = \sum_i b_i f_i$. If α is a root that satisfies the criteria of Vinberg's algorithm then $(\alpha, e_i) \leq 0$ for $i = 1, ..., 10$. From this it follows that all $b_i \leq 0$. Since also $f_i^2 \leq 0$ for all i and thus also $(f_i, f_j) \leq 0$ for all i, j then

$$\alpha^2 = \sum_i \sum_j b_i b_j (f_i, f_j) \leq 0.$$

So there can be no root that satisfies the criteria, the sequence of simple roots determined by Vinberg's algorithm breaks off at the 10th root.
The Coxeter diagram corresponding to the simple roots $e_1, ..., e_{10}$ is the diagram E_{10} in Figure 5.1.

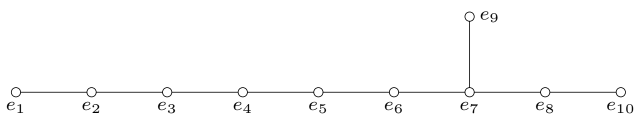


Figure 5.1: E_{10}.

The diagram E_{10} contains the elliptic subdiagram A_9 and $\det(E_{10}) = -1$ (calculation using Lemma 5.1 in [20]: $\det(E_{10}) = \det(A_3)\det(E_7) - 4\cos^2(\frac{\pi}{3})\det(A_2)\det(E_6) = 4 \times 2 - 4 \times \frac{1}{4} \times 3 \times 3 = -1$), so the signature of the Gram matrix of E_{10} is $(9, 1)$ and it is the diagram of a Coxeter polyhedron $\tilde{D} \subset H^9$. Removing the leftmost node in the diagram gives the parabolic subdiagram $\tilde{E}_8$, removing any other node gives elliptic subdiagrams. So $\tilde{D}$ has one vertex at infinity corresponding with E_8. The diagram has no symmetries so the group of orthochronous automorphisms of $\mathrm{II}_{9,1}$ is the reflection group determined by E_{10}, $W(E_{10})$.

Example 3. $\mathrm{II}_{17,1}$
Now suppose $n = 16$. Let $c = (0, \ldots, 0, 2) \in \mathbb{R}^{17,1}$ be the controlling vector and $\{e_1, \ldots, e_{17}\}$ the above described basis for $D_{17} = R \cap c^{\perp}$. Suppose that $\alpha = \sum_i a_i \epsilon_i$ is a root of height 1 that satisfies the criteria of Vinberg's algorithm. Then $\mathrm{ht}(\alpha) = 2a_{18} = 1 \Rightarrow a_{18} = \frac{1}{2}$ and thus $\sum_{i=1}^{17} a_i^2 = 2\frac{1}{4}$. Because $a_{18} = \frac{1}{2}$ and α is in $\mathrm{II}_{17,1}$ all $a_i - \frac{1}{2} \in \mathbb{Z}$. But then $\sum_{i=1}^{17} a_i^2 \geq 4\frac{1}{4}$ so there can be no root of height 1.
Suppose now that α is a root of height 2. Then $a_{18} = 1$ and thus $a_i \in \mathbb{Z}$ for all i. Furthermore, $\alpha^2 = \sum_{i+1}^{17} a_i - 1 = 2$ implies that $\sum_{i+1}^{17} a_i = 3$ and thus the only possibility is that three of the a_i $(i = 1, ..., 17)$ are ± 1 and the others are zero. Vinberg's algorithm imposes the following conditions on the a_i:

$$\begin{aligned}
a_i - a_{i+1} &\leq 0 \text{ for } i = 1, ..., 16 \\
a_{16} + a_{17} &\leq 0
\end{aligned}$$

and thus for $a_i \in \{-1, 0, 1\}$:

$$\begin{aligned} a_i &= 1 \Rightarrow a_{i+1} = 1 \text{ for } i = 1, ..., 16 \\ a_i &= -1 \Rightarrow a_{i-1} = -1 \text{ for } i = 2, ..., 17. \end{aligned}$$

The first property together with $a_{16} + a_{17} \le 0$ implies that $a_i \neq 1$ for all i. The second property implies that the only possible root is

$$e_{18} = -\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_{18}$$

Continuing with Vinberg's algorithm we now consider roots α of height 3. Then $a_{18} = \frac{3}{2}$ and $\alpha^2 = \sum_{i=1}^{17} a_i^2 - \frac{9}{4} = 2$ so $\sum_{i=1}^{17} a_i^2 = 4\frac{1}{4}$. Since $a_{18} = \frac{3}{2}$ all $a_i - \frac{1}{2} \in \mathbb{Z}$ and it follows that the only possibility is that all a_i are $\pm\frac{1}{2}$. From the criteria imposed by Vinberg's algorithm it then follows that there is just one root of height 3 accepted, namely

$$e_{19} = -\frac{1}{2} \sum_{i=1}^{17} \epsilon_i + \frac{3}{2} \epsilon_{18}.$$

The Coxeter diagram determined by the simple roots $e_1, ..., e_{19}$ is shown in Figure 5.2.

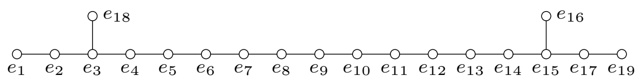


Figure 5.2: Coxeter diagram of $\mathrm{II}_{17,1}$.

This is in fact the Coxeter diagram of the lattice $\mathrm{II}_{17,1}$ which we will now prove by showing that the sequence of simple roots determined by Vinberg's algorithm breaks off at e_{19}. Suppose that there is one more simple root α with $\mathrm{ht}(\alpha) \ge 4$. Then α is an element of the convex polyhedral cone $D = \{\lambda \in V = \mathbb{R}^{17,1} \mid (\lambda, e_i) \le 0 \text{ for all } i = 1, ..., 19\}$. A ray $\mathbb{R}_{<0}\lambda \in D$ is an extremal ray if μ, ν rays in D such that $\mathbb{R}_{<0}\lambda = \mu + \nu$ implies $\mu, \nu = \mathbb{R}_{<0}\lambda$. By the Krein-Milman theorem (for convex cones [15] appendix 2, also see [1]) D is the convex hull of its extremal rays. In the case of $n = 8$ the extremal rays are the $\mathbb{R}_{<0} f_i$. But for $n = 16$ the α_i are not linearly independent so we can not simply take the dual basis.
Let U be a 19 dimensional real vector space with basis $\{\alpha_i\}_{i=1}^{19}$ and equipped with bilinear form given by $(\alpha_i, \alpha_j) = (e_i, e_j)$. This bilinear form has signature $(17, 1, 1)$. Let $\{\lambda_i\}_{i=1}^{19}$ be the dual basis of U^*. Define $D' = \{\lambda \in U^* \mid (\lambda, \alpha_i) \le 0 \text{ for all } i = 1, ...19\} = \{\sum_i k_i \lambda_i \mid k_i \le 0\}$, where (λ, α_i) denotes the natural pairing of U^* and U. D' has 19 extremal rays given by $\mathbb{R}_{<0}\lambda_i$.
The one-dimensional kernel of the bilinear form on U is spanned by $\delta = \sum_{i=1}^{19} k_i \alpha_i$, where the k_i are the numbers next to the corresponding node i in Figure 5.3 ($\delta^2 = 0$ since $\sum_{j \in \{k \mid (\alpha_k, \alpha_i) = -1\}} k_j = 2k_i$ and $\delta \neq 0$).

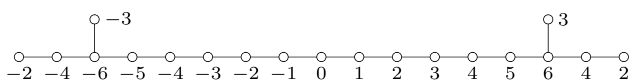


Figure 5.3: Coxeter diagram of $\mathrm{II}_{17,1}$ labelled with the coefficients k_i.

Write $L = \ker_U(\cdot, \cdot) = \mathbb{R}\delta$. Now $V \cong U \setminus L$ and $(U \setminus L)^* \cong L^\perp$ since $U \setminus L$ is equipped with a nondegenerate bilinear form. So $K = D' \cap L^\perp$. Thus the extremal rays of K are either extremal rays of D' that are contained in $L^\perp$ (i.e. perpendicular to δ) or they are the intersection of a 2-dimensional face of D' with $L^\perp$. The only extremal ray of D' perpendicular to δ is $\mathbb{R}_{<0}\lambda_9$ since $k_9 = 0$. The 2-dimensional faces are given by $\mathbb{R}_{\le 0}\lambda_i + \mathbb{R}_{\le 0}\lambda_j$. Now, if for some $t, s \in \mathbb{R}_{<0}$, $(t\lambda_i + s\lambda_j, \delta) = tk_i + sk_j = 0$ then k_i and k_j both have to be nonzero and they can't have the

same sign. So $L^{\perp}$ has a one-dimensional intersection with the 2-dimensional faces $\mathbb{R}_{\leq 0}\lambda_i + \mathbb{R}_{\leq 0}\lambda_j$ such that node i is to the left of the middle node 9 and node j is to the right of node 9. Now the diagram obtained by deleting two such nodes is either positive definite of rank 17 or nonnegative definite and degenerate of rank 18. It follows from Proposition 1 that they correspond to (possibly infinitely distant) vertices of $\tilde{D}$ and hence that the extremal rays of D are all inside of the closure of V_+. Since D is the convex hull of these extremal rays any element of D is in the closure of V_+ and thus cannot be a root. Hence there is no more root that satisfies the criteria of Vinberg's algorithm.
We can also determine directly from the diagram that it corresponds to a reflection group of $H^{17,1}$ with a fundamental domain $\tilde{D}$ of finite volume. Write G for the Gram matrix that corresponds to the diagram above, which we denote by S. Then $\det(G) = \det(E_{10})\det(\tilde{E}_8) - 4\cos^2(\frac{\pi}{3})\det(\tilde{E}_8)\det(E_8) = 0$ but S is not parabolic. Furthermore, S contains the elliptic subdiagram $2E_8 + A_1$ of rank 17. Thus the signature of G is $(17, 1, 1)$ and S is the diagram of a Coxeter polyhedron $\tilde{D} \subset H^{17}$. Removing the middle node gives the parabolic subdiagram $2\tilde{E}_8$ of rank 16, so this subdiagram corresponds to a vertex q at infinity. Removing the leftmost and the rightmost node also gives a parabolic subdiagram of rank 16, namely $\tilde{D}_{16}$. So this subdiagram also corresponds to a vertex at infinity. All the other vertices are obtained by removing one node left of the middle vertex and one node right of the middle vertex, the subdiagrams that are obtained are all elliptic. So $\tilde{D} \subset H^n$ is a polyhedron with finite volume and two vertices at infinity. Topologically, $\tilde{D}$ is a pyramid with vertex at q, constructed on the direct product of two 9-simplexes.

The only symmetry of the diagram is a reflection through the middle. So $\mathrm{Sym}(\tilde{D}) = \mathbb{Z}_2$ and the group of orthochronous automorphisms of $\mathrm{II}_{17,1}$ is the semidirect product of $W(S) =< r_{e_i} \mid i = 1, ..., 19 >$ with $\mathbb{Z}_2$.

5.3 Niemeier lattices

In 1935 Witt found more than 10 of the 24-dimensional even unimodular lattices, now called Niemeier lattices. In 1965 Leech found such a lattice without roots, now called the Leech lattice. The list of 24-dimensional even unimodular lattices was completed by Niemeier in 1967. It turned out that there exist twenty-four Niemeier lattices. Niemeier's proof was later simplified by Boris Venkov ([6], Chapter 18). His proof consists of two parts. First, using modular forms, he determines a list of possible root systems for a Niemeier lattice. These are the twenty-four root systems listed in Table 5.1. Then by a case by case verification it is shown that for each root system there exists a unique even unimodular lattice. We will discuss this proof in this section.

First of all, the list of possible root systems is determined by showing that a Niemeier lattice N has to satisfy:

1. $R(N) = \emptyset$ or rank $R(N) = 24$,
2. All irreducible components of $R(N)$ have the same Coxeter number h,
3. $|R(N)| = 24h$.

To show this the next proposition is crucial. It is proved using the theory of modular forms, see [6], chapter 18.

Proposition 2 (Venkov 1980). *If N is a Niemeier lattice and $x \in \mathbb{R}^{24}$ then*

$$\sum_{\alpha \in R(N)} (\alpha, x)^2 = \frac{1}{12}(x, x)|R(N)| \tag{5.2}$$

Corollary 1. *If N is a Niemeier lattice then either $R(N) = \emptyset$ or rank $R(N)$=24.*

Proof. Suppose that rank $R(N) < 24$. Then there is a nonzero vector $y \in \mathbb{R}^{24}$ that is orthogonal to all $\alpha \in R(N)$. If we now set $x = y$ in 5.2 then it follows that $0 = \frac{1}{12}(y, y)|R(N)|$. So $|R(N)| = 0$, i.e. $R(N) = \emptyset$. □

Table 5.1: Niemeier lattices listed by their root systems, h is the Coxeter number and V is the number of Leech lattice points around the corresponding deep hole (see section 6.5).

Root system	h	V	Root system	h	V
D_{24}	46	25	$2A_9 + D_6$	10	27
$D_{16} + E_8$	30	26	$4D_6$	10	28
$3E_8$	30	27	$3A_8$	9	27
A_{24}	25	25	$2A_7 + 2D_5$	8	28
$2D_{12}$	22	26	$4A_6$	7	28
$A_{17} + E_7$	18	26	$4A_5 + D_4$	6	29
$D_{10} + 2E_7$	18	27	$6D_4$	6	30
$A_{15} + D_9$	16	26	$6A_4$	5	30
$3D_8$	14	27	$8A_3$	4	32
$2A_{12}$	13	26	$12A_2$	3	36
$A_{11} + D_7 + E_6$	12	27	$24A_1$	2	48
$4E_6$	12	28	$\emptyset$	0	-

Corollary 2. *If N is a Niemeier lattice then all irreducible components of $R(N)$ have the same Coxeter number h and $|R(N)| = 24h$.*

Proof. If we set $x = \beta \in R(N)$ in equation 5.2 then it follows that

$$\sum_{\alpha \in R(N)} (\alpha, \beta)^2 = \frac{1}{6}|R(N)|. \tag{5.3}$$

Denote by R_β the irreducible component of $R(N)$ that contains β. Then if $\alpha \notin R_\beta$ clearly $(\alpha, \beta) = 0$. So only $\alpha \in R_\beta$ contribute to the sum on the left side of the equation above. Since R_β is a simply laced root system for $\alpha, \alpha' \in R_\beta$ we have $(\alpha, \alpha') \in \{0, \pm 1, \pm 2\}$ and $(\alpha, \alpha') = \pm 2$ if and only if $\alpha' = \pm\alpha$. Let $\gamma(R_\beta) = \sharp\{\alpha \in R_\beta \mid (\alpha, \beta) = 1\}$. Then it follows that

$$\sum_{\alpha \in R(N)} (\alpha, \beta)^2 = 2 \times 2^2 + 2\gamma(R_\beta). \tag{5.4}$$

The Coxeter number h has the following property: If R is a simply laced root system then the number of elements of R not orthogonal to a fixed $\alpha \in R$ is equal to $4h - 6$ ([4], chapter VI, prop. 32). Hence $2 + 2\gamma(R_\beta) = 4h(R_\beta) - 6$ and thus $\gamma(R_\beta) = 2h(R_\beta) - 4$. So combining this with 5.3 and 5.4 we get

$$2 \times 2^2 + 2(2h(R_\beta) - 4) = \frac{1}{6}|R(N)|.$$

Hence $|R(N)| = 24h(R_\beta)$ and since $\beta \in R(N)$ is chosen arbitrarily both claims in the corollary now follow. □

Proposition 3. *The root system of a Niemeier lattice is one of those in Table 5.1.*

Proof. Suppose $R := R(N) \neq \emptyset$. Then R is isometric to an orthogonal sum of root lattices of the types A_i, D_j and E_k. Say

$$R = \sum_{i=1}^{24} p_i A_i + \sum_{j=4}^{24} q_j D_j + \sum_{k=6}^{8} r_k E_k.$$

Since all components of R have the same Coxeter number and $h(A_n) = n + 1$, $h(D_n) = 2n - 2$, $h(E_6) = 12$, $h(E_7) = 18$ and $h(E_8) = 30$ it follows that R is of type $pA_i + qD_j + rE_k$. By Corollary 1, rank $R = 24$ and thus $pi + qj + rk = 24$. So we have the following possibilities for R:

1. $R = pA_i$: Then it follows from $pi = 24$ that the possibilities are:
   A_{24}, $2A_{12}$, $3A_8$, $4A_6$, $6A_4$, $8A_3$, $12A_2$ and $24A_1$.

2. $R = qD_j$: Then it follows from $qj = 24$ that the possibilities are:
   D_{24}, $2D_{12}$, $3D_8$, $4D_6$ and $6D_4$.

3. $R = rE_k$: Then it follows from $rk = 24$ that the possibilities are: $4E_6$ and $3E_8$.
4. $R = pA_i + qD_j$ $(p, q \neq 0)$: From $h(D_j) = 2j - 2 = h(A_i) = i + 1$ it follows that $i = 2j - 3$. Hence $p(2j-3) + qj = 24$ with $j \geq 4$ and this equation has the following solutions: $(j, p, q) = (9, 1, 1)$, $(6, 2, 1)$, $(5, 2, 2)$ and $(4, 4, 1)$.
   So in this case the possibilities for R are: $A_{15} + D_9$, $2A_9 + D_6$, $2A_7 + 2D_5$ and $4A_5 + D_4$.
5. $R = pA_i + qD_j + rE_k$ with $r \neq 0$ and q and p not both zero:
   (a) If $k = 6$ then $h = 12$ and thus $j = 7$ and $i = 11$ (if $p, q \neq 0$). So $11p + 7q + 6r = 24$. Now r can be one, two or three and it easily follows that there is only a solution for $r = 1$, namely $(p, q) = (1, 1)$. So this gives $R = A_{11} + D_7 + E_6$.
   (b) If $k = 7$ then $h = 18$ and thus $j = 10$ and $i = 17$ (if $p, q \neq 0$). There are two solutions for $17p + 10q + 7r = 24$, namely $(p, q, r) = (1, 0, 1)$ or $(p, q, r) = (0, 1, 2)$. Hence we get the following two possibilities for R: $A_{17} + E_7$ and $D_{10} + 2E_7$.
   (c) If $k = 8$ then $h = 30$ and thus $j = 16$ and $i = 29$ (if $p, q \neq 0$). Now $i = 29$ is not possible, so $p = 0$ and $16q + 8r = 24$. This yields as only solution $R = D_{16} + E_8$.

We see that the possibilities that are found are indeed exactly those listed in Table 5.1.

□

The next step is to determine the existence and uniqueness of a lattice of type II_{24} for all the root systems in Table 5.1. We postpone the proof of the existence and uniqueness of the lattice with empty root system, the Leech lattice, to Chapter 6.
Let $Q = \mathbb{Z}R$ be a root lattice generated by one of the nonempty root systems in Table 5.1. If $R = 3E_8$ then Q is unimodular and we are finished. In the other cases Q is not unimodular so we then need to determine a glue code and show that it is unique. Recall that for a lattice L the glue group is L^*/L. So for $R = \sum_i n_i R_i$ (with the R_i irreducible) the glue vectors are elements of the group

$$T(R) = P(R)/Q(R) = \bigoplus_i (P(R_i)/Q(R_i))^{n_i} .$$

Hence the glue code is an additive subgroup $A < T(R)$. As coset representatives for $T(R_i)$ we pick the $w_i^n \in P(R_i)$ that satisfy $(w_i^n, \alpha) \in \{0, 1\}$ for all $\alpha > 0$ in R_i. It is well known that these vectors form a complete set of coset representatives, they correspond to the special vertices of $\tilde{R}_i$. Furthermore, for $g_i^j \in T(R_i)$, let $l(g_i^j)$ be the norm of the corresponding coset representative w_i^n. Finally for a glue vector $g^j = (g_1^j, ...g_k^j) \in T(R)$ (with k the number of irreducible components of R) the function l is extended to be:

$$l(g^j) = \sum_{i=1}^{k} l(g_i^j).$$

Since the glue code must give a unimodular lattice, A has to satisfy $|A|^2 = |T(R)|$. Furthermore, since the lattice must be even and with root lattice R all $g \in A \setminus \{0\}$ must have the norm $l(g)$ equal to an even integer that is > 2. We call a subgroup A that satisfies these two conditions even and self-dual.
If R_i is an irreducible root system then the group $G(R_i) := \mathrm{Aut}(R_i)/W(R_i)$ acts transitively on $T(R_i)$. This group $G(R_i)$ is the symmetry group of the Coxeter diagram of R_i. For $R = n_i R_i$ the corresponding symmetry group of the Coxeter diagram of R acting on $T(R)$ is the wreath product $G(R_i)^{n_i} \rtimes S_{n_i}$. For $R = \sum_i n_i R_i$ it is

$$G(R) = \prod_i (G(R_i)^{n_i} \rtimes S_{n_i}) .$$

Clearly, the norm $l(g)$ corresponding to $g \in A$ is invariant under the action of $G(R)$. Hence the set of even and self-dual subgroups $A < T(R)$ is $G(R)$-invariant. The Niemeier lattice constructed from the glue code A is the lattice $N =< R, A >$ generated by the root system R and the glue code A. Conversely, given a Niemeier lattice N with root system R a glue code for this lattice is $A = N/Q(R(N)) < P(R(N))/Q(R(N))$. It follows that there is a natural one-to-one correspondence between Niemeier lattices with nonempty root system isomorphic to R (up to isomorphism) and even self-dual subgroups $A < T(R)$ (up to the action of $G(R)$).

Example 4. Let $R = 4E_6$. We have $T(E_6) = \mathbb{Z}/3\mathbb{Z} = \{0, \pm 1\}$ and $G(E_6) = \mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ with $\sigma(\pm 1) = \mp 1$. Furthermore, $l(0) = 0$ and $l(\pm 1) = 4/3$. It follows that $T(R) = (\mathbb{Z}/3\mathbb{Z})^4$ and $G(R) = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$. So a Niemeier lattice with root system $4E_6$ corresponds to an even selfdual subgroup $A < \{x = (x_1, x_2, x_3, x_4) \mid x_i = 0, \pm 1\}$ of order 9. The only elements that are even with norm > 2 have three coordinates equal to 1 or -1 and the other one equal to 0. We can assume that $x = (1, 1, 1, 0) \in A$. This is an element of order 3 so we need to add more elements to get a subgroup of order 9. Because the sum of two elements has to be even and with norm > 2 it follows that the only elements that can be added are of the form $y = (y_1, y_2, y_3, y_4)$ with $\{y_1, y_2, y_3\} = \{0, 1, -1\}$ and $y_4 = \pm 1$. Any element of this form can be changed into any other element of this form by an element of $G(R)$ that leaves x invariant. So without loss of generality we can assume that $y = (0, -1, 1, 1) \in A$. Now $A =< x, y >$ is an even self-dual subgroup of order 9 and this group is unique op to action of $G(R)$. The glue code A is in fact equal to the *tetracode* (see for example [6], Chapter 3).

For most of the possible root systems the proof of the existence and uniqueness of a Niemeier lattice with this root system comes down to an easy verification like the one in the example above. Particularly easy cases are the ones $R = D_{24}$ and $R = E_8 + D_{16}$ for which the corresponding Niemeier lattices are isomorphic to D_{24}^+ respectively $E_8 \oplus D_{16}^+$.
The cases $R = 24A_1$ and $R = 12A_2$ are not that easy. They depend on the existence and uniqueness of the (extended) binary and ternary Golay codes.

Example 5. Let $R = 24A_1$. We have $T(A_1) = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, $G(A_1) = 1$, $l(0) = 0$ and $l(1) = 1/2$. So $T(R) = (\mathbb{Z}/2\mathbb{Z})^{24}$ and $G(R) = S_{24}$. Furthermore, for $x \in T(R)$ we have $l(x) = \frac{1}{2}\mathrm{wt}(x)$. Here $\mathrm{wt}(x)$ is the *(Hamming) weight* of the codeword x that is defined to be the number of nonzero coordinates of x. It follows that any x in an even self-dual subgroup $A < T(R)$ must satisfy $\mathrm{wt}(x) \geq 8$. So an even self-dual subgroup $A < T(R)$ corresponds to a binary self-dual code with minimum distance 8. The unique such code is the (extended) binary Golay code, see [11] Chapter 10 for a proof of the existence and uniqueness of this code.

Example 6. Let $R = 12A_2$. We have $T(A_2) = \mathbb{Z}/3\mathbb{Z} = \{0, \pm 1\}$, $G(A_2) = \mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ with $\sigma(\pm 1) = \mp 1$, $l(0) = 0$ and $l(\pm 1) = 2/3$. So $T(R) = (\mathbb{Z}/3\mathbb{Z})^{12}$ and $G(R) = (\mathbb{Z}/2\mathbb{Z})^{12} \rtimes S_{12}$. Furthermore, for $x \in T(R), l(x) = \frac{2}{3}\mathrm{wt}(x)$. It follows that any glue vector x in an even self-dual subgroup $A < T(R)$ must satisfy $\mathrm{wt}(x) \geq 6$. So an even self-dual subgroup $A < T(R)$ corresponds to a self-dual code in $\mathbb{F}_3^{12}$ with minimum distance 6. The (extended) ternary Golay code is the unique such code, again see [11] Chapter 10.

As noted in the beginning of this chapter, another way to prove the correctness of the classification of the Niemeier lattices is by using the Minkowski-Siegel mass formula ([6], Chapter 16). Yet another proof can be given using the list of deep holes of the Leech lattice as determined in [6] Chapter 23. As we will see in the next chapter, the Leech lattice corresponds to the Coxeter diagram of the unique lattice of type $\mathrm{II}_{25,1}$ and deep holes of the Leech lattice correspond to subdiagrams of this Coxeter diagram that are unions of the extended Coxeter diagrams $\tilde{A}_n (n \geq 1)$, $\tilde{D}_n (n \geq 4)$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$. But as seen in section 5.1 these subdiagrams correspond exactly to the Niemeier lattices with roots. Indeed, the diagrams found in the enumeration of the deep holes of the Leech lattice correspond exactly to the root systems of the Niemeier lattices with roots. The determination of the deep holes is done by extensive computations so this provides by no means a more straightforward proof of the classification. However, in the cases $R = 24A_1$ and $R = 12A_2$ a corresponding deep hole can be constructed quite easily as is shown at the end of the next chapter.

We end this section with some results on Niemeier lattices we need later. They are all taken from [3] section 2.

Lemma 5. *Let R be a simply laced irreducible root system and $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$ its Weyl vector. Then $\rho^2 = \frac{1}{12}nh(h+1)$, where h is the Coxeter number of R and n its rank.*

Proof. Let α_i be a simple root. Then $\rho(\alpha_i) = -1$ and it thus follows that $\rho = -\sum_i \omega_i$ where the ω_i are the fundamental weights, i.e. the dual basis for the basis of simple roots. So if $\rho = \sum_i c_i\alpha_i$

(where $c_i < 0$) then $\rho^2 = (\sum_i c_i\alpha_i, -\sum_i \omega_i) = -\sum_i c_i$. The Weyl vectors ρ_R for the simply laced irreducible root systems are:

$$\begin{aligned}
\rho_{A_n} &= -\frac{1}{2}\left(n\alpha_1 + 2(n-1)\alpha_2 + \ldots + i(n-i+1)\alpha_i + \ldots + n\alpha_n\right), \\
\rho_{D_n} &= -\left((n-1)\alpha_1 + (2n-3)\alpha_2 + \ldots + (in - \frac{i(i+1)}{2})\alpha_i + \ldots + \frac{n(n-1)}{4}(\alpha_{n-1}+\alpha_n)\right), \\
\rho_{E_6} &= -\left(8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6\right), \\
\rho_{E_7} &= -\frac{1}{2}\left(34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7\right), \\
\rho_{E_8} &= -\left(46\alpha_1 + 68\alpha_2 + 91\alpha_3 + 135\alpha_4 + 110\alpha_5 + 84\alpha_6 + 57\alpha_7 + 29\alpha_8\right),
\end{aligned}$$

see for example [4] (note that we have a different sign convention). Furthermore, the Coxeter numbers are $h(A_n) = n+1$, $h(D_n) = 2n-2$, $h(E_6) = 12$, $h(E_7) = 18$, and $h(E_8) = 30$. Now

$$\begin{aligned}
\rho_{A_n}^2 &= \frac{1}{2}\sum_{i=1}^{n} i(n-i+1) = \frac{1}{2}\left[n\sum_{i=1}^{n} i - \sum_{i=1}^{n} i(i-1)\right] \\
&= \frac{1}{2}\left[n\frac{n(n+1)}{2} - \frac{1}{3}n(n^2-1)\right] = \frac{1}{12}n(n+1)(n+2), \\
\rho_{D_n}^2 &= \sum_{i=1}^{n-2}\left(in - \frac{i(i+1)}{2}\right) + \frac{n(n-1)}{2} = n\sum_{i=1}^{n-2} i - \frac{1}{2}\sum_{i=1}^{n-1} i(i-1) + \frac{n(n-1)}{2} \\
&= n\frac{(n-2)(n-2+1)}{2} - \frac{1}{2}\cdot\frac{1}{3}(n-1)((n-1)^2-1) + \frac{n(n-1)}{2} \\
&= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{12}n(2n-2)(2n-1) \\
\rho_{E_6}^2 &= 8+11+15+21+15+8 = 78 = \frac{1}{12}\cdot 6\cdot 12\cdot 13, \\
\rho_{E_7}^2 &= \frac{1}{2}(34+49+66+96+75+52+27) = \frac{399}{2} = \frac{1}{12}\cdot 7\cdot 18\cdot 19, \\
\rho_{E_8}^2 &= 46+68+91+135+110+84+57+29 = 620 = \frac{1}{12}\cdot 8\cdot 30\cdot 31,
\end{aligned}$$

where we have used the easily verifiable identities $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ and $\sum_{i=1}^{n} i(i-1) = \frac{1}{3}n(n^2-1)$. We see that the Weyl vectors do indeed satisfy $\rho^2 = \frac{1}{12}nh(h+1)$. □

Note that this formula for the norm of the Weyl vector was not discovered by just calculating ρ^2 like in the proof above. It can be proved using the strange formula of Freudenthal and de Vries that was proven by them in [8].

Lemma 6 (Venkov 1980)**.** *Let N be a Niemeier lattice with roots and ρ its Weyl vector. Then* $\rho^2 = 2h(h+1)$

Proof. By Corollary 1 rank $R(N) = 24$, say $R(N) = \sum_{i=1}^{k} R_i$ where the R_i are irreducible root systems and $\sum_{i=1}^{k} n_i = 24$ with $n_i = \text{rank } R_i$. Also, by Corollary 2 all irreducible components of $R(N)$ have the same Coxeter number $h = \frac{1}{24}|R(N)|$. Furthermore, for all irreducible components R_i of $R(N)$ with Weyl vector ρ_{R_i} it follows from Lemma 5 that $\rho_{R_i}^2 = \frac{1}{12}n_i h(h+1)$. So $\rho^2 = \sum_{i=1}^{k}\rho_{R_i}^2 = \sum_{i=1}^{k}\frac{1}{12}n_i h(h+1) = 2h(h+1)$. □

Lemma 7. *If N is a Niemeier lattice and $\nu \in N$ then*

$$\sum_{\alpha\in R(N)} (\nu,\alpha)^2 = 2h\nu^2.$$

Proof. By Proposition 2 and Corollary 2 we have:

$$\begin{aligned}
\sum_{\alpha\in R(N)} (\nu,\alpha)^2 &= \frac{1}{12}\nu^2|R(N)| \\
&= 2h\nu^2.
\end{aligned}$$

□

Lemma 8. *If ρ is the Weyl vector of a Niemeier lattice N then ρ lies in N.*

Proof. We will show that if $\nu \in N$ then $(\rho, \nu) \in \mathbb{Z}$. Because N is unimodular it then follows that $\rho \in N$. So let $\nu \in N$. Then

$$\begin{aligned}
(2\rho, \nu)^2 &= (\sum_{\alpha>0} \alpha, \nu)^2 \\
&= (\sum_{\alpha>0} (\alpha, \nu))^2 \\
&\equiv \sum_{\alpha>0} (\alpha, \nu)^2 \bmod 2 \\
&= \nu^2 h \qquad \text{by Lemma 7} \\
&\equiv 0 \bmod 2 \qquad \text{since } N \text{ is even.}
\end{aligned}$$

So $(2\rho, \nu)^2$ is an even integer. Since $(2\rho, \nu) = \sum_{\alpha>0}(\alpha, \nu) \in \mathbb{Z}$ it follows that $(2\rho, \nu) \in 2\mathbb{Z}$ and hence $(\rho, \nu) \in \mathbb{Z}$. □

Lemma 9. *Let N be a Niemeier lattice with roots. Denote by $Q = \mathbb{Z}R(N)$ the root lattice of N and by P the corresponding dual weight lattice. For all $\lambda \in P$ we have*

$$\left(\frac{\rho}{h} - \lambda\right)^2 \geq 2\left(1 + \frac{1}{h}\right),$$

and the λ for which equality holds form a complete set of representatives for the glue group P/Q.

Proof. Since $\rho^2 = 2h(h+1)$, for all $\lambda \in P$ we have:

$$\begin{aligned}
\left(\frac{\rho}{h} - \lambda\right)^2 - 2\left(1 + \frac{1}{h}\right) &= \left(\frac{\rho}{h} - \lambda\right)^2 - \left(\frac{\rho}{h}\right)^2 \\
&= \lambda^2 - \frac{2}{h}(\rho, \lambda) \\
&= \frac{1}{h}\left(\sum_{\alpha>0} (\lambda, \alpha)^2 - \sum_{\alpha>0} (\lambda, \alpha)\right) \qquad \text{by Lemma 7} \\
&= \frac{1}{h}\sum_{\alpha>0} (\lambda, \alpha)\left[(\lambda, \alpha) - 1\right] \geq 0 \qquad \text{since } (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha > 0.
\end{aligned}$$

All terms in the above sum are in fact ≥ 0 so there is an equality if and only if $(\lambda, \alpha) \in \{0, 1\}$ for all $\alpha > 0$. It is well known that the $\lambda \in P$ for which this holds form a complete set of coset representatives for P/Q. □

The lemma implies that the covering radius of P, and hence also of N, is at least $\sqrt{2(1 + 1/h)}$. It is not necessarily equal to this. For example, the covering radius of $N = 3E_8$ is $\sqrt{3} > \sqrt{2(1 + 1/30)}$. So the covering radius of a Niemeier lattice with roots is always greater than $\sqrt{2}$. Later on we will see that the covering radius of the Leech lattice is exactly $\sqrt{2}$.

Chapter 6

The Leech Lattice

The Leech lattice is the unique Niemeier lattice without roots. It was discovered by Leech in 1965. Soon after this discovery, Leech conjectured that the lattice had covering radius $\sqrt{2}$ because there were several known holes of this radius. He failed to find a proof. Parker later noticed that the known holes all seemed to correspond to a Niemeier lattice. Inspired by this, Conway, Parker and Sloane found all the holes of this radius ([6], Chapter 23). There turned out to be 23 classes of deep holes corresponding in a natural way with the 23 Niemeier lattices with roots. Using the fact that the covering radius of the Leech lattice is $\sqrt{2}$ Conway later proved that the lattice of type $\mathrm{II}_{25,1}$ has a Weyl vector and that its Coxeter diagram can be identified with the Leech lattice ([6], Chapter 27).
Most of the proofs of these results involved long calculations and case by case verifications. In his paper from 1985 on the Leech lattice ([3]), Richard Borcherds gave new more conceptual proofs of the existence and uniqueness of the Leech lattice and of the fact that it has covering radius $\sqrt{2}$. He also gave a uniform proof of the correctness of the "holy constructions" of the Leech lattice which are described in [6], Chapter 24. In this chapter we present these proofs.
We will first give the proof of the existence and uniqueness of the Leech lattice and then turn to the covering radius and the deep holes of the Leech lattice. Although this seems like the most natural order one must note that the proof we give in section 6.2 of the uniqueness of the Leech lattice does depend on the fact that the Leech lattice has covering radius $\sqrt{2}$. This will be proved in section 6.4 without using any results from section 6.2.

6.1 The existence of the Leech lattice

Here we prove the existence of a Niemeier lattice with no roots following section 4 in [3]. The existence is proved by showing that given any Niemeier lattice with roots we can construct another Niemeier lattice with half as many roots.
Let N be a Niemeier lattice with Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and Coxeter number h. Let $L = N \oplus U$ be the lattice of type $\mathrm{II}_{25,1}$ with coordinates (ν, m, n), where $\nu \in N$, $m, m \in \mathbb{Z}$ and $(\nu, m, n)^2 = \nu^2 - 2mn$. Let $z = (0, 0, 1)$. Then z is the primitive norm zero vector corresponding to the Niemeier lattice N. The roots in $z^{\perp}$, that is the elements in the set $\{\alpha \in R(L) \mid (\alpha, z) = 0\}$, form the affine root system associated with N.
Let $z' = (\rho, h, h+1)$. It follows from Lemma 7 that $z' \in N$. Furthermore, by Lemma 5 it has norm 0. Hence z' is also a primitive norm zero vector in $\mathrm{II}_{25,1}$ and therefore corresponds to a Niemeier lattice, say N'. Note that z and z' are in the same connected component of the set of non zero norm zero vectors since $((1-t)z + tz')^2 = 2(1-t)t(z, z') = -2h(1-t)t < 0$ for $t \in (0, 1)$. We will show that the Niemeier lattice N' has at most half as many roots as N.

Lemma 10. *If $\alpha \in R(L)$ is in $z'^{\perp}$ and $(z, \alpha) \leq 0$ then $(z, \alpha) \leq -2$.*

Proof. Let $\alpha = (\nu, m, n) \in R(L)$ be a root perpendicular to z' and suppose that $(z, \alpha) = -m = 0$.

Then α has to be of the form $(\nu, 0, n)$ with $\nu \in R(N)$. Thus

$$\begin{aligned}(\alpha, z') &= ((\nu, 0, n), (\rho, h, h+1)) \\ &= (\nu, \rho) - nh = 0.\end{aligned}$$

But for $\nu \in R(N)$ we have $1 \leq |(\nu, \rho)| \leq h - 1$ and thus $(\nu, \rho) = nh$ is not possible.
Suppose now that $(z, \alpha) = -1$. Then α is of the form $(\nu, 1, n)$ with $\alpha^2 = \nu^2 - 2n = 2$. Furthermore,

$$\begin{aligned}(\alpha, z') &= ((\nu, 1, n), (\rho, h, h+1)) \\ &= (\nu, \rho) - (h+1) - nh = 0,\end{aligned}$$

and thus

$$\begin{aligned}\left(\frac{\rho}{h} - \nu\right)^2 &= \frac{\rho^2}{h^2} - \frac{2}{h}(\rho, \nu) + \nu^2 \\ &= \frac{2h(h+1)}{h^2} - \frac{2}{h}(h + 1 + nh) + 2n + 2 \\ &= 2.\end{aligned}$$

This is in contradiction with Lemma 9 so it follows that $(\alpha, z) \leq -2$. □

Lemma 11. *The Coxeter number h' of the Niemeier lattice N' is at most $\frac{1}{2}h$.*

Proof. If $h' = 0$ then the inequality holds so we can assume $h' \neq 0$. So N' is a Niemeier lattice with roots. The roots $\alpha \in R(L)$ that are in $z'^{\perp}$ form the affine root system associated with $R(N')$. Let R' be an irreducible component of this root system and pick a set of simple roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ for R' with $(\alpha_i, z) \leq 0$. Here r is the rank of the irreducible component of $R(N')$ corresponding to R'. Denote by k_i the weights of the roots α_i. Then $\sum_{i=0}^{r} k_i \alpha_i = z'$ and $\sum_{i=0}^{r} k_i = h'$. By Lemma 10, $(z, \alpha_i) \leq -2$ for $i = 0, \ldots r$ and thus

$$(z', z) = (\sum_{i=0}^{r} k_i \alpha_i, c) \leq -2h'.$$

Since also

$$(z', z) = ((\rho, h, h+1), (0, 0, 1)) = -h,$$

it follows that $h' \leq \frac{1}{2}h$. □

Theorem 2. *There exists a Niemeier lattice with no roots.*

Proof. Start with an arbitrary Niemeier lattice N with Coxeter number $h \neq 0$. By Lemma 11 we can find a Niemeier lattice N' with Coxeter number $\leq \frac{1}{2}h$. By repeating this process we will eventually find a Niemeier lattice with Coxeter number equal to zero and hence with no roots. □

In fact, there are no roots in $z'^{\perp}$ and this gives a direct construction of a Niemeier lattice with no roots after one step. The proof of this will be given in Section 6.5.

6.2 The uniqueness of the Leech lattice

Again denote by Λ a Niemeier lattice without roots. We will show that Λ is unique following the proof in section 6 of [3]. let $L = \Lambda \oplus U$ be the lattice of type $\mathrm{II}_{25,1}$ with the same coordinates as before and let $\tilde{D}$ be the fundamental domain of L that contains the controlling vector $z = (0, 0, 1) \in L$. Denote by V_+ the open cone of vectors with negative norm that contains z in its closure. In the proof of the next theorem we use the fact that the covering radius of Λ is $\sqrt{2}$. This will be proved later.

Theorem 3. *The simple roots of the fundamental domain $\tilde{D}$ of L are just the simple roots $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height one with respect to z.*

Proof. Obviously there can be no roots of height zero since Λ has no roots. By Vinberg's algorithm it then follows that all roots $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height one are simple roots. Now suppose that these are not all roots and let $\alpha = (\mu, m, \frac{1}{2m}(\mu^2 - 1))$ be a simple root of height $m \geq 2$. Because α is a simple root, it satisfies $(\alpha, \alpha_\lambda) \leq 0$ for all simple roots $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height one. Also, since Λ has covering radius $\sqrt{2}$ there is a vector $\lambda \in \Lambda$ such that $(\lambda - \mu/m)^2 \leq 2$. But then

$$\begin{aligned}
(\alpha, \alpha_\lambda) &= (\mu, \lambda) - m(\frac{1}{2}\lambda^2 - 1) - \frac{1}{2m}(\mu^2 - 2) \\
&= m + \frac{1}{m} + (\mu, \lambda) - \frac{m}{2}\lambda^2 - \frac{1}{2m}\mu^2 \\
&= m + \frac{1}{m} - \frac{m}{2}\left(\lambda - \frac{\mu}{m}\right)^2 \\
&\geq m + \frac{1}{m} - m = \frac{1}{m} > 0,
\end{aligned}$$

which is a contradiction with $(\alpha, \alpha_\lambda) \leq 0$. □

Corollary 3. *The Leech lattice is the unique (up to isomorphism) Niemeier lattice with no roots.*

Proof. A Niemeier lattice Λ without roots corresponds (up to isomorphism) to a primitive norm zero vector w in L (up to the action of $\mathrm{Aut}(L)$) with no roots perpendicular to it. So it suffices to show that any two such vectors are conjugate under $\mathrm{Aut}(L)$. Hence if the fundamental domain $\tilde{D}$ of $W < \mathrm{Aut}(L)$ contains only one such vector then the claim follows. So suppose that $w \in L \cap \tilde{D}$ is a primitive norm zero vector with no roots perpendicular to it. Then by Theorem 4 w has inner product -1 with all simple roots of $\tilde{D}$. Since these roots span L it follows that the vector w is unique. □

6.3 Theorem of Conway

In this section Λ will denote a Niemeier lattice without roots. Note that in this and the next section we will not assume that such a lattice is unique up to isomorphism since the prove of this depends on the fact that the covering radius of such a lattice is $\sqrt{2}$. The purpose of this and the next section is to prove that the covering radius is indeed $\sqrt{2}$.
Let $N_m = \sharp\{\lambda \in \Lambda \mid \lambda^2 = m\}$, the number of vectors in Λ with norm equal to m. Two vectors $\lambda, \mu \in \Lambda$ are called *equivalent* if $\lambda - \mu = 2\nu$ for some $\nu \in \Lambda$. That is, we consider $\Lambda/2\Lambda$. Clearly, the number of equivalence classes is $|\Lambda/2\Lambda| = 2^{24}$. We call a vector $\lambda \in \Lambda$ *short* if $\lambda^2 \leq 8$. It turns out that every element of Λ is equivalent to a short vector. Since λ and $-\lambda$ are equivalent, nonzero short vectors in an equivalence class always occur in pairs.

Theorem 4 (Conway)**.** *Each equivalence class contains a short vector. The equivalence classes that contain more than a single pair of short vectors are precisely those that contain vectors of norm* 8*, and these classes all contain exactly* 24 *mutually orthogonal pairs of vectors of that norm.*

Proof. Let $\lambda, \mu \in \Lambda$ be equivalent short vectors that do not form a pair, i.e. $\lambda \neq \pm\mu$. By replacing μ with $-\mu$ if necessary we can assume that $(\lambda, \mu) \geq 0$. Since λ, μ are equivalent there is a $\nu \in \Lambda$ such that $\lambda - \mu = 2\nu$. Hence $(\lambda - \mu)^2 = 4\nu^2 \geq 16$. Now $(\lambda - \mu)^2 = \lambda^2 + \mu^2 - 2(\lambda, \mu)$ and we know that $\lambda^2 \leq 8$, $\mu^2 \leq 8$ and $(\lambda, \mu) \geq 0$. Together this implies that $\lambda^2 = \mu^2 = 8$ and $(\lambda, \mu) = 0$. So two short vectors $\lambda \neq \mu$ that are equivalent but do not form a pair are always orthogonal and of norm 8. It thus follows that an equivalence class with more than one pair of short vectors at most contains 24 pairs of short vectors that furthermore necessarily have norm 8. Hence the number of equivalence classes that contain short vectors is at least

$$N_0 + \frac{N_4}{2} + \frac{N_6}{2} + \frac{N_8}{48}.$$

By calculating the coefficients N_m of the theta series of Λ it can be shown that this sum is in fact equal to 2^{24}. Indeed, since Λ has no roots we can set $N_2 = 0$ in equation (5.1). So

$$\begin{aligned}\theta_\Lambda(z) &= E_6 - \frac{65520}{691}\Delta \\ &= 1 + \frac{65520}{691}(2073q^2 + 176896q^3 + 4197825q^4 + \ldots) \\ &= 1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \ldots,\end{aligned}$$

and thus

$$N_0 + \frac{N_4}{2} + \frac{N_6}{2} + \frac{N_8}{48} = 1 + \frac{196560}{2} + \frac{16773120}{2} + \frac{398034000}{48} = 16777216 = 2^{24}.$$

Hence each equivalence class contains a short vector. This can be either the norm 0 vector, or an opposite pair of vectors of norm 4 or 6, or a collection of 24 mutually orthogonal pairs of vectors of norm 8. □

Corollary 4. *The distance between any two vertices of a hole of the Leech lattice Λ is at most $2\sqrt{2}$.*

Proof. (following the proof in [10]). Suppose that v_i and v_j are two vertices of a hole c such that $(v_i - v_j)^2 \geq 10$. By Theorem 4 the equivalence class of $v_i - v_j$ contains a short vector $\mu \in \Lambda$. Then $v_i - v_j - \mu = 2\nu$ for a $\nu \in \Lambda$ and thus

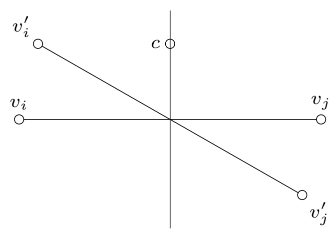
$$(v_i - v_j - 2\nu)^2 = \mu^2 \leq 8.$$

Hence the distance between the vectors $v_i' := v_i - \nu$ and $v_j' := v_j + \nu$ in Λ is at most $2\sqrt{2}$ and they have the same midpoint $(v_i' + v_j')/2 = (v_i + v_j)/2$ as v_i and v_j. It then easily follows that either v_i' or v_j' is closer to c then v_i and v_j are (also see Figure 6.1):

Suppose that

$$\left(\frac{v_i' - v_j'}{2}, c - \frac{v_i' + v_j'}{2}\right) \geq 0. \tag{6.1}$$

Figure 6.1:



Then

$$\begin{aligned}(v_i - c)^2 &= \left(\frac{v_i - v_j}{2}\right)^2 + \left(c - \frac{v_i + v_j}{2}\right)^2, \text{ by Pythagoras} \\ &> \left(\frac{v_i' - v_j'}{2}\right)^2 + \left(c - \frac{v_i' + v_j'}{2}\right)^2, \text{ since } (v_i - v_j)^2 > (v_i' - v_j')^2 \\ &\geq (v_i' - c)^2, \text{ by 6.1.}\end{aligned}$$

□

As before, denote by Λ a Niemeier lattice without roots and let $L = \Lambda \oplus U$ be the lattice of type $\mathrm{II}_{25,1}$. Also, let $z = (0, 0, 1)$ and let $\tilde{D}$ be the fundamental domain of the reflection group $W(L)$ that contains z. As seen in Theorem 3, the simple roots are the roots $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ with

$\lambda \in \Lambda$ (this does depend on the fact that the covering radius of Λ is $\sqrt{2}$, which we will prove in the next section without assuming that all simple roots have height 1). We can therefore identify the simple roots of L with the vectors of Λ and thus the Coxeter diagram of L is indexed by the vectors of Λ. The Gram matrix $G = (g_{\lambda\mu})$ of the simple roots $\alpha_\lambda, \lambda \in \Lambda$ is given by

$$\begin{aligned} g_{\lambda\mu} &= (\alpha_\lambda, \alpha_\mu) \\ &= (\lambda, \mu) - (\frac{1}{2}\mu^2 - 1) - (\frac{1}{2}\lambda^2 - 1) \\ &= 2 - \frac{1}{2}(\lambda - \mu)^2. \end{aligned}$$

Two points λ, μ of Λ are joined by $0, 1, 2, \ldots$ edges if the norm of their difference is $4, 6, 8, \ldots$. By Corollary 4, vertices $\lambda, \mu \in \Lambda(c)$ of a hole c always satisfy $(\lambda - \mu)^2 \leq 8$. Thus the corresponding nodes in the hole diagram are always joined by 0, 1, or 2 edges.
Now embed $\Lambda \otimes \mathbb{R}$ in $\mathbb{R}^{25}$ by identifying $\Lambda \otimes \mathbb{R}$ with the hyperplane $H = \{(x_1, \ldots, x_{25}) \in \mathbb{R}^{25} \mid x_{25} = 0\}$ in $\mathbb{R}^{25}$.Suppose that $R(c) \leq \sqrt{2}$ and let c' be the point on the line perpendicular to H that goes through c that has distance $\sqrt{2}$ from all vertices of the hole (see Figure 6.3). So if $R(c) = \sqrt{2}$ then $c = c'$. If $R(c) > \sqrt{2}$ we define $c' = c$. If the radius of a hole c is $\leq \sqrt{2}$ then for $\lambda, \mu \in \Lambda$:

$$(\lambda - \mu)^2 = 4 - 2(\lambda - c', \mu - c'),$$

so two nodes λ, μ of the hole diagram are joined by 0, 1 or 2 edges if $(\lambda - c', \mu - c') = 0, -1$ or -2 respectively (see Figure 6.2).

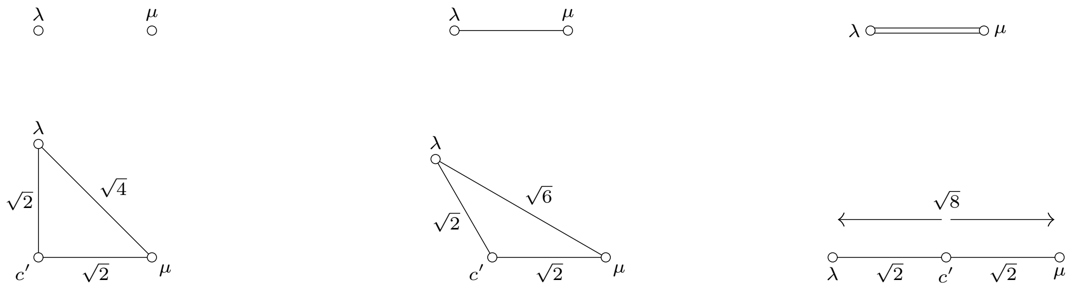


Figure 6.2: Conditions for two vertices λ, μ of a hole c of radius$\leq \sqrt{2}$ to be joined by 0, 1 or 2 edges in the hole diagram.

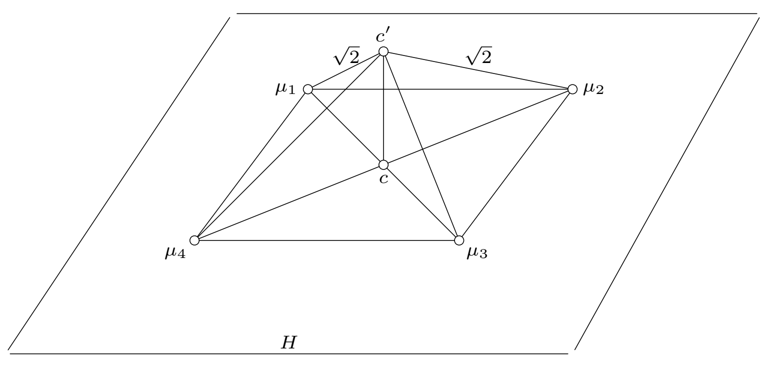


Figure 6.3: Construction of the point c'.

6.4 Covering radius

In this section we prove that the covering radius of a Niemeier lattice Λ with no roots is $\sqrt{2}$. This was first proved by Conway *et al* ([6], Chapter 23) by explicit calculation, the proof below is due to Borcherds [3]. As before, let $L = \Lambda \oplus U$ be the lattice of type $\mathrm{II}_{25,1}$. Also, let $z = (0,0,1)$ and let $\tilde{D}$ be the fundamental domain of the reflection group $W(L)$ that contains z.

Proposition 4. *Any connected extended Coxeter subdiagram of Λ is contained in a subdiagram of Λ which is a disjoint union of extended Coxeter diagrams of total rank 24 (so this subdiagram has a total of $24 + l$ nodes, where l is the number of connected extended Coxeter diagrams in the disjoint union).*

Proof. Let X be an extended Coxeter diagram in Λ. Suppose that $v_0, v_1, \dots, v_r \in \Lambda$ are the nodes of this diagram and let $\alpha_i = (v_i, 1, \frac{1}{2}v_i^2 - 1)$ be the corresponding simple roots of L. They determine a primitive norm zero vector $w = \sum_{i=0}^{r} k_i \alpha_i$ (with k_i the "weights" of the extended diagram) that is contained in $\tilde{D}$. Hence w corresponds to a Niemeier lattice N with roots. By Corollary 1 $R(N)$ has rank 24 and by Corollary 2 all its irreducible components have the same Coxeter number

$$h = \sum_{i=0}^{r} k_i = -(z, w).$$

This last equality holds because $w = \sum_{i=0}^{r} k_i \alpha_i$ and $(z, \alpha_i) = -1$ for all α_i.
Hence $R \cap w^\perp$ is an affine root system of rank 24. The proposition will follow if we show that all simple roots of this root system have height 1 with respect to z and thus correspond to vectors of the Leech lattice. So suppose that $\beta_0, \beta_1, \dots, \beta_s$ are the simple roots of another connected component of the Coxeter diagram of $w^\perp$ with weights k_i' so that $\sum_{i=0}^{s} k_i' \beta_i = w$. Then

$$-h = (z, w) = \sum_{i=0}^{s} k_i'(z, \beta_i).$$

Since this component has the same Coxeter number h as X, also $\sum_{i=0}^{s} k_i' = h$. Furthermore, because Λ has no roots it follows immediately that there are no simple roots β_i with $(z, \beta_i) = 0$. So we have $(z, \beta_i) \le -1$ and thus the above formula implies that $(z, \beta_i) = -1$ for all β_i. Hence all simple roots in $R \cap w^\perp$ have height 1. □

When changing from the hyperboloid model of hyperbolic space to the upper half-space model the rational lines of primitive norm zero vectors in L are mapped bijectively to the points of $\Lambda \otimes \mathbb{Q} \cup \infty$ as follows: Let x be a norm zero vector of L representing a rational line in $L \otimes \mathbb{Q}$. If $x = (0,0,1)$ then x is mapped to ∞ in $\Lambda \otimes \mathbb{Q} \cup \infty$. Otherwise, $x = (\lambda, m, \lambda^2/2m)$ with $m \neq 0$ and x is mapped to λ/m in $\Lambda \otimes \mathbb{Q} \cup \infty$.

Lemma 12. *The reflection in the simple root $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of $\tilde{D}$ acts on $\Lambda \otimes \mathbb{Q} \cup \infty$ as inversion in the sphere of radius $\sqrt{2}$ with center λ.*

Proof. Let ξ be a point of $\Lambda \otimes \mathbb{Q}$ that corresponds to the rational line in $L \otimes \mathbb{Q}$ generated by the norm zero vector $x = (\xi, 1, \frac{1}{2}\xi^2)$. Let $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ be a simple root. Then

$$\begin{aligned}(x, \alpha_\lambda) &= (\xi, \lambda) - (\frac{1}{2}\lambda^2 - 1) - \frac{1}{2}\xi^2 \\ &= 1 - \frac{1}{2}(\xi - \lambda)^2,\end{aligned}$$

and thus $(x, \alpha_\lambda) = 0 \Leftrightarrow (\xi - \lambda)^2 = 2$. □

Theorem 5. *The covering radius of Λ is $\sqrt{2}$.*

Proof. Let $c \in \Lambda \otimes \mathbb{Q}$ be a hole of Λ. Denote by $R(c)$ the radius of this hole given by $R(c)^2 = \inf\{(\lambda - c)^2 \mid \lambda \in \Lambda\}$. By Corollary 4 we know that the distance between any two vertices of c is at most $2\sqrt{2}$. So the nodes of the hole diagram of c are joined by 0, 1 or 2 edges. Now there are the following two possibilities:

- The hole diagram contains no extended diagram:
  In this case the hole diagram contains no double bonds because if it did it would contain the extended diagram $\tilde{A}_1$. So the diagram is simply laced and it follows that it can contain only simply laced elliptic diagrams, i.e. the diagrams in Figure 1. Indeed, otherwise one could delete nodes from the diagram until in the next step such an elliptic diagram would be obtained and thus there would be an extended diagram contained in the hole diagram. We now show that a hole whose diagram contains only elliptic diagrams has radius $R < \sqrt{2}$:
  A simply laced elliptic diagram corresponds to a root system in which all the roots have the same length (see [4]). Suppose that the rank of the diagram is n. Let $\{\mu_i\}_{i=1}^n$ be the set of vertices of the hole c. Furthermore, let $\{\alpha_i\}_{i=1}^n$ be a set of fundamental roots in $\mathbb{R}^n$ corresponding to the diagram. They all have the same length and they have the same mutual distances as the μ_i. Hence the set of vectors $\{\alpha_i \mid i = 1, \ldots, n\}$ is isometric to the set of vectors $\{\mu_i - c' \mid i = 1, \ldots, n\}$. Furthermore, since the origin in $\mathbb{R}^n$ is not a linear combination of the α_i it follows that c' cannot be a linear combination of the μ_i. So it follows that c' is not in the hyperplane that contains the μ_i and thus by definition of c' we have $R < \sqrt{2}$.

- The hole diagram of c contains an extended diagram:
  Let $\{v_i\}$ be the set of vertices corresponding to this diagram. By Proposition 4 this diagram is contained in a disjoint union of extended diagrams of total rank 24. Let $\{\alpha_i\}$ be the set of simple roots that correspond to the vertices $\{v_i\}$ (i.e. $\alpha_i = (v_i, 1, \frac{1}{2}v_i^2 - 1)$). Furthermore, let $\{v_i'\}$ be a set of vertices of one of the other extended diagrams of the hole c with a corresponding set of simple roots $\{\alpha_i'\}$. Let $w \in \tilde{D}$ be the norm zero vector corresponding to this set of extended diagrams, i.e. $w = \sum_i k_i\alpha_i = \sum k_i'\alpha_i'$ with k_i, k_i' the weights of the extended diagrams formed by the simple roots α_i, α_i'. Then $w = (\sum_i k_i v_i, h, \ldots) = (\sum_i k_i' v_i', h, \ldots)$. So the norm zero vector w corresponds to the vector $\tilde{c} = \frac{1}{h}\sum k_i v_i = \frac{1}{h}\sum k_i' v_i' \in \Lambda \otimes \mathbb{Q}$. The diagrams then form the vertices of a hole $\tilde{c}$ of radius $\sqrt{2}$ whose center is the center of any of the components of this set of vertices. Suppose that $\hat{c}$ is another hole with the vertices $\{v_i\}$ among its vertices. Since $\tilde{c}$ is the center of this set of vertices this is only possible if $\hat{c} = \tilde{c} + v$ with $v \in \Lambda \otimes \mathbb{Q}$ a vector perpendicular to all the v_i. Then the radius of this hole is $R(\hat{c}) = (\hat{c} - v_i)^2 = R(\tilde{c}) + v^2 + 2(c, v)$. Since the extended diagrams that form the hole diagram of $\tilde{c}$ have total rank 24 there is a vector v_i' among its vertices that is not perpendicular to v. But then $(\hat{c} - v_i')^2 = R(\tilde{c}) + v^2 + 2(c, v) - (v_i', c)$ and we see that the distance of $\hat{c}$ to either v_i' or $-v_i'$ is smaller than its distance to the vertices v_i. So $\hat{c}$ is not a hole with the vertices v_i among its vertices. It follows that any hole which has the $\{v_i\}$ among its vertices must be equal to $\tilde{c}$. So in particular $c = \tilde{c}$ has radius $\sqrt{2}$.

It follows that Λ has covering radius $\sqrt{2}$.

□

6.5 Deep holes

Again denote by Λ the Leech lattice, which we now know to be the unique Niemeier lattice without roots. In this section we give a proof that the construction in section 6.1 gives the Leech lattice after one step. We also describe the "holy constructions" of the Leech lattice as described in [6], Chapter 24 and give Borcherds proof that these constructions work. We elaborate on the deep holes, especially those corresponding to the Niemeier lattices with root system $24A_1$ and $12A_2$.

It follows from Lemma 12 that in the upper half-space model $U^{25} = \{(x, y) \mid x \in \Lambda \otimes \mathbb{R}, y \in \mathbb{R}_{\geq 0}\}$ the fundamental domain $\tilde{D}$ of $L = \mathrm{II}_{25,1}$ is given by

$$\tilde{D} = \{(x, y) \in U^{25} \mid (x - \lambda)^2 + y^2 \geq 2 \text{ for all } \lambda \in \Lambda\} \cup \{\infty\}.$$

Furthermore, $\mathrm{Aut}_+(L) = W(L) \rtimes \mathrm{Sym}(\tilde{D})$ with $\mathrm{Sym}(\tilde{D})$ equal to the group of diagram automorphisms of Λ. So $\mathrm{Sym}(\tilde{D}) = T(\Lambda) \rtimes \mathrm{Aut}(\Lambda)$ with $T(\Lambda) \cong \Lambda$ the translation group of Λ. This group $T(\Lambda) \rtimes \mathrm{Aut}(\Lambda)$ is called Co_∞ or $\cdot\infty$. $\mathrm{Aut}(\Lambda)$ is also called Co_0 or $\cdot 0$, it is the double cover of Conway's simple group $\mathrm{Co}_1 = \cdot 1$ of order $2^{21}3^9 5^4 7^2 11 \cdot 13 \cdot 23$.

The statement in the Proposition below was first proved by Conway *et al* ([6], Chapter 23) by explicit calculation of all the deep holes of the Leech lattice. The proof below is due to Borcherds ([3] Corollary 7.2).

Proposition 5. *The Niemeier lattices with roots are in natural bijection with the orbits of deep holes of Λ under $\cdot\infty$. The vertices of a deep hole form the extended Coxeter diagram of the corresponding Niemeier lattice.*

Proof. As discussed in section 5.1, the Niemeier lattices correspond to orbits of primitive norm zero vectors in L. By Lemma 12, the point $\xi \in \Lambda \otimes \mathbb{Q}$ corresponds to a norm zero vector z of $\tilde{D}$ if and only if $(\xi - \lambda)^2 \geq 2$ for all $\lambda \in \Lambda$. If this is the case then ξ is a deep hole and its vertices are the $\lambda \in \Lambda$ at distance $\sqrt{2}$ from ξ. These vertices λ correspond to the simple roots $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of $\tilde{D}$ in $z^\perp$. So Niemeier lattices with roots correspond to the orbits of primitive norm zero vectors other than $(0,0,1)$ and thus to deep holes ξ of L. The vertices of this deep hole ξ form the extended Coxeter diagram of the corresponding Niemeier lattice. $\square$

So up to the action of $\cdot\infty$ there are 23 types of deep holes in the Leech lattice each corresponding to a different Niemeier lattice with roots. There are 284 types of shallow holes (up to the action of $\cdot\infty$), for a classification see [6] Chapter 25. A shallow hole has 25 vertices and the corresponding hole diagram is a union of elliptic Coxeter diagrams. The radius of a shallow hole is $\sqrt{2 - 1/\rho^2}$ with ρ the Weyl vector of the hole diagram.

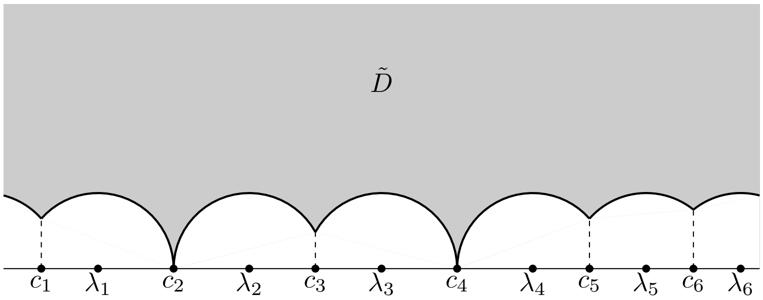


Figure 6.4: Fundamental domain $\tilde{D}$ of $L = \Lambda \oplus U$ in upper half-space model. With $\lambda_i \in \Lambda$, c_2 and c_4 deep holes and c_1, c_3, c_5 and c_6 shallow holes.

We can now show that the construction described in Section 6.1 indeed always gives the Leech lattice after one step:

Theorem 6. *Let N be a Niemeier lattice with roots and with Weyl vector $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and Coxeter number h. Let $L = N \oplus U$ be the lattice of type $II_{25,1}$ with coordinates (ν, m, n), where $\nu \in N$, $m, m \in \mathbb{Z}$ and $(\nu, m, n)^2 = \nu^2 - 2mn$. Then $(\rho, h, h+1)$ is a primitive norm zero vector in L that corresponds to the Leech lattice Λ.*

Proof. Let $V = N \otimes \mathbb{R}$ and let $A = \{\xi \in V \mid -1 \leq (\xi, \alpha) \leq 0 \text{ for all } \alpha < 0\}$ be the standard alcove. The sphere $S\left(\rho/h, 1/(\sqrt{2}h)\right)$ with center ρ/h and radius $1/(\sqrt{2}h)$ is the inscribed sphere of this simplex:
Indeed, if $\alpha < 0$ is a simple root then $(\rho/h, \alpha) = -1/h$. Furthermore, if $\theta < 0$ is a highest root then $(\rho/h, \theta) = (h-1)/h$ and hence $-1 + (\rho/h, \theta) = -1/h$. So ρ/h has distance $1/(\sqrt{2}h)$ from the planes H_α with $\alpha < 0$ a simple root and $H_{\theta,-1} = \{\xi \in V \mid (\xi, \theta) = -1\}$ with $\theta < 0$ a highest root bounding C. In particular, it follows that ρ/h lies on the middle plane of any two of the (possibly affine) hyperplanes bounding C.
Now change to coordinates $(\lambda, m, n) \in \Lambda \oplus U = L$ with $(\lambda, m, n)^2 = \lambda^2 - 2mn$, so the point at ∞ in the upper half-space model corresponds to the Leech lattice Λ. Let $\xi \in \tilde{D}$ be a deep hole of Λ that corresponds to the Niemeier lattice N. The vertices of the deep hole ξ are those $\lambda \in \Lambda$ with $(\lambda - \xi)^2 = 2$. They correspond to simple roots $\alpha_\lambda = (\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of L and form the extended Coxeter diagram of N. By Lemma 12 the reflections corresponding to these simple roots act on $\Lambda \otimes \mathbb{Q} \cup \infty$ as inversion in the sphere $S(\lambda, \sqrt{2})$ with center λ and radius $\sqrt{2}$. The geodesic γ from

ξ to ∞ is the intersection of the middle planes of these spheres (that contain ξ).
Now change back coordinates by moving ξ to the point at ∞. Then the geodesic γ is the intersection of the middle planes of the hyperplanes corresponding to the simple and highest roots of N. By the above discussion we can conclude that this geodesic goes from ∞ to the point $\rho/h \in N \otimes \mathbb{Q}$. Hence the Leech lattice corresponds to the primitive norm zero vector $(\rho, h, h+1) \in N \oplus U$. $\square$

Lemma 13. *Let c be the center of a deep hole that corresponds to a Niemeier lattice N with roots and with Coxeter number h. Then $c \in \frac{1}{h}\Lambda$.*

Proof. Let $\tilde{R}_j$ be an irreducible component of the extended root system $\tilde{R}(N)$. If $\{\alpha_0, \ldots, \alpha_r\}$ is a set of simple roots corresponding to the extended diagram of $\tilde{R}_j$ then there are positive integers $k_0, \ldots, k_r$ such that $\sum_{i=0}^{r} k_i\alpha_i = 0$ and $\sum_{i=0}^{r} k_i = h(R_j) = h$. Now let $v_0, \ldots, v_r$ be the corresponding vertices of the deep hole with center c whose vertices form the extended Coxeter diagram of N. Then $\sum_{i=0}^{r} k_i(v_i - c) = 0$. So the center of the hole is

$$c = \sum_{i=0}^{r} k_i v_i \Big/ \sum_{i=0}^{r} k_i = \frac{1}{h}\sum_{i=0}^{r} k_i v_i \ .$$

Since the $v_i \in \Lambda$ and the k_i are positive integers, $\sum k_i v_i \in \Lambda$ and thus $hc \in \Lambda$. $\square$

The "holy constructions"

The so called "holy constructions" of the Leech lattice are described by Conway and Sloane in [6] Chapter 24. They give a construction for the Leech lattice from each of the Niemeier lattices with roots and remark: "The fact that this construction always gives the Leech lattice still quite astonishes us, and we have only been able to give a case-by-case verification, as follows... We would like to see a more uniform proof." Such a proof was found by Borcherds ([3]). Below we will follow his proof while explaining things in more detail.
Let N be a Niemeier lattice with Coxeter number $h \neq 0$. Suppose $R(N) = \sum_{i=1}^{k} R_i$, where the R_i are irreducible components of rank r_i of $R(N)$. Let $\{\alpha_i^j \mid j = 0, \ldots, r_i\}$ be a set of simple roots of R_i together with the highest root, so that the α_i^j form the extended diagram of R_i. Let ρ be the Weyl vector of N and define the glue vectors g_i to be the vectors $g_i = \nu_i - \rho/h$ where $\nu_i \in N$ such that $g_i^2 = 2(1 + 1/h)$. By Lemma 9 the vectors ν_i are the lattice vectors closest to ρ/h and they form a complete set of coset representatives of N/Q, where Q is the root lattice $\mathbb{Z}R(N)$. One of these ν_i is the zero vector, say $\nu_0 = 0$. So $g_0 = -\rho/h$.
Then the "holy construction" is as follows:
The Niemeier lattice N is the set of all integer combinations

$$\sum m_i^j \alpha_i^j + \sum n_i g_i \text{ with } \sum n_i = 0, \tag{6.2}$$

while the set of all integer combinations

$$\sum m_i^j \alpha_i^j + \sum n_i g_i \text{ with } \sum m_i^j + \sum n_i = 0 \tag{6.3}$$

is a copy of the Leech lattice.
We call two sets isometric if they are isomorphic as metric spaces after identifying pairs of points whose distance apart is 0. So for example N is isometric to $N \oplus 0$.

Lemma 14. *Let z be a primitive norm zero vector in $\tilde{D}$ that corresponds to the Niemeier lattice N. The set of vectors α_i^j and g_i is isometric to the simple roots of $\tilde{D}$ of height* 0 *and* 1 *with respect to z.*

Proof. Use coordinates $(\nu, m, n) \in N \oplus U = L$ with $z = (0, 0, 1)$ so that z is the point at infinity in the upper half-space model. If we now apply Vinberg's algorithm with z as controlling vector then the roots of height 0 are the roots in $z^\perp$. This is the affine root system of N, so a set of simple roots of height 0 is given by roots

$$\begin{aligned} f_i^j &= (\alpha_i^j, 0, 0), \text{for } j \geq 1, \text{and} \\ f_i^0 &= (\alpha_i^0, 0, 1). \end{aligned}$$

Then $z = \sum_j k_j f_i^j$ for all i, where the k_j are positive integers (the "weights" of the roots). If $x = (\nu, m, n)$ is a simple root of height 1 then x has to satisfy:

$$\begin{aligned} (x, f_i^j) &\leq 0, \text{ for all } f_i^j, \text{ and} && (6.4)\\ (x, z) &= -1. && (6.5) \end{aligned}$$

We have $z = \sum_j k_j f_i^j$ for all i and $(x, f_i^j) \in \mathbb{Z}$ since L is an even lattice. It then follows from the equations 6.4 and 6.5 above that for each i

$$\begin{aligned} (x, f_i^j) &= -1, \text{ for one } j \text{ with } k_j = 1,\\ (x, f_i^j) &= 0, \text{ for all other } j. \end{aligned}$$

The only roots that satisfy these criteria are:

$$h_i = (\nu_i, 1, \frac{1}{2}\nu_i^2 - 1).$$

The set consisting of the vectors f_i^j and h_i is easily seen to be isometric to the set of vectors α_i^j and g_i. □

Lemma 15. *The vectors f_i^j and h_i generate L.*

Proof. Let $\Gamma = \sum m_i^j f_i^j + \sum n_i h_i$ be the lattice generated by the vectors f_i^j and h_i. Since the simple roots α_i^j generate $Q = \mathbb{Z}R(N)$ and the vectors $\nu_i \in N$ are a complete set of representatives of N/Q we have $\Gamma = N \oplus A$ with $A \subset U$. Since furthermore $z = (0, 0, 1)$ and $h_0 = (0, 1, -1)$ are elements of Γ we see that in fact $\Gamma = N \oplus U = L$.

□

We can now give a proof that the holy constructions indeed all give the Leech lattice.

Proof. (Holy constructions)
The first claim immediately follows from the fact that the roots α_i^j generate the root lattice $Q = \mathbb{Z}R(N)$ of N and the vectors ν_i form a complete set of coset representatives for N/Q.
Now suppose that w is a primitive norm zero vector that corresponds to the Leech lattice Λ. Then, by Theorem 3, $(w, f_i^j) = (w, h_i) = -1$ for all f_i^j and h_i. In fact, it is easily calculated that for $w = (\rho, h, h+1)$ we have $(w, f_i^j) = (w, h_i) = -1$ for all f_i^j and h_i. Hence with Lemma 15 it follows that $w^\perp = \Lambda \oplus 0$ is equal to $\sum m_i^j f_i^j + \sum n_i h_i$ with $\sum m_i^j + \sum n_l = 0$. So $\sum m_i^j \alpha_i^j + \sum n_i g_i$ with $\sum m_i^j + \sum n_i = 0$ is isometric to $\Lambda \oplus 0$. Since it is contained in the positive definite space $N \otimes \mathbb{Q}$ it follows that it is isomorphic to Λ. □

The holy construction is equivalent to the $(\rho, h, h+1)^\perp$ construction so this also gives another proof that this construction works. The following Lemma is a remark in Chapter 24 of [6].

Lemma 16. *Let N be a Niemeier lattice with roots and h its Coxeter number. Then $[\Lambda : \Lambda \cap N] = [N : \Lambda \cap N] = h$.*

Proof. Denote by $\{\alpha_i^j \mid j = 0, \dots, r\}$ a set of simple roots of a component R_i of $R(N)$ together with the highest root of this component, so that the α_i^j form the extended diagram of R_i. Then there are positive integers k_j such that $\sum_{j=0}^r k_j \alpha_i^j = 0$ and $\sum_{j=0}^r k_j = h$. So the equality in equation 6.3 can be replaced by $\sum m_i^j + \sum n_w \equiv 0$ (modulo h). It then follows that $\Lambda \cap N$ is the set of all integer combinations

$$\sum m_i^j \alpha_i^j + \sum n_w g_w \text{ with } \sum m_i^j \equiv 0 \text{ (modulo } h) \text{ and } \sum n_w = 0.$$

From this it immediately follows that $[N : \Lambda \cap N] = h$. Since N and Λ are both unimodular then also $[\Lambda : \Lambda \cap N] = h$. □

Suppose that c is the center of a deep hole that corresponds to a Niemeier lattice N with Coxeter number h. Set

$$\Lambda' = \{\lambda \in \Lambda \mid (\lambda, c) \in \mathbb{Z}\},$$

and let $\Gamma =< \Lambda', c >$ be the lattice generated by Λ' and c. Then $[\Lambda : \Lambda'] = [\Gamma, \Lambda']$ and hence Γ is a unimodular lattice. It also follows from its construction that Γ is an even lattice, so we see that Γ is also a Niemeier lattice. Furthermore, $R(\Gamma)$ contains the elements c and $c - v_i$ where the v_i are the vertices of the deep hole. Hence $R(\Gamma) = R(N)$ and thus $\Gamma \cong N$. It follows from the above Lemma that $[\Lambda : \Lambda'] = [\Gamma : \Lambda'] = h$. Hence $kc \notin \Lambda$ for $1 < k < h$.

Example 7. Suppose that N is a Niemeier lattice with root system $R(N) = 24A_1$ and let c be a deep hole corresponding to N. Since $h = 2$ we have $\mu := 2c \in \Lambda$. Then $\mu^2 = 8$.
In fact, starting with a norm 8 vector in Λ we can construct a deep hole that corresponds to a Niemeier lattice with root system $24A_1$. Let μ be such a vector of norm 8 and consider the equivalence relation $\mu \sim \nu \Leftrightarrow \mu - \nu \in 2\Lambda$. Then by Theorem 4 the equivalence class $[\mu] \in \Lambda/2\Lambda$ contains besides $\pm\mu$ 23 other pairs $\nu, -\nu$ of norm 8 vectors. These pairs are mutually orthogonal. Now the vertices of a deep hole with center $\frac{1}{2}\mu$ are $0, \mu$ and the pairs $\frac{1}{2}(\mu + \nu), \frac{1}{2}(\mu - \nu)$, where $\pm\nu \in [\mu]$ are norm 8 vectors. Indeed, since $\pm\nu \in [\mu]$ we have $\frac{1}{2}(\mu \pm \nu) \in \Lambda$ and $(\frac{1}{2}\mu - (\frac{1}{2}(\mu \pm \nu)))^2 = (\pm\frac{1}{2}\nu)^2 = 2$ so all these vectors indeed have distance 2 from the vector $\frac{1}{2}\mu$. Furthermore, $(\frac{1}{2}(\mu + \nu) - \frac{1}{2}(\mu - \nu))^2 = \nu^2 = 8$. So the nodes corresponding to the vectors $\frac{1}{2}(\mu \pm \nu)$ are joined by two edges. Also, if $\nu_1 \neq \pm\nu_2$ and both are norm 8 vectors in $[\mu]$ not equal to $\pm\mu$ then ν_1, ν_2 are orthogonal and thus $(\frac{1}{2}(\mu \pm \nu_1) - \frac{1}{2}(\mu \pm \nu_2))^2 = \frac{1}{2}\mu^2 = 4$. So the nodes corresponding to $\frac{1}{2}(\mu \pm \nu_1)$ are not joined to the nodes corresponding to $\frac{1}{2}(\mu \pm \nu_2)$. Hence we have constructed a deep hole with hole diagram $24\tilde{A}_1$ and have thus proven the existence of a Niemeier lattice with root system $24A_1$.
Since the group $Co_0 = \cdot 0 = \mathrm{Aut}(\Lambda)$ acts transitively on vectors of norm 8 ([6], Chapter 10, Theorem 27) it also follows that there can only be one Niemeier lattice with root system $24A_1$ (up to isomorphism).

Example 8. Suppose that N is a Niemeier lattice with root system $R(N) = 12A_2$ and let c be a deep hole corresponding to N. Since the Coxeter number $h = 3$ in this case we have $\mu := 3c \in \Lambda$. Then $\mu^2 = 18$. We can divide the vectors with norm 18 in Λ into two types. They are either the sum of two vectors in Λ of norm 6 or the sum of one vector of norm 4 and one vector of norm 8. The vector $0 \in \Lambda$ is one of the vertices of this deep hole. Suppose that $\mu_1, \mu_2 \in \Lambda$ are the two vertices of this hole that together with 0 correspond to a subdiagram $\tilde{A}_2$ of the hole diagram $12\tilde{A}_2$. Then

$$\begin{aligned}
\mu_1^2 &= \mu_2^2 = 6, \text{ and} \\
c &= \frac{1}{h}\sum_{i=0}^{r} k_i v_i = \frac{1}{3}(\mu_1 + \mu_2).
\end{aligned}$$

So it follows that $\mu = \mu_1 + \mu_2$ is a sum of two vectors of norm 6 and the case that is the sum of a norm 4 and a norm 8 vector does not occur here. The Conway group Co_0 acts transitively on vectors of this type ([6], Chapter 10, Theorem 29) and it thus follows that up to isomorphism there can be at most one Niemeier lattice with root system $12A_2$.
As before, we can now prove the existence of a Niemeier lattice with root system $12A_2$. Let $\mu_1, \mu_2 \in \Lambda$ be two norm 6 vectors with $(\mu_1, \mu_2) = 3$. So $\mu = \mu_1 + \mu_2$ is a vector of norm 18. Set $c := \frac{1}{3}\mu$ so $c^2 = 2$ and $2c \notin \Lambda$ since otherwise also the norm 2 vector $c = 3c - 2c$ would be an element of Λ. Now set

$$\Lambda' = \{\lambda \in \Lambda \mid (\lambda, c) \in \mathbb{Z}\},$$

and define $\Gamma =< \Lambda', c >$ to be the lattice generated by Λ' and c. Then $[\Lambda : \Lambda'] = [\Gamma : \Lambda'] = 3$ and together with the definition of Λ' this implies that Γ is an even unimodular lattice in $\mathbb{R}^{24}$. Furthermore, as $\Lambda' = \Lambda \cap \Gamma$ it follows that Γ has to be the Niemeier lattice with $R(\Gamma) = 12A_2$ because this is the only root system listed in Table 5.1 with $h = 3$.

In general, suppose that $\lambda \in \Lambda$ is a primitive vector of norm $2 \cdot h^2$ with $h \in \mathbb{N}$. Then we can construct a Niemeier lattice with Coxeter number equal to h as follows:

Let

$$\Lambda' = \{\mu \in \Lambda \mid (\mu, \lambda/h) \in \mathbb{Z}\},$$

and let $\Gamma =< \Lambda', \lambda/h >$ be the lattice generated by Λ' and λ/h. This is a Niemeier lattice with Coxeter number equal to h. If there is only one root system in Table 5.1 with this Coxeter number then this would show the existence of a Niemeier lattice with this root system. Note that in the approach we have taken here the proof of existence and uniqueness becomes harder as the Coxeter number h increases contrary to Venkov's proof where it is the other way around.

The uniqueness of a Niemeier lattice with a given root system in this approach depends on the transitivity of the Conway group Co_0 on vectors of Λ of a certain type. However, the proof of, for example, Theorem 29 in [6] Chapter 10 depends on a construction of the Leech lattice that uses the binary Golay code. So the proof of the uniqueness of the Niemeier lattices with root system $24A_1$ and $12A_2$ still depends on the existence and uniqueness of this code. It remains to be seen if one can construct a proof of the uniqueness of these lattices that completely avoids the use of the Golay codes.

The $(\rho, h, h+1)^\perp$ construction of the Leech lattice Λ from an arbitrary Niemeier lattice N with roots gives a geodesic from the primitive norm zero vector in $\tilde{D}$ corresponding to N to the primitive norm zero vector in $\tilde{D}$ corresponding to Λ. Another approach for proving the existence of a Niemeier lattice would be to find such a geodesic from the primitive norm zero vector corresponding to a Niemeier lattice whose existence is easy to prove (like $3E_8$) to the primitive norm zero vector corresponding to one for which it is harder (for example the one with root system $24A_1$). We did not obtain any results in this direction.

Of course it would even be better to find a uniform proof of the uniqueness and existence of a Niemeier lattice with given root system for all the root systems in Table 5.1. That is, for root systems of rank 24 whose irreducible components all have the same Coxeter number.

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