Lie Algebras in Mathematics and Physics

Gert Heckman

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Contents

Preface

Sophus Lie (1842-1899) was a Norwegian mathematician, who created an algebraic language (Lie algebras) to deal with the notion of continuous symmetry (Lie groups). Let us consider $GL_n(\mathbb{R})$ as an open subset of $Mat_n(\mathbb{R})$, which we identify with the Cartesian product of n^2 copies of $\mathbb{R}$. A linear Lie group G is by definition a closed subgroup $G \subset GL_n(\mathbb{R})$ for some $n \in \mathbb{N}$. Examples of linear Lie groups are $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $SO_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $SU_n(\mathbb{C})$ with these last three closed subgroups of $GL_{2n}(\mathbb{R})$.

Let us be given a linear Lie group $G \subset GL_n(\mathbb{R})$. A smooth curve in G through the identity element is a smooth map $(-\epsilon, \epsilon) \to G$, denoted $t \mapsto g(t)$, for some $\epsilon > 0$ and $g(0)$ is equal to the identity of G. For n a positive integer we can stretch such a smooth curve by a factor n, defined by $t \mapsto g(t/n)^n$. Because G is a closed subgroup of $GL_n(\mathbb{R})$ the limit for $n \to \infty$ exists, and can be shown to be of the form $t \mapsto \exp(tx)$ for some $x \in Mat_n(\mathbb{R})$. In fact x equals the velocity vector of the original curve $t \to G$ at time $t = 0$. The Lie algebra $\mathfrak{g}$ of the linear Lie group $G \subset GL_n(\mathbb{R})$ is by definition the set of all such velocity vectors $x \in Mat_n(\mathbb{R})$. It is clear that $\mathfrak{g}$ is invariant under scalar multiplication. Just rescale the time of the defining curve, and use the chain rule. It is also clear that $\mathfrak{g}$ is invariant under addition of two elements. Just consider the product of the two curves, and use the Leibniz product rule. Hence $\mathfrak{g}$ is a linear subspace of $Mat_n(\mathbb{R})$.

For $x, y \in Mat_n(\mathbb{R})$ we define the commutator bracket by $[x, y] = xy - yx$. It is straightforward to check that the commutator bracket is antisymmetric

$$[y, x] = -[x, y]$$

and satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in Mat_n(\mathbb{R})$. A bilinear form $[\cdot, \cdot]$ on a vector space with these two properties is called a Lie bracket, and a(n abstract) Lie algebra is a vector space equiped with a Lie bracket. For example $Mat_n(\mathbb{R})$ is a Lie algebra with respect to the commutator bracket.

We claim that $[x, y] \in \mathfrak{g}$ for all $x, y \in \mathfrak{g}$. Consider therefore the following curve through the identity element of G defined for positive time by

$$[0, \epsilon) \ni t \mapsto g(t) = exp(t^{1/2}x)exp(t^{1/2}y)exp(-t^{1/2}x)exp(-t^{1/2}y) ,$$

and for negative time by

$$(-\epsilon, 0] \ni t \mapsto g(-t)^{-1} .$$

One can show that $t \mapsto g(t)$ is a smooth curve in G (just expand g(t) in a power series of t) with velocity vector at time $t = 0$ equal to $[x, y] \in \mathfrak{g}$. Therefore $[x, y] \in \mathfrak{g}$ for all $x, y \in \mathfrak{g}$. Hence $\mathfrak{g}$ becomes a Lie algebra relative to the Lie bracket $[\cdot, \cdot]$.

The Lie algebra $\mathfrak{g}$ of a linear Lie group $G \subset GL_n(\mathbb{R})$ only sees an infinitesimal neighborhood of the identity element $e \in G$. In particular G and the connected component G° of the identity have the same lie algebra $\mathfrak{g}$. It is a truely remarkable fact the the Lie algebra captures most of the structure of a connected linear Lie group. In these lectures we forget the linear Lie group G altogether, and only work with the Lie algebra $\mathfrak{g}$.

1 Associative algebras and Lie algebras

Unless otherwise stated all vector spaces are defined over the field $\mathbb{C}$ of complex numbers. This text is about algebra, and if we use the word Hilbert space we usually shall mean pre Hilbert space, that is a suitable dense subspace of the Hilbert space completion. Likewise we write $B(\mathcal{H})$ for the bounded operators on $\mathcal{H}$, although sometimes operators are also unbounded.

Definition 1.1. *An associative algebra $\mathcal{A}$ is a vector space with a product rule $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \mapsto ab$ which is associative*

$$(ab)c = a(bc)$$

and compatible with the vector space structure

$$(a + b)c = ac + bc \ , \ a(b + c) = ab + ac$$

$$(\lambda a)b = a(\lambda b) = \lambda(ab)$$

for all $a, b, c \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

A star structure on $\mathcal{A}$ is an antilinear antiinvolution, that is a map $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^\star$ with

$$(a + b)^\star = a^\star + b^\star \ , \ (\lambda a)^\star = \overline{\lambda} a^\star$$

$$(ab)^\star = b^\star a^\star \ , \ a^{\star\star} = a$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. An associative algebra together with a star structure is called a star algebra.

Example 1.2. *If V is a vector space then $End(V) = \{$linear operators on $V\}$ is the standard example of an associative algebra with product rule the composition of linear operators. If $V = \mathcal{H}$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ then $B(\mathcal{H}) = \{$bounded linear operators on $\mathcal{H}\}$ is an associative algebra with star structure $a^\star = a^\dagger$ with the dagger defined by $\langle a\phi, \psi \rangle = \langle \phi, a^\dagger \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$.*

Example 1.3. *If G is a finite group then the space $L(G) = \{$complex valued functions on $G\}$ is an associative algebra with respect to the convolution product*

$$\phi * \psi(x) = |G|^{-1} \sum_y \phi(xy^{-1}) \psi(y) \ .$$

The associativety of the convolution product follows easily from the associativety of the product rule on the group G. A star structure on $L(G)$ is defined by $\phi^\star(x) = \overline{\phi(x^{-1})}$. Indeed one easily checks that $(\phi * \psi)^\star = \psi^\star * \phi^\star$.

Example 1.4. *On the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ the operator $p = -i\hbar\partial$ (with $\partial = d/dx$) and the multiplication by x operator q are selfadjoint and satisfy the Heisenberg commutation relation*

$$[p, q] = -i\hbar$$

with $\hbar > 0$. The "creation" and "annihilation" operators are defined by

$$a_- = (q - ip)/\sqrt{2}\ ,\ a_+ = (q + ip)/\sqrt{2}$$

and satisfy the commutation relation

$$[a_+, a_-] = \hbar\ .$$

We have $q^\dagger = q$ and $p^\dagger = p$, or equivalently $a_+^\dagger = a_-$ and $a_-^\dagger = a_+$. The energy operator

$$H = (a_+a_- + a_-a_+)/2 = (p^2 + q^2)/2 = a_-a_+ + \hbar/2$$

satisfies the commutation rules

$$[H, a_-] = \hbar a_-\ ,\ [H, a_+] = -\hbar a_+\ .$$

The vector $v_0(x) = e^{-x^2/2\hbar}$ in $\mathcal{H}$ satisfies $a_+v_0 = 0$, and is characterized by this equation upto a scalar multiple. This implies $Hv_0 = \hbar v_0/2$. If we write $v_n = a_-^n v_0$ then the commutation relation $[H, a_-] = \hbar a_-$ yields $[H, a_-^n] = n\hbar a_-^n$, which in turn implies that $Hv_n = \hbar(n + 1/2)v_n$. So the eigenvalue spectrum of H is equal to $\hbar(\mathbb{N} + 1/2)$. This is Dirac's treatment of the quantum mechanical harmonic oscillator. The star algebra $\mathcal{A}$ with generators the creation operator a_- and the annihilation operator a_+ has a vector space basis $a_-^m a_+^n$ with $m, n \in \mathbb{N}$, and is called the oscillator algebra. The basis $a_-^m a_+^n$ is called in normal ordering, which means creation operators on the left and annihilation operators on the right.

Definition 1.5. *A representation of an associative algebra $\mathcal{A}$ on a vector space V is an associative algebra homomorphism $\mathcal{A} \to End(V)$. If $V = \mathcal{H}$ is a Hilbert space then a unitary representation of a star algebra $\mathcal{A}$ on $\mathcal{H}$ is a representation of $\mathcal{A}$ on $\mathcal{H}$ such that the star on $\mathcal{A}$ corresponds to the dagger on $B(\mathcal{H})$.*

Suppose we have a representation of an associative algebra $\mathcal{A}$ on a vector space V. A linear subspace $U \subset V$ is called invariant if $au \in U$ for all $a \in \mathcal{A}$ and $u \in U$. The representation of $\mathcal{A}$ on V is called irreducible if the trivial subspaces 0 and V are the only two invariant linear subspaces. Remark that by definition the vector space 0 is not irreducible, since there is just one invariant subspace 0.

Suppose G is a finite group. A unitary representation $U : G \to U(\mathcal{H})$ extends to a unitary representation $U : L(G) \to B(\mathcal{H})$ by

$$U(\phi) = |G|^{-1} \sum_x \phi(x) U(x) \ .$$

Indeed one easily verifies $U(\phi * \psi) = U(\phi)U(\psi), U(\phi^\star) = U(\phi)^\dagger$. The conclusion is that a unitary group representation of G is equivalent to a unitary associative algebra representation of $L(G)$. However L(G) is more flexible than G itself, e.g. the center C(G) of (class functions in) L(G) yields intertwiners in representation spaces.

Definition 1.6. *A Lie algebra $\mathfrak{g}$ is a vector space equiped with a product rule $x, y \in \mathfrak{g} \mapsto [x, y] \in \mathfrak{g}$ which is bilinear*

$$[x + y, z] = [x, z] + [y, z] \ , \ [x, y + z] = [x, y] + [x, z]$$

$$[\lambda x, y] = [x, \lambda y] = \lambda [x, y]$$

and which is a Lie bracket (i.e. antisymmetry and Jacobi identity holds)

$$[y, x] = -[x, y]$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

for all $x, y, z \in \mathfrak{g}$ and all scalars λ.

The three Lie algebras (with $\mathfrak{gl}_n$ for n by n matrices) with Lie bracket the commutator bracket of square matrices

$$\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n ;\ tr(x) = 0\}$$

$$\mathfrak{so}_n = \{x \in \mathfrak{gl}_n ;\ x^t + x = 0\}$$

$$\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} ;\ x^t J + J x = 0\}$$

are called the special linear algebra, the orthogonal algebra and the symplectic algebra respectively. Together they constitute the classical Lie algebras. The verification of the Lie algebra properties is outlined in the exercises. Here J is the $2n$ by $2n$ matrix

$$\begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

so that $\omega(u,v) = (u, Jv)$ in the notation of the exercises.

If $\mathfrak{g}_0 \subset \mathfrak{g}$ is a real form (i.e. $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0, \mathfrak{g}_0 \cap i\mathfrak{g}_0 = 0$ and $[x,y] \in \mathfrak{g}_0$ for $x, y \in \mathfrak{g}_0$) then put

$$(x + iy)^\star = -x + iy$$

for $x, y \in \mathfrak{g}_0$. It is easy to check that that the star operator is an antilinear antiinvolution on $\mathfrak{g}$. Any antilinear antiinvolution of a Lie algebra $\mathfrak{g}$ arises in this way for some real form $\mathfrak{g}_0 \subset \mathfrak{g}$.

An associative algebra $\mathcal{A}$ is in a natural way a Lie algebra, with Lie bracket $[x,y] = xy - yx$ the commutator bracket in $\mathcal{A}$. The antisymmetry of the Lie bracket is obvious and the Jacobi identity is a matter of spelling out.

A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\mathfrak{g} \mapsto \mathfrak{gl}(V)$ with $\mathfrak{gl}(V) = End(V)$ as a vector space, but $End(V)$ is an associative algebra with respect to composition of linear operators, whereas $\mathfrak{gl}(V)$ is the associated Lie algebra with respect to the commutator bracket. The concepts of invariant subspace and irreducibility for a representation of a Lie algebra $\mathfrak{g}$ on a vector space V are defined in the same way as for associative algebra representations.

A unitary representation of a real Lie algebra $\mathfrak{g}_0$ on a Hilbert space $\mathcal{H}$ is a representation of the complexification $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ on $\mathcal{H}$ such that the star on $\mathfrak{g}$ corresponds to the dagger in the representation space.

Our goal is to associate to a Lie algebra $\mathfrak{g}$ in a canonical way an associative algebra $U\mathfrak{g}$ (called the universal enveloping algebra) such that Lie algebra representations of $\mathfrak{g}$ on V give rise to associative algebra representations of $U\mathfrak{g}$ on V, and vice versa. The construction of $U\mathfrak{g}$ is rather abstract, but in the end it will pay back greatly, because associative algebra representations are easier and more fundamental than Lie algebra representations. Indeed the associativity condition $(ab)c = a(bc)$ is more basic and easier than the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

This correspondence between Lie algebra representations of $\mathfrak{g}$ and associative algebra representations of $U\mathfrak{g}$ should be thought of as analogous to the transition from groups representations of a finite group G to associative algebra representations of the convolution algebra $L(G)$.

Proposition 1.7. *Let V be a representation of a Lie algebra $\mathfrak{g}$. Suppose $h \in \mathfrak{g}$ (or h a polynomial expression in some elements of $\mathfrak{g}$) is a diagonalisable operator in V so that*

$$V = \bigoplus_\lambda Ker(h - \lambda)$$

algebraically (any $v \in V$ is a finite sum $v = \sum_\lambda v_\lambda$ with $hv_\lambda = \lambda v_\lambda$). Then any subrepresentation U of V respects this decomposition so that

$$U = \bigoplus_\lambda (U \cap Ker(h - \lambda)) \ .$$

Proof. Write $u \in U$ as $u = \sum_\lambda u_\lambda$ with $hu_\lambda = \lambda u_\lambda$. Then $h^n u = \sum_\lambda \lambda^n u_\lambda$ lie in U for all $n \in \mathbb{N}$. Using the nonvanishing of a Vandermonde determinant we can write each u_λ as a linear combination of $h^n u$ with $n \in \mathbb{N}$. □

Exercise 1.1. *Prove that $a_-^m a_+^n$ with $m, n \in \mathbb{N}$ is a vector space basis of the oscillator algebra $\mathbb{C}[a_-, a_+]$ with relation $[a_+, a_-] = \hbar$.*

Exercise 1.2. *Check the relations $U(\phi * \psi) = U(\phi)U(\psi), U(\phi^\star) = U(\phi)^\dagger$ for a unitary representation U of a finite group G on a Hilbert space $\mathcal{H}$.*

Exercise 1.3. *Check that the commutator bracket on an associative algebra satisfies the Jacobi identity. Conclude that $\mathfrak{gl}_n$ is a Lie algebra with Lie bracket the commutator bracket.*

Exercise 1.4. *Check that for V a finite dimensional vector space*

$$\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V); tr(x) = 0\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$, called the special linear algebra.

Exercise 1.5. *Let V be a finite dimensional vector space with a nondegenerate symmetric biliniear form $(\cdot, \cdot)$. Check that*

$$\mathfrak{so}(V) = \{x \in \mathfrak{gl}(V); (xu, v) + (u, xv) = 0 \ \forall u, v \in V\}$$

is a Lie subalgebra of $\mathfrak{gl}(V)$. The Lie algebra $\mathfrak{so}(V)$ is called the orthogonal algebra of V with $(\cdot, \cdot)$.

Exercise 1.6. *Answer the same question as in the previous exercise but with the form $(\cdot,\cdot)$ replaced by a symplectic form ω on V, which is a nondegenerate antisymmetric bilinear form. The Lie algebra*

$$\mathfrak{sp}(V) = \{x \in \mathfrak{gl}(V); \omega(xu,v) + \omega(u,xv) = 0 \ \forall u,v \in V\}$$

is called the symplectic algebra of V with ω.

Exercise 1.7. *Check that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of traceless two by two matrices with basis*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

has commutation relations

$$[h,e] = 2e \ , \ [h,f] = -2f \ , \ [e,f] = h \ .$$

Exercise 1.8. *Prove that the antilinear antiinvolution on $\mathfrak{sl}_2(\mathbb{C})$ defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

corresponds to the real form $\mathfrak{su}(2)$ of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 1.9. *Prove that the antilinear antiinvolution on $\mathfrak{sl}_2(\mathbb{C})$ defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}$$

corresponds to the real form $\mathfrak{su}(1,1)$ of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 1.10. *If in the notation of Example 1.4 we define*

$$e = a_+^2/2\hbar \ , \ h = -(a_+a_- + a_-a_+)/2\hbar \ , \ f = -a_-^2/2\hbar$$

then verify that the operators $\{e,h,f\}$ satisfy the commutation relations

$$[h,e] = 2e \ , \ [h,f] = -2f \ , \ [e,f] = h$$

of $\mathfrak{sl}_2(\mathbb{C})$. This representation of $\mathfrak{sl}_2(\mathbb{C})$ is called the oscillator representation. Compute the eigenvalue spectrum of the operator h in the oscillator representation. Check that $e^\star = -f, h^\star = h$, $f^\star = -e$. Conclude that the oscillator representation of $\mathfrak{sl}_2(\mathbb{C})$ is unitary for $\mathfrak{su}(1,1)$.

Exercise 1.11. *Show that the oscillator representation of $\mathfrak{sl}_2(\mathbb{C})$ decomposes as a direct sum of two irreducible representations. Hint: Use Proposition 1.7. Explain these two subrepresentations in terms of the original Hilbert space $\mathcal{H} = L^2(\mathbb{R})$.*

2 Poisson algebras and universal enveloping algebras

Suppose U, V are finite dimensional complex vector spaces. Define

$$U \oplus V = U \times V \ , \ U^* = Hom(U, \mathbb{C}) \ ,$$

$$Hom(U, V) = \{linear\ maps\ U \mapsto V\} \ ,$$
$$Bil(U \oplus V, W) = \{bilinear\ maps\ U \oplus V \mapsto W\}$$

with $U \oplus V$ the direct sum of U and V, and U^* the dual vector space of U. The tensor product $U \otimes V$ of U and V is defined by the property

$$Bil(U \oplus V, W) = Hom(U \otimes V, W)$$

so any bilinear map B on $U \oplus V$ with values in W factorizes trough $U \otimes V$ as a linear map b to W, in a commutative diagram

$$\begin{array}{ccc} U \oplus V & \xrightarrow{B} & W \\ \Big\downarrow \mathrm{i} & & \Big\downarrow \mathrm{id} \\ U \otimes V & \xrightarrow{b} & W \end{array}$$

with $B = b \circ i$ and $i : U \oplus V \longrightarrow U \otimes V, \ i(u, v) = u \otimes v$ the natural bilinear map.

If $\{u_i\}$ and $\{v_j\}$ are bases of U and V then $\{u_i \otimes v_j\}$ is a basis of $U \otimes V$. Hence $\dim(U \otimes V) = \dim U \cdot \dim V$, whereas $\dim(U \oplus V) = \dim U + \dim V$.

Definition 2.1. *The tensor algebra on V is equal to $TV = \oplus T^k V$ (sum over $k \in \mathbb{N}$) with $T^0 V = \mathbb{C}, \ T^1 V = V, \ T^2 V = V \otimes V, \ \ldots$ so $T^k V = V \otimes \cdots \otimes V$ (with k factors).*

The tensor product operation turns TV into an associative algebra

$$(u_1 \otimes \cdots \otimes u_k) \otimes (v_1 \otimes \cdots \otimes v_l) = (u_1 \otimes \cdots \otimes u_k \otimes v_1 \otimes \cdots \otimes v_l)$$

so the product is a linear map $T^k V \otimes T^l V \to T^{k+l} V$. Any linear map $\pi : V \to \mathcal{A}$ with $\mathcal{A}$ an associative algebra extends uniquely to an associative algebra homomorphism $T\pi : TV \to \mathcal{A}$ by $T\pi(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \pi(v_1)\pi(v_2) \cdots \pi(v_k)$.

Let us write

$$IV = span\{TV \otimes (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes TV;\ v_1, v_2 \in V\}\ .$$

This is a linear subspace of TV closed under multiplication by arbitrary tensors on the left and on the right (so a two sided ideal of TV). The factor space $SV = TV/IV$ is called the symmetric algebra of V. The space SV inherits a natural product rule from TV turning SV in a commutative associative algebra. The product of $s, t \in SV$ is denoted st. If $\{v_i\}$ is a basis of V then

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k};\ 1 \le i_1, \cdots, i_k \le n = dimV\},\ k \in \mathbb{N}$$

is a basis of TV while

$$\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n};\ 0 \le m_1, m_2, \cdots, m_n < \infty\}$$

is a basis of SV. Note that

$$dim(T^k V) = n^k\ ,\ dim(S^k V) = \binom{n+k-1}{k}\ .$$

The symmetric algebra can be canonically identified with the commutative algebra PV^* of polynomial functions on the dual vector space V^*. The above basis of SV is just the basis of monomials of PV^* relative to a fixed basis $\{v_i\}$ of $V = V^{**}$. This ends our discussion of the linear algebra constructions of the tensor algebra TV and the symmetric algebra $SV = PV^*$ associated with any vector space V.

Now take $V = \mathfrak{g}$ to be a Lie algebra with Lie bracket $[\cdot, \cdot]$. The Lie bracket on $\mathfrak{g}$ extends to a bilinear Poisson bracket on $S\mathfrak{g} = P\mathfrak{g}^*$ via the Leibniz rule

$$\{\cdot, \cdot\} : P^k\mathfrak{g}^* \oplus P^l\mathfrak{g}^* \to P^{k+l-1}\mathfrak{g}^*\ , \{\cdot, \cdot\} = [\cdot, \cdot]\ on\ \mathfrak{g} \oplus \mathfrak{g}\ ,$$

$$\{r, st\} = \{r, s\}t + s\{r, t\}\ ,\ \{rs, t\} = \{r, t\}s + r\{s, t\}\ \forall\ r, s, t \in P\mathfrak{g}^*\ .$$

Equipped with this Poisson bracket $P\mathfrak{g}^*$ is called the Poisson algebra of $\mathfrak{g}$. It is a graded algebra, i.e. $P\mathfrak{g}^* = \oplus P^k\mathfrak{g}^*$ and $P^k\mathfrak{g}^* P^l\mathfrak{g}^* \subset P^{k+l}\mathfrak{g}^*$. A Poisson bracket reminds us of classical mechanics. The quantization of the Poisson algebra $P\mathfrak{g}^*$ will be the universal enveloping algebra.

Let us denote

$$J\mathfrak{g} = span\{T\mathfrak{g} \otimes (x \otimes y - y \otimes x - [x, y]) \otimes T\mathfrak{g};\ x, y \in \mathfrak{g}\ \};.$$

This is again a two sided ideal of $T\mathfrak{g}$, and therefore the quotient $U\mathfrak{g} = T\mathfrak{g}/J\mathfrak{g}$, called the universal enveloping algebra of $\mathfrak{g}$, becomes an associative algebra. We have natural linear maps $i : \mathfrak{g} \hookrightarrow T\mathfrak{g}$ and $j : \mathfrak{g} \to U\mathfrak{g}$. Note that

$$j([x, y]) = j(x)j(y) - j(y)j(x) \ \forall x, y \in \mathfrak{g}$$

so that under j the Lie bracket becomes the commutator bracket.

Theorem 2.2. *Let $\mathfrak{g}$ be a Lie algebra and $j : \mathfrak{g} \to U\mathfrak{g}$ its universal enveloping algebra. Given an associative algebra $\mathcal{A}$ and a linear map $\pi : \mathfrak{g} \to \mathcal{A}$ such that*

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x) \ \forall x, y \in \mathfrak{g}$$

then there exists a unique associative algebra homomorphism $U\pi : U\mathfrak{g} \to \mathcal{A}$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{j} & U\mathfrak{g} \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle U\pi} \\ \mathfrak{g} & \xrightarrow{\pi} & \mathcal{A} \end{array}$$

is commutative. It is common to denote the associative algebra homomorphism $U\pi$ also by π, if no confusion arises.

Proof. Any linear map $\pi : \mathfrak{g} \to \mathcal{A}$ extends to a unique associative algebra homomorphism $T\pi : T\mathfrak{g} \to \mathcal{A}$ by the very definition of the tensor algebra. By assumption we have

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$$

and therefore

$$T\pi(x \otimes y - y \otimes x - [x, y]) = 0$$

$\forall x, y \in \mathfrak{g}$, which in turn implies that $T\pi(J\mathfrak{g}) = 0$. Hence $T\pi$ induces an associative algebra homomorphism $U\pi : U\mathfrak{g} \to \mathcal{A}$. Since $T\pi = \pi$ on $T^1\mathfrak{g} = \mathfrak{g}$ we conclude that $T\pi \circ i = \pi$ and $U\pi \circ j = \pi$. The uniqueness of $U\pi$ is obvious from $U\pi \circ j = \pi$. $\square$

Corollary 2.3. *Every Lie algebra representation π of $\mathfrak{g}$ on a vector space V yields a unique associative algebra homomorphism $U\pi : U\mathfrak{g} \to End(V)$ such that $U\pi \circ j = \pi$.*

If $\mathfrak{g}$ is an Abelian Lie algebra then $J\mathfrak{g} = I\mathfrak{g}$ and $U\mathfrak{g} = S\mathfrak{g}$ is a commutative associative algebra. But for $\mathfrak{g}$ not Abelian the universal enveloping algebra $U\mathfrak{g}$ is a noncommutative associative algebra, which can be thought of as a quantization of the Poisson algebra $S\mathfrak{g} = P\mathfrak{g}^*$. We shall make this phrase more precise in an exercise of the next section. One should think of the associative algebra $U\mathfrak{g}$ as some universal operator algebra attached to $\mathfrak{g}$, in which associative algebra computations can be universaly performed without being limited to $\mathfrak{g}$ with its Lie bracket and nasty Jacobi identity.

For the moment $U\mathfrak{g}$ is a rather unapproachable abstract object. A fundamental theorem about the structure of $U\mathfrak{g}$ is the Poincaré-Birkhof-Witt Theorem, which gives a vector space basis for $U\mathfrak{g}$ from a vector space basis of $\mathfrak{g}$. The idea behind the PBW Theorem is reminiscent to the normal ordering basis for the oscillator algebra.

Exercise 2.1. *Check that the Heisenberg algebra $\mathfrak{a}$ with basis $\{p, q, z\}$ given by*

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \ z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a Lie algebra with commutation relations

$$[p, q] = z \ , \ [z, p] = [z, q] = 0 \ .$$

Exercise 2.2. *In the new basis $\{a_-, a_+, z\}$ of the Heisenberg algebra $\mathfrak{a}$ given by*

$$a_- = (q - ip)/\sqrt{2} \ , \ a_+ = (q + ip)/\sqrt{2}$$

the commutation relations become

$$[a_+, a_-] = iz \ , \ [z, a_-] = [z, a_+] = 0 \ .$$

Exercise 2.3. *Show that for any Lie algebra $\mathfrak{g}$ the Poisson bracket on $S\mathfrak{g} = P\mathfrak{g}^*$ is a Lie bracket.*

Exercise 2.4. *Let $\mathfrak{g}$ be a Lie algebra with a real form $\mathfrak{g}_0 \subset \mathfrak{g}$. Prove that the associated antilinear antiinvolution of $\mathfrak{g}$ extends in a unique way to a star structure on $U\mathfrak{g}$.*

3 The Poincaré-Birkhof-Witt Theorem

Let $\mathfrak{g}$ be a Lie algebra of dimension n with a basis $x_1, \cdots, x_n$. The tensor algebra $\mathrm{T}\mathfrak{g} = \oplus T^d\mathfrak{g}$ is a graded algebra with corresponding basis of $T^d\mathfrak{g}$ given by

$$\{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_d};\ 1 \leq i_1, \cdots, i_d \leq n = dimV\},\ d \in \mathbb{N}\ .$$

The symmetric algebra $S\mathfrak{g} = \oplus S^d\mathfrak{g} = T\mathfrak{g}/I\mathfrak{g} = P\mathfrak{g}^*$ is also graded with degree d part $P^d\mathfrak{g}^* = T^d\mathfrak{g}/I^d\mathfrak{g}$ with

$$I^d\mathfrak{g} = I\mathfrak{g} \cap T^d\mathfrak{g} = \bigoplus_{k+l=d-2} \bigoplus_{i<j} \{T^k\mathfrak{g} \otimes (x_i \otimes x_j - x_j \otimes x_i) \otimes T^l\mathfrak{g}\}\ .$$

In turn this implies that $S^d\mathfrak{g}$ has basis

$$\{x_{i_1} x_{i_2} \cdots x_{i_d};\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}\ .$$

The universal enveloping algebra $U\mathfrak{g} = T\mathfrak{g}/J\mathfrak{g}$ is no longer graded because the ideal $J\mathfrak{g}$, defined by

$$J\mathfrak{g} = \bigoplus_{i<j} \{T\mathfrak{g} \otimes (x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]) \otimes T\mathfrak{g}\}\ ,$$

is no longer homogeneous. What remains is a filtration $U\mathfrak{g} = \cup U_d\mathfrak{g}$ with

$$T_d\mathfrak{g} = \bigoplus_{k \leq d} T^k\mathfrak{g} \quad \text{tensors of degree} \leq d\ ,$$

$$J_d\mathfrak{g} = \bigoplus_{k+l=d-2} \bigoplus_{i<j} \{T_k\mathfrak{g} \otimes (x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]) \otimes T_l\mathfrak{g}\}\ ,$$

$$U_d\mathfrak{g} = T_d\mathfrak{g}/J_d\mathfrak{g}\ ,\ J_d\mathfrak{g} = J\mathfrak{g} \cap T_d\mathfrak{g}\ .$$

Clearly multiplication in $U\mathfrak{g}$ defines a linear map $U_d\mathfrak{g} \otimes U_e\mathfrak{g} \mapsto U_{d+e}\mathfrak{g}$.

Definition 3.1. *The excess e of a basis vector $x_{i_1} \otimes \cdots \otimes x_{i_d}$ of $T^d\mathfrak{g}$ is given by*

$$e = \#\ \{(p, q); p < q, i_p > i_q\}$$

We write ${}^eT^d\mathfrak{g} = \mathrm{span}\{x_{i_1} \otimes \cdots \otimes x_{i_d};\ \text{excess} = e\}$ and ${}^eT\mathfrak{g} = \oplus_d\, {}^eT^d\mathfrak{g}$. A monomial $x_{i_1} \otimes \cdots \otimes x_{i_d}$ has excess equal to 0 if and only if $i_1 \leq i_2 \leq \cdots \leq i_d$ in which case we also speak of a standard monomial. The standard monomials in the basis of ${}^0T$ descend to the basis $\{x_{i_1} x_{i_2} \cdots x_{i_d}\}$ of $S\mathfrak{g} = T\mathfrak{g}/I\mathfrak{g}$. In other words $T\mathfrak{g} = {}^0T\mathfrak{g} \oplus I\mathfrak{g}$, i.e. $T\mathfrak{g} = {}^0T\mathfrak{g} + I\mathfrak{g}$ and ${}^0T\mathfrak{g} \cap I\mathfrak{g} = 0$.

A similar result holds for $U\mathfrak{g} = T\mathfrak{g}/J\mathfrak{g}$, and is called the PBW theorem.

Theorem 3.2. *We have $T\mathfrak{g} = {}^0T\mathfrak{g} \oplus J\mathfrak{g}$.*

Proof. We will first show that $T\mathfrak{g} = {}^0T\mathfrak{g} + J\mathfrak{g}$.

Consider a monomial $x_{i_1} \otimes \cdots \otimes x_{i_d} \in {}^eT^d\mathfrak{g}$ of degree d and excess e. We shall prove that $x_{i_1} \otimes \cdots \otimes x_{i_d} \in {}^0T\mathfrak{g} + J\mathfrak{g}$ by induction on the degree d and the excess e. If $d = 0$ then $e = 0$, and there is nothing to prove. Likewise if $d \geq 0$ and $e = 0$ then $x_{i_1} \otimes \cdots \otimes x_{i_d} \in {}^0T\mathfrak{g} \subset {}^0T\mathfrak{g} + J\mathfrak{g}$. Now suppose that $e \geq 1$. Then we write (with say $i_p > i_{p+1}$)

$$x_{i_1} \otimes \cdots \otimes x_{i_d} = x_{i_1} \otimes \cdots \otimes x_{i_p} \otimes x_{i_{p+1}} \otimes \cdots \otimes x_{i_d} =$$

$$x_{i_1} \otimes \cdots \otimes x_{i_{p+1}} \otimes x_{i_p} \otimes \cdots \otimes x_{i_d} + x_{i_1} \otimes \cdots \otimes [x_{i_p}, x_{i_{p+1}}] \otimes \cdots \otimes x_{i_d} +$$
$$x_{i_1} \otimes \cdots \otimes (x_{i_p} \otimes x_{i_{p+1}} - x_{i_{p+1}} \otimes x_{i_p} - [x_{i_p}, x_{i_{p+1}}]) \otimes \cdots \otimes x_{i_d}$$

which lies in ${}^0T\mathfrak{g} + J\mathfrak{g}$, because the first term has degree d and excess $(e-1)$, the second term has degree $(d-1)$, while the third term lies in $J\mathfrak{g}$.

The second claim ${}^0T\mathfrak{g} \cap J\mathfrak{g} = 0$ is proved by a similar induction, but the details are more complicated.

Lemma 3.3. *There exists a (unique) linear map $L : T\mathfrak{g} \mapsto T\mathfrak{g}$ such that L is the identity on ${}^0T\mathfrak{g}$ and (in case $i_p > i_{p+1}$)*

$$L(x_{i_1} \otimes \cdots \otimes x_{i_p} \otimes x_{i_{p+1}} \otimes \cdots \otimes x_{i_d}) =$$

$$L(x_{i_1} \otimes \cdots \otimes x_{i_{p+1}} \otimes x_{i_p} \otimes \cdots \otimes x_{i_d}) + L(x_{i_1} \otimes \cdots \otimes [x_{i_p}, x_{i_{p+1}}] \otimes \cdots \otimes x_{i_d}) \ .$$

We shall refer to this formula as the ordering formula. Note that the ordering formula amounts to

$$L(x_{i_1} \otimes \cdots \otimes (x_{i_p} \otimes x_{i_{p+1}} - x_{i_{p+1}} \otimes x_{i_p} - [x_{i_p}, x_{i_{p+1}}]) \otimes \cdots \otimes x_{i_d})$$

so that $L = 0$ on $J\mathfrak{g}$. Hence ${}^0T\mathfrak{g} \cap J\mathfrak{g} \subset Ker(L-1) \cap Ker(L) = 0$. Therefore the second claim is a direct consequence of the existence of the linear map L of the lemma.

Proof. We prove the lemma by a double induction on the degree d and the excess e. The problem is to define L on the monomial basis $x_{i_1} \otimes \cdots \otimes x_{i_d}$ of ${}^eT^d\mathfrak{g}$. If $e = 0$ then $L(x_{i_1} \otimes \cdots \otimes x_{i_d}) = x_{i_1} \otimes \cdots \otimes x_{i_d}$ by assumption. Now suppose $e \geq 1$. If we have initially (with say $i_p > i_{p+1}$ and $i_q > i_{q+1}$)

$$I = L(x_{i_1} \otimes \cdots \otimes x_{i_p} \otimes x_{i_{p+1}} \otimes \cdots \otimes x_{i_q} \otimes x_{i_{q+1}} \cdots \otimes x_{i_d})$$

and then rewrite it using the the ordering formula of the lemma as a linear combination (involving the structure constants of $\mathfrak{g}$ for the given basis) of the image under L of monomials of lower excess or degree. However the rewriting can be done using the ordering formula at place $(p, p+1)$ or at place $(q, q+1)$. We have to check that the outcome is the same.

In the first case we assume that $p+1 < q$. Let the final expression F be obtained from I by using the ordering formula at place $(p, p+1)$, and let F' be obtained from I by using the ordering formula at place $(q, q+1)$. Then we get $F = F'$ by using the above lemma for F at place $(q, q+1)$ and for F' at place $(p, p+1)$, which is allowed by induction on the excess and the degree. Explicitly

$$F = L(x_{i_1} \otimes \cdots \otimes x_{i_{p+1}} \otimes x_{i_p} \otimes \cdots \otimes x_{i_q} \otimes x_{i_{q+1}} \cdots \otimes x_{i_d})$$

$$+ L(x_{i_1} \otimes \cdots \otimes [x_{i_p}, x_{i_{p+1}}] \otimes \cdots \otimes x_{i_q} \otimes x_{i_{q+1}} \cdots \otimes x_{i_d})$$

which can be rewritten as

$$F = L(x_{i_1} \otimes \cdots \otimes x_{i_{p+1}} \otimes x_{i_p} \otimes \cdots \otimes x_{i_{q+1}} \otimes x_{i_q} \cdots \otimes x_{i_d})$$

$$+ L(x_{i_1} \otimes \cdots \otimes x_{i_{p+1}} \otimes x_{i_p} \otimes \cdots \otimes [x_{i_q}, x_{i_{q+1}}] \cdots \otimes x_{i_d})$$

$$+ L(x_{i_1} \otimes \cdots \otimes [x_{i_p}, x_{i_{p+1}}] \otimes \cdots \otimes x_{i_{q+1}} \otimes x_{i_q} \cdots \otimes x_{i_d})$$

$$+ L(x_{i_1} \otimes \cdots \otimes [x_{i_p}, x_{i_{p+1}}] \otimes \cdots \otimes [x_{i_q}, x_{i_{q+1}}] \cdots \otimes x_{i_d})\ .$$

The outcome has a symmetric role in p and q, and hence $F = F'$.

In the second case we assume that $p+1 = q$. Our initial expression is

$$I = L(x_{i_1} \otimes \cdots \otimes x_{i_p} \otimes x_{i_{p+1}} \otimes x_{i_{p+2}} \otimes \cdots \otimes x_{i_d})$$

or more simply (but without loss of generality)

$$I = L(x_{i_1} \otimes x_{i_2} \otimes x_{i_3})$$

with $i_1 > i_2 > i_3$. So we have

$$F = L(x_{i_2} \otimes x_{i_1} \otimes x_{i_3}) + L([x_{i_1}, x_{i_2}] \otimes x_{i_3}) =$$

$$L(x_{i_2} \otimes x_{i_3} \otimes x_{i_1}) + L(x_{i_2} \otimes [x_{i_1}, x_{i_3}]) + L([x_{i_1}, x_{i_2}] \otimes x_{i_3}) =$$

$$L(x_{i_3} \otimes x_{i_2} \otimes x_{i_1}) + L([x_{i_2}, x_{i_3}] \otimes x_{i_1}) + L(x_{i_2} \otimes [x_{i_1}, x_{i_3}]) + L([x_{i_1}, x_{i_2}] \otimes x_{i_3})$$

and similarly

$$F' = L(x_{i_1} \otimes x_{i_3} \otimes x_{i_2}) + L(x_{i_1} \otimes [x_{i_2}, x_{i_3}]) =$$

$$L(x_{i_3} \otimes x_{i_1} \otimes x_{i_2}) + L([x_{i_1}, x_{i_3}] \otimes x_{i_2}) + L(x_{i_1} \otimes [x_{i_2}, x_{i_3}]) =$$

$$L(x_{i_3} \otimes x_{i_2} \otimes x_{i_1}) + L(x_{i_3} \otimes [x_{i_1}, x_{i_2}]) + L([x_{i_1}, x_{i_3}] \otimes x_{i_2}) + L(x_{i_1} \otimes [x_{i_2}, x_{i_3}]) \ .$$

Hence we find

$$F - F' = L([x_{i_2}, x_{i_3}] \otimes x_{i_1} - x_{i_1} \otimes [x_{i_2}, x_{i_3}]) +$$

$$L(x_{i_2} \otimes [x_{i_1}, x_{i_3}] - [x_{i_1}, x_{i_3}] \otimes x_{i_2}) + L([x_{i_1}, x_{i_2}] \otimes x_{i_3} - x_{i_3} \otimes [x_{i_1}, x_{i_2}])$$
$$= L([[x_{i_2}, x_{i_3}], x_{i_1}] + [x_{i_2}, [x_{i_1}, x_{i_3}]] + [[x_{i_1}, x_{i_2}], x_{i_3}])$$
$$= L([[x_{i_2}, x_{i_3}], x_{i_1}] + [[x_{i_3}, x_{i_1}], x_{i_2}] + [[x_{i_1}, x_{i_2}], x_{i_3}]) = L(0) = 0$$

by the antisymmetry of the bracket and the Jacobi identity. This concludes the proof that $F = F'$ in the second case, and finishes the proof of the lemma. □

As mentioned before the ordering formula of the lemma has the theorem as immediate consequence. This finishes the proof of the PBW theorem. □

Corollary 3.4. *Given a basis $x_1, \cdots, x_n$ of $\mathfrak{g}$ the universal enveloping algebra $U\mathfrak{g}$ has PBW basis*

$$\{x_{i_1} x_{i_2} \cdots x_{i_d};\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n = dimV,\ d \in \mathbb{N}\}\ ,$$

or equivalently

$$\{x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n};\ 0 \leq m_1, m_2 \cdots, m_n < \infty\}\ .$$

Corollary 3.5. *The natural map $j : \mathfrak{g} \to U\mathfrak{g}$ is an injection. From now on we write $x \mapsto x$ for the canonical injection $\mathfrak{g} \to U\mathfrak{g}$.*

This corollary justifies the terminology universal enveloping algebra for $U\mathfrak{g}$, since it is truely an enveloping associative algebra of $\mathfrak{g}$. Moreover $U\mathfrak{g}$ is generated as an associative algebra by $\mathfrak{g}$. Finally it is universal with respect to these two properties.

Exercise 3.1. *Let $\mathfrak{a}$ be the Heisenberg algebra with basis a_-, a_+, z and commutation relations $[a_+, a_-] = iz,\ [z, a_+] = [z, a_-] = 0$. Prove that the algebra $U\mathfrak{a}/U\mathfrak{a}(z + i\hbar)$ becomes the associative algebra $\mathcal{A}$ with generators a_- and a_+ and with relation $[a_+, a_-] = \hbar$. So the normal ordering basis $a_-^m a_+^n$ with $m, n \in \mathbb{N}$ of $\mathcal{A}$ is nothing but the PBW basis of $U\mathfrak{a}/U\mathfrak{a}(z + i\hbar)$.*

Exercise 3.2. *Show that the natural linear bijection $T_d\mathfrak{g}/T_{d-1}\mathfrak{g} \to T^d\mathfrak{g}$ induces a linear bijection $U_d\mathfrak{g}/U_{d-1}\mathfrak{g} \to S^d\mathfrak{g}$ that is compatible with the multiplication in $U\mathfrak{g}$ and $S\mathfrak{g}$. Show that the Poisson bracket $\{\cdot,\cdot\}$*

$$S^d\mathfrak{g} \otimes S^{d'}\mathfrak{g} \to S^{d+d'-1}\mathfrak{g}$$

amounts to

$$U_d\mathfrak{g}/U_{d-1}\mathfrak{g} \otimes U_{d'}\mathfrak{g}/U_{d'-1}\mathfrak{g} \to U_{d+d'-1}\mathfrak{g}/U_{d+d'-2}\mathfrak{g}$$

sending $u \otimes v$ to $uv - vu$ (multiplication in $U\mathfrak{g}$). The conclusion is that the Poisson bracket is the leading term of the commutator bracket. This is entirely analoguous to the way classical mechanics can be obtained from quantum mechanics as the classical limit.

4 The Lie algebra $\mathfrak{sl}_2$ and its Verma representations

The Lie algebra $\mathfrak{sl}_2$ has Chevalley basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;,\; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,\; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutation relations

$$[h,e] = 2e \;,\; [h,f] = -2f \;,\; [e,f] = h \;.$$

The universal enveloping algebra $U\mathfrak{sl}_2$ has the associated PBW basis $f^i h^j e^k$ for $i,j,k \in \mathbb{N}$.

Lemma 4.1. *In $U\mathfrak{sl}_2$ the following relations hold*

1. $[h, f^{k+1}] = -2(k+1)f^{k+1}$
2. $[e, f^{k+1}] = (k+1)f^k(h-k)$

for all $k \in \mathbb{N}$.

Proof. The proof of the first formula goes by induction on $k \in \mathbb{N}$. The case $k = 0$ is clear, since $[h,f] = -2f$. Now for $k \geq 1$ we have

$$[h, f^{k+1}] = [h, ff^k] = [h,f]f^k + f[h,f^k] = -2ff^k - 2kff^k = -2(k+1)f^{k+1}$$

which proves the first formula.

Likewise the proof of the second formula goes by induction on $k \in \mathbb{N}$. The case $k = 0$ is clear, since $[e,f] = h$. Now for $k \geq 1$ we have

$$[e, f^{k+1}] = [e,f]f^k + f[e,f^k] = hf^k + kff^{k-1}(h-(k-1))$$

by using the induction hypothesis. Using the first formula we get

$$[e, f^{k+1}] = [h, f^k] + f^k h + kf^k(h-(k-1)) = -2kf^k + f^k h + kf^k(h-(k-1))$$

and therefore

$$[e, f^{k+1}] = (k+1)f^k h - f^k(2k + k(k-1)) = (k+1)f^k(h-k)$$

which proves the second formula. $\square$

The universal enveloping algebra $U\mathfrak{sl}_2$ has a natural (associative algebra) representation on the infinite dimensional vector space $U\mathfrak{sl}_2$ by left multiplication. For $\lambda \in \mathbb{C}$ the linear subspace

$$J(\lambda) = U\mathfrak{sl}_2(h-\lambda) + (U\mathfrak{sl}_2)e$$

is clearly an invariant linear subspace, and hence the quotient

$$M(\lambda) = U\mathfrak{sl}_2/J(\lambda)$$

also becomes an associative algebra representation space of $U\mathfrak{sl}_2$, called the Verma representation of $U\mathfrak{sl}_2$ with parameter $\lambda \in \mathbb{C}$. By restriction from $U\mathfrak{sl}_2$ to $\mathfrak{sl}_2$ the Verma representation $M(\lambda)$ can be equally well considered as a Lie algebra representation of $\mathfrak{sl}_2$.

Lemma 4.2. *The Verma representation $M(\lambda)$ of $\mathfrak{sl}_2$ has basis $v_k = f^k + J(\lambda)$ for $k \in \mathbb{N}$, and the generators $\{e, h, f\}$ act on this basis as*

$$hv_k = (\lambda - 2k)v_k\ ,\ ev_k = k(\lambda - (k-1))v_{k-1}\ ,\ fv_k = v_{k+1}$$

with $v_{-1} = 0$. The operator h in $M(\lambda)$ is diagonal in the basis v_k for $k \in \mathbb{N}$. The operators e and f in $M(\lambda)$ are called "ladder" operators.

Proof. The PBW basis vector $f^i h^j e^k$ can be reduced modulo $J(\lambda)$ to a multiple of f^i. Hence $v_k,\ k \in \mathbb{N}$ span $M(\lambda)$, and $v_k \neq 0\ \forall k \in \mathbb{N}$.

By definition $v_0 = 1 + J(\lambda)$, hence

$$hv_0 \equiv h = \lambda + (h-\lambda) \equiv \lambda \equiv \lambda v_0$$

with $\equiv$ denoting equality modulo $J(\lambda)$. Moreover

$$hv_k = hf^k v_0 = [h, f^k]v_0 + f^k h v_0 = -2kf^k v_0 + \lambda f^k v_0 = (\lambda - 2k)v_k$$

and therefore $v_k,\ k \in \mathbb{N}$ is a basis of $M(\lambda)$. Clearly

$$ev_0 \equiv e \equiv 0$$

so that

$$ev_k = ef^k v_0 = [e, f^k]v_0 + f^k e v_0 = k(\lambda - (k-1))v_{k-1}\ .$$

The last formula $fv_k = v_{k+1}$ is trivial. $\square$

Definition 4.3. *The eigenvalues of the operator h in a representation space of $\mathfrak{sl}_2$ are called the "weights" of that representation. For example, the weights of the Verma representation $M(\lambda)$ are $\{\lambda - 2k;\ k \in \mathbb{N}\}$ and the parameter $\lambda \in \mathbb{C}$ is therefore called the "highest weight" of $M(\lambda)$.*

Lemma 4.4. *A linear subspace of $M(\lambda)$ that is invariant under h is the linear span of some of the v_k. A nonzero linear subspace of $M(\lambda)$ that is invariant under h and f is the linear span of the v_k for $k \geq k_0$, for some $k_0 \in \mathbb{N}$.*

Proof. The first statement follows from Proposition 1.7, and the second statement is obvious from the first statement. □

Corollary 4.5. *A nontrivial (distinct from 0 and the full space) linear subspace of $M(\lambda)$ that is invariant under $\mathfrak{sl}_2$ exists if an only if $ev_{k_0} = 0$ for some $k_0 \geq 1$, i.e. $\lambda = n \in \mathbb{N}$ and $k_0 = (n+1)$. In this case we have $M(-n-2) \hookrightarrow M(n)$ with irreducible quotient representation $L(n) = M(n)/M(-n-2)$ of dimension $(n+1)$, and weights $\{n, n-2, \cdots, -n\}$.*

We shall denote by $L(\lambda)$ the irreducible quotient of $M(\lambda)$. In other words $L(\lambda) = M(n)/M(-n-2)$ if $\lambda = n \in \mathbb{N}$ while $L(\lambda) = M(\lambda)$ if $\lambda \notin \mathbb{N}$.

We have seen that $\mathfrak{sl}_2(\mathbb{C})$ has two real forms $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ with corresponding antilinear antiinvolutions

$$h^\star = h\ ,\ e^\star = f\ ,\ f^\star = e$$

$$h^\star = h\ ,\ e^\star = -f\ ,\ f^\star = -e$$

respectively. A Hermitian form $\langle \cdot, \cdot \rangle$ on V assigns to each pair $u, v \in V$ a complex number $\langle u, v \rangle$ which is linear in $u \in V$ and antilinear in $v \in V$. We do not require the Hermitian form to be unitary (which means $\langle v, v \rangle$ positive for all $v \neq 0$), and even the kernel of the Hermitian form (which consists of all $u \in V$ with $\langle u, v \rangle = 0$ for all $v \in V$) might be a nonzero linear subspace. Now let V also be a representation space for $\mathfrak{sl}_2$. The Hermitian form $\langle \cdot, \cdot \rangle$ is called invariant for $\mathfrak{sl}_2$ with respect to the given star structure if

$$\langle xu, v \rangle = \langle u, x^\star v \rangle\ \forall u, v \in V\ ,\ \forall x \in \mathfrak{sl}_2\ .$$

In particular for $V = M(\lambda)$ an invariant Hermitian form $\langle \cdot, \cdot \rangle$ satisfies

$$(\lambda - 2k)\langle v_k, v_l \rangle = \langle hv_k, v_l \rangle = \langle v_k, hv_l \rangle = (\overline{\lambda} - 2l)\langle v_k, v_l \rangle$$

and hence $((\lambda-\overline{\lambda})-2(k-l))\langle v_k, v_l\rangle = 0$. Since $(\lambda-\overline{\lambda}) \in i\mathbb{R}$ and $(k-l) \in \mathbb{Z}$ we get $\langle v_k, v_l\rangle = 0$ if $k \neq l$. Furthermore we find

$$\langle v_{k+1}, v_{k+1}\rangle = \langle v_{k+1}, f v_k\rangle = \pm\langle e v_{k+1}, v_k\rangle = \pm(k+1)(\lambda-k)\langle v_k, v_k\rangle$$

with $\pm = +$ for $\mathfrak{su}(2)$ and $\pm = -$ for $\mathfrak{su}(1,1)$.

Theorem 4.6. *The irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2$ is unitary for $\mathfrak{su}(2)$ if and only if $\lambda = n \in \mathbb{N}$, i.e. $L(\lambda = n)$ has finite dimension $(n+1)$.*

Proof. We may assume that $\langle v_0, v_0\rangle$ is positive. If $\lambda = n \in \mathbb{N}$ then in $M(n)$ we see that

$$\langle v_k, v_k\rangle = (k!)^2 \binom{n}{k} \langle v_0, v_0\rangle$$

is positive for $0 \leq k \leq n$, while $\langle v_{k+1}, v_{k+1}\rangle = 0 \ \forall k \geq n$. Hence the kernel of the invariant Hermitian form on $M(n)$ is exactly equal to the maximal proper subrepresentation $M(-n-2)$, and the invariant Hermitian form on $M(n)$ descends to an invariant unitary structure on $L(n)$. If $\lambda \notin \mathbb{N}$ then $M(\lambda) = L(\lambda)$ is a unitary representation if and only if $(\lambda - k) > 0 \ \forall k \in \mathbb{N}$. These inequalities have no solution. $\square$

All unitary representations $L(n)$ for $n \in \mathbb{N}$ of $\mathfrak{su}(2)$ can be integrated to unitary representations of the group $SU(2)$. In fact these are all the unitary irreducible representations of $SU(2)$.

Theorem 4.7. *The irreducible representation $L(\lambda)$ of $\mathfrak{sl}_2$ is unitary for $\mathfrak{su}(1,1)$ if and only if $\lambda \leq 0$.*

Proof. Assume that $\langle v_0, v_0\rangle$ is positive. Like in the above proof we get

$$\langle v_k, v_k\rangle = (k!)^2(-1)^k \binom{\lambda}{k} \langle v_0, v_0\rangle \ .$$

Clearly $\langle v_1, v_1\rangle = (-\lambda)\langle v_0, v_0\rangle$ to be nonnegative is a necessary condition for unitarity of $L(\lambda)$ is that , i.e. $\lambda \leq 0$. If $\lambda = 0$ then $L(0)$ is the one dimensional trivial representation, which is always unitary. If $\lambda < 0$ then $L(\lambda) = M(\lambda)$ is indeed unitary since $-(\lambda - k) > 0 \ \forall k \in \mathbb{N}$. $\square$

All unitary representations $L(-n)$ for $n \in \mathbb{N}$ of $\mathfrak{su}(1,1)$ can be integrated (after a Hilbert space completion) to unitary representations of the group $SU(1,1)$. For $n < -1$ these representations are so called "discrete series" representations of $SU(1,1)$, because all the matrix coëfficients are square integrable with respect to a biinvariant measure on $SU(1,1)$. These (and their lowest weight companions in the first exercise below) are not all the unitary irreducible representations of $SU(1,1)$. The family of unitary representations of $\mathfrak{su}(1,1)$ described in the above theorem is ususally referred to as the analytic continuation of the discrete series representations of $SU(1,1)$.

Exercise 4.1. *The Verma representations $M(\lambda)$ of $\mathfrak{sl}_2$ are sometimes called the highest weight Verma representations to distinguish them from the lowest weight Verma representations $M'(\lambda) = U\mathfrak{sl}_2/J'(\lambda)$ defined by*

$$J'(\lambda) = U\mathfrak{sl}_2(h-\lambda) + U\mathfrak{sl}_2 f \ .$$

Sow that for $n \in \mathbb{N}$ the irreducible quotient $L'(-n)$ has finite dimension $(n+1)$, and is unitary for $\mathfrak{su}(2)$. Show that $L(n)$ and $L'(-n)$ are equivalent representations. Show that the irreducible quotient $L'(\lambda)$ is unitary for $\mathfrak{su}(1,1)$ in case $\lambda \geq 0$.

Exercise 4.2. *Write formally $exp(x) = \sum_{n\geq 0} \ x^n/n!$ for the exponential series.*

1. *Show that $exp(x)exp(y) = exp(x+y)$ if $xy = yx$.*
2. *Deduce that $exp(x)$ is invertible with inverse $exp(-x)$.*
3. *Show that $exp(e)$ is a well defined operator in the Verma representation $M(\lambda)$ of $\mathfrak{sl}_2$.*
4. *Show that $exp(f)$ is a well defined operator in the finite dimensional representation $L(n)$, $n \in \mathbb{N}$ of $\mathfrak{sl}_2$.*
5. *Show that in the adjoint representation $ad : \mathfrak{sl}_2 \to End(\mathfrak{sl}_2)$, $ad(x)y = [x,y]$ the operator $s = exp(e)exp(-f)exp(e)$ satisfies*

$$sh = -h \ , \ se = -f \ , \ sf = -e \ .$$

6. *Show that $(exp(x))y(exp(-x)) = exp(ad(x))y$ in any representation space of a Lie algebra $\mathfrak{g}$.*
7. *Conclude that for a finite dimensional representation $\mathfrak{sl}_2 \to End(L(n))$ the operator s is well defined, and maps vectors of weight k to vectors of weight $-k$.*

5 Lie algebra representations via linear algebra constructions

Suppose $\mathfrak{g}$ is a Lie algebra, and $U, V, W, \cdots$ are representation spaces for $\mathfrak{g}$. The tensor product $U \otimes V$ becomes a representation space for $\mathfrak{g}$ by defining

$$x(u \otimes v) = (xu) \otimes v + u \otimes (xv) \;\forall\, x \in \mathfrak{g},\; \forall\, u \in U,\; \forall\, v \in V$$

because $[x, y](u \otimes v) = x(y(u \otimes v)) - y(x(u \otimes v))$ by a straightforward computation. Likewise the dual vector space $U^* = Hom(U, \mathbb{C})$ becomes a representation space for $\mathfrak{g}$ by defining

$$xf(u) = -f(xu) \;,$$

and more generally the vector space $Hom(U, V)$ becomes a representation space for $\mathfrak{g}$ by defining

$$xf(u) = x(f(u)) - f(xu) \;.$$

The tensor algebra $TV = \oplus T^k V$ is also in a natural way a representation space for $\mathfrak{g}$, and the subspaces

$$IV = span\{TV \otimes (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes TV;\; v_1, v_2 \in V\} \;,$$

$$JV = span\{TV \otimes (v_1 \otimes v_2 + v_2 \otimes v_1) \otimes TV;\; v_1, v_2 \in V\}$$

are easily seen to be invariant linear subspaces for $\mathfrak{g}$. Hence the quotient spaces $SV = TV/IV$ (the symmetric algebra on V) and $AV = TV/JV$ (the antisymmetric algebra on V) become in this way natural representation spaces of $\mathfrak{g}$. Written out explicitly in a basis $\{v_i\}$ of V we have for the symmetric algebra

$$x(v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n}) = m_1 v_1^{m_1-1}(xv_1) v_2^{m_2} \cdots v_n^{m_n} +$$

$$m_2 v_1^{m_1} v_2^{m_2-1}(xv_2) \cdots v_n^{m_n} + \cdots + m_n v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n-1}(xv_n)$$

and for the antisymmetric algebra (with the product denoted by $\wedge$)

$$x(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}) = (xv_{i_1}) \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} -$$

$$v_{i_1} \wedge (xv_{i_2}) \wedge \cdots \wedge v_{i_k} + \cdots + (-1)^{k+1} v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge (xv_{i_k})$$

which are called the Leibniz product rules. If the dimension of V is equal to n then the dimension of S^kV is equal to $\binom{n+k-1}{k}$ while the dimension of A^kV is equal to $\binom{n}{k}$. In particular SV is infinite dimensional (if $n \neq 0$) while the dimension of AV is equal to 2^n.

We write $V^{\mathfrak{g}} = \{v \in V;\ xv = 0\ \forall\ x \in \mathfrak{g}\}$ for the space of invariants in V. In the particular case of $Hom(U,V)$ the linear operators in $Hom(U,V)^{\mathfrak{g}}$ are called intertwiners. If $A \in Hom(U,V)^{\mathfrak{g}}$ is in addition a bijection then we say that U and V are equivalent representations, and we denote $U \cong V$. Here is Schur's Lemma, which is easy and fundamental!

Lemma 5.1. *If U and V are irreducible finite dimensional representations of $\mathfrak{g}$ then the space $Hom(U,V)^{\mathfrak{g}}$ of intertwiners has dimension 1 if $U \cong V$ and dimension 0 otherwise.*

Proof. Suppose $A \in Hom(U,V)^{\mathfrak{g}}$ is a nonzero intertwiner. Then the kernel of A is a proper invariant subspace of U, hence equal to 0. Likewise the image of A is a nonzero invariant subspace of V, hence equal to V. Hence $A \neq 0$ is equivalent to A being a bijection, which in turn is equivalent to $U \cong V$. Therefore for nonequivalent U and V we get $Hom(U,V)^{\mathfrak{g}} = 0$.

Now suppose that U and V are equivalent, and fix a nonzero (hence bijective) intertwiner $A \in Hom(U,V)^{\mathfrak{g}}$. For any $B \in Hom(U,V)^{\mathfrak{g}}$ we see $BA^{-1} \in End(V)^{\mathfrak{g}}$ is a selfintertwiner on V. Pick an eigenvalue λ of BA^{-1} (which is always possible because V is finite dimensional by assumption). Hence the selfintertwiner $(BA^{-1} - \lambda) \in End(V)^{\mathfrak{g}}$ has a nonzero kernel, which by irreducibility of V implies that $(BA^{-1} - \lambda) = 0$, or equivalently $B = \lambda A$. The conclusion is that for U and V equivalent the intertwiner space $Hom(U,V)^{\mathfrak{g}}$ is one dimensional. $\square$

Suppose $\mathfrak{g}_0$ is a real form of the Lie algebra $\mathfrak{g}$ with corresponding antilinear antiinvolution $(x + iy)^\star = -x + iy$ for $x, y \in \mathfrak{g}_0$. If $U, V, W, \cdots$ are unitary representations of $\mathfrak{g}$ then the tensor product $U \otimes V$ and the dual space V^* carry natural Hermitian inner products, defined by

$$\langle u \otimes v, u' \otimes v' \rangle = \langle u, u' \rangle \langle v, v' \rangle\ ,\ \langle f, g \rangle = \sum f(v_i)\overline{g(v_i)}$$

with $\{v_i\}$ an orthonormal basis of V. It is easy to check that $U \otimes V$ and V^* become unitary representations of $\mathfrak{g}$. Likewise the vector space $Hom(U,V)$, the tensor algebra TV, the symmetric algebra SV and the antisymmetric algebra AV all become unitary representations of $\mathfrak{g}$.

Theorem 5.2. *A finite dimensional unitary representation V of a Lie algebra $\mathfrak{g}$ with star operation $x \mapsto x^\star$ is completely reducible, in the sense that V can be written as a direct sum of irreducible subrepresentations. If each unitary irreducible representation of $\mathfrak{g}$ occurs with multiplicity at most one then we say that the representation is multiplicity free.*

Proof. If $U \subset V$ is an invariant linear subspace for $\mathfrak{g}$ then the orthogonal complement $U^\perp = \{v \in V; \langle u, v\rangle = 0 \ \forall u \in U\}$ is easily seen to be invariant as well, and $V = U \oplus U^\perp$. A required orthogonal direct sum decomposition of V into irreducible subrepresentations can be found by induction on the dimension of V. $\square$

Remark that the above proof does not come with an algorithm how to find a direct sum decomposition into irreducibles. In actual examples finding an explicit decomposition of a unitary representation into irreducible subrepresentations can be an outstandingly complicated task!

An important remark is that constructions of linear algebra with unitary representations of $\mathfrak{g}$ with star structure $x \mapsto x^\star$ produce again unitary representations. The proof is easy and given as an exercise.

A last remark is that the decomposition of V as a direct sum of irreducible subrepresentations is in general not unique. However what is unique is the direct sum decomposition

$$V = V^{\mathfrak{g}} \oplus \mathfrak{g}V$$

with $V^{\mathfrak{g}}$ the direct sum of all the trivial subrepresentations and $\mathfrak{g}V$ the direct sum of all the nontrivial subrepresentations. The selfadjoint projection operator $R \in End(V)$ with image $V^{\mathfrak{g}}$ and with kernel $\mathfrak{g}V$ is called the Reynolds operator. Clearly $R \in End(V)^{\mathfrak{g}}$.

Corollary 5.3. *If V is a finite dimensional unitary representation of $\mathfrak{g}$ then the the tensor algebra TV and the symmetric algebra $SV = PV^*$ are completely reducible.*

Proof. Since $TV = \oplus T^kV$ and $SV = \oplus S^kV$ are direct sum decompositions with finite dimensional components invariant under $\mathfrak{g}$ this follows indeed from the previous theorem. $\square$

Lemma 5.4. *In the notation of the above corollary the Reynolds operator R on $SV = PV^*$ satisfies $R(fg) = fR(g)$ for all $f \in (SV)^{\mathfrak{g}}$ and $g \in SV$. Moreover the Reynolds operator can not raise the degree.*

Proof. Observe that multiplication by $f \in (SV)^{\mathfrak{g}}$ preserves both $(SV)^{\mathfrak{g}}$ and $\mathfrak{g}(SV)$. If we write $g = Rg + (g - Rg)$ then $fg = fRg + f(g - Rg)$ with $fRg \in (SV)^{\mathfrak{g}}$ and $f(g - Rg) \in \mathfrak{g}(SV)$. Hence $R(fg) = fR(g)$ for all $f \in (SV)^{\mathfrak{g}}$ and $g \in SV$. $\square$

Example 5.5. *The standard representation of $\mathfrak{sl}_2$ on $\mathbb{C}^2$ is by definition unitary for $\mathfrak{su}(2)$. The representation space $S^n(\mathbb{C}^2)$ has dimension $(n+1)$ with basis $x^{n-m}y^m$ for $m = 0, 1, \cdots, n$. The Chevalley basis $\{e, h, f\}$ acts in this representation space via the first order differential operators*

$$e = x\partial/\partial y \ , \ f = y\partial/\partial x \ , \ h = [e, f] = x\partial/\partial x - y\partial/\partial y \ .$$

The representation $S^n(\mathbb{C}^2)$ of $\mathfrak{sl}_2$ is called the degree n binary forms representation. The monomial x^n is the (up to a multiplicative constant unique) highest weight vector of weight n, and therefore $S^n(\mathbb{C}^2)$ is equivalent to $L(n)$. Now the corollary implies that $L(n) \cong S^n(\mathbb{C}^2)$ is also unitary for $\mathfrak{su}(2)$. This proof is a quick alternative for Theorem 4.6. Proving unitarity of a given representation via constructions of linear algebra somehow not only proves but also explains why the given representation is unitary.

Example 5.6. *The decomposition of $L(m) \otimes L(n)$ into irreducible subrepresentations is multiplicity free and given by the Clebsch-Gordan rule*

$$L(m) \otimes L(n) \cong L(m+n) \oplus L(m+n-2) \oplus \cdots \oplus L(|m-n|) \ .$$

By Theorem 5.2 we can write $L(m) \otimes L(n)$ as a direct sum over $k \in \mathbb{N}$ of the $L(k)$ with certain multiplicities $m_k \in \mathbb{N}$. The weights of $L(m) \otimes L(n)$ are on the one hand (as a set with multiplicities) equal to

$$\{i + j; |i| \leq m, \ |j| \leq n, \ (n-i) \in 2\mathbb{N}, \ (m-j) \in 2\mathbb{N}\}$$

and on the other hand equal to the union over $k \in \mathbb{N}$ of $\{k, k-2, \cdots, -k\}$, counted with multiplicity m_k. A bit of combinatorial puzzling gives the above Clebsch-Gordan rule.

Example 5.7. *A special case of the Clebsch-Gordan rule gives*

$$L(n) \otimes L(n) = L(2n) \oplus L(2n-2) \cdots \oplus L(2) \oplus L(0) \ .$$

Note that the occurence of $L(0)$ with multiplicity one is also clear from Schur's Lemma. On the other hand we also have the decomposition in symmetric and antisymmetric tensors

$$L(n) \otimes L(n) = S^2(L(n)) \oplus A^2(L(n)) \ ,$$

and one can ask how the two decompositions match? The answer to this question is

$$S^2(L(n)) = L(2n) \oplus L(2n-4) \oplus \cdots ,$$

$$A^2(L(n)) = L(2n-2) \oplus L(2n-6) \oplus \cdots .$$

One can mimic the argument of the previous example. Indeed the weights of $S^2(L(n))$ *are given by (as a set with multiplicities)*

$$\{i+j; |i| \leq m,\ |j| \leq n,\ (n-i) \in 2\mathbb{N},\ (m-j) \in 2\mathbb{N},\ i \leq j\} ,$$

while the weights of $A^2(L(n))$ *are given by (as a set with multiplicities)*

$$\{i+j; |i| \leq m,\ |j| \leq n,\ (n-i) \in 2\mathbb{N},\ (m-j) \in 2\mathbb{N},\ i < j\} .$$

We leave the combinatorial details to the reader.

If V is a representation of $\mathfrak{g}$ then the vectorspace $SV^{\mathfrak{g}}$ is in fact a subalgebra of SV, called the algebra of invariants for $\mathfrak{g}$. Indeed if $f, g \in SV^{\mathfrak{g}}$ then $fg \in SV^{\mathfrak{g}}$ by the Leibniz product rule. The next theorem is one of the fundamental results of invariant theory due to Hilbert and Weyl.

Theorem 5.8. *If V is a finite dimensional unitary representation of a Lie algebra $\mathfrak{g}$ with a star structure then the algebra of invariants $(SV)^{\mathfrak{g}}$ is a finitely generated algebra.*

Proof. The theorem will follow from the Hilbert basis theorem. Let V be a finite dimensional vector space. An ideal $I \subset SV$ is a vector subspace with the property that $fg \in I$ for all $f \in I,\ g \in SV$. The Hilbert basis theorem says that any ideal I in SV is finitely generated, so

$$I = f_1 SV + f_2 SV + \cdots + f_n SV$$

for some $f_1, f_2 \cdots, f_n \in I$. The proof of the Hilbert basis theorem can be found in any text book on (commutative) algebra.

We shall apply the Hilbert basis theorem to the homogeneous ideal generated by $(SV)^{\mathfrak{g}}_+$, consisting of all homogeneous invariants of positive degree. Suppose $f_1, f_2 \cdots f_n \in (SV)^{\mathfrak{g}}_+$ are homogeneous ideal generators of degree $1 \leq d_1 \leq d_2 \leq \cdots \leq d_n$. Let $f \in (SV)^{\mathfrak{g}}_+$ be a homogeneous invariant of degree $d \geq 1$. We shall prove that $f \in \mathbb{C}[f_1, f_2, \cdots, f_n]$ by induction on the degree d.

Write $f = f_1 g_1 + f_2 g_2 + \cdots + f_n g_n$ with $g_1, g_2, \cdots, g_n \in SV$ homogeneous of degree strictly smaller than d. Application of the Reynolds operator and using Lemma 5.4 gives

$$f = R(f) = f_1 R(g_1) + f_2 R(g_2) + \cdots + f_n R(g_n)$$

and because $R(g_1), R(g_2), \cdots, R(g_n) \in (SV)^{\mathfrak{g}}$ are all homogeneous of degree strictly smaller than d the theorem follows from the induction hypothesis. $\square$

Example 5.9. *The algebra of invariants $S(L(n))^{\mathfrak{sl}_2}$ for degree n binary forms has a (upto multiplicative scalar unique) generator f_2 in degree 2 if $n \geq 2$ is even, and has no generator in degree 2 if $n \geq 2$ is odd. This is clear from Example 5.7. In the special case of the adjoint representation $L(2) = \mathfrak{sl}_2$ one can show that $S(L(2))^{\mathfrak{sl}_2}$ is isomorphic to $\mathbb{C}[f_2]$.*

Example 5.10. *In the next section we will show that the algebra of invariants $S(L(4))^{\mathfrak{sl}_2}$ for degree 4 binary forms is isomorphic to $\mathbb{C}[f_2, f_3]$ with generators f_2 in degree 2 and f_3 in degree 3.*

Describing in an explicit way generators (and relations) for the algebra of invariants $SV^{\mathfrak{g}}$ is an extremely hard problem. Even in the case of invariants $S(L(n))^{\mathfrak{sl}_2}$ of $\mathfrak{sl}_2$ for degree n binary forms the problem is still hopeless, and even with the help of modern computers beyond comprehension for say n greater than 10.

Exercise 5.1. *Show that the linear isomorphism of vector spaces*

$$U^* \otimes V \to Hom(U, V),\ f \otimes v \mapsto \{u \mapsto f(u)v\}$$

is a natural isomorphism of representation spaces for the Lie algebra $\mathfrak{g}$.

Exercise 5.2. *If U and V are unitary representations of $\mathfrak{g}$ then check that $U \otimes V$ and V^* are unitary representations of $\mathfrak{g}$ as well.*

Exercise 5.3. *Show that the Chevalley basis $\{e, h, f\}$ of $\mathfrak{sl}_2$ acting in the representation space $S^n(\mathbb{C}^2)$ with basis $x^{n-m}y^m$ for $m = 0, 1, \cdots, n$ via the first order differential operators*

$$e = x\partial/\partial y\ ,\ f = y\partial/\partial x\ ,\ h = [e, f] = x\partial/\partial x - y\partial/\partial y$$

defines a representation of $\mathfrak{sl}_2$.

Exercise 5.4. *Show that the degree* 2 *invariant in* $S(\mathfrak{sl}_2)^{\mathfrak{sl}_2}$ *is given by the expression* $h^2 + 4ef$.

Exercise 5.5. *Show that the Casimir operator* $C = h^2 + 2ef + 2fe$ *is central in* $U\mathfrak{sl}_2$. *By Schur's Lemma the Casimir operator* C *acts in* $L(n)$ *as a scalar operator. Rewriting* $C = h^2 + 2h + 4fe$ *check that this scalar equals* $n^2 + 2n = (n+1)^2 - 1$.

Exercise 5.6. *Show that the Casimir operator* C *in a unitary representation* V *of* $\mathfrak{su}_2$ *is selfadjoint. The eigenspace decomposition* $V = \oplus V(n)$ *with* $V(n) = Ker(C - (n^2 + 2n))$ *is called the direct sum decomposition of* V *in isotypical components. Apparently the direct sum decomposition of* V *in irreducible components is not unique, but the direct sum decomposition of* V *in isotypical components (so we take irreducible components of the same type together) is unique by the above argument.*

Exercise 5.7. *Suppse we have given a representation of* $\mathfrak{sl}_2$ *on a finite dimensional Hilbert space* V *that is unitary for* $\mathfrak{su}_2$. *The eigenvalues of* h *on* V *are called the weights of* V. *Suppose* $k \in \mathbb{Z}$ *occurs as weight of* V *with multiplicity* $m_k \in \mathbb{N}$.

1. *Show that* $m_{-k} = m_k$ *for all* $k \in \mathbb{Z}$.

2. *Show that* $m_0 \geq m_2 \geq m_4 \geq \cdots$ *and* $m_1 \geq m_3 \geq m_5 \geq \cdots$. *The first and second items together are rephrased by saying that the multiplicities of the even weights and of the odd weights form so called palindromic sequences.*

3. *Suppose all weights of* V *are even and* $m_0 = 5, m_2 = 3, m_4 = 3, m_8 = 2, m_{12} = 1, m_{14} = 0$. *Describe the decomposition of* V *into irreducible components.*

4. *Suppose all weights of* V *are odd and* $m_5 = 4, m_7 = 1$ *and the dimension of* V *is equal to* 28. *Describe the decomposition of* V *into irreducible components. These data do not quite suffice for a unique solution, but there are two possibilities.*

6 Formal characters

Suppose we have an Abelian Lie algebra $\mathfrak{h}$ and a representation $\mathfrak{h} \to \mathfrak{gl}(V)$ of $\mathfrak{h}$ on a vector space V. For $\mu \in \mathfrak{h}^*$ we shall write

$$V_\mu = \{v \in V; hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$$

for the linear subspace of vectors of weight μ, and we shall call V_μ the weight space of weight μ. All weights μ with $V_\mu \neq 0$ are called the weights of the given representation. We say that the representation is $\mathfrak{h}$-diagonalizable if

$$V = \bigoplus_\mu V_\mu \ ,$$

and $\mathfrak{h}$-admissable if in addition all weights spaces are finite dimensional. For example, Verma representations for $\mathfrak{sl}_2$ restricted to $\mathfrak{h} = \mathbb{C}h$ are $\mathfrak{h}$-admissable. If $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$, then any finite dimensional representation of $\mathfrak{h}$ that is unitary for $\mathfrak{h}_0$ is indeed $\mathfrak{h}$-diagonalizable. Moreover all weights of such a unitary representation are real, i.e. satisfy $\mu^\star = \mu$.

Suppose we have given a basis $\mathbf{h} = \{h_1, \cdots, h_n\}$ of $\mathfrak{h}$. This allows us to identify $\mathfrak{h}^*$ with $\mathbb{C}^n$ via $\mu \mapsto (\mu(h_1), \cdots, \mu(h_n))$. A vector $\mu \in \mathfrak{h}^*$ is called $\mathbf{h}$-integral if μ maps into $\mathbb{Z}^n$.

Suppose we have given a Lie algebra $\mathfrak{g}$ with a fixed Abelian subalgebra $\mathfrak{h}$, together with a fixed basis $\mathbf{h} = \{h_1, \cdots, h_n\}$ of $\mathfrak{h}$. Suppose we have given a representation $\mathfrak{g} \to \mathfrak{gl}(V)$, that is $\mathfrak{h}$-admissable and $\mathbf{h}$-integral, in the sense that all weights of V are $\mathbf{h}$-integral. Under these assumptions we can define the formal charachter by

$$char_V(\mathfrak{g}, \mathbf{h}, \mathbf{q}) = tr_V(\mathbf{q}^{\mathbf{h}}) = tr_V(q_1^{h_1} \cdots q_n^{h_n})$$

which lies in $\mathbb{Z}[[\mathbf{q}^{\pm 1}]] = \mathbb{Z}[[q_1^{\pm 1}, \cdots, q_n^{\pm 1}]]$. Clearly the familiar relations of characters for the dual representation

$$char_{V^*}(\mathfrak{g}, \mathbf{h}, \mathbf{q}) = char_V(\mathfrak{g}, \mathbf{h}, \mathbf{q}^{-1}) \ ,$$

and for the tensor product of two representations

$$char_{U \otimes V}(\mathfrak{g}, \mathbf{h}, \mathbf{q}) = char_U(\mathfrak{g}, \mathbf{h}, \mathbf{q}) char_V(\mathfrak{g}, \mathbf{h}, \mathbf{q})$$

do hold, like in the case of finite groups. However the multiplication in $\mathbb{Z}[[\mathbf{q}^{\pm 1}]]$ is not well defined, and one should either restrict to $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ for finite dimensional representations, or possibly work with $\mathbb{Z}[\mathbf{q}^{\pm 1}]]$ and $\mathbb{Z}[[\mathbf{q}^{\pm 1}]$ for Verma representations and their duals.

For example, for $\mathfrak{g} = \mathfrak{sl}_2$ with $\mathfrak{h} = \mathbb{C}h$ we have for $n \in \mathbb{Z}$

$$char_{M(n)}(\mathfrak{sl}_2, h, q^{1/2}) = q^{n/2}/(1 - q^{-1}) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]] \ ,$$

which in turn implies for $n \in \mathbb{N}$

$$char_{L(n)}(\mathfrak{sl}_2, h, q^{1/2}) = char_{M(n)}(\mathfrak{sl}_2, h, q^{1/2}) - char_{M(-n-2)}(\mathfrak{sl}_2, h, q^{1/2}) \ .$$

Therefore we find

$$char_{L(n)}(\mathfrak{sl}_2, h, q^{1/2}) = (q^{n/2} - q^{-(n+2)/2})/(1 - q^{-1}) =$$

$$(q^{(n+1)/2} - q^{-(n+1)/2})/(q^{1/2} - q^{-1/2}) \in \mathbb{Z}[q^{1/2}, q^{-1/2}] \ ,$$

which is called the Weyl character formula for $\mathfrak{sl}_2$. Observe that

$$char_{L(n)}(\mathfrak{sl}_2, h, q^{1/2} = 1) = (n + 1) \ ,$$

which is the dimension of $L(n)$ as should.

For $f(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}^{\pm 1}]$ written out in multiindex notation $f(\mathbf{q}) = \sum_{\mathbf{m}} a_{\mathbf{m}} q^{\mathbf{m}}$ we denote the constant term by $CT_{\mathbf{q}}(f(\mathbf{q})) = a_0$. The ring of Laurent polynomials $\mathbb{Z}[\mathbf{q}^{\pm 1}]$ with integral coëfficients has a natural star structure defined by $f^\star(\mathbf{q}) = f(\mathbf{q}^{-1})$. The finite dimensional irreducible characters of $\mathfrak{sl}_2$ are invariant under this star structure on $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, which is a reflection of the fact that the representations $L(n)$ are selfdual. The bilinear form

$$\langle f, g \rangle = CT_{\mathbf{q}}(f(\mathbf{q}) g^\star(\mathbf{q}))$$

is an inner product on $\mathbb{Z}[\mathbf{q}^{\pm 1}]$. The Schur orthonormality relations for an Abelian Lie algebra $\mathfrak{h}$ with fixed basis $\mathbf{h}$ are really trivial, and take the following form.

Theorem 6.1. *The irreducible* $\mathbf{h}$*-integral characters* $\mathbf{q}^{\mathbf{m}}$ *with* $\mathbf{m} \in \mathbb{Z}^n$ *are an orthonormal basis of* $\mathbb{Z}[\mathbf{q}^{\pm 1}]$*.*

This theorem is nothing but the familiar orthonormality relation from the theory of Fourier series. The Schur orthonormality relations for $\mathfrak{su}_2$ take the following form.

Theorem 6.2. *The characters of the finite dimensional irreducible unitary representations of $\mathfrak{su}_2$ in $\mathbb{Z}[q^{1/2},q^{-1/2}]$ are orthonormal with respect to the inner product*

$$\langle f,g\rangle_{\mathfrak{su}_2}=\langle f(1-q^{-1}),g(1-q^{-1})\rangle/2=CT_q(fg^\star(1-q)(1-q^{-1}))/2$$

on $\mathbb{Z}[q^{1/2},q^{-1/2}]$.

Indeed, this is obvious from the Weyl character formula for $\mathfrak{sl}_2$, since

$$\langle char_{L(m)}(q),char_{L(n)}(q)\rangle_{\mathfrak{su}_2}=$$
$$CT_q((q^{m/2}-q^{-(m+2)/2})(q^{-n/2}-q^{(n+2)/2}))/2$$

is equal to δ_{mn} for all $m,n\in\mathbb{Z}$.

We shall illustrate the power of formal characters in the example of the algebra of invariants $V=S(L(4))^{\mathfrak{sl}_2}$ for degree 4 binary forms of $\mathfrak{sl}_2$. In the basis $\{x^4,x^3y,x^2y^2,xy^3,y^4\}$ of $L(4)$ the operator h of $\mathfrak{sl}_2$ has eigenvalues $4,2,0,-2,-4$. Let d be the identity operator on $L(4)$. The formal charachter $char_{S(L(4))}(p,q)$ of $S(L(4))$ is given by

$$tr_{S(L(4))}(p^{Sd}q^{Sh/2})=1/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2}))\ .$$

We should expand this expression as a power series in p, with coëfficients Laurent polynomials in q. So we can take $0<p<1$ and q a complex number on the unit circle. Using the Schur orthonormality relations for $\mathfrak{su}_2$ the formal character $char_V(p)$ of $V=S(L(4))^{\mathfrak{sl}_2}$ viewed as a representation space for $\mathbb{C}d$ becomes

$$CT_q[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2}))]/2\ ,$$

which is equal to $1/2\pi i$ times

$$\oint[(1-q)(1-q^{-1})dq/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2})q)]/2$$

with $\oint$ the contour integral over the unit circle. Using the residue theorem we can rewrite $char_V(p)$ as

$$\sum res_q[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2})q)]/2$$

with $\sum$ the sum over the singular points q inside the unit circle.

The singular points inside the unit circle are $q = 0, p, \pm p^{1/2}$. The residue at $q = 0$ becomes

$$res_0[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2})q)]/2 =$$

$$res_0[(1-q)(q-1)q/((1-pq^2)(1-pq)(1-p)(q-p)(q^2-p))]/2 = 0 ,$$

while the residue at $q = p$ becomes

$$res_p[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2})q)]/2 =$$

$$res_p[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(q-p)(1-pq^{-2}))]/2 =$$

$$(1-p)(1-p^{-1})/(2(1-p^3)(1-p^2)(1-p)(1-p^{-1})) = 1/(2(1-p^2)(1-p^3)) .$$

Finally we have to compute the residue at $q = \pm p^{1/2}$ of the function

$$[(1-q)(1-q^{-1})/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(1-pq^{-2})q)]/2 =$$

$$[(1-q)(1-q^{-1})q/((1-pq^2)(1-pq)(1-p)(1-pq^{-1})(q-p^{1/2})(q+p^{1/2}))]/2.$$

The residue at $q = p^{1/2}$ becomes

$$(1-p^{1/2})(1-p^{-1/2})p^{1/2}/(2(1-p^2)(1-p^{3/2})(1-p)(1-p^{1/2})2p^{1/2}) =$$

$$(1-p^{-1/2})(1+p^{3/2})/(4(1-p)(1-p^2)(1-p^3)) =$$
$$(1-p-p^{-1/2}+p^{3/2})/(4(1-p)(1-p^2)(1-p^3)) ,$$

and likewise the residue at $q = -p^{1/2}$ becomes

$$(1-p+p^{-1/2}-p^{3/2})/(4(1-p)(1-p^2)(1-p^3)) .$$

Summing up the four residues we find

$$char_V(p) = 1/((1-p^2)(1-p^3)) .$$

This character formula is in accordance with our claim in Example 5.10. Indeed we do get nonzero invariants $f_2 \in S^2(L(4))^{\mathfrak{sl}_2}$ of degree 2 and $f_3 \in S^3(L(4))^{\mathfrak{sl}_2}$ of degree 3.

The three dimensional linear subspace U of $L(4)$ given by

$$U = \{p(x,y) = (a_0x^3 + a_2xy^2 + a_3y^3)y\}$$

is called the standard slice. Let the group $\mathbb{C}^\times$ act on U via $q \cdot (a_0, a_2, a_3) = (qa_0, q^{-1}a_2, q^{-2}a_3)$ for all $q \in \mathbb{C}^\times$ and $(a_0, a_2, a_3) \in U$. This action is nothing but the natural action of the diagonal subgroup (with diagonal entries $q^{1/2}$ and $q^{-1/2}$) of $SL_2(\mathbb{C})$ on $U \subset L(4)$, because the weights of $h/2$ on U are equal to $1, -1, -2$. Almost all orbits of $SL_2(\mathbb{C})$ on $L(4)$ intersect the standard slice U in exactly one orbit of $\mathbb{C}^\times$ on U.

The restriction of the invariants f_2, f_3 to the standard slice U have the simple form

$$f_2(a_0, a_2, a_3) = a_0 a_2 \ , \ f_3(a_0, a_2, a_3) = a_0^2 a_3$$

after a possible rescaling. Hence the invariants f_2, f_3 are algebraically independent. The final conclusion is that $S(L(4))^{\mathfrak{sl}_2} = \mathbb{C}[f_2, f_3]$ as stated in Example 5.10. If the leading coëfficient a_0 of $p(x, y) \in U$ is nonzero, then we can write $p(x, y) = a_0(x - z_1 y)(x - z_2 y)(x - z_3 y)y$ for some $z_1, z_2, z_3 \in \mathbb{C}$ with

$$z_1 + z_2 + z_3 = 0 \ , \ a_0(z_1 z_2 + z_2 z_3 + z_1 z_3) = a_2 \ , \ a_0 z_1 z_2 z_3 = -a_3 \ .$$

The fourth zero of the degree 4 binary form $p(x, y) \in U$ is to be thought of as ∞. In other words the invariants f_2, f_3 become

$$f_2 = a_0^2(z_1 z_2 + z_2 z_3 + z_1 z_3) \ , \ f_3 = -a_0^3 z_1 z_2 z_3$$

and are essentially the elementary symmetric functions of degree 2 and 3 in the three roots $z_1, z_2, z_3 \in \mathbb{C}$.

A particular degree 6 invariant is the discriminant D whose restriction to U is given by by

$$D = -4f_2^3 - 27f_3^2 = a_0^6(z_1 - z_2)^2(z_2 - z_3)^2(z_1 - z_3)^2 \ .$$

The "discriminant variety" $D = 0$ on $L(4)$ consists of those degree 4 binary forms with a multiple root.

We finish this discussion by describing the orbit structure of $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/(\pm e)$ on the projective space $\mathbb{P}(L(4))$ in geometric terms. There is a unique closed orbit C consisting of degree 4 binary forms with a quadruple root. This orbit has dimension one, and is a smooth rational curve of degree 4, in the sense that a generic hyperplane in $\mathbb{P}(L(4))$ intersects C in 4 points.

The tangent bundle TC of C is by definition the union of all tangent lines to C. It has dimension two, and is a surface of degree 6 in the sense that a generic plane in $\mathbb{P}(L(4))$ intersects T in 6 points. The surface $TC = C \sqcup (TC - C)$ is singular along the curve C and smooth at $TC - C$. The smooth part $TC - C$ is an orbit of dimension two consisting of degree 4 binary forms with a triple root. The defining equations for TC are $f_2 = 0, f_3 = 0$.

The chordal variety CC consists of all chords of C, so it is the union of all lines through any two points of C. It has dimension three, and is a threefold of degree 3 in the sense that a generic line in $\mathbb{P}(L(4))$ intersects T in 3 points. The threefold CC is singular along the curve C and smooth at $CC - C$. The subset $CC - TC$ of CC is an orbit of dimension three. The defining equation for CC is $f_3 = 0$, and CC is referred to as the "chordal cubic".

The Hessian variety H of the chordal cubic CC is a threefold of degree 5, and therefore $H = Q \cup CC$ with Q the "invariant quadric", with defining equation $f_2 = 0$. A generic line in $\mathbb{P}(L(4))$ intersects H in 5 points, with 2 of them on the quadric Q and the remaining 3 on the cordal cubic CC. The quadric Q is smooth, and $Q - TC$ consists of one orbit of dimension three. The intersection $Q \cap CC$ equals TC, as mentioned above.

The discriminant variety DV with defining equation $D = 0$ consists of degree 4 binary forms with a multiple root. It has dimension three, and is a threefold of degree 6. The discriminant variety DV is smooth outside the surface $C \sqcup (DD - C) \sqcup (TC - C)$. This surface has two components, one is DD and the other is TC. The smooth part $DV - (DD \cup TC)$ of the discriminant variety consists of degree 4 binary forms with one double root. The singular part $DD \cup TC$ of the discriminant variety consists of degree 4 binary forms, with two double roots $DD - C$, with a triple root $TC - C$, and with a quadruple root C.

It can be shown that all remaining orbits have dimension three, and the stabilizer in $PSL_2(\mathbb{C})$ of any point on such an orbit is the Klein fourgroup V_4. We have discussed above 3 special orbits of dimension three, namely the open orbit $Q - TC$ in the invariant quadric, the open orbit $CC - TC$ in the chordal cubic, and the open orbit $DV - (DD \cup TC)$ in the discriminant variety. One can check that the stabilizer in $PSL_2(\mathbb{C})$ of any point on such an orbit is isomorphic to the alternating group A_4 of order 12, the dihedral group D_4 of order 8, and the cyclic group C_2 of order 2 respectively. This ends our discussion of the invariant theory of degree 4 binary forms.

Exercise 6.1. *Check that the relation*

$$-4(z_1z_2 + z_2z_3 + z_1z_3)^3 - 27(z_1z_2z_3)^2 = (z_1 - z_2)^2(z_2 - z_3)^2(z_1 - z_3)^2$$

holds if $(z_1 + z_2 + z_3) = 0$. *Hint: Check the identity both in case* $z_1 = z_2 = z, z_3 = -2z$ *and in case* $z_1 = 1, z_2 = 0, z_3 = -1$, *and use the symmetry under the symmetric group* S_3.

Exercise 6.2. *Which orbits of $PSL_2(\mathbb{C})$ on $\mathbb{P}(L(4))$ do not intersects the standard slice $\mathbb{P}U$? Answer: The two orbits C and $DD - C$.*

Exercise 6.3. *The subset of nonzero binary forms $p(x,y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ in $L(n)$ with a root of multiplicity $m \geq 1$ has as standard slice*

$$U = \{p(x,y) = (a_mx^{n-m} + a_{m+2}x^{n-m-2}y^2 + \cdots + a_ny^{n-m})y^m; a_m \neq 0\} .$$

Each orbit under $SL_2(\mathbb{C})$ of such a binary form in $L(n)$ intersects U in an orbit of $\mathbb{C}^\times$. Here $\mathbb{C}^\times$ acts on U by

$$q \cdot (a_m, a_{m+2}, \cdots, a_n) = (q^{n/2-m}a_m, q^{n/2-m-2}a_{m+2}, \cdots, q^{-n/2}a_n)$$

and this is just the action of the diagonal subgroup of $SL_2(\mathbb{C})$ on U.

Show that for $p \in U$ we have $\lim_{q\to\infty} q \cdot p = 0 \Leftrightarrow m > n/2$. This enables one to conclude that the orbit under $SL_2(\mathbb{C})$ of a nonzero $p \in L(n)$ contains the origin 0 in its closure if and only if p has a root of multiplicity $m > n/2$. Such binary forms are called unstable. The complement of the unstable locus in $L(n)$ is called the set of semistable binary forms. For semistable binary forms all roots have multiplicity $m \leq n/2$. The stable locus consists of those nonzero binary forms with all roots of multiplicity $m < n/2$. Semistable but not stable is called strictly semistable. So we have

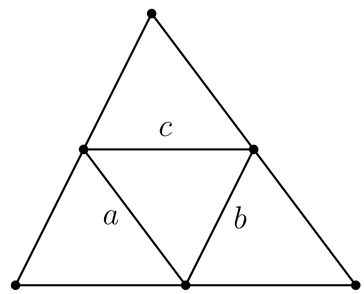
$$\mathbb{P}(L(n)) = \mathbb{P}(L(n))^{us} \sqcup \mathbb{P}(L(n))^{ss} \ , \ \mathbb{P}(L(n))^{ss} = \mathbb{P}(L(n))^{sss} \sqcup \mathbb{P}(L(n))^{s}$$

as partitions of $\mathbb{P}(L(n))$ invariant under $SL_2(\mathbb{C})$. A degree n binary form is called minimal semistable if it is either stable or strictly semistable with just two roots of multiplicity $m = n/2$. Consider stereographic projection of $\mathbb{C} \cup \infty$ on the unit sphere $\mathbb{S} = \{u^2 + v^2 + w^2 = 1\}$. For $p \in U$ project the n roots $z_1, \cdots, z_n$ stereographically on the unit sphere $\mathbb{S}$, to obtain n points $s_1, \cdots, s_n$ on $\mathbb{S}$ among which the north pole occurs with multiplicity m.

Show that for $p \in U$ the center of gravity $(s_1 + \cdots + s_n)/n$ lies on the axis from north to south pole, and p is minimal semistable if and only upto action of $\mathbb{R}^\times$ on U the center of gravity $(s_1 + \cdots + s_n)/n$ can be taken in the origin 0. The conclusion is that the minimal semistable locus in $\mathbb{P}(L(n))$ is represented by those configurations of n points on $\mathbb{S}$ with possible collisions for which the center of gravity lies at the origin.

Exercise 6.4. *Let us consider a sharp angled triangle with sides of lengths $2a, 2b, 2c > 0$. Fold this triangle along the three midparallels (of lengths a, b, c)*

upwards to form a tetrahedron T with four congruent faces. Indeed all four faces of T are triangles with lengths of edges equal to a, b, c. Note that each pair of opposite edges of T has equal length. If the original triangle is right angled then the tetrahedron T degenerates into a flat figure.



Show that the line trough the midpoints of opposite edges of T is perpendicular to both these edges. Conclude that the rotation over π with axis the line through the midpoints of opposite edges of T lies in the rotation symmetry group $G(T)$ of T. Conclude that $G(T) \cong V_4$ in general, except for the cases $a = \sqrt{2}b = \sqrt{2}c$ and $G(T) \cong D_4$, or $a = b = c$ and $G(T) \cong A_4$. Conclude that the center of gravity of the four vertices of T is equal to the center of the circumscribed sphere of T.

7 The bound states for the Kepler problem

In this section we will discuss the Kepler problem from the viewpoint of both classical mechanics (planetary motion aroun the sun) and quantum mechanics (the hydrogen atom). Our discussion of the classical mechanics of the Kepler problem is taken from an article "Teaching the Kepler laws for freshmen" by Maris van Haandel and myself from 2008.

We shall use inner products $\mathbf{u} \cdot \mathbf{v}$ and outer products $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the compatibility conditions

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

and the Leibniz product rules

$$(\mathbf{u} \cdot \mathbf{v})^{\cdot} = \dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}}$$

$$(\mathbf{u} \times \mathbf{v})^{\cdot} = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}}$$

without further explanation.

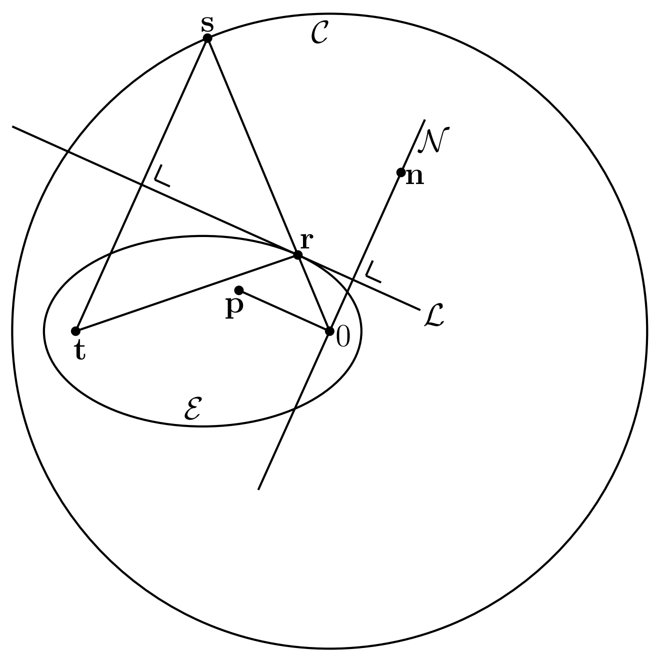
For a central force field $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}/r$ the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved by Newton's law of motion $\mathbf{F} = \dot{\mathbf{p}}$, thereby leading to Kepler's second law. For a spherically symmetric central force field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}/r$ the energy

$$H = p^2/2m + V(r)\ ,\ V(r) = -\int f(r)\,dr$$

is conserved as well. These are the general initial remarks.

From now on consider the Kepler problem $f(r) = -k/r^2$ en $V(r) = -k/r$ with $k > 0$ a coupling constant. More precisely, we consider the reduced Kepler problem for the sun S and a planet P with $\mathbf{r} = \mathbf{r}_P - \mathbf{r}_S$ the relative position of the planet. Let m_S and m_P be the masses of the sun S and the planet P respectively. According to Newton the reduced mass becomes $m = m_S m_P/(m_S + m_P)$ while the coupling constant is given by $k = G m_S m_P$ with G the universal gravitational constant. By conservation of energy the motion for fixed energy $H < 0$ is bounded inside a sphere with center 0 and radius $-k/H$.

Consider the following picture of the plane perpendicular to $\mathbf{L}$. The circle $\mathcal{C}$ with center 0 and radius $-k/H$ is the boundary of a disc where motion with energy $H < 0$ takes place. Let $\mathbf{s} = -k\mathbf{r}/rH$ be the projection of $\mathbf{r}$ from the center 0 on this circle $\mathcal{C}$. The line $\mathcal{L}$ through $\mathbf{r}$ with direction vector $\mathbf{p}$ is the tangent line of the orbit $\mathcal{E}$ at position $\mathbf{r}$ with velocity $\mathbf{v}$. Let $\mathbf{t}$ be the orthogonal reflection of the point $\mathbf{s}$ in the line $\mathcal{L}$. As time varies, the position vector $\mathbf{r}$ moves along the orbit $\mathcal{E}$, and likewise $\mathbf{s}$ moves along the circle $\mathcal{C}$. It is a good question to investigate how the point $\mathbf{t}$ moves.



Theorem 7.1. *The point* $\mathbf{t}$ *equals* $\mathbf{K}/mH$ *and therefore is conserved.*

Proof. The line $\mathcal{N}$ spanned by $\mathbf{n} = \mathbf{p} \times \mathbf{L}$ is perpendicular to $\mathcal{L}$. The point $\mathbf{t}$ is obtained from $\mathbf{s}$ by subtracting twice the orthogonal projection of $\mathbf{s} - \mathbf{r}$ on the line $\mathcal{N}$, and therefore

$$\mathbf{t} = \mathbf{s} - 2((\mathbf{s} - \mathbf{r}) \cdot \mathbf{n})\mathbf{n}/n^2.$$

Now

$$\mathbf{s} = -k\mathbf{r}/rH$$

$$(\mathbf{s} - \mathbf{r}) \cdot \mathbf{n} = -(H + k/r)\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L})/H = -(H + k/r)L^2/H$$

$$n^2 = p^2L^2 = 2m(H + k/r)L^2$$

and therefore

$$\mathbf{t} = -k\mathbf{r}/rH + \mathbf{n}/mH = \mathbf{K}/mH$$

with $\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$ the Lenz vector. The final step $\dot{\mathbf{K}} = 0$ is derived by a straightforward computation using the compatibility relations and the Leibniz product rules for inner and outer products of vectors in $\mathbb{R}^3$. $\square$

Corollary 7.2. *The orbit $\mathcal{E}$ is an ellipse with foci* 0 *and* $\mathbf{t}$*, and major axis equal to* $2a = -k/H$*.*

Proof. Indeed we have

$$|\mathbf{t} - \mathbf{r}| + |\mathbf{r} - 0| = |\mathbf{s} - \mathbf{r}| + |\mathbf{r} - 0| = |\mathbf{s} - 0| = -k/H.$$

Hence $\mathcal{E}$ is an ellipse with foci 0 and $\mathbf{t}$, and major axis $2a = -k/H$. $\square$

The conserved vector $\mathbf{t} = \mathbf{K}/mH$ is a priori well motivated in Euclidean geometric terms. In most text books on classical mechanics (e.g. H. Goldstein, Classical Mechanics) the Lenz vector K is written down, and the motivation comes only a posteriori from $\dot{\mathbf{K}} = \mathbf{0}$ as a vector in the direction of the long axis of the elliptical orbit. The Lenz vector goes already back to Lagrange in his article "Théorie des variations séculaires des élements des planètes" from 1781.

It is easy to check the relations $\mathbf{K} \cdot \mathbf{L} = 0$ and $K^2 = 2mHL^2 + k^2m^2$. Hence besides the familiar conserved quantities angular momentum $\mathbf{L}$ and energy H only the direction of the Lenz vector $\mathbf{K}$ is a new independent conserved quantity. Altogether there are $3 + 1 + 1 = 5$ independent conserved quantities, whose level curves in the phase space $\mathbb{R}^6$ are the Kepler ellipses, at least for $H < 0$ corresponding to bounded motion.

We now turn to the quantum mechanics of the Kepler problem. The first discussion of this question was given by Pauli in an article "On the hydrogen spectrum from the standpoint of the new quantum mechanics" from January 1926. Pauli's solution is a beautiful piece of Lie algebra theory, which made it difficult to digest for the average physicist of that time, who happily adopted the shortly after found solution by Schrödinger. The latter method rewrites the Schrödinger eigenvalue equation in spherical coördinates, and subsequently consults a book on Special Functions (for Laguerre and Legendre polynomials). This method of Schrödinger is just a lot of calculations. In the proof by Pauli one has to do also a fair amount of calculations. But the proof by Pauli is by far the better, because it explains the natural degeneration of the hydrogen spectrum, while in the proof by Schrödinger this spectrum has still accidental degeneration. The hidden conserved Lenz vector is what is missing in the treatment by Schrödinger.

We consider Pauli's proof below and the proof above of Kepler's law of ellipses as proofs from "The Book". This is an expression of the remarkable mathematician Paul Erdös, who supposed that God has a book in which he keeps only the most beautiful proofs of mathematical theorems. One of the joyful parts of mathematics is to strive for proofs from "The Book".

We shall only outline the method of Pauli, and leave the proofs of the formulas to the interested reader. In quantum mechanics the three components of the position $\mathbf{r}$ and the momentum $\mathbf{p}$ are selfadjoint operators on a Hilbert space $\mathcal{H}$, satisfying the Heisenberg commutation relations

$$[r_i, r_j] = [p_i, p_j] = 0 \ , \ [p_i, r_j] = -i\hbar\delta_{ij} \ .$$

We wish to quantize the energy H, the angular momentum $\mathbf{L}$ and the Lenz vector $\mathbf{K}$. For H and $\mathbf{L}$ we can just take the classical formulas

$$H = p^2/2m - k/r \ , \ \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

but for the Lenz vector $\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{r}/r$ there is an ambiguity with the definition of $\mathbf{p}\times\mathbf{L}$. Indeed, for the first component should we take $p_2L_3 - p_3L_2$ or $L_3p_2 - L_2p_3$? For the solution of this ambiguity Pauli chose the average

$$\mathbf{K} = (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p})/2 - km\mathbf{r}/r \ ,$$

which is easily seen to be selfadjoint. The following formulas

$$[L_i, r_j] = i\hbar\epsilon_{ijk}r_k \ , \ [L_i, p_j] = i\hbar\epsilon_{ijk}p_k$$

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \ , \ [L_i, K_j] = i\hbar\epsilon_{ijk}K_k$$

mean that $\mathbf{r}, \mathbf{p}, \mathbf{L}$ and $\mathbf{K}$ are so called vector operators. Here ϵ_{ijk} is the totally antisymmetic ϵ-tensor. The conservation of $\mathbf{L}$ and $\mathbf{K}$ amounts to the relations

$$[H, \mathbf{L}] = [H, \mathbf{K}] = 0 \ .$$

The first relation is immediate because H has spherical symmetry, but the second relation requires some calculation. The following formulas are in perfect analogy with classical mechanics

$$\mathbf{L} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{L} = 0 \ ,$$

but the formula

$$K^2 = 2mH(L^2 + \hbar^2) + k^2m^2$$

has a "quantum correction" compared with the classical formula, vanishing in the limit for $\hbar \to 0$. Our last formula

$$[K_i, K_j] = i\hbar\epsilon_{ijk}(-2mH)L_k$$

is obtained after a careful calculation.

The sometimes cumbersome calculations behind the above formulas shall now be rewarded by an elegant and conceptually clear method for finding the hydrogen spectrum. We wish to compute the dimension of the eigenspace

$$\{\psi \in \mathcal{H}; H\psi = E\psi\}$$

with a fixed eigenvalue $E < 0$. If we denote

$$\mathbf{I} = (\mathbf{L} + (-2mE)^{-1/2}\mathbf{K})/2\ ,\ \mathbf{J} = (\mathbf{L} - (-2mE)^{-1/2}\mathbf{K})/2$$

then it is easy to see that $\mathbf{I}$ and $\mathbf{J}$ satisfy the commutation relations

$$[I_i, I_j] = i\hbar\epsilon_{ijk}I_k\ ,\ [J_i, J_j] = i\hbar\epsilon_{ijk}J_k\ ,\ [I_i, J_j] = 0\ .$$

In other words the six dimensional vector space spanned by the components of $\mathbf{L}$ and $\mathbf{K}$ is a Lie algebra isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ commuting with H. The first $\mathfrak{sl}_2$ has basis I_i and the second $\mathfrak{sl}_2$ has basis J_j. From $\mathbf{L}^\star = \mathbf{L}, \mathbf{K}^\star = \mathbf{K}$ and $E < 0$ we deduce that $\mathbf{I}^\star = \mathbf{I}, \mathbf{J}^\star = \mathbf{J}$. In other words the real six dimensional vector space spanned by the components of $\mathbf{L}$ and $\mathbf{K}$ (or equivalently spanned by the components of $\mathbf{I}$ and $\mathbf{J}$) becomes identified with $i\mathfrak{su}_2 \oplus i\mathfrak{su}_2$.

Assuming that the spectrum has natural degeneration with respect to angular momentum and Lenz vector we can conclude

$$\{\psi \in \mathcal{H}; H\psi = E\psi\} = L(m) \otimes L(n)$$

for some $m, n \in \mathbb{N}$. Here I_i and J_j work in the first and the second factor of $L(m) \otimes L(n)$ respectively. Because

$$2(I^2 - J^2) = (\mathbf{I} + \mathbf{J}) \cdot (\mathbf{I} - \mathbf{J}) + (\mathbf{I} - \mathbf{J}) \cdot (\mathbf{I} + \mathbf{J}) =$$

$$(-2mE)^{-1/2}(\mathbf{L} \cdot \mathbf{K} + \mathbf{K} \cdot \mathbf{L}) = 0\ ,$$

and by Exercise 5.5 and Exercise 7.2 below

$$I^2|_{L(m)\otimes L(n)} = m(m+2)\hbar^2/4 \ , \ J^2|_{L(m)\otimes L(n)} = n(n+2)\hbar^2/4$$

we conclude that $m = n$ and

$$\{\psi \in \mathcal{H}; H\psi = E\psi\} = L(n) \otimes L(n)$$

for some $n \in \mathbb{N}$. Finally we shall derive a formula for E_n as function of n. Rewriting the above formula $K^2 = 2mH(L^2 + \hbar^2) + k^2m^2$ in the form

$$L^2 + (-2mE)^{-1}K^2 + \hbar^2 = -k^2m/2E$$

and because on the space $L(n) \otimes L(n)$

$$L^2 + (-2mE)^{-1}K^2 + \hbar^2 = 2(I^2 + J^2) + \hbar^2 = n(n+2)\hbar^2 + \hbar^2 = (n+1)^2\hbar^2$$

we arrive at

$$E = E_n = -k^2m/2(n+1)^2\hbar^2$$

with n running over the set $\mathbb{N}$. The energy level E_n has multiplicity $(n+1)^2$. As a representation for $\mathbf{I}$ and $\mathbf{J}$ it is irreducible of the form $L(n) \otimes L(n)$, but restricting to the diagonal subalgebra $\mathbf{L}$ we have the Clebsch-Gordan rule

$$L(n) \otimes L(n) = L(0) \oplus L(2) \oplus \cdots \oplus L(2n) \ .$$

If we forget the symmetry of the Lenz vector $\mathbf{K}$ and take into account only the spherical symmetry of the angular momentum vector $\mathbf{L} = \mathbf{I} + \mathbf{J}$ the n^{th} energy level has accidental degeneration of multiplicity $(n+1)$.

Exercise 7.1. *Show that for A, B selfadjoint operators on a Hilbert space $\mathcal{H}$ the operator $(AB+BA)/2$ is again selfadjoint.*

Exercise 7.2. *The Pauli spin matrices are defined by*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

so $h = \sigma_3, e = (\sigma_1 + i\sigma_2)/2, f = (\sigma_1 - i\sigma_2)/2$ gives the relation between the Chevalley basis and the Pauli matrices. In a unitary representation of $\mathfrak{su}_2$ the Pauli matrices act as selfadjoint operators. Check the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

and show that the Casimir operator $C = h^2 + 2ef + 2fe$ is equal to $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$. The conclusion is that $L^2 = \hbar^2C/4$, so apart from a factor $\hbar^2/4$ the Casimir operator C is just the square length of the the angular momentum vector.

Exercise 7.3. *Show that the Cartesian space* $\mathbb{R}^3$ *with Lie bracket* $[\cdot,\cdot]$ *equal to the vector product* $\times$ *is a Lie algebra. Show that this Lie algebra* $\mathfrak{g}_0$ *is isomorphic to* $\mathfrak{so}_3$*. Show that* $\mathfrak{g}_0$ *has a basis* e_i *with* $[e_i, e_j] = \epsilon_{ijk}e_k$*.*

Exercise 7.4. *Suppose we are in the regime where* $H = E > 0$*. Show that* $\mathbf{L}^\star = \mathbf{L}, \mathbf{K}^\star = \mathbf{K}$ *implies that* $\mathbf{I}^\star = \mathbf{J}, \mathbf{J}^\star = \mathbf{I}$*. Hence the Casimir operator* $I^2 + J^2$ *acts on the eigenspace*

$$\{\psi \in \mathcal{H}; H\psi = E\psi\}$$

as a selfadjoint operator. If we assume this eigenspace to be an irreducible representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ *then the operator* $I^2 + J^2$ *acts by multiplication with a real number* $s(s+2)\hbar^2/2$ *by Schur's Lemma with* $s \in -1 + i\mathbb{R}$ *or* $s \in \mathbb{R}$*. Because* $E = -k^2m/2(s+1)^2\hbar^2 > 0$ *we have* $s \in -1 + i\mathbb{R}$*. The parametrization of the nonpositive real numbers by* $(s+1)^2\hbar^2/2$ *with* $s \in -1 + i\mathbb{R}$ *will become clear in a later section on representations of the Lorentz algebra* $\mathfrak{so}_{3,1}(\mathbb{R})$*.*

Exercise 7.5. *Suppose* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are two Lie algebras. Suppose* V_1 *and* V_2 *are representations of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *respectively. We define the outer tensor product representation* $V_1 \boxtimes V_2$ *of the direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *as follows: As a vector space* $V_1 \boxtimes V_2 = V_1 \otimes V_2$ *and the representation is defined by*

$$(x_1, x_2)(v_1 \otimes v_2) = (x_1v_1) \otimes v_2 + v_1 \otimes (x_2v_2)$$

for all $x_1 \in \mathfrak{g}_1$*,* $x_2 \in \mathfrak{g}_2$*,* $v_1 \in V_1$ *and* $v_2 \in V_2$*. Show that the outer tensor product is indeed a representation of the direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$*. Show that the outer tensor product of two irreducible representations is again irreducible, and each irredicible representation of the direct sum* $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ *arises this way.*

8 Spinning elementary particles

One can imagine an elementary particle as a small spherically symmetric ball, so with internal symmetry Lie algebra $\mathfrak{so}_3 \cong \mathfrak{sl}_2$. This internal symmetry can be quantized by the unitary irreducible representation $L(2s)$ of dimension $(2s+1)$ and highest weight $2s \in \mathbb{N}$. The parameter $s \in \mathbb{N}/2$ is called the spin J of the irreducible representation $L(2s)$ of $\mathfrak{so}_3 \cong \mathfrak{sl}_2$. The faster the elementary particle spins the larger the spin quantum number $s \in \mathbb{N}/2$. Hence an elementary particle with spin $J = s \in \mathbb{N}/2$ can occupy $(2s+1)$ states indexed by a quantum number J_3 taken from the set $\{s, s-1, \cdots, -s\}$, which is just the set of eigenvalues of the Lie algebra element $h/2$ from $\mathfrak{sl}_2$. So the spin J is an index for the irreducible representation of $\mathfrak{sl}_2$, while the spin around the (third) axis J_3 is an index for a basis in this representation space. These notations are fairly standard.

Examples are the Higgs particle with spin 0, the electron with spin 1/2, the photon with spin 1 or the graviton with spin 2. For example, the electron with spin 1/2 has two quantum numbers, taken from the set $\{1/2, -1/2\}$. Equivalently an electron has spin up $(u \leftrightarrow 1/2)$ or spin down $(d \leftrightarrow -1/2)$. The notion of electron spin was proposed by Goudsmit and Uhlenbeck in 1926, at a time they were still graduate students under Paul Ehrenfest in Leiden. There are no elementary particles known to exist with spin greater than 2. Apparently if an elementary particle spins too fast it becomes unstable, preventing its existence.

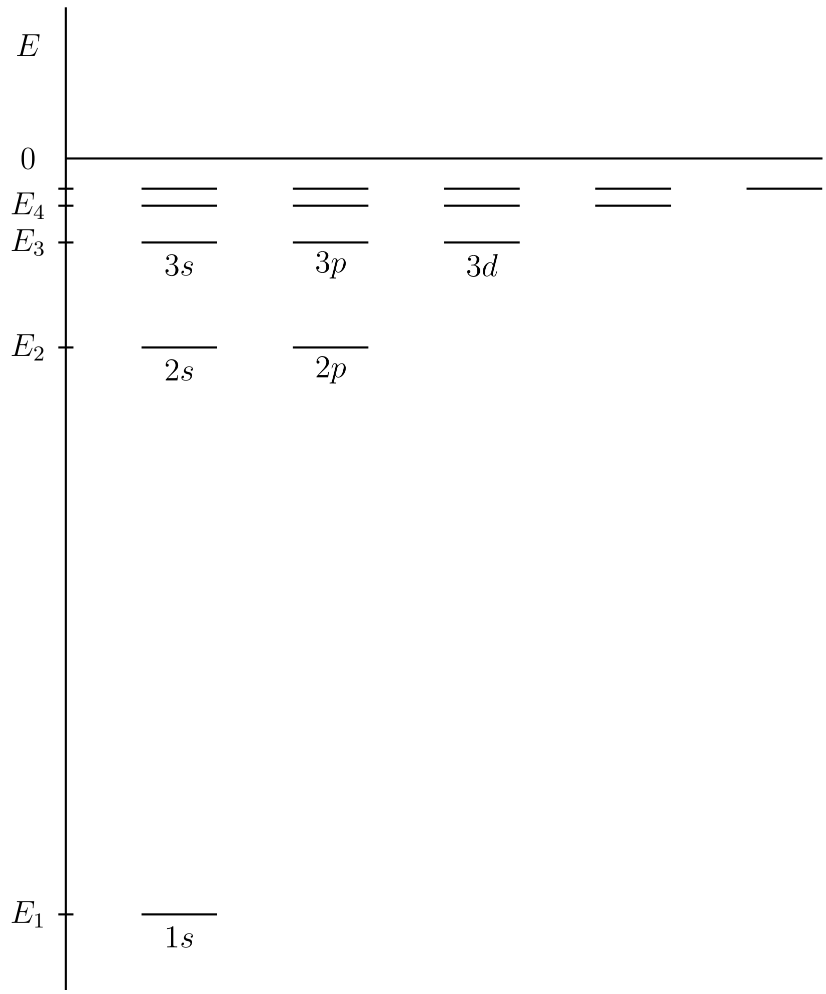
Particles with integral spin are called bosons, and particles with half integral spin are called fermions. Suppose an elementary particle with spin s has as state space a Hilbert space $\mathcal{H}$. The "spin statistics theorem" states that a system of n such identical particles has as state space the n^{th} symmetric power $S^n(\mathcal{H})$ or the n^{th} antisymmetric power $A^n(\mathcal{H})$, depending on whether the particle has integral spin or half integral spin respectively. An identical system of n bosons can live happily together in a same state $\psi^n \in S^n(\mathcal{H})$, but for an identical system of n fermions no two particles can live together in the same state, because $\psi \wedge \psi \wedge \psi_3 \wedge \cdots \wedge \psi_n$ vanishes in $A^n(\mathcal{H})$. This is the famous Pauli exclusion principle: two electrons can never occupy the same state.

The spectrum of the Kepler problem has been computed before using the angular momentum vector $\mathbf{L}$ and the Lenz vector $\mathbf{K}$, which together generate for negative energy $E < 0$ a Lie algebra $\mathfrak{so}_4$. The discrete energy spectrum is located at $E_{n+1} = -1/(n+1)^2$ for $n \in \mathbb{N}$ in suitable units. The $(n+1)^{st}$

energy level is degenerated with multiplicity $(n+1)^2$, and transforms under $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as the irreducible representation $L(n) \otimes L(n)$. The $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ corresponding to the angular momentum vector $\mathbf{L}$ lies inside $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as the diagonal, and the restriction of $L(n) \otimes L(n)$ to this diagonal is given by the Clebsch-Gordan rule

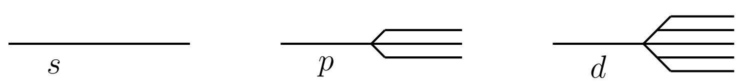
$$L(n) \otimes L(n) = L(0) \oplus L(2) \oplus \cdots \oplus L(2n) \ .$$

Therefore the spectrum of the Kepler problem can be pictorially described by the following figure.



The shells indicated with an s correspond to the one dimensional irreducible representation $L(0)$. The shells indicated with a p correspond to the three dimensional irreducible representation $L(2)$. The shells indicated with a d correspond to the five dimensional irreducible representation $L(4)$. The shells indicated with an f correspond to the seven dimensional irreducible representation $L(6)$, and so on. The elements of the periodic system can be understood to consist of a nucleus (consisting of protons and neutrons) surrounded by a cloud of electrons. Let us assume for simplicity that the electrons only interact with the nucleus, so the mutual interaction among the electrons is neglected. Then the energy levels of our Kepler problem are filled with electrons according to Pauli's exclusion principle from the bottom up, and for fixed energy first the shell s, subsequently the shell p, then the shell d, and so on. For example, the hydrogen atom H has one electron in the shell $1s$, leaving one open place in shell $1s$. The carbon atom C occurs in the periodic system on place number 6. The shells $1s, 2s$ are completely filled with $2 + 2 = 4$ electrons, and the shell $2p$ is occupied with 2 electrons, leaving four open places in shell $2p$. This is the reason for the existence of the chemical binding CH_4.

In the situation of a constant electric or magnetic field along some axis the energy levels will split up according to their degeneration, so



This splitting of spectral lines is called the Stark-Zeeman effect. It is a consequence of symmetry breaking from $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ to $\mathfrak{gl}_1$.

There is also the concept of isospin, that was introduced by Heisenberg in the thirties. It is usually denoted by I (for the irreducible representation of $\mathfrak{sl}_2$) and I_3 for an element of the basis of weight vectors in this representation space. The atomic nucleus consists of protons and neutrons. Heisenberg proposed to view the proton p and the neutron n as the two states of the spin 1/2 irreducible representation. A nucleus with k protons and l neutrons is then viewed as $p^k n^l$, and therefore the nucleus is considered as a particular state of one particle with isospin $m = (k + l)/2$. Given $m > 0$ there are $(m + 1)$ possible nuclei with isopspin $m \in \mathbb{N}/2$.

9 Reductive Lie algebras

Suppose we have given a complex Lie algebra $\mathfrak{g}$ together with a real form $\mathfrak{g}_0$. A representation $\mathfrak{g} \to End(V)$ is called faithful if the only $x \in \mathfrak{g}$ that maps to 0 in $End(V)$ is equal to $0 \in \mathfrak{g}$.

Definition 9.1. *The Lie algebra $\mathfrak{g}$ is called reductive with compact real form $\mathfrak{g}_0$ if there exists a faithful representation $\mathfrak{g} \to End(\mathcal{H})$ on a finite dimensional Hilbert space $\mathcal{H}$ that is unitary for $\mathfrak{g}_0$.*

Throughout this section $\mathfrak{g}$ will be a reductive Lie algebra with compact real form $\mathfrak{g}_0$. As before let $x \mapsto x^\star$ be the antilinear antiinvolution corresponding to $\mathfrak{g}_0$. Examples of reductive Lie algebras are $\mathfrak{gl}_n(\mathbb{C})$ with compact real form $\mathfrak{u}_n(\mathbb{C})$, or $\mathfrak{sl}_n(\mathbb{C})$ with compact real form $\mathfrak{su}_n(\mathbb{C})$, or $\mathfrak{so}_n(\mathbb{C})$ with compact real form $\mathfrak{so}_n(\mathbb{R})$. All three examples have a faithful unitary representation on $\mathbb{C}^n$ with Hermitian form $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ for z, w in $\mathbb{C}^n$.

It is our goal to classify all finite dimensional irreducible representations of $\mathfrak{g}$, that are unitary with respect to the given compact real form $\mathfrak{g}_0$.

Definition 9.2. *A Cartan subalgebra of a reductive Lie algebra $\mathfrak{g}$ with compact form $\mathfrak{g}_0$ is a maximal Abelian subalgebra $\mathfrak{h}$ that is invariant under the antilinear antiinvolution.*

Fix once and for all a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ invariant under the star structure. The dimension of $\mathfrak{h}$ is called the rank. The standard choice for a Cartan subalgebra in $\mathfrak{gl}_n$ or $\mathfrak{sl}_n$ is the linear space $\mathfrak{h}$ of all diagonal matrices $\mathbf{d}$ in $\mathfrak{gl}_n$ or the traceless diagonal matrices $\mathbf{d}$ in $\mathfrak{sl}_n$. The standard choice for a Cartan subalgebra in $\mathfrak{so}_{2n}$ is given by

$$\mathfrak{h} = \left\{ h = \begin{pmatrix} 0 & -i\mathbf{d} \\ i\mathbf{d} & 0 \end{pmatrix} \right\}$$

with $\mathbf{d} \in \mathfrak{gl}_n$ a diagonal matrix with diagonal entries $d_1, \cdots, d_n$. For these choices of Cartan subalgebras the ranks of $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_{2n}$ are $n, n-1, n$ respectively. Note that in all three cases $\mathbf{d}^\star$ is just complex conjugation on the components $d_1, \cdots, d_n$ of $\mathbf{d}$.

Suppose we have another finite dimensional representation of $\mathfrak{g}$ on a Hibert space V, which is unitary with respect to a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V. For $\mu \in \mathfrak{h}^*$ we have called the linear subspace

$$V_\mu = \{v \in V; hv = \mu(h)v \ \forall h \in \mathfrak{h}\}$$

the weight space in V of weight μ relative to $\mathfrak{h}$. A standard result in linear algebra says that eigenvalues of Hermitian operators are real numbers. In turn this implies that a weight μ of V (with $V_\mu \neq 0$) is real in the sense that $\mu^\star = \mu$. Moreover $V = \oplus V_\mu$ is a direct sum of weight spaces, and weight spaces for different weights are orthogonal. Weights are usually written as $\lambda, \mu, \nu, \dots \in \mathfrak{h}^*$.

Any Lie algebra $\mathfrak{g}$ has a natural representation on the vector space $\mathfrak{g}$. It is called the adjoint representation, and is denoted $ad : \mathfrak{g} \to End(\mathfrak{g})$, and is defined by $ad(x)y = [x, y]$ for all $x, y \in \mathfrak{g}$. This is an easy verification using the Jacobi identity.

If $\mathfrak{g}$ is reductive with compact real form $\mathfrak{g}_0$ and faithful unitary representation $\mathfrak{g} \to End(\mathcal{H})$ then the adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ becomes unitary with respect to the Hermitian trace form

$$\langle x, y\rangle = tr_{\mathcal{H}}(xy^\star)$$

on $\mathfrak{g}$. Indeed $\langle [z, x], y\rangle = \langle x, [z^\star, y]\rangle$ for all $x, y, z \in \mathfrak{g}$ by direct computation. The symmetric trace form on $\mathfrak{g}$ is defined by

$$(x, y) = tr_{\mathcal{H}}(xy) \ ,$$

and the relation between the symmetric and the Hermitian trace form becomes

$$\langle x, y\rangle = (x, y^\star) \ \forall x, y \in \mathfrak{g} \ .$$

Both trace forms are nondegenerate on $\mathfrak{g}$. The restriction of the Hermitian trace form $\langle \cdot, \cdot\rangle$ to any linear subspace of $\mathfrak{g}$ is nondegenerate, which in turn implies that the restriction of the symmetric trace form $(\cdot, \cdot)$ to any star invariant linear subspace of $\mathfrak{g}$ is also nondegenerate. For example the restriction of the symmetric form $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate.

Corollary 9.3. *The restriction of the symmetric form $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate. This gives rise to a linear isomorphism $\mathfrak{h}^* \to \mathfrak{h}$, denoted $\lambda \mapsto h_\lambda$, via*

$$\lambda(h) = (h_\lambda, h)$$

for all $h \in \mathfrak{h}$. We transport the symmetric form $(\cdot, \cdot)$ from $\mathfrak{h}$ to the dual space $\mathfrak{h}^*$ via this linear isomorphism, so $(\lambda, \mu) = (h_\lambda, h_\mu)$ for all $\lambda, \mu \in \mathfrak{h}^*$. We also transport the star structure from $\mathfrak{h}$ to the dual space $\mathfrak{h}^*$ via $\lambda^\star(h) = \overline{\lambda(h^\star)}$. It is easy to check that $h_{\lambda^\star} = (h_\lambda)^\star$. We write $\mathfrak{a} = \{h \in \mathfrak{h}; h^\star = h\}$ and

$\mathfrak{a}^* = \{\lambda \in \mathfrak{h}^*; \lambda^\star = \lambda\}$. *Both* $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *are Euclidean vector spaces with respect to* $(\cdot,\cdot)$. *The weights of any unitary representation are real, so they are contained in* $\mathfrak{a}^*$.

Suppose V is a finite dimensional unitary representation of $\mathfrak{g}$. The set of weights of V, viewed as a subset of $\mathfrak{a}^*$ with multiplicities (i.e. $m_\mu = \dim V_\mu$ is called the multiplicity of the weight μ of V), is called the weight diagram of V with respect to $\mathfrak{h}$. Clearly the direct sum $V_1 \oplus V_2$ has for weight diagram the union, while the tensor product $V_1 \otimes V_2$ has for weight diagram the Minkowski sum of the two weight diagrams.

Lemma 9.4. *The symmetric trace form on* $\mathfrak{g}$ *satisfies* $([x,y],z) = (x,[y,z])$ *for all* $x,y,z \in \mathfrak{g}$.

Proof. The Hermitian trace form on $\mathfrak{g}$ satisfies $\langle [z,x],y\rangle = \langle x,[z^\star,y]\rangle$, and is related to the symmetric trace form by $(x,y) = \langle x,y^\star\rangle$. Hence

$$([x,y],z) = \langle -[y,x],z^\star\rangle = \langle x,-[y^\star,z^\star]\rangle = (x,-[y^\star,z^\star]^\star) = (x,[y,z])$$

for all $x,y,z \in \mathfrak{g}$. □

An inner product on $\mathfrak{g}$ with this property is called associative or invariant. The zero weight space of the adjoint representation is equal to the centralizer

$$\mathfrak{z} = \{x \in \mathfrak{g}; [h,x] = 0 \ \forall h \in \mathfrak{h}\}$$

of $\mathfrak{h}$ in $\mathfrak{g}$. It is clear that $\mathfrak{z} \supset \mathfrak{h}$ is again invariant under the star structure.

Lemma 9.5. *We have* $\mathfrak{z} = \mathfrak{h}$.

Proof. Suppose on the contrary that the inclusion $\mathfrak{h} \subset \mathfrak{z}$ is proper. Because $\mathfrak{z}^\star = \mathfrak{z}$ we can pick $z \in \mathfrak{z} - \mathfrak{h}$ with $z^\star = z$. Clearly $\mathfrak{h} + \mathbb{C}z$ is an Abelian subalgebra of $\mathfrak{g}$ invariant under the star structure, contradicting the maximality of the Cartan subalgebra $\mathfrak{h}$. Hence $\mathfrak{z} = \mathfrak{h}$. □

Definition 9.6. *The set of nonzero weights of the adjoint representation representation is called the root system of the pair* $(\mathfrak{g},\mathfrak{h})$, *and is denoted* $R(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{a}^*$. *Elements of* $R = R(\mathfrak{g},\mathfrak{h})$ *are called roots, and usually written as* $\alpha,\beta,\gamma,\dots \in R \subset \mathfrak{a}^*$.

The weight space decomposition of the adjoint representation of $\mathfrak{g}$ takes the form

$$\mathfrak{g} \ominus \mathfrak{h} = \oplus \mathfrak{g}_\alpha$$

with the direct sum over all roots in R. This decomposition is orthogonal with respect to the Hermitian trace form on $\mathfrak{g}$. The nonzero spaces $\mathfrak{g}_\alpha$ are called the root spaces.

In the example of $\mathfrak{gl}_n$ the Cartan subalgebra $\mathfrak{h}$ was taken to be

$$\{\mathbf{d} = diag(d_1, \cdots, d_n)); (d_1, \cdots, d_n) \in \mathbb{C}^n)\} \ .$$

The symmetric trace form on $\mathfrak{h}$ is just the standard inner product $(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_n y_n$. Let us write e_{ij} for the matrix with 1 on the place (i,j) and 0 elsewhere. Clearly $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$ and since $\mathbf{d} = \sum d_i e_{ii}$ we find

$$[\mathbf{d}, e_{ij}] = (d_i - d_j) e_{ij} \ .$$

The conclusion is that under the identification of $\mathfrak{a}^*$ with $\mathbb{R}^n$ with canonical basis $\{e_1, \cdots, e_n\}$ the root sytem R becomes

$$R = \{e_i - e_j ; 1 \leq i \neq j \leq n\} \ .$$

For $\mathfrak{sl}_n$ the discussion is essentially the same except that $\mathfrak{a}^*$ becomes identified with $\{\mathbf{x} \in \mathbb{R}^n ; \sum x_i = 0\}$. The root system R of $\mathfrak{gl}_n$ was already contained in this subspace, and becomes also the root system for $\mathfrak{sl}_n$.

An element of the standard Cartan subalgebra in $\mathfrak{so}_{2n}$ was taken of the form

$$h = \begin{pmatrix} 0 & -i\mathbf{d} \\ i\mathbf{d} & 0 \end{pmatrix}$$

with $\mathbf{d} \in \mathfrak{gl}_n$ a diagonal matrix with diagonal entries $d_1, \cdots, d_n$. In particular $\mathbf{d}^t = \mathbf{d}$. An element x of $\mathfrak{so}_{2n}$ is an antisymmetric matrix, and hence of the form

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in \mathfrak{gl}_n$ and $a^t = -a, b^t = -c, c^t = -b, d^t = -d$. The commutator $[h, x]$ takes the form

$$[h, x] = i \begin{pmatrix} -\mathbf{d}c - b\mathbf{d} & -\mathbf{d}d + a\mathbf{d} \\ +\mathbf{d}a - d\mathbf{d} & +\mathbf{d}b + c\mathbf{d} \end{pmatrix} \ ,$$

and the equation $[h, x] = \alpha(\mathbf{d})x$ becomes

$$i \begin{pmatrix} -\mathbf{d}c - b\mathbf{d} & -\mathbf{d}d + a\mathbf{d} \\ +\mathbf{d}a - d\mathbf{d} & +\mathbf{d}b + c\mathbf{d} \end{pmatrix} = \begin{pmatrix} \alpha(\mathbf{d})a & \alpha(\mathbf{d})b \\ \alpha(\mathbf{d})c & \alpha(\mathbf{d})d \end{pmatrix} .$$

This amounts to the three equations

$$i(\mathbf{d}b^t - b\mathbf{d}) = \alpha(\mathbf{d})a \ , \ i(\mathbf{d}a - d\mathbf{d}) = -\alpha(\mathbf{d})b^t \ , \ i(\mathbf{d}b - b^t\mathbf{d}) = \alpha(\mathbf{d})d$$

and we look for solutions $\alpha \in R$ with $\alpha(\mathbf{d}) \neq 0$. Either $b^t = b, a = d$ and the three equations reduce to the two equations

$$i[\mathbf{d}, b] = \alpha(\mathbf{d})a \ , \ i[\mathbf{d}, a] = -\alpha(\mathbf{d})b \ ,$$

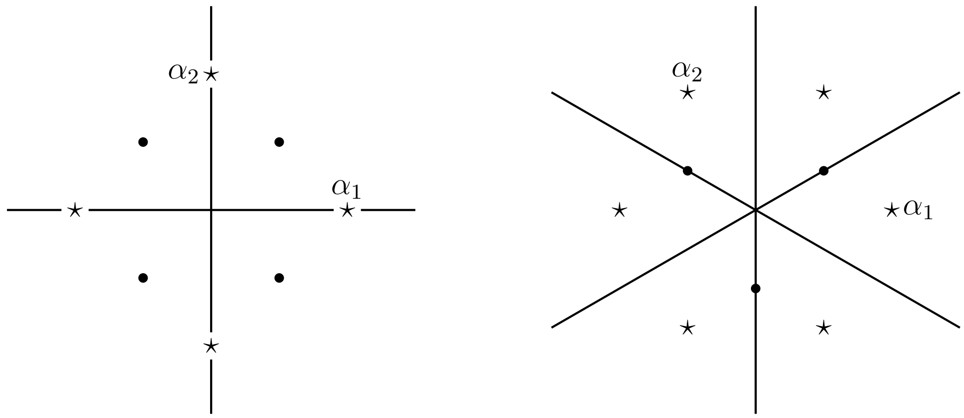
or or $b^t = -b, a = -d$ and the three equations reduce to the two equations

$$i(\mathbf{d}b + b\mathbf{d}) = -\alpha(\mathbf{d})a \ , \ i(\mathbf{d}a + a\mathbf{d}) = \alpha(\mathbf{d})b \ .$$

In the first case we take $a = i(e_{ij} - e_{ji}), b = (e_{ij} + e_{ji})$ and get the roots $\alpha(\mathbf{d}) = (d_i - d_j)$ for $1 \leq i \neq j \leq n$. These are $n(n-1)$ roots. In the second case we take $a = i(e_{ij} - e_{ji}), b = \pm(e_{ij} - e_{ji})$ and get the roots $\alpha(\mathbf{d}) = \pm(d_i + d_j)$ for $1 \leq i < j \leq n$. Again these are $n(n-1)$ roots. The dimension of $\mathfrak{so}_{2n}$ is equal to $(2n(n-1) + n) = n(2n-1)$ as should.

The symmetric trace form on $\mathfrak{h}$ is twice the standard inner product on $\mathbf{d} \in \mathbb{C}^n$. For this reason we shall renormalize the Hermitian and symmetric trace form on $\mathfrak{so}_{2n}$ by dividing by 2. In all three cases $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}_{2n}$ the symmetric trace form on $\mathfrak{h}$ is now just the standard inner product on $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{C}^n$. Consider the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$. In the case of $\mathfrak{so}_{2n}$ the roots are all norm 2 vectors in $\mathbb{Z}^n$, and in the case of $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ the roots are all norm 2 vectors in $\mathbb{Z}^n$ with sum of coordinates equal to 0.

For the rank 2 Lie algebras $\mathfrak{so}_4$ of dimension 6 and $\mathfrak{sl}_3$ of dimension 8 we can make Euclidean pictures of the roots (denoted by a $\star$), and of the weights (denoted by a $\bullet$) of the defining representation of $\mathfrak{so}_4$ on $\mathbb{C}^4$ and of $\mathfrak{sl}_3$ on $\mathbb{C}^3$.



For each root we have drawn the line perpendicular to the root. Note that the pictures are symmetric under the orthogonal reflection with mirror the perpendicular of a root. The group generated by all these reflections is called the Weyl group W. For $\mathfrak{so}_4$ the Weyl group is the dihedral group of order 4, and its action on $\mathfrak{a}^* \simeq \mathbb{R}^2$ is reducible. This comes from the fact that the adjoint representation of $\mathfrak{so}_4$ is reducible, with $\mathfrak{so}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. For $\mathfrak{sl}_3$ the Weyl group is the dihedral group of order 6, and its action on $\mathfrak{a}^* \simeq \mathbb{R}^2$ is irreducible. This explains why $\mathfrak{sl}_3$ is a simple Lie algebra in the sense that it is not isomorphic to a direct sum of smaller Lie algebras.

We return to the case of a general reductive Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and associative symmetric trace form $(\cdot,\cdot)$. We have the root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$

$$\mathfrak{g} \ominus \mathfrak{h} = \oplus \mathfrak{g}_\alpha$$

with the direct sum over all roots in R.

Lemma 9.7. *For $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ we have $[x,y] \in \mathfrak{g}_{\alpha+\beta}$. For $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ we have*

$$[x,y] = (x,y)h_\alpha$$

with $h_\alpha \in \mathfrak{h}$ dual to α, that is $(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$.

Proof. The first relation is clear since

$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] = (\alpha(h) + \beta(h))[x,y]$$

for all $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. For the second relation observe that for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ the bracket $[x, y]$ lies in the centralizer $\mathfrak{z}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Hence $[x, y] - (x, y)h_\alpha$ lies in $\mathfrak{z} = \mathfrak{h}$. Moreover

$$(h, [x, y] - (x, y)h_\alpha) = (h, [x, y]) - \alpha(h)[x, y] = ([h, x], y) - \alpha(h)[x, y] = 0$$

for all $h \in \mathfrak{h}$. The lemma follows since the symmetric form $(\cdot, \cdot)$ is nondegenerate on $\mathfrak{h}$. □

Lemma 9.8. *Consider the symmetric trace form $(\cdot, \cdot)$ on the reductive Lie algebra $\mathfrak{g}$. For α, β roots we have $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$, and the bilinear map $(\cdot, \cdot) : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ gives a nondegenerate pairing.*

Proof. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ it is clear that elements of $\mathfrak{g}_\alpha$ are nilpotent for $\alpha \neq 0$. Hence $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$. Because $(\cdot, \cdot)$ is nondegenerate on $\mathfrak{g}$ it is also clear that $(\cdot, \cdot) : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ gives a nondegenerate pairing. □

Corollary 9.9. *For each $\alpha \in R$ choose $x_\alpha \in \mathfrak{g}_\alpha, y_\alpha \in \mathfrak{g}_{-\alpha}$ with $(x_\alpha, y_\alpha) = 2/(h_\alpha, h_\alpha)$, and put $z_\alpha = 2h_\alpha/(h_\alpha, h_\alpha)$. The elements $x_\alpha, y_\alpha, z_\alpha$ satisfy the commutation relations*

$$[z_\alpha, x_\alpha] = 2x_\alpha \ , \ [z_\alpha, y_\alpha] = -2y_\alpha \ , \ [x_\alpha, y_\alpha] = z_\alpha$$

making the vector space $\mathfrak{s}_\alpha$ spanned by $x_\alpha, y_\alpha, z_\alpha$ a Lie algebra isomorphic to $\mathfrak{sl}_2$. Finally we can choose $x_\alpha, y_\alpha, z_\alpha$ in such a way that $x_\alpha^\star = y_\alpha, y_\alpha^\star = x_\alpha, z_\alpha^\star = z_\alpha$.

For $\alpha \in R$ let $V_\alpha = (\oplus \mathfrak{g}_{k\alpha}) \oplus \mathbb{C}h_\alpha$ with the direct sum over all nonzero $k \in \mathbb{Z}$. Clearly V_α is a representation space for $\mathfrak{s}_\alpha = \mathbb{C}x_\alpha + \mathbb{C}y_\alpha + \mathbb{C}z_\alpha$, and all weights are even integers. Because the zero weight space is one dimensional this representation is irreducible by standard representation theory of $\mathfrak{sl}_2$. Hence $V_\alpha = \mathfrak{s}_\alpha$.

Corollary 9.10. *All root spaces $\mathfrak{g}_\alpha$ are one dimensional, and the only roots that are a multiple of α are $\pm\alpha$.*

Suppose V is a finite dimensional representation of $\mathfrak{g}$ that is unitary for the compact real form $\mathfrak{g}_0$. For $\mu \in \mathfrak{h}^*$ a weight of V the subspace $\oplus V_{\mu+k\alpha}$ with the sum over $k \in \mathbb{Z}$ is a representation space for $\mathfrak{s}_\alpha$. In turn this implies that the weight multiplicities $\dim V_{\mu+k\alpha}$ are palindromic. The set of weights $\{\mu + k\alpha; k \in \mathbb{Z}, \dim V_{\mu+k\alpha} > 0\}$ of V is called the α-ladder through μ. The palindromic nature of weight multiplicities shows that root ladders are unbroken.

Theorem 9.11. *The subspace $\mathfrak{a} = \{h \in \mathfrak{h}; h^\star = h\}$ is a real form of the vector space $\mathfrak{h}$. The symmetric trace form $(\cdot,\cdot)$ is an inner product on $\mathfrak{a}$, turning $\mathfrak{a}$ into a Euclidean vector space. We have a linear isomorphism $h_\lambda \mapsto \lambda$ from $\mathfrak{a}$ to $\mathfrak{a}^*$, which allows us to transport the inner product from $\mathfrak{a}$ to $\mathfrak{a}^*$, so $(\lambda,\mu) = (h_\lambda, h_\mu)$. For $\alpha \in R$ let $s_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ be the orthogonal reflection with mirror $\alpha^\perp$, so in a formula*

$$s_\alpha(\lambda) = \lambda - 2(\alpha,\lambda)\alpha/(\alpha,\alpha)$$

for all $\lambda \in \mathfrak{a}^*$. Then the root system R satisfies the following axioms:

1. *The only multiples in R of $\alpha \in R$ are $\pm\alpha$.*
2. *We have $s_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.*
3. *We have $2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$.*

These three axioms say that R is a root system in the Euclidean space $\mathfrak{a}^*$.

Exercise 9.1. *Show that the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is indeed a representation.*

Exercise 9.2. *Show that for a reductive Lie algebra $\mathfrak{g}$ with compact real form $\mathfrak{g}_0$ and faithful unitary representation $\mathfrak{g} \to End(\mathcal{H})$ on a finite dimensional Hilbert space $\mathcal{H}$ the adjoint representation is unitary in the sense that $\langle [z,x], y\rangle = \langle x, [z^\star, y]\rangle$ for all $x, y, z \in \mathfrak{g}$.*

Exercise 9.3. *In the notation of Corollary 9.3 show that $h_{\lambda^\star} = (h_\lambda)^\star$.*

Exercise 9.4. *A Lie subalgebra $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is called normal if $[x,y] \in \mathfrak{n}$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{n}$. A Lie algebra $\mathfrak{g}$ is called simple if the only normal subalgebras are the two trivial ones 0 and $\mathfrak{g}$ itself. Show that a reductive Lie algebra $\mathfrak{g}$ is a unique direct sum*

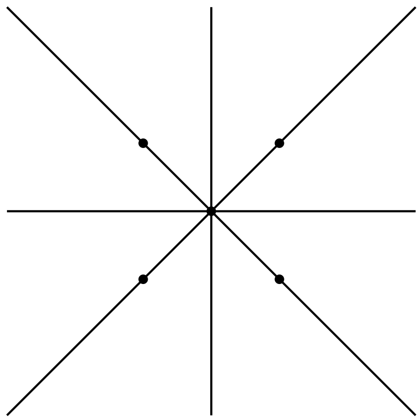
$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

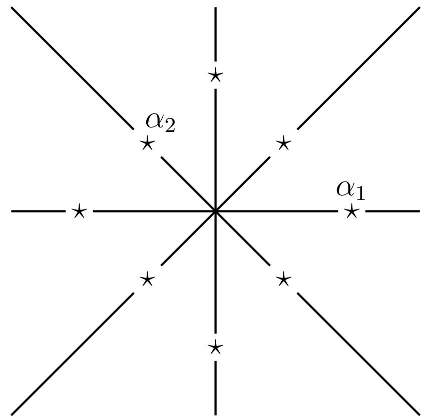
with $\mathfrak{z}$ the Abelian center and the $\mathfrak{g}_i$ all simple Lie algebras.

Exercise 9.5. *Show that R spans $\mathfrak{a}^*$ if and only if the center $\mathfrak{z}$ of $\mathfrak{g}$ is zero.*

Exercise 9.6. *Show that in Corollary 9.9 we can choose $x_\alpha, y_\alpha, z_\alpha$ in such a way that $x_\alpha^\star = y_\alpha, y_\alpha^\star = x_\alpha, z_\alpha^\star = z_\alpha$.*

Exercise 9.7. *Consider the natural inclusion $\mathfrak{so}_{2n} \hookrightarrow \mathfrak{so}_{2n+1}$ by adding zeros in the last column and row. Show that the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_{2n}$ is also a Cartan subalgebra for $\mathfrak{so}_{2n+1}$. Determine the root system for $\mathfrak{so}_{2n+1}$. Show that for the weights of the defining representation and the root system of $\mathfrak{so}_5$ are given by the pictures*





respectively. The lines are the mirrors perpendicular to the roots.

Exercise 9.8. *Show that the matrices*

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ L_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \ , \ K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \ , \ K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k \ , \ [L_i, K_j] = i\epsilon_{ijk} K_k \ , \ [K_i, K_j] = -i\epsilon_{ijk} L_k \ ,$$

and form a real basis of $i\mathfrak{so}_{3,1}(\mathbb{R})$, with $\mathfrak{so}_{3,1}(\mathbb{R})$ the Lorentz algebra defined by

$$\mathfrak{so}_{3,1}(\mathbb{R}) = \{x \in \mathfrak{gl}_4(\mathbb{R}); x^t I + I x = 0\}$$

with $I \in \mathfrak{gl}_4(\mathbb{R})$ *the diagonal matrix with diagonal entries* $1, 1, 1, -1$. *We conclude that for the hydrogen atom the coherent positive energy* $E > 0$ *eigenspaces for* H *have natural degeneration according to an irreducible representation of the Lorentz algebra.*

10 Triangular decomposition and Verma representations

Suppose we have given a reductive Lie algebra $\mathfrak{g}$ with compact real form $\mathfrak{g}_0$, with defining unitary representation $\mathfrak{g} \to End(\mathcal{H})$ and associated symmetric trace form $(\cdot,\cdot)$ on $\mathfrak{g}$. Suppose we have chosen a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ with associated root system $R = R(\mathfrak{g},\mathfrak{h})$ in the Euclidean space $\mathfrak{a}^*$. We shall assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ is zero, or equivalently that the root system R spans $\mathfrak{a}^*$.

Let $\mathfrak{a}^*_\circ$ be the complement in $\mathfrak{a}^*$ of the mirror hyperplanes perpendicular to the roots. We fix troughout this section a connected component $\mathfrak{a}^*_+$ of $\mathfrak{a}^*_\circ$ and call it the positive Weyl chamber. We have a corresponding partition

$$R = R_+ \sqcup R_-$$

into positive roots $R_+ = \{\alpha \in R; (\alpha,\lambda) > 0\ \forall\lambda \in \mathfrak{a}^*_+\}$ and negative roots $R_- = \{\alpha \in R; (\alpha,\lambda) < 0\ \forall\lambda \in \mathfrak{a}^*_+\}$. We also write $\alpha > 0$ if $\alpha \in R_+$ and $\alpha < 0$ if $\alpha \in R_-$. The decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\ ,\ \mathfrak{n}_- = \oplus_{\alpha<0}\mathfrak{g}_\alpha\ ,\ \mathfrak{n}_+ = \oplus_{\alpha>0}\mathfrak{g}_\alpha$$

is called the triangular decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $\mathfrak{a}^*_+$.

For the example of $\mathfrak{sl}_n$ with Cartan subalgebra $\mathfrak{h}$ of diagonal matrices identified with $\{\mathbf{d} \in \mathbb{C}^n; d_1 + \cdots + d_n = 0\}$, and the mirror complement $\mathfrak{a}^*_\circ$ identified with $\{\mathbf{d} \in \mathbb{R}^n; d_1 + \cdots + d_n = 0, d_i \neq d_j\ \forall i \neq j\}$ we take for the standard open Weyl chamber

$$\mathfrak{a}^*_+ = \{\mathbf{d} \in \mathbb{R}^n; d_1 + \cdots + d_n = 0, d_1 > \cdots > d_n\}\ .$$

The corresponding positive roots have the form

$$R_+ = \{\alpha = e_i - e_j; 1 \leq i < j \leq n\}\ .$$

In turn the triangular decomposition of $\mathfrak{sl}_n$ is just the direct sum decomposition in lower triangular matrices, trace zero diagonal matrices and upper triangular matrices.

Definition 10.1. *A positive root $\alpha \in R_+$ is called simple if α is not of the form $\alpha = \beta + \gamma$ with $\beta,\gamma \in R_+$.*

Lemma 10.2. *If $\alpha, \beta \in R_+$ are distinct simple roots then $(\alpha, \beta) \leq 0$.*

Proof. If $(\alpha, \beta) > 0$ then the ladder $\beta + \mathbb{Z}\alpha$ contains the root $\beta - \alpha$ by representation theory of $\mathfrak{s}_\alpha \simeq \mathfrak{sl}_2$. If $(\beta - \alpha) > 0$ then $\beta = \alpha + (\beta - \alpha)$ contradicting the simplicity of β, and if $(\beta - \alpha) < 0$ then $\alpha = (\alpha - \beta) + \beta$ contradicting the simplicity of α. Hence $(\alpha, \beta) \leq 0$ as stated. $\square$

Corollary 10.3. *The simple roots in R_+ are linearly independent, and each positive root in R_+ is a nonnegative integral linear combination of simple roots.*

Proof. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots in R_+. Suppose $x_1\alpha_1 + \cdots + x_n\alpha_n = 0$. Bringing the terms with $x_i < 0$ to the other side gives us

$$\sum y_i\alpha_i = \sum z_i\alpha_i$$

with $y_i, z_i \geq 0$ and $y_i z_i = 0$ for all i. Since

$$(\sum y_i\alpha_i, \sum z_j\alpha_j) = \sum y_i z_j(\alpha_i, \alpha_j) \leq 0$$

we conclude that

$$\sum y_i\alpha_i = \sum z_i\alpha_i = 0 \ .$$

Taking the inner product with $\lambda \in \mathfrak{a}_+^*$ gives $y_i = z_i = 0$ for all i. The second statement is clear by induction on the height $ht(\alpha) = \sum k_i$ of the positive root $\alpha = \sum k_i\alpha_i$ $\square$

The conclusion is that inside the positive roots R_+ we have a basis of simple roots $\alpha_1, \cdots, \alpha_n$ with n equal to the rank of R. Given R_+ the basis of simple roots is canonical up to ordering. It is easy to check that in the rank two pictures of the previous section $\{\alpha_1, \alpha_2\}$ is a basis of simple roots for a suitably chosen positive Weyl chamber.

Lemma 10.4. *The upper triangular subalgebra $\mathfrak{n}_+$ is generated as a Lie algebra by the root spaces $\mathfrak{g}_{\alpha_i}$ for $i = 1, \cdots, n$, and likewise the lower triangular subalgebra $\mathfrak{n}_-$ is generated by the root spaces $\mathfrak{g}_{-\alpha_i}$ for $i = 1, \cdots, n$.*

Proof. If $\alpha > 0$ is a positive root, which is not simple, then $(\alpha, \alpha_i) > 0$ for some i. If this is not the case then $\{\alpha, \alpha_1, \cdots, \alpha_n\}$ is an obtuse set in R_+, and hence linearly independent. This is a contradiction with n being the rank of R. Hence $\alpha = \alpha_i + (\alpha - \alpha_i)$ and therefore $\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{\alpha - \alpha_i}]$ by representation theory of $\mathfrak{s}_i \simeq \mathfrak{sl}_2$. The lemma follows by induction on the height $ht(\alpha) = \sum k_i$ if $\alpha = \sum k_i\alpha_i$. $\square$

Corollary 10.5. *We can choose a basis $\{h_1, \cdots, h_n\}$ of $\mathfrak{a}$, and $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ such that*

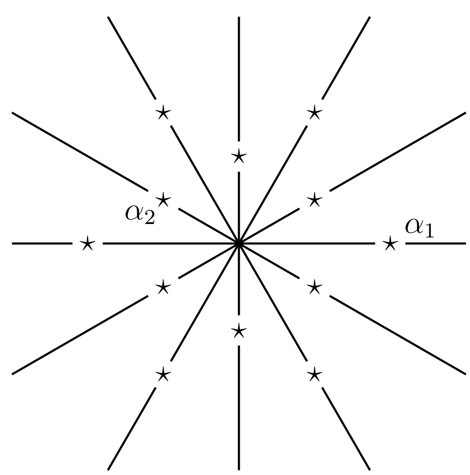
$$[h_i, e_j] = a_{ij}e_j \ , \ [h_i, f_j] = -a_{ij}f_j \ , \ [e_i, f_j] = \delta_{ij}h_i$$

with $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbb{Z}$ the so called Cartan matrix of R. Finally we can choose this basis such that $e_i^\star = f_i, f_i^\star = e_i, h_i^\star = h_i$.

The Gram matrix (α_i, α_j) of the basis of simple roots is a positive definite matrix, while the Cartan matrix $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ has a 2 for the diagonal entries and nonpositive integers for the off diagonal entries. These conditions are very restrictive, and allow for a classification of the (indecomposable) Cartan matrices. For example in the rank 2 case with say $(\alpha_1, \alpha_1) \geq (\alpha_2, \alpha_2)$ the possible Cartan matrices are

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} , \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} , \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} , \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} .$$

Indeed these are the only integral solutions of the equations $a_{11} = a_{22} = 2$, $a_{11}a_{22} - a_{12}a_{21} > 0$, $a_{21} \leq a_{12} \leq 0$ and $a_{12} = 0 \Leftrightarrow a_{21} = 0$. The root system with the most right Cartan matrix has a following Euclidean picture.



As before the roots are denoted by $\star$ and the lines are the mirrors perpendicular to the roots. The elements $\{e_i, f_i, h_i; i = 1, \cdots, n\}$ are called the Chevalley generators. Using the Chevalley generators we are in the situation to define the Verma representations, and derive their basic properties.

Definition 10.6. *The Weyl group W is the subgroup of the orthogonal group $O(\mathfrak{a}^*)$ generated by the simple reflections $s_1, \cdots, s_n$ defined by*

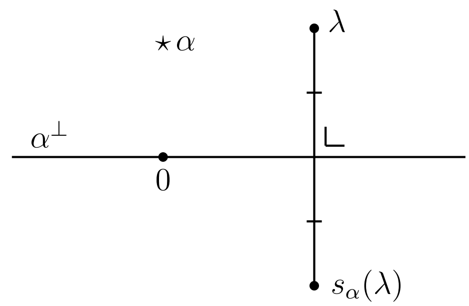
$$s_i(\lambda) = \lambda - 2(\lambda, \alpha_i)\alpha_i/(\alpha_i, \alpha_i)$$

for $\lambda \in \mathfrak{a}^*$.

Lemma 10.7. *For each root $\alpha \in R$ the orthogonal reflection*

$$s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)\alpha/(\alpha, \alpha)$$

with mirror the hyperplane $\alpha^\perp$ perpendicular to α lies in the Weyl group W.



It is easy to check that $s_{w\alpha} = ws_\alpha w^{-1}$ for all $w \in W$ and $\alpha \in R$. Indeed evaluate both expressions on $\lambda \in \mathfrak{a}^*$ and check that the outcome is the same.

Proof. Because $s_{-\alpha} = s_\alpha$ we may assume that $\alpha \in R_+$. The height of $\alpha = \sum_i k_i\alpha_i \in R_+$ is defined as $ht(\alpha) = \sum_i k_i \in \mathbb{N}$. We prove the lemma by induction on the height $ht(\alpha)$. If $ht(\alpha) = 1$ then $\alpha = \alpha_i$ for some i and $s_\alpha = s_i \in W$. Now suppose $\alpha \in R_+$ with $ht(\alpha) > 1$. Then there exists an index i with $(\alpha, \alpha_i) > 0$. In turn this implies that $\beta = s_i(\alpha)$ has height strictly smaller than $ht(\alpha)$. Therefore $s_\alpha = s_is_\beta s_i$ lies in W by the induction hypothesis. $\square$

Definition 10.8. *If $\mathfrak{g}$ is a Lie algebra with triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

with a basis $\{h_1, \cdots, h_n\}$ of $\mathfrak{h}$ and Lie algebra generators $\{e_1, \cdots, e_n\}$ for $\mathfrak{n}_+$ and $\{f_1, \cdots, f_n\}$ for $\mathfrak{n}_-$ such that

$$[h_i, e_j] = a_{ij}e_j\ ,\ [h_i, f_j] = -a_{ij}f_j\ ,\ [e_i, f_j] = \delta_{ij}h_i\ ,$$

then the Verma representation $M(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$ is defined by

$$M(\lambda) = U\mathfrak{g}/J(\lambda)$$

with $J(\lambda) = \sum_h U\mathfrak{g}(h - \lambda(h)) + (U\mathfrak{g})\mathfrak{n}_+ = \sum_i U\mathfrak{g}(h_i - \lambda(h_i)) + \sum_i (U\mathfrak{g})e_i$ *a left ideal of* $U\mathfrak{g}$*. It is the universal representation of* $\mathfrak{g}$ *with a nonzero generating highest weight vector* $v_0 = 1 + J(\lambda)$ *with*

$$hv_0 = \lambda(h)v_0 \;\forall h \in \mathfrak{h} \;,\; xv_0 = 0 \;\forall x \in \mathfrak{n}_+ \;.$$

The Verma representation $M(\lambda)$ can be viewed as a representation on a fixed vector space $U\mathfrak{n}_-$ with the representation of $\mathfrak{n}_-$ acting by left multiplication and extending to a representation of $\mathfrak{g}$ depending on $\lambda \in \mathfrak{h}^*$ in a polynomial way. The Verma representation $M(\lambda)$ has a unique maximal proper subrepresentation, and the quotient $L(\lambda)$ is the unique irreducible representation of $\mathfrak{g}$ with a generating vector (by abuse of notation) denoted $v_0 \in L(\lambda)$ with

$$hv_0 = \lambda(h)v_0 \;\forall h \in \mathfrak{h} \;,\; xv_0 = 0 \;\forall x \in \mathfrak{n}_+ \;.$$

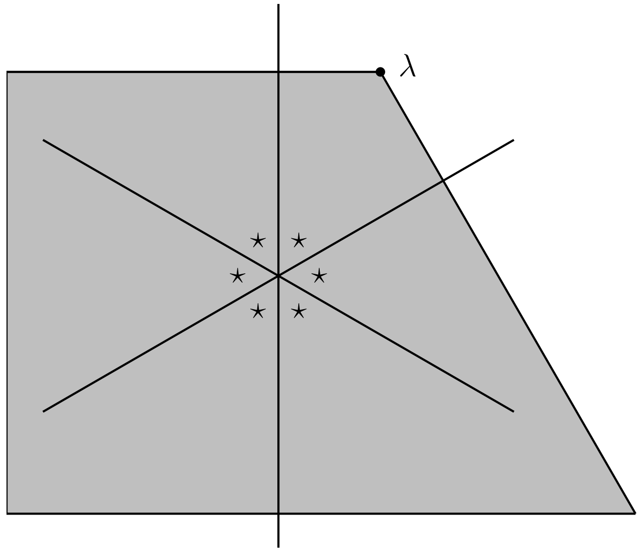
The vector $v_0 \in L(\lambda)$ is called the (up to a constant unique) highest weight vector of weight $\lambda \in \mathfrak{h}^*$. The two basic questions are to characterize those $\lambda \in \mathfrak{h}^*$ for which $L(\lambda)$ is finite dimensional, and for which $L(\lambda)$ is unitary with respect to the given star structure.

Theorem 10.9. *The representation $L(\lambda)$ of $\mathfrak{g}$ is finite dimensional if and only if $\lambda(h_i) \in \mathbb{N}$ for $i = 1, \cdots, n$.*

Proof. The vector v_0 is a highest weight vector in $L(\lambda)$ of weight $\lambda \in \mathfrak{h}^*$. For fixed $i = 1, \cdots, n$ and $k \in \mathbb{N}$ the vector $f_i^k v_+$ has weight $\lambda - k\alpha_i$. If $L(\lambda)$ is finite dimensional then $f_i^{k_i+1}v_+ = 0$ while $f_i^{k_i}v_+ \neq 0$ for a unique $k_i \in \mathbb{N}$. Hence $f_i^{k_i+1}v_+$ is a highest weight vector in $M(\lambda)$, or equivalently $f_i^{k_i+1}v_+$ is annihilated by e_i. It is trivially annihilated by the e_j with $j \neq i$. By $\mathfrak{sl}_2$ representation theory this amounts to $\lambda(h_i) = k_i$ for $i = 1, \cdots, n$.

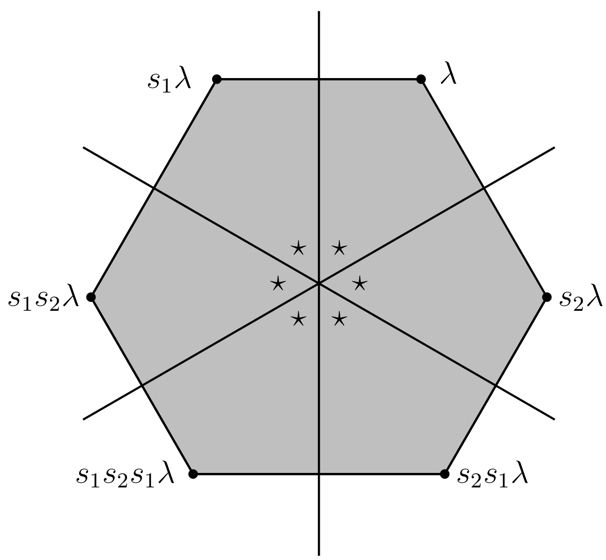
Conversely, suppose $\lambda(h_i) \in \mathbb{N}$ for $i = 1, \cdots, n$. In turn (using the PBW theorem) this implies that the operators f_i act locally nilpotently on $L(\lambda)$. Since the operators e_i also act locally nilpotently on $L(\lambda)$ (even on $M(\lambda)$ for any $\lambda \in \mathfrak{h}^*$) we conclude using Exercise 4.2 that the operators $s_i = exp(e_i)exp(-f_i)exp(e_i)$ are well defined on $L(\lambda)$ and map $L(\lambda)_\mu$ to $L(\lambda)_{s_i\mu}$. Hence the weight multiplicities $m(\lambda)_\mu$, which are by definition the dimensions of the weight spaces $L(\lambda)_\mu$, are invariant under the Weyl group W as a function of $\mu \in \mathfrak{h}^*$. The weights of $M(\lambda)$ are contained in the set

$$\{\mu \in \mathfrak{h}^*; \lambda - \mu \in \mathbb{N}R_+\} \ ,$$



which in turn implies that the weights of $L(\lambda)$ are contained in the set

$$\{\mu \in \mathfrak{h}^*; \lambda - w(\mu) \in \mathbb{N}R_+, \forall\, w \in W\} \ .$$



But this last set can be shown to be equal to

$$CH(W\lambda) \cap \{\mu \in \mathfrak{h}^*; \lambda - \mu \in \mathbb{Z}R\}$$

with $CH(W\lambda)$ denoting the convex hull of the Weyl group orbit $W\lambda$. The set of points in $\mathfrak{h}^*$ given by

$$CH(W\lambda) \cap \{\mu \in \mathfrak{h}^*; \lambda - \mu \in \mathbb{Z}R\}$$

is called the integral convex hull of $W\lambda$. Note that $s_2s_1s_2\lambda = s_1s_2s_1\lambda$. $\square$

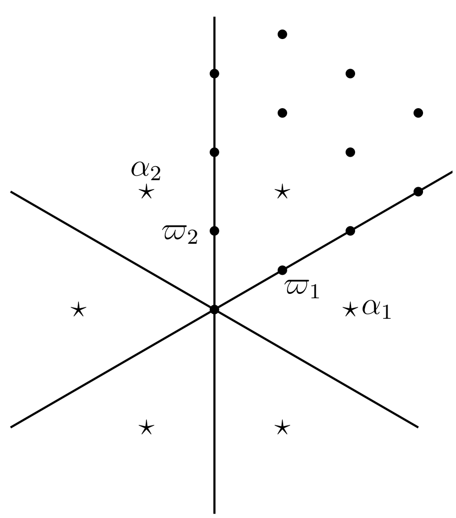
Definition 10.10. *The set $Q = \mathbb{Z}R$ of integral linear combinations of roots is called the root lattice of R. The set of points*

$$P = \{\lambda \in \mathfrak{h}^*; \lambda(h_i) \in \mathbb{Z}\}$$

is called the weight lattice. Clearly the root lattice Q is a sublattice of the weight lattice P. The subset of the weight lattice

$$P_+ = \{\lambda \in \mathfrak{h}^*; \lambda(h_i) \in \mathbb{N}\}$$

is called the integral cone of dominant weights. The cone of dominant weights is spanned over the natural numbers $\mathbb{N}$ by the set of fundamental weights $\{\varpi_1, \cdots, \varpi_n\}$ defined by $(\varpi_i, \alpha_j) = \delta_{ij}$. So the basis of fundamental weights and the basis of simple roots are dual bases.



So we can rephrase the above theorem by saying that the irreducible representation $L(\lambda)$ is finite dimensional if and only if $\lambda \in P_+$. The set of weights of $L(\lambda)$ is in fact equal to

$$CH(W\lambda) \cap \{\mu \in \mathfrak{h}^*; \lambda - \mu \in \mathbb{Z}R\} \ .$$

In the above theorem we have shown that the weighths of $L(\lambda)$ are contained in the integral convex hull of $W\lambda$. The converse that all points in this integral convex hull have multiplicity ≥ 1 follows from the palindromic nature of weight multiplicities along root ladders. Indeed the weights of $W\lambda$ have multiplicity equal to 1, and one proceeds by induction on the dimension of the faces of the convex hull of $W\lambda$ to conclude that all weights in the integral convex hull of $W\lambda$ have multiplicity ≥ 1.

We still have to discuss the question of unitarity for the finite dimensional irreducible representations $L(\lambda)$ for $\lambda \in P_+$. It turns out that all irreducible representations $L(\lambda)$ for $\lambda \in P_+$ are unitarizable for $\mathfrak{g}_0$, so they are unitary with respect to a suitable Hermitian inner product $\langle \cdot, \cdot \rangle$ on $L(\lambda)$. This is a general theorem, but the general proof is not so easy. For an easy argument that works for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g}_0 = \mathfrak{su}_n$ we refer to the exercises.

Exercise 10.1. *Check that in the rank two pictures of the previous section* $\{\alpha_1, \alpha_2\}$ *is a basis of simple roots for a suitably chosen positive Weyl chamber.*

Exercise 10.2. *Show that for* $v \in M(\lambda)$ *and* $x \in \mathfrak{g}$ *the expression* xv *depends in a polynomial way on* $\lambda \in \mathfrak{h}^*$. *Hint: For* $h \in \mathfrak{h}$ *the expression* hv *varies linearly with* $\lambda \in \mathfrak{h}^*$. *Analyse the case* $x = e_i$ *and* $v = f_{i_1} f_{i_2} \cdots f_{i_r} v_0$ *with* $1 \leq i_1, i_2, \cdots, i_r \leq n$.

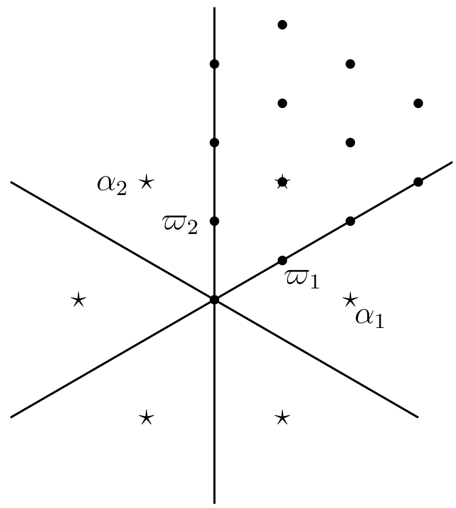
Exercise 10.3. *Suppose that* V *is a finite dimensional irreducible representation of* $\mathfrak{g}$ *whose weights are contained in* $\lambda - \mathbb{N}R_+$ *for some* $\lambda \in P_+$, *and with the multiplicity of* λ *equal to* 1. *Show that* V *is equivalent to* $L(\lambda)$.

Exercise 10.4. *Suppose that for* $\lambda, \mu \in P_+$ *the irreducible finite dimensional representations* $L(\lambda), L(\mu)$ *are also unitary. Show that the irreducible representation* $L(\lambda + \mu)$ *occurs in the decomposition of the tensor product* $L(\lambda) \otimes L(\mu)$ *with multiplicity equal to one. This result is called Cartan's Theorem. Conclude that* $L(\lambda + \mu)$ *is unitary as well. Hint: Show that tensor product* $u_0 \otimes v_0$ *of the two highest weight vectors* $u_0 \in L(\lambda)$ *and* $v_0 \in L(\mu)$ *is again a highest weight vector of weight* $\lambda + \mu$.

Exercise 10.5. *For the Lie algebra* $\mathfrak{g} = \mathfrak{sl}_n$ *the standard representation* $\mathbb{C}^n = L(\varpi_1)$ *is unitary for* $\mathfrak{g}_0 = \mathfrak{su}_n$. *Check that the* k^{th} *antisymmetric power* $A^k L(\varpi_1) = L(\varpi_k)$ *for* $k = 1, \cdots, n-1$. *In turns this implies that* $L(\varpi_k)$ *is unitary for* $\mathfrak{su}_n$, *as a linear algebra construction of a unitary representation. Using the previous exercise show that all irreducible representations* $L(\lambda)$ *for* $\lambda \in P_+$ *are unitarizable for* $\mathfrak{su}_n$.

11 The Lie algebra $\mathfrak{sl}_3$

In this chapter we shall discuss the finite dimensional unitary representation theory of the Lie algebra $\mathfrak{sl}_3$ by using pictures of weight diagrams. With our standard choice of Cartan subalgebra $\mathfrak{h}$ the root system R has the following picture.



We have the positive roots $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, the simple roots $\{\alpha_1, \alpha_2\}$ and the fundamental weights $\{\varpi_1, \varpi_2\}$ satisfying

$$\varpi_1 = (2\alpha_1 + \alpha_2)/3 \ , \ \varpi_2 = (\alpha_1 + 2\alpha_2)/3 \ ,$$

$$\alpha_1 = 2\varpi_1 - \varpi_2 \ , \ \alpha_2 = -\varpi_1 + 2\varpi_2 \ .$$

The cone of dominant integral weights becomes $P_+ = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2$. For $\lambda = l_1\varpi_1 + l_2\varpi_2 \in P_+$ we write $L(\lambda) = L(l_1, l_2)$. With this notation the one dimensional trivial representation becomes $L(0,0)$.

The standard three dimensional irreducible representation $L(1,0)$ has a basis $\{x, y, z\}$ of weights $\varpi_1, \varpi_1 - \alpha_1, \varpi_1 - \alpha_1 - \alpha_2$ respectively. It is evidently unitary for $\mathfrak{su}_3$. The dual representation $L(1,0)^*$ of $L(1,0)$ has dual basis $\{\xi, \eta, \zeta\}$ of weights $-\varpi_1, -\varpi_1 + \alpha_1, -\varpi_1 + \alpha_1 + \alpha_2$ respectively, which in reversed order are equal to $\varpi_2, \varpi_2 - \alpha_2, \varpi_2 - \alpha_1 - \alpha_2$. Therefore $L(0,1) = L(1,0)^*$ is unitary as well.

The n^{th} symmetric power $S^n(L(1,0))$ has as basis $\{x^i y^j z^k; i,j,k \in \mathbb{N}, i+j+k=n\}$, so the weights are the integral convex hull of the triangle with vertices $\{n\varpi_1, n(\varpi_1-\alpha_1), n(\varpi_1-\alpha_1-\alpha_2)\}$, all with multiplicity one. In turn this implies that $S^n(L(1,0)) = L(n,0)$ has dimension $(n+2)(n+1)/2$. Likewise the n^{th} symmetric power $S^n(L(0,1))$ has as basis $\{\xi^i\eta^j\zeta^k; i,j,k \in \mathbb{N}, i+j+k=n\}$, so the weights are the integral convex hull of the triangle with vertices $\{n\varpi_2, n(\varpi_2-\alpha_2), n(\varpi_2-\alpha_1-\alpha_2)\}$, all with multiplicity one. Hence $S^n(L(0,1)) = L(0,n)$ has dimension $(n+2)(n+1)/2$ as well. Clearly both these symmetric powers are also unitary, being realized as a construction of linear algebra of unitary representations.

Using Cartan's Theorem (the last exercise of the previous chapter) we conclude that $L(l_1,l_2)$ for $l_1,l_2 \in \mathbb{N}$ occurs as a multiplicity one constituent of the tensor product $L(l_1,0)\otimes L(0,l_2)$, and therefore is always a unitary irreducible finite dimensional representation. So we have shown that all finite dimensional irreducible representations of $\mathfrak{sl}_3$ are unitary.

In order to understand the weight diagram of $L(l_1,l_2)$ we derive an explicit tensor product decomposition, which allows proofs by induction on l_1+l_2.

Theorem 11.1. *If both integers l_1,l_2 are at least 1 then*

$$L(1,0)\otimes L(l_1,l_2) = L(l_1+1,l_2)\oplus L(l_1-1,l_2+1)\oplus L(l_1,l_2-1)$$

is a multiplicity free direct sum decomposition.

Proof. Let us introduce a partial ordering $\leq$ on the weight lattice $P = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$ by

$$\mu \leq \lambda \Leftrightarrow (\lambda-\mu) \in \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2\ .$$

Indeed, one easily checks that the relation $\leq$ is a partial ordering, i.e. $\mu \leq \lambda, \nu \leq \mu \Rightarrow \nu \leq \lambda$ and $\mu \leq \lambda, \lambda \leq \mu \Rightarrow \lambda = \mu$. For example, the weights of $L(\lambda)$ are all smaller or equal to λ in this partial ordering.

We claim that for $\lambda = l_1\varpi_1 + l_2\varpi_2$ with l_1,l_2 both at least 1 the weights

$$\lambda, \lambda-\alpha_1, \lambda-\alpha_2, \lambda-\alpha_1-\alpha_2$$

of $L(\lambda)$ have multiplicities $1,1,1,2$ respectively. We prove this by induction on l_1+l_2. If $l_1=l_2=1$ then $L(1,1)$ is the adjoint representation. All roots are weights of multiplicity 1 while 0 is a weight of multipicity 2. Hence the induction hypothesis holds if $l_1+l_2=2$. Now suppose $l \geq 2$ and the claim holds for all integers $l_1,l_2 \geq 1$ with $l_1+l_2 \leq l$. Then the weights

$$\varpi_1+\lambda, \varpi_1+\lambda-\alpha_1, \varpi_1+\lambda-\alpha_2, \varpi_1+\lambda-\alpha_1-\alpha_2$$

of the tensor product $L(1,0) \otimes L(l_1, l_2)$ have multiplicities $1, 2 = 1+1, 1, 4 = 2+1+1$ respectively. In turn this implies

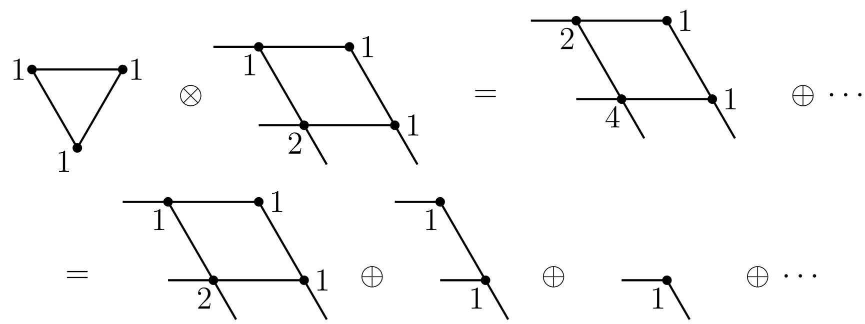
$$L(1,0) \otimes L(l_1, l_2) = L(l_1+1, l_2) \oplus L(l_1-1, l_2+1) \oplus L(l_1, l_2-1) \oplus \cdots$$

with $\cdots$ denoting representations with highest weight smaller in our partial ordering. In addition this completes the inductive proof of the claim that the multiplicities of the weights

$$\lambda, \lambda - \alpha_1, \lambda - \alpha_2, \lambda - \alpha_1 - \alpha_2$$

of the irreducible representation $L(\lambda) = L(l_1, l_2)$ with $l_1, l_2 \geq 1$ are equal to $1, 1, 1, 2$ respectively.

Here are some pictures of (pieces of) weight diagrams together with their weight multiplicities



that illustrate a pictorial proof of the decomposition

$$L(1,0) \otimes L(l_1, l_2) = L(l_1+1, l_2) \oplus L(l_1-1, l_2+1) \oplus L(l_1, l_2-1) \oplus \cdots$$

of the tensor product in irreducible components.

In order to finish the proof of the theorem we have to eliminate these $\cdots$. Let us write $[L(\lambda) \otimes L(\mu) : L(\nu)] \in \mathbb{N}$ for the multiplicity with which $L(\nu)$ occurs in the tensor product $L(\lambda) \otimes L(\mu)$. Using Schur's Lemma one can show that

$$[L(\varpi_1) \otimes L(\lambda) : L(\mu)] = [L(\varpi_2) \otimes L(\mu) : L(\lambda)]$$

is equal to 0 unless $\lambda \leq (\varpi_2 + \mu) \Leftrightarrow ((\lambda + \varpi_1) - (\alpha_1 + \alpha_2)) \leq \mu$. Drawing a picture illustrates that the $\cdots$ are eliminated. $\square$

Corollary 11.2. *The dimension of the unitary irreducible representation $L(\lambda) = L(l_1, l_2)$ is equal to $(l_1 + 1)(l_2 + 1)(l_1 + l_2 + 2)/2$ for all $l_1, l_2 \in \mathbb{N}$.*

Proof. The proof is by induction on the natural number $l_1 + l_2$. We have already seen that the formula is correct if either $l_1 = 0$ or $l_2 = 0$ or for the adjoint representation with $l_1 = l_2 = 1$. Using that

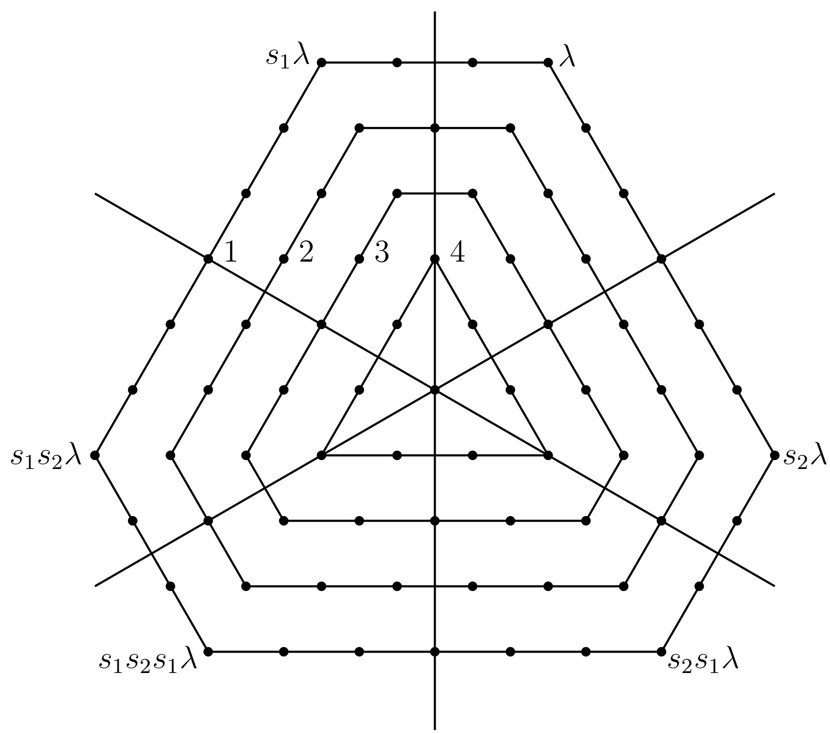
$$L(l_1 + 1, l_2) = L(1, 0) \otimes L(l_1, l_2) \ominus L(l_1 - 1, l_2 + 1) \ominus L(l_1, l_2 - 1)$$

for $l_1, l_2 \geq 1$ we obtain for the dimension of $L(l_1 + 1, l_2)$ the formula

$$(3(l_1 + 1)(l_2 + 1) - l_1(l_2 + 2))(l_1 + l_2 + 2)/2 - (l_1 + 1)l_2(l_1 + l_2 + 1)/2$$

which by a direct calculation equals $(l_1 + 2)(l_2 + 1)(l_1 + l_2 + 3)/2$. $\square$

Corollary 11.3. *The multiplicities of the weights of the irreducible representation $L(\lambda) = L(l_1, l_2)$ are 1 in the outer hexagonal shell, and increase steadily by 1 if we pass from outer hexagonal shell to inner shell, untill the shells become triangles after which the multiplicity stabilizes.*



In the figure we have drawn the weights of the irreducible representation $L(3,6)$. *The number of weights in the hexagonal shells is subsequently equal to* $3(3+6)$, $3(2+5)$, $3(1+4)$ *and inside the triangle is equal to* 10, *making up for dimension* $27+42+45+40$ *which is equal to* $154 = 4\times 7\times 11/2$ *as should.*

Proof. If $l_1 = 0$ or $l_2 = 0$ then the outer shell is already a triangle, and all weights have multiplicity 1 in accordance with $S^n(L(1,0)) = L(n,0)$ or $S^n(L(0,1)) = L(0,n)$. Now use the inductive formula

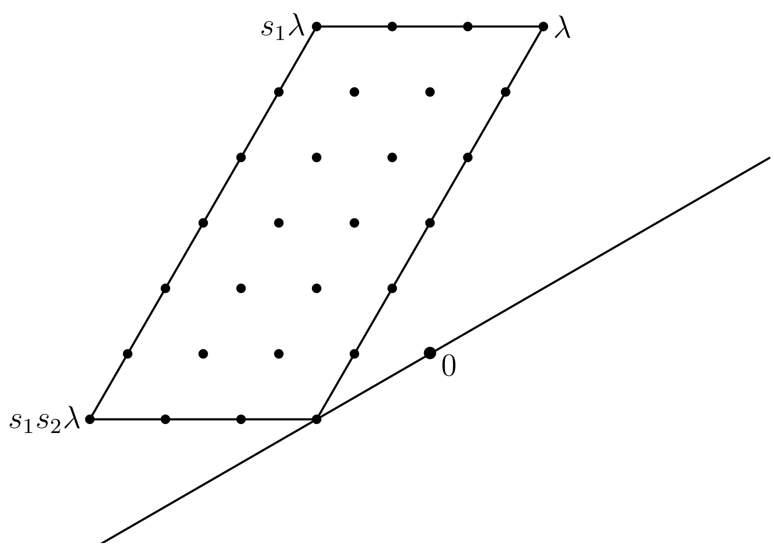
$$L(l_1+1,l_2) = L(1,0)\otimes L(l_1,l_2)\ominus L(l_1-1,l_2+1)\ominus L(l_1,l_2-1)$$

of Theorem 11.1. We leave the details to the reader. $\square$

The Lie algebra $\mathfrak{sl}_3$ has a subalgebra $\mathfrak{gl}_1\oplus\mathfrak{sl}_2$ consisting of the standard Cartan subalgebra $\mathfrak{h}$ and the two root spaces corresponding to $\pm\alpha_2$. For $\lambda = (l_1\varpi_1 + l_2\varpi_2)\in P_+$ the restriction of an irreducible representation $L(\lambda)$ of $\mathfrak{sl}_3$ to this subalgebra $\mathfrak{gl}_1\oplus\mathfrak{sl}_2$ decomposes as a multiplicity free direct sum of $(l_1+1)(l_2+1)$ irreducible representations with highest weights from the integral parallellogram

$$CH(\lambda,\lambda-l_1\alpha_1,\lambda-(l_1+l_2)\alpha_1-l_2\alpha_2,\lambda-l_2\alpha_1-l_2\alpha_2)\cap\{\lambda+Q\}$$

with $Q=\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$ the root lattice. As before the abbreviation CH stands for convex hull. Here is a figure for $\lambda = 3\varpi_1+6\varpi_2$.



We have only drawn the mirror corresponding to the root α_2 of the subalgebra. This picture is easily obtained from the previous one by representation theory of $\mathfrak{sl}_2$. The decomposition into unitary irreducible representations of the restriction of an irreducible unitary representation to a subalgebra is called a branching rule. Explicit branching rules are important for symmetry breaking.

Exercise 11.1. *Show that in the proof of Theorem 11.1 we have*

$$[L(\varpi_1) \otimes L(\lambda) : L(\mu)] = [L(\varpi_2) \otimes L(\mu) : L(\lambda)]$$

using Schur's Lemma.

Exercise 11.2. *Check that*

$$(3(l_1+1)(l_2+1) - l_1(l_2+2))(l_1+l_2+2)/2 - (l_1+1)l_2(l_1+l_2+1)/2$$

is equal to $(l_1+2)(l_2+1)(l_1+l_2+3)/2$.

Exercise 11.3. *Show that the tensor product decomposition*

$$L(1,0) \otimes L(l_1,l_2) = L(l_1+1,l_2) \oplus L(l_1-1,l_2+1) \oplus L(l_1,l_2-1)$$

remains valid for all $l_1, l_2 \in \mathbb{N}$ *if we denote* $L(-1,n) = L(n,-1) = 0$ *for* $n = -1, 0, 1, 2, \cdots$.

Exercise 11.4. *Show that for integers* $n_1 \geq n_2 \geq 1$ *we have the decomposition in irreducible components*

$$L(n_1,0) \otimes L(n_2,0) = \oplus L(n_1+n_2-2k,k)$$

with the direct sum over integers k *from* 0 *to* n_2*. Hint: Use representation theory of the* $\mathfrak{sl}_2$ *corresponding to the roots* $\pm\alpha_1$ *to show an inclusion* $\supset$*. Subsequently derive the equality* $=$ *using the dimension formula for* $L(\lambda)$*.*

12 Quarks

We have seen that for each pair $(l_1, l_2) \in \mathbb{N}^2$ there exists an irreducible representation $L(l_1, l_2)$ of $\mathfrak{sl}_3$ with highest weight $\lambda = l_1\varpi_1 + l_2\varpi_2 \in P_+$. Moreover the representation $L(l_1, l_2)$ is unitary for $\mathfrak{su}_3$. The irreducible representation $L(0,0)$ is the trivial one dimensional representation. The irreducible representation $L(1,0)$ is the standard three dimensional representation with weights $\varpi_1, \varpi_1 - \alpha_1, \varpi_1 - \alpha_1 - \alpha_2$. The irreducible representation $L(0,1)$ is the dual of the standard reprsentation, and has weights $\varpi_2, \varpi_2 - \alpha_2, \varpi_2 - \alpha_1 - \alpha_2$. The irreducible representation $L(1,1)$ is the adjoint representation of dimension eight. The six roots are the multiplicity one weights, and the zero weight has multiplicity two.

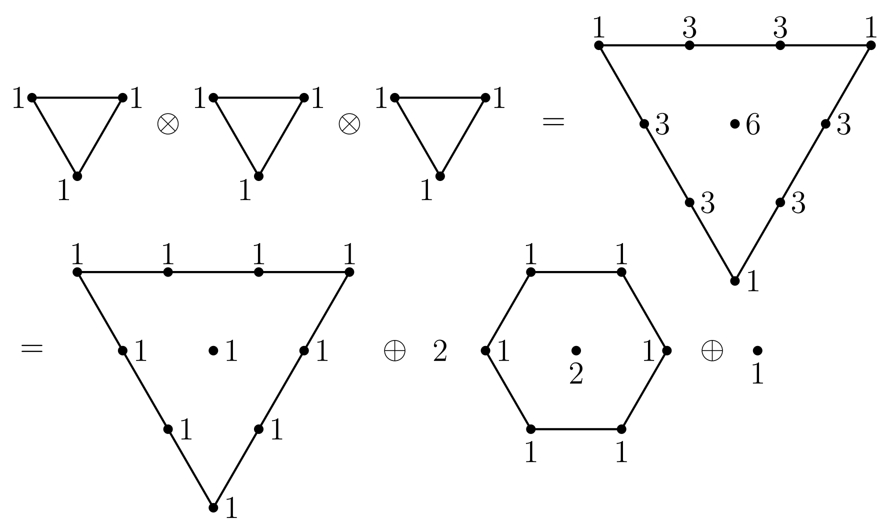
Theorem 12.1. *We have the decompositions in irreducible subrepresentations*

$$L(1,0)^{\otimes 3} = L(3,0) \oplus 2L(1,1) \oplus L(0,0)$$

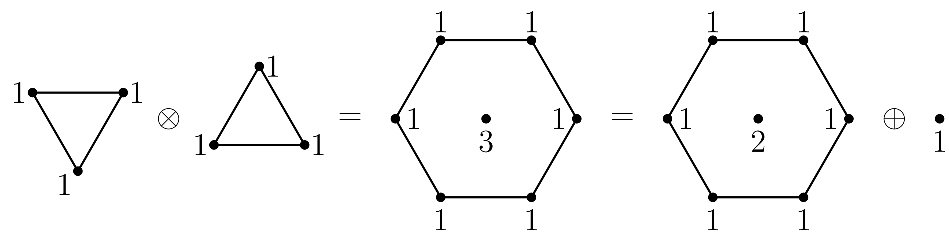
$$L(1,0) \otimes L(0,1) = L(1,1) \oplus L(0,0)$$

for the Lie algebra $\mathfrak{sl}_3$.

Proof. This can be derived using Theorem 11.1. The details of this derivation are left to the reader. A direct pictorial proof with weight multiplicity diagrams goes along the following lines. For the first relation we get



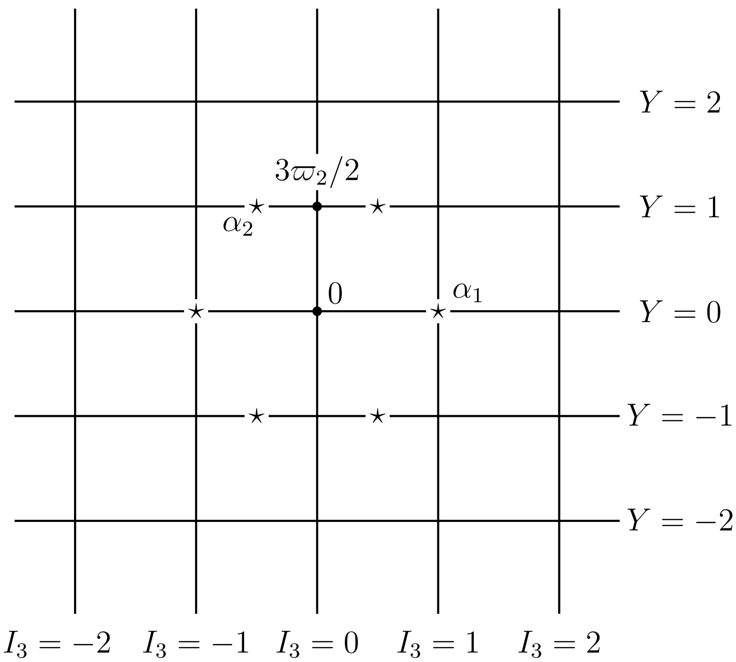
Likewise for the second relation



This completes the proof of the theorem. □

In order to go from the mathematics of weight diagrams of unitary irreducible representations of $\mathfrak{su}_3$ to the physics of elementary particles we have to switch coordinates. A weight vector $\lambda \in P$ is written as

$$\lambda = l_1\varpi_1 + l_2\varpi_2 = I_3\alpha_1 + Y(3\varpi_2/2) \ .$$



The numbers $l_1, l_2 \in \mathbb{Z}$ are the coefficients of λ relative to the basis of fundamental weights $\{\varpi_1, \varpi_2\}$. The numbers

$$I_3 \in \mathbb{Z}/2 \ , \ Y \in \mathbb{Z}/3$$

are the coefficients of the weight λ relative to the orthogonal basis $\{\alpha_1, 3\varpi_2/2\}$, and are called "isospin around the axis" and "hypercharge" respectively.

With our normalisation of the inner product the square norms of these basis vectors are given by $(\alpha_1, \alpha_1) = 2$ and $(3\varpi_2/2, 3\varpi_2/2) = 3/2$. Since $2\varpi_1 = \alpha_1 + \varpi_2$ we get

$$I_3 = l_1/2 \ , \ Y = (l_1 + 2l_2)/3$$

for isospin around the axis and hypercharge in terms of $l_1, l_2 \in \mathbb{Z}$. Note that for λ in the weight lattice $P = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$ the hypercharge Y takes integral values exactly on the root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$.

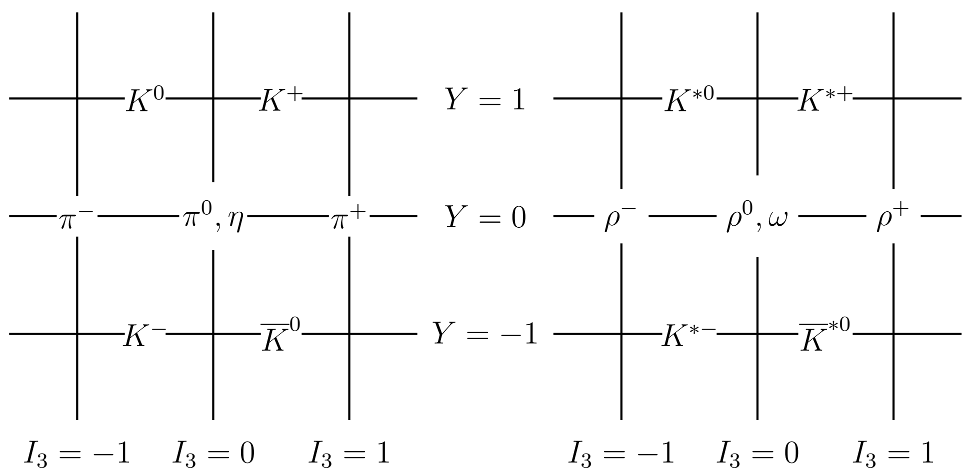
Since 1960 a whole zoo of new elementary particles were found in collider experiments. The lightest of these new particles were the spin 0 mesons

$$K^+, K^0, \pi^+, \pi^0, \eta, \pi^-, \overline{K}^0, K^-$$

and the spin 1 mesons

$$K^{*+}, K^{*0}, \rho^+, \rho^0, \omega, \rho^-, \overline{K}^{*0}, K^{*-}$$

with isospin around the axis and hypercharge according to the similar octet diagrams below.

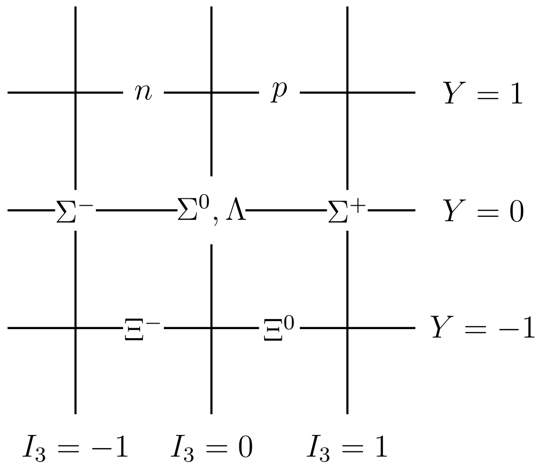


In addition there were found a singlet spin 0 meson η' and a singlet spin 1 meson ϕ. The word singlet means that they satisfied $I_3 = Y = 0$.

More strange particles were found with similar patterns. There are the heavier spin 1/2 baryons

$$p, n, \Sigma^+, \Sigma^0, \Lambda, \Sigma^-, \Xi^0, \Xi^-$$

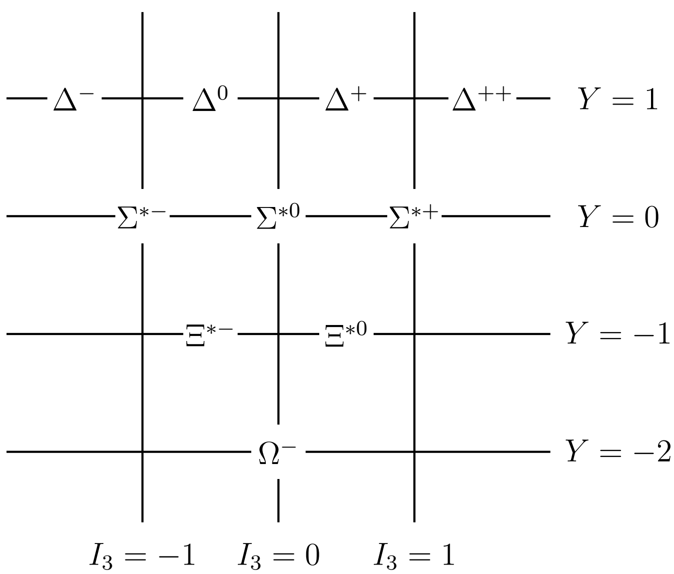
with isospin around the axis and hypercharge according to the octet diagram below.



In addition there were also found the heavier spin 3/2 baryons

$$\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}, \Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}, \Xi^{*0}, \Xi^{*-}, \Omega^{-}$$

which have isospin around the axis and hypercharge according to the decuplet diagram below.



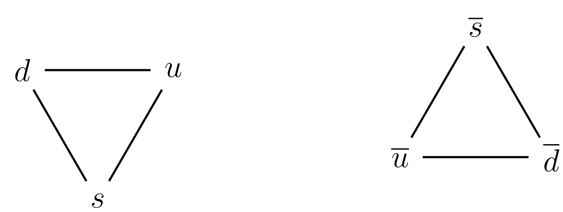
All these particles with the exception of the proton and neutron have a very short lifetime of a minuscule (yet observable) fraction of a second. The lifetime of the neutron is about 10^3 seconds, and the proton is stable.

When Gell-Mann first discussed the above relation between the weight diagrams for $\mathfrak{sl}_3$ and the singlet, octet and decuplet diagrams of mesons and baryons, the Ω^- had not yet been observed. On the basis of these weight diagrams Gell-Mann was able to predict not only the existence of Ω^-, but also its mass. It turned out that the experimentally found masses (in MeV) were

$$M_\Delta = 1230 \; , \; M_{\Sigma^\star} = 1385 \; , \; M_{\Xi^\star} = 1530$$

and so the spacings between the masses in the rows are nearly equal, and about 150. This led Gell-Mann to the prediction of the Ω^- with mass about 1680. Shortly after the Ω^- was found with a mass of 1672 MeV. This was convincing evidence on the $\mathfrak{su}_3$ symmetry behind the multiplet diagrams of the hadrons. The word hadron stands for either meson or baryon.

The next step taken by Gell-Mann was the hypothesis that all hadrons discussed so far are not yet elementary particles, but compositions of smaller objects, which he called quarks and antiquarks. Quarks and antiquarks are postulated to have spin 1/2. Presumably there are 3 quarks with the names up, down and strange, and abbreviated u, d, s. The corresponding antiquarks are denoted $\overline{u}, \overline{d}, \overline{s}$. The quarks have isospin around the axis and hypercharge according to the triplet weight diagram, and the antiquarks likewise according to the dual triplet weight diagram below.



So the u and $\overline{d}$ have $I_3 = 1/2$, the s and the $\overline{s}$ have $I_3 = 0$, and the d and $\overline{u}$ have $I_3 = -1/2$. Similarly the $\overline{s}$ has $Y = 2/3$, the u and d have $Y = 1/3$, the $\overline{u}$ and $\overline{d}$ have $Y = -1/3$, and the s has $Y = -2/3$.

The basic axiom underlying the quark formalism was postulated by Gell-Mann in the following way. The only way quarks and antiquarks can bind together into a bound hadron state, is either a triplet of quarks or a pair of a quark and an antiquark. A pair of quark and antiquark can bind into a meson of spin 0 or spin 1. A triplet of quarks can bind together into a baryon of spin 1/2 or spin 3/2. The tensor product decompositions of Theorem 12.1, and in particular the underlying pictures of weight diagrams in the proof of that theorem, can now be matched with the formation of hadrons out of quarks and antiquarks. The conclusion is the following matching for the spin 0 mesons

$$K^+, K^0, \pi^+, \pi^-, \overline{K}^0, K^-$$

$$u\overline{s}, d\overline{s}, u\overline{d}, d\overline{u}, s\overline{d}, s\overline{u}$$

and the spin 1 mesons

$$K^{*+}, K^{*0}, \rho^+, \rho^-, \overline{K}^{*0}, K^{*-}$$

$$u\overline{s}, d\overline{s}, u\overline{d}, d\overline{u}, s\overline{d}, s\overline{u}$$

for the quark-antiquark combinations. The spin 0 mesons π^0, η and η' are linear combinations of $u\overline{u}, d\overline{d}, s\overline{s}$, and the same is true for the spin 1 mesons ρ^0, ω and ϕ. Similarly we get for the spin 1/2 baryons

$$p, n, \Sigma^+, \Sigma^0, \Lambda, \Sigma^-, \Xi^0, \Xi^-$$

$$uud, udd, uus, uds(twice), dds, uss, dss$$

and for the spin 3/2 baryons

$$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-, \Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}, \Xi^{*0}, \Xi^{*-}, \Omega^-$$

$$uuu, uud, udd, ddd, uus, uds, dds, uss, dss, sss$$

for the triple quark combinations.

The combination of three u quarks into $\Delta^{++} = uuu$ conflicts the Pauli exclusion principle: At least two identical half integral spin particles can not occupy the same state. To fix this all quarks were postulated to have three colours, say red, green and blue. This breaks the symmetry, and so three u quarks, one in red, one in green and one in blue, can combine into a particle $\Delta^{++} = uuu$. The "visible" particles made out of three quarks are the white ones, being made up of all three colours. Likewise the "visible" particles made out of a quark and an antiquark of the same colour are the black ones, with no colour. Gell-Mann was awarded the Nobel prize of physics in 1969 for his work on elementary particles.

From the seventies on with experiments in stronger colliders much more new particles were found. The ultimate idea behind this zoo of particles was the hypothesis of three families of quarks, which go under the names up and down, strange and charm, bottom and top. The abreviations are u and d, s and c, b and t respectively. Their masses are in this order 5 and 10, 200 and 1300, 4500 and 175000 MeV. The heaviest top quark was observed at Fermilab (near Chicago) in 1995. All six quarks (and their antiquarks) have spin 1/2. The charges of the quarks u and d, c and s, t and b are 2/3 and $-1/3$, in this order. There are six more spin 1/2 elementary particles, the so called leptons. To each family of two quarks there are associated two leptons. To the u and d family there are associated the electron e and the e-neutrino ν_e, to the c and s family there are associated the muon μ and the μ-neutrino ν_μ, and to the t and b family there are associated the tau τ and the τ-neutrino ν_τ. The three families of two quarks, each with their associated two leptons, form the sector of the spin 1/2 elementary particles of the so called Standard Model.

For further reading about the Standard Model we refer the reader to two text books, one written by the mathematician Shlomo Sternberg, Group Theory and Physics from 1994, and the other written by the physicist Howard Georgi, Lie Algebras in Particle Physics from 1999. Many historical comments can be found in the beautiful book of Martinus Veldman, Facts and Mysteries in Elementary Particle Physics from 2003. All three books make a fairly complementary reading of the story.

13 Spherical harmonics

Fix a positive integer n. Let us denote by

$$P = P(\mathbb{R}^n) = \mathbb{C}[x_1, \cdots, x_n]$$

the commutative algebra of complex valued polynomial functions on $\mathbb{R}^n$. It is a representation space for the orthogonal Lie algebra $\mathfrak{so}_n$ acting by the linear first order differential operators

$$x_i\partial_j - x_j\partial_i$$

with ∂_i standing for $\partial/\partial x_i$. Of course, there are also compatible representations of the Lie groups O_n and SO_n on P.

Note that P is a unitary representation of $\mathfrak{so}_n$ and likewise of O_n and SO_n. Indeed P is obtained as a linear algebra construction from the standard (unitary) representation on the space of linear functions on $\mathbb{R}^n$. More concretely, the Hermitian inner product on P is obtained from

$$\langle p_1, p_2\rangle = \int p_1(x)\overline{p_2(x)}d\mu(x)$$

with μ the Euclidean volume element on the unit sphere S^{n-1} in $\mathbb{R}^n$ given by

$$S^{n-1} = \{x \in \mathbb{R}^n; x^2 = 1\} \ .$$

The Euler operator, the Lapace operator and the square length operator

$$E = \sum x_i\partial_i \ , \ \Delta = \sum \partial_i^2 \ , \ x^2 = \sum x_i^2$$

all act on P, and commute with the actions of $\mathfrak{so}_n$, O_n and SO_n. Let us write $P^m = ker(E - m)$ for the space of homogeneous polynomials of degree m. Clearly $P = \oplus P^m$.

Lemma 13.1. *As operators on P we have $[\Delta, x^2] = 4E + 2n$.*

Proof. Indeed we have

$$[\partial_i, x_i^2] = 2x_i$$

and therefore also

$$[\partial_i^2, x_i^2] = \partial_i 2x_i + 2x_i\partial_i = 4x_i\partial_i + 2$$

which gives the desired formula. □

Corollary 13.2. *The linear operators* $e = -\Delta/2, h = -(E + n/2), f = x^2/2$ *satisfy the usual commutation relations of* $\mathfrak{sl}_2$*.*

So we get a representation of $\mathfrak{so}_n \oplus \mathfrak{sl}_2$ on the polynomial space P. We denote $H = ker(\Delta)$ for the space of harmonic polynomials, and also $H^m = H \cap P^m$ for the space of harmonic polynomials of degree m. Because $\Delta : P^m \to P^{(m-2)}$ we get $H = \oplus H^m$.

Lemma 13.3. *We have* $H^m \cap x^2 P^{(m-2)} = 0$*.*

Proof. Write $p \in H^m, p \neq 0$ in the form $p = x^{2k} q$ with $k \in \mathbb{N}$ maximal. Let us assume that $k \geq 1$. Then we have (using the familiar relation $[e, f^k] = kf^{k-1}(h - k + 1)$ of $\mathfrak{sl}_2$)

$$\begin{aligned} 0 = \Delta p &= -2^{k+1} e f^k q = -2^{k+1}(f^k eq + kf^{k-1}(h - k + 1)q) \\ &= x^{2k} \Delta q + 2k(2m - 4k + n + 2k - 2)x^{2k-2} q \ . \end{aligned}$$

Since $m \geq 2k$ and $k, n \geq 1$ imply that $2k(2m-4k+n+2k-2) > 0$ we conclude that q is divisible by x^2. This contradicts the maximality assumption of $k \geq 1$, and therefore $k = 0$. $\square$

We have $\Delta : P^m \to P^{(m-2)}$ with kernel equal to H^m. Hence by standard linear algebra we get

$$dim(H^m) + dim(P^{(m-2)}) \geq dim(P^m) \ ,$$

while by the above lemma

$$dim(H^m) + dim(P^{(m-2)}) \leq dim(P^m) \ .$$

We therefore have proven the following result.

Theorem 13.4. *We have the direct sum decomposition* $P^m = H^m \oplus x^2 P^{(m-2)}$ *and the Laplace operator* $\Delta : P^m \to P^{(m-2)}$ *is onto.*

Corollary 13.5. *The dimension of* H^m *is given by the formula*

$$dim(H^m) = \frac{(m + n - 3)!}{(n - 2)!m!}(2m + n - 2)$$

and so for fixed dimension n *it is a polynomial in* m *of degree* $(n - 2)$ *with leading coefficient equal to* 2*.*

Proof. Indeed we have

$$dim(H^m) = \binom{m+n-1}{n-1} - \binom{m+n-3}{n-1} =$$
$$\frac{(m+n-1)!}{(n-1)!m!} - \frac{(m+n-3)!}{(n-1)!(m-2)!} =$$
$$\frac{(m+n-3)!}{(n-1)!m!}\{(m+n-1)(m+n-2) - m(m-1)\} =$$
$$\frac{(m+n-3)!}{(n-1)!m!}(n-1)(2m+n-2) = \frac{(m+n-3)!}{(n-2)!m!}(2m+n-2)\ .$$

□

Example 13.6. *For $n = 1$ we have $dim(H^0) = dim(H^1) = 1$ and $H^m = 0$ for $m \geq 2$. For $n = 2$ we have $dim(H^0) = 1$ while $dim(H^m) = 2$ for $m \geq 1$. In this case H^m is spanned by the functions $(x_1+ix_2)^m, (x_1-ix_2)^m$. For $n = 3$ we have $dim(H^m) = 2m + 1$, while for $n = 4$ we have $dim(H^m) = (m + 1)^2$. For $n \geq 3$ we have $dim(H^m) < dim(H^{(m+1)})$ for all $m \in \mathbb{N}$.*

Since the action of the orthogonal group O_n on P commutes with the action of the $\mathfrak{sl}_2$ spanned by e, h, f the space H^m of harmonic polynomials becomes a representation space for O_n.

Theorem 13.7. *If nonzero the space H^m is an irreducible representation of the group O_n.*

Let O_{n-1} be the stabilizer in O_n of the north pole $(0, 0, \cdots, 0, 1)$. Consider the space of zonal polynomials

$$Z(\mathbb{R}^n) = P(\mathbb{R}^n)^{\mathrm{O}_{n-1}} = \mathbb{C}[x_1^2 + \cdots + x_{n-1}^2, x_n]$$

on $\mathbb{R}^n$. Averaging over the compact group O_{n-1} gives a surjective mapping $P^m \to Z^m$, that commutes with $\Delta : Z^m \to Z^{(m-2)}$. Because

$$dim(Z^m) = 1\ (n = 1),\ dim(Z^m) = [m/2] + 1\ (n \geq 2)$$

we arrive at

$$dim(H^m \cap Z^m) = 1$$

at least if $H^m \neq 0$. In terms of representation theory this says that the space H^m, at least if nonzero, as a representation space of O_{n-1} contains the trivial representation with multiplicity one.

Lemma 13.8. *Every nonzero invariant subspace of H^m contains $H^m \cap Z^m$.*

Proof. Suppose $V \subset H^m$ is a nonzero subspace, that is invariant under O_n. If $0 \neq p \in V$ then there exists $a \in \mathrm{O}_n$ with $ap(0, 0, \cdots, 0, 1) \neq 0$. Averaging ap over O_{n-1} gives a zonal harmonic polynomial, which is nonzero at the point $(0, 0, \cdots, 0, 1)$, and therefore the one dimensional space $H^m \cap Z^m$ is contained in V. □

Now we can finish the proof of the above theorem. Suppose that H^m is a reducible representation of O_n, say $H^m = V_1 \oplus V_2$ is a direct sum of two nonzero invariant subspaces. Then

$$(H^m \cap Z^m) \subset (V_1 \cap V_2) = \{0\}$$

contradicting that $dim(H^m \cap Z^m) = 1$.

Remark 13.9. *Under the assumption $n \geq 3$ one can show with similar arguments that H^m is irreducible as a representation of SO_n or equivalently of $\mathfrak{so}_n$. However for $n = 2$ we get*

$$H^m = \mathbb{C}(x_1 + ix_2)^m \oplus \mathbb{C}(x_1 - ix_2)^m$$

as a direct sum of two irreducible representations for SO_2.

Remark 13.10. *From the above theorem it follows that the restriction from $\mathbb{R}^n$ to S^{n-1} induces an injection $H^m \hookrightarrow C^\infty(S^{n-1})$. Indeed, the kernel of this restriction map would be a proper invariant subspace of H^m, and hence it is zero. In spherical coordinates the Laplace operator $\Delta = \Delta_{\mathbb{R}^n}$ takes the form*

$$\Delta_{\mathbb{R}^n} = \frac{d^2}{dr^2} + (n-1)\frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_{S^{n-1}}$$

with $\Delta_{S^{n-1}}$ the Laplacian on the unit sphere S^{n-1}. A harmonic polynomial $h \in H^m$ gives via restriction to S^{n-1} an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $-m(m+n-2)$. The conclusion is that

$$C^\infty(S^{n-1}) \simeq \oplus H^m$$

which is the development of smooth functions on S^{n-1} in spherical harmonics. It is the spectral resolution of the Laplacian on S^{n-1}, and is a higher dimensional analogue of the expansion of a smooth function on S^1 in a Fourier series.

Exercise 13.1. *Show that as a representation of* $\mathfrak{so}_n \oplus \mathfrak{sl}_2$ *we have the decomposition*

$$P = \oplus[H^m \otimes M(-(m+n/2))]$$

with $M(s)$ *the Verma representation of* $\mathfrak{sl}_2$ *with highest weight* s*.*

Exercise 13.2. *Check that the restriction of a harmonic polynomial* $h \in H^m$ *to the unit sphere* S^{n-1} *is an eigenfunction of the spherical Laplacian* $\Delta_{S^{n-1}}$ *with eigenvalue* $-m(m+n-2)$*.*

Exercise 13.3. *There exists a unique zonal harmonic polynomial, that is homogeneous of degree* m *and takes the value* 1 *at the point* $(0,0,\cdots,0,1)$*. Its restriction to the unit sphere* S^{n-1} *is a polynomial function of the height* $z = x_n$ *only, and as such is called the Gegenbauer polynomial of degree* m*. Show that the Gegenbauer polynomials are orthogonal on the interval* $[-1,1]$ *with respect to the measure* $(1-z^2)^{(n-3)/2}dz$*. In case* $n=3$ *the Gegenbauer polynomials are also called the Legendre polynomials.*

14 Spherical representations of the Lorentz group

For a homogeneous polynomial $p \in P^m$ of degree m there exist homogeneous polynomials $p_+ \in H^m$ and $p_- \in P^{(m-2)}$ with

$$p = p_+ + x^2 p_-$$

and such a decomposition is unique. Observe that this decomposition commutes with the representation of the Lie group O_n and its Lie algebra $\mathfrak{so}_n$ on the vector space P^m. This is the content of Theorem 13.4.

Lemma 14.1. *If $p \in P^m$ with $\Delta^2(p) = 0$ then $p_- = \frac{\Delta(p)}{4(m-2)+2n}$.*

Proof. Suppose $p \in P^m$ with $\Delta^2(p) = 0$. Let us compute

$$\Delta(x^2\Delta(p)) = (4E + 2n)\Delta(p) + x^2\Delta^2(p) = (4(m-2) + 2n)\Delta(p) \ ,$$

and therefore

$$\Delta(p - \frac{x^2\Delta(p)}{4(m-2)+2n}) = \Delta(p) - \Delta(p) = 0 \ .$$

Hence $p = p_+ + x^2 p_-$ with $p_+ \in H^m$ and $p_- = \Delta(p)/(4(m-2)+2n)$. $\square$

Corollary 14.2. *If $h \in H^m$ then $(x_i h)_- = (2m - 2 + n)^{-1}\partial_i(h) \in H^{(m-1)}$.*

Proof. Apply the above lemma with $p = x_i h$. Let us compute

$$\Delta(x_i h) = [\Delta, x_i](h) + x_i\Delta(h) = 2\partial_i(h)$$

which in turn implies that $\Delta^2(x_i h) = 2\Delta(\partial_i(h)) = 2\partial_i(\Delta(h)) = 0$, as required. Moreover we see that

$$(x_i h)_- = (4(m-1) + 2n)^{-1}\Delta(x_i h) = (2m - 2 + n)^{-1}\partial_i(h)$$

using the above lemma. Finally note that $\partial_i(h)$ is harmonic if h is harmonic. Indeed the Laplacian Δ and the derivative ∂_i commute, as do all constant coefficients linear differential operators. $\square$

Lemma 14.3. *For $h \in H^m, k \in H^{(m+1)}$ we get $\langle (x_i h)_+, k\rangle = \langle h, (x_i k)_-\rangle$ with $\langle\cdot,\cdot\rangle$ the Hermitian inner product on P as defined in the previous section (namely by integrating over the unit sphere).*

Proof. For $f, g \in C^\infty(S^{n-1}) \simeq \oplus H^m$ we have

$$\langle f, g\rangle = \int_{S^{n-1}} f(x)\overline{g(x)}d\mu(x)$$

with μ the Euclidean measure on S^{n-1}. Moreover

$$\langle x_i f, g\rangle = \langle f, x_i g\rangle$$

since x_i is real valued on S^{n-1}. Now take for f the restriction of $h \in H^m$ and for g the restriction of $k \in H^{(m+1)}$ to the unit sphere S^{n-1}. Then we arrive at

$$\langle (x_i h)_+, k\rangle = \langle x_i h, k\rangle = \langle h, x_i k\rangle = \langle h, (x_i k)_-\rangle$$

since the restriction to S^{n-1} of two homogeneous harmonic polynomials of different degrees are orthogonal. □

Corollary 14.4. *The linear maps $T^m_\pm : H^1 \otimes H^m \to H^{(m\pm1)}$ defined by*

$$T^m_\pm(l \otimes h) = (lh)_\pm$$

are intertwining operators for the action of O_n and $\mathfrak{so}_n$. Moreover $T^m_\pm$ is surjective for all $m \in \mathbb{N}$.

Proof. Clearly $T^m_\pm : H^1 \otimes H^m \to H^{(m\pm1)}$ are intertwining operators because the decomposition in Theorem13.4 commutes with the action of O_n and $\mathfrak{so}_n$. Since the range of the intertwining operators $T^m_\pm$ is an irreducible representation (for $(m\pm1) \in \mathbb{N}$) the surjectivety is equivalent with being a nonzero map, by Schur's Lemma. But being a nonzero map under the given conditions is easily checked, and left to the reader. □

This ends our discussion of the theory of spherical harmonics on $\mathbb{R}^n$. We shall now apply this theory for the construction of a one parameter family of representation of the Lorentz group and Lorentz algebra. Subsequently we shall analyze the question of unitarity for this family, by infinitesimal methods (so using the Lie algebra rather than the Lie group).

Consider $\mathbb{R}^{n,1} = \{(x, y); x = (x_1, \cdots, x_n) \in \mathbb{R}^n, y \in \mathbb{R}\}$, so as a vector space it is just $\mathbb{R}^{n+1}$, but with Lorentz square length

$$(x, y)^2 = x^2 - y^2$$

with x^2 and y^2 the Euclidean square lengths on $\mathbb{R}^n$ and $\mathbb{R}$ respectively. The Lorentz group $\mathrm{G} = \mathrm{O}_{n,1}(\mathbb{R})$ has Lie algebra $\mathfrak{g} = \mathfrak{so}_{n,1}$ with decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{k} = \mathfrak{so}_n$ the Lie algebra of $\mathrm{K} = \mathrm{O}_n(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})$, and $\mathfrak{p}$ the orthogonal complement with respect to the trace form on $\mathfrak{g}$ derived from the standard representation of $\mathfrak{g}$ on $\mathbb{C}^{n,1}$. The group K is a so called maximal compact subgroup of G. The real Lorentz algebra $\mathfrak{so}_{n,1}(\mathbb{R})$ consists of the real matrices in $\mathfrak{so}_{n,1}$, and is spanned over $\mathbb{R}$ by the vector fields

$$x_i\partial_j - x_j\partial_i \ , \ x_i\partial + y\partial_i$$

in $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Here ∂_i stands for $\partial/\partial x_i$, while ∂ stands for $\partial/\partial y$.

The forward light cone L^+ is defined by

$$L^+ = \{(x,y) \in \mathbb{R}^{n,1}; x^2 = y^2, y > 0\} \ .$$

Its symmetry group $G^+ = \mathrm{O}^+_{n,1}(\mathbb{R})$ is an index 2 subgroup of $\mathrm{O}_{n,1}(\mathbb{R})$. Note that G^+ has still two connected components, which are distinguished by the sign of the determinant, and so $G^\circ = \mathrm{SO}^+_{n,1}(\mathbb{R})$ is the connected symmetry group of L^+ with Lie algebra $\mathfrak{g} = \mathfrak{so}_{n,1} \simeq \mathfrak{so}_{n+1}$. The next lemma is obvious.

Lemma 14.5. *The map $L^+ \ni (x,y) \mapsto (x/y, y) \in S^{n-1} \times \mathbb{R}_+$ induces a diffeomorphism (a bijective map, both smooth and the inverse smooth) of L^+ onto $S^{n-1} \times \mathbb{R}_+$.*

Definition 14.6. *Let $s \in \mathbb{C}$ be a complex parameter. The vector space of smooth complex valued functions on the forward light cone L^+ that are homogeneous of degree $-s$ is denoted by $S_s = S_s(L^+)$, and viewed as a representation space of the proper Lorentz group $O^\circ_{n,1}$ and the corresponding Lorentz Lie algebra $\mathfrak{so}_{n,1}$, is called the spherical principal series representation with spectral parameter s. Note that by the above lemma and the theory of spherical harmonics we have the isomorphism*

$$S_s = S_s(L^+) \simeq \oplus H^m(\mathbb{R}^n) y^{-(s+m)} \ .$$

Note that the restriction of the spherical principal series representation to the Lie subalgebra $\mathfrak{k} = \mathfrak{so}_n$ of the Lorentz algebra $\mathfrak{g} = \mathfrak{so}_{n,1}$ decomposes in a multiplicity free way as

$$S_s|_{\mathfrak{k}} = \oplus H^m(\mathbb{R}^n) \ .$$

In particular the action of $\mathfrak{k}$ on S_s is independent of the parameter $s \in \mathbb{C}$. Indeed the Lie algebra $\mathfrak{k}$ has basis $x_i\partial_j - x_j\partial_i$ and therefore only acts on $h \in H^m(\mathbb{R}^n)$ and not on the factor $y^{-(s+m)}$. However the action of $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ does depend on s, since $\mathfrak{p}$ has basis $x_i\partial + y\partial_i$ with $\partial = \partial/\partial y$.

Theorem 14.7. *The action of $\mathfrak{p}$ on the spherical principal series S_s is given by*

$$(x_i\partial + y\partial_i)(h(x)y^{-(s+m)}) =$$
$$-(m+s)(x_ih(x))_+y^{-(s+m+1)} + (m-s+n-2)(x_ih(x))_-y^{-(s+m-1)} ,$$

or written in an equivalent way we find for all $z \in \mathfrak{p}$ that

$$z(h(x)y^{-(s+m)}) =$$
$$-(m+s)T_+^m(z \otimes h(x))y^{-(s+m+1)} + (m-s+n-2)T_-^m(z \otimes h(x))y^{-(s+m-1)} .$$

This theorem is an immediate consequence of Corollary 14.2. Note that the dependence of the spherical principal series representation of $\mathfrak{g} = \mathfrak{so}_{n,1}$ on the parameter $s \in \mathbb{C}$ is explicit, and in fact quite manageable.

Corollary 14.8. *The spherical principal series representation S_s of $\mathfrak{so}_{n,1}$ is reducible if $s = -m$ or $s = (m+n-2)$ for some $m \in \mathbb{N}$, and irreducible otherwise. For $s = -m \in -\mathbb{N}$ the representation S_s has a finite dimensional subrepresentation equal to $H^m(\mathbb{R}^{n,1})$, while for $s = (m+n-2)$ with $m \in \mathbb{N}$ we get an infinite dimensional subrepresentation with a finite dimensional quotient isomorphic to $H^m(\mathbb{R}^{n,1})$.*

For which values of the parameter $s \in \mathbb{C}$ is the spherical principal series S_s as a representation of the real Lorentz algebra $\mathfrak{so}_{n,1}(\mathbb{R})$ unitarizable? The word unitarizable means unitary with respect to some Hermitian inner product. Suppose there exists a Hermitian from

$$H_s = \oplus\, c_m(s)\langle\cdot,\cdot\rangle_m$$

with $\langle\cdot,\cdot\rangle_m$ the standard Hermitian form on $H^m(\mathbb{R}^n)$ which is unitary for the representation of $\mathfrak{k} = \mathfrak{so}_n$, and with $c_m(s)$ suitable real numbers depending on $m \in \mathbb{N}, s \in \mathbb{C}$.

The form H_s is invariant for $\mathfrak{so}_{n,1}(\mathbb{R})$ if and only if for all $z = x_i\partial + y\partial_i$ and for all $h \in H^m, k \in H^{(m+1)}$ we have

$$H_s(z(h(x)y^{-(s+m)}), k(x)y^{-(s+m+1)}) + H_s(h(x)y^{-(s+m)}, z(k(x)y^{-(s+m+1)})) = 0 ,$$

which in turn amounts to

$$c_{m+1}(s)(m+s)\langle (x_i h(x))_+, k(x)\rangle + c_m(s)(m-\overline{s}+n-1)\langle h(x), (x_i k(x))_-\rangle = 0\ .$$

Using Lemma 14.3 we arrive at the following result.

Theorem 14.9. *The spherical principal series S_s is a irreducible Hermitian representation of the real Lorentz algebra $\mathfrak{so}_{n,1}(\mathbb{R})$ if and only if the equation*

$$c_{m+1}(s)(m+s) + c_m(s)(m-\overline{s}+n-1) = 0$$

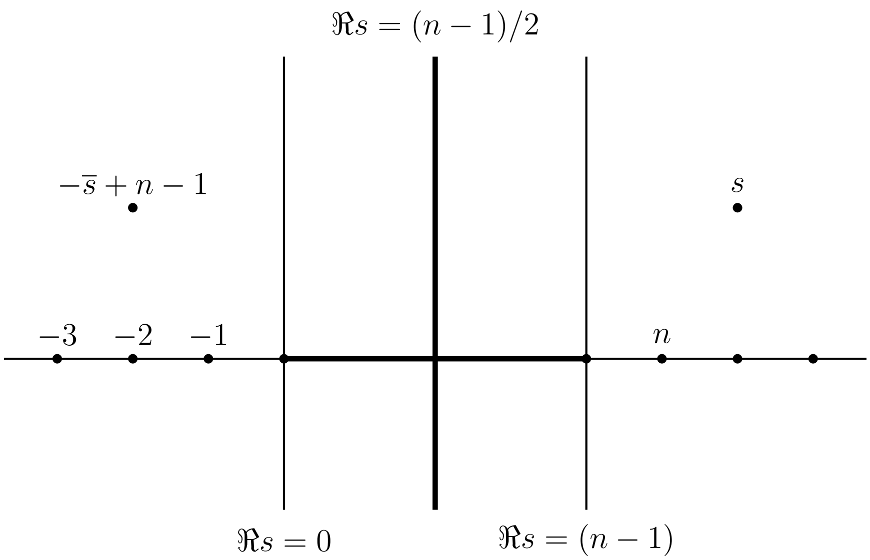
has a solution $c_m(s) \in \mathbb{R}^\times = \{c \in \mathbb{R}; c \neq 0\}$ for all $m \in \mathbb{N}$.

Corollary 14.10. *The spherical principal series S_s is a irreducible unitary representation of the real Lorentz algebra $\mathfrak{so}_{n,1}(\mathbb{R})$ if and only if*

$$c_{m+1}(s)(m+s) + c_m(s)(m-\overline{s}+n-1) = 0$$

has a solution $c_m(s) \in \mathbb{R}_+ = \{c \in \mathbb{R}; c > 0\}$ for all $m \in \mathbb{N}$. In turn this condition is equivalent to $\Re(s) = (n-1)/2$ or $0 < s < (n-1)$.

Proof. Observe that s and $(-\overline{s}+n-1)$ are mirror images of one another in the line $\Re(s) = (n-1)/2$. Hence the ratio $(m+s)/(m-\overline{s}+n-1)$ is positive for all $m \in \mathbb{N}$ if and only either this ratio equals 1 and $\Re(s) = (n-1)/2$ or this ratio lies in $\mathbb{R}_+ - \{1\}$ and $0 < s < (n-1), s \neq (n-1)/2$. This easily follows since the points s and $(-\overline{s}+n-1)$ in the complex plane are mirror images with the line $\Re(s) = (n-1)/2$ as mirror. $\square$



For $\Re(s) = (n-1)/2$ the unitary spherical principal series S_s is called the tempered spherical principal series, while for $0 < s < (n-1), s \neq (n-1)/2$ the unitary spherical principal series S_s is called the complementary spherical principal series.

The methods of this section can be extended to some other groups, like the complex Lorentz group $\mathrm{U}_{n,1}$. For further details we refer to the article by Roger Howe and Eng-Chye Tan in the Bulletin of the American Mathematical Society, Volume 28, January 1993.

Exercise 14.1. *In the notation of Exercise 9.8 show that the quadratic expression (called the Casimir operator of $\mathfrak{so}_{3,1}$)*

$$C = (K_1^2 + K_2^2 + K_3^2) - (L_1^2 + L_2^2 + L_3^2)$$

lies in the center of the universal enveloping algebra $U\mathfrak{so}_{3,1}$ of the Lorentz algebra. In turn this implies by Schur's Lemma that C acts in an irreducible representation space by some scalar. Check that the scalar by which C acts in the spherical principal series S_s equals $-s(s-2)$. Hint: We have

$$L_1 = i(x_2\partial_3 - x_3\partial_2), \cdots , \; K_1 = i(x_1\partial - y\partial_1), \cdots$$

in the usual cyclic notation. Now check that $C\varphi_s = -s(s-2)\varphi_s$ with $\varphi_s(x,y) = y^{-s}$ as vector in S_s. The vector $\varphi_s \in S_s$ is called spherical because $(L_1^2 + L_2^2 + L_3^2)\varphi_s = 0$. Finally observe that $(U\mathfrak{so}_{3,1})\varphi_s$ equals all of S_s.

Exercise 14.2. *In the notation of the previous exercise let us replace*

$$L_i \mapsto \hbar L_i, K_i \mapsto \hbar\sqrt{2mE}K_i$$

with $\hbar, m, E$ all three positive parameters. Check that the new elements L_i, K_i satisfy the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \; , \; [L_i, K_j] = i\hbar\epsilon_{ijk}K_k \; , \; [K_i, K_j] = -i\hbar\epsilon_{ijk}(2mE)L_k$$

which we recognize from our discussion of the quantization of the Kepler problem (equivalently from our discussion of the computation of the spectrum of the hydrogen atom). Using the previous exercise show that the operator

$$(2mE)^{-1}K^2 - L^2$$

acts on the spherical principal series S_s by multiplication with the scalar $-\hbar^2 s(s-2)$.

Exercise 14.3. *In our discussion of the quantization of the Kepler problem we have seen the key identity*

$$K^2 = 2mH(L^2 + \hbar^2) + k^2m^2 \ ,$$

which we can also write as

$$H = k^2m/2(K^2/(2mH) - L^2 - \hbar^2) \ .$$

Show that for positive energie $H = E > 0$ the coherent eigenstate of the hydrogen atom is degenerated according to the spherical principal series S_s of $\mathfrak{so}_{3,1}$ with energy $E = E_s$ given by

$$E = E_s = -k^2m/2(s-1)^2\hbar^2$$

which in turn implies that the tempered spherical principal series occurs in the continuous spectrum of the hydrogen atom, while the complementary spherical principal series is absent. What is the meaning for physics of the complementary series?

Exercise 14.4. *Analyse in case of reducibility of the spherical principal series S_s the unitarity of the sub and quotient representations of S_s.*