Moduli of real curves of genus three

Proefschrift

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. dr. Th. L. M. Engelen,
volgens besluit van het college van decanen
in het openbaar te verdedigen op donderdag 12 februari 2015
om 12:30 uur precies

door

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geboren op 19 juli 1985
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The research in this thesis was supported by NWO grant 613.000.909: "Reality questions for some period mappings".

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Chapter 1

Introduction

This is a thesis on *real* algebraic geometry. This means that we study objects that are defined by polynomial equations with coefficients in the field of real numbers. Specifically we will be concerned with the study of real quartic curves; a subject with a rich 19th century history. A gem from this period is the topological classification of smooth real plane quartic curves by the Danish mathematician Zeuthen [46] in 1874. The set of real points of such a curve consists of at most four ovals in the real projective plane and the six possible configurations are shown in Figure 1.1. It was proved by Klein [24] that each of them determines a connected component in the space of smooth real plane quartic curves. Consequently the moduli space of such quartics consists of six connected components.

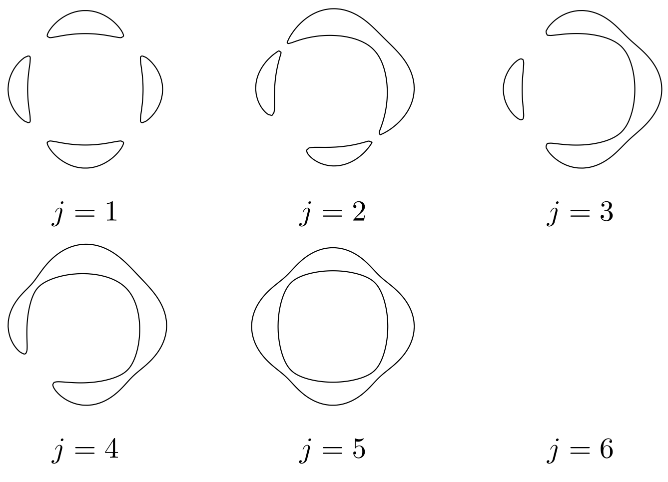


Figure 1.1: The six topological types of smooth real plane quartic curves by deforming the union of two ellipses.

In this thesis we study the moduli space of smooth real plane quartics and other moduli spaces of real objects using modern techniques of reflections in algebraic geometry. An excellent survey article on this material was written by Dolgachev [18]. It touches on many of the subjects we encounter in this thesis but does not discuss questions of reality. For this we refer to the survey written by Degtyarev and Kharlamov [14] on the topological properties of real algebraic varieties. We now give an overview of this thesis with an emphasis on the new results. We start with the final chapter and go backwards; this corresponds to the order in which this thesis was conceived.

Chapter 6. In this chapter we consider a variation on the classification by Zeuthen of Figure 1.1. We study smooth real plane quartic curves equipped with a real point such that the tangent line at that point intersects the curve in two other distinct points. These two points can be both real or form a pair of complex conjugate points. Furthermore since by definition such a curve is smooth and contains a real point it consists of at least one oval. We will prove that the moduli space of these pointed curves consists of twenty connected components. For each of these components a representative curve along with the tangent line at the chosen point is shown in Figure 1.2. This classification is more subtle then the topological one; the second and fourth picture from the first row are topologically equivalent but represent different components in the moduli space. At this point we would like to mention Appendix A of [13] in which similar real moduli and deformation problems are discussed.

We start by extending work of Looijenga [29], [30] on moduli spaces of complex del Pezzo pairs to the real setting. To formulate this result we introduce some notation. Let Q be a root lattice of type E_7 and define the complex torus $\mathbb{T} = \mathrm{Hom}(Q, \mathbb{C}^*)$. The Weyl group W of type E_7 acts by reflections on Q and thus on $\mathbb{T}$. We denote by $\mathbb{T}^\circ$ the complement of the toric mirrors for this action.

Theorem 1.0.1. *Let $(\mathcal{Q}_1^\circ)^\mathbb{R}$ be the moduli space of pairs (C, p) with C a smooth real plane quartic curve and $p \in C(\mathbb{R})$ such that T_pC intersects C in two other distinct points. There is an isomorphism of real orbifolds:*

$$(\mathcal{Q}_1^\circ)^\mathbb{R} \longrightarrow (W \backslash \mathbb{T}^\circ)(\mathbb{R}) \tag{1.1}$$

where the right hand side consists of all W-orbits of $t \in \mathbb{T}^\circ$ such that $w \cdot t = \bar{t}$ for some $w \in W$.

The isomorphism of Equation 1.1 is similar to a period map. It allows us to study the moduli space $(\mathcal{Q}_1^\circ)^\mathbb{R}$ using the theory of root systems and involutions in Weyl groups on the right hand side. We study the real points of such torus quotients and their connected components for general root system of type ADE. For type E_7 we then prove there are twenty connected components and relate them to the pictures of Figure 1.2.

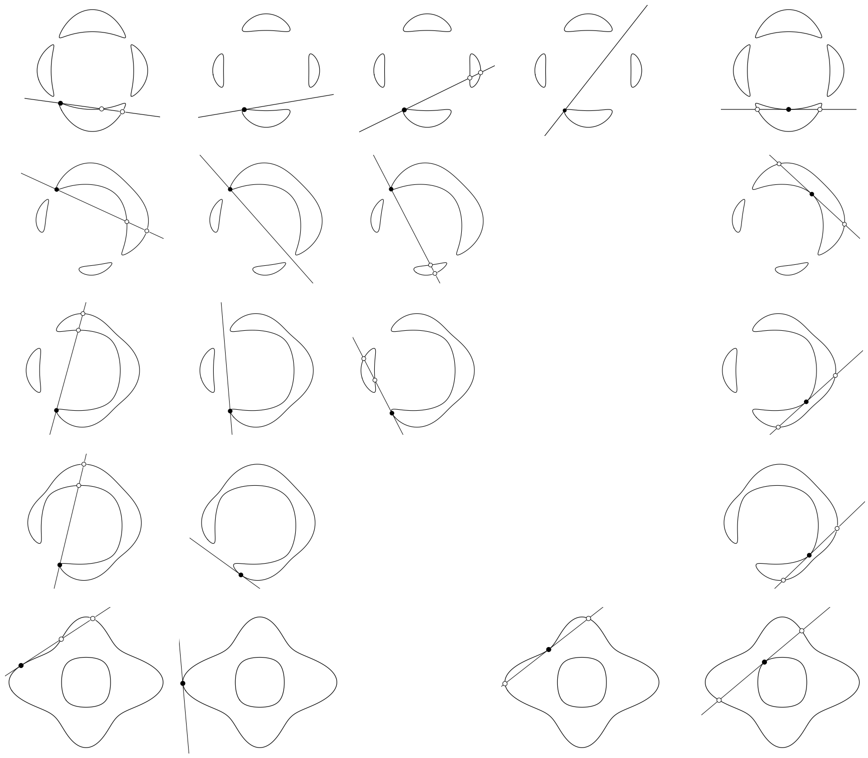


Figure 1.2: Representatives for the 20 connected components of the moduli space $(\mathcal{Q}_1^\circ)^{\mathbb{R}}$.

Chapter 5. Recently there has been a great deal of progress in the construction of period maps from moduli spaces to ball quotients. This offers a new approach to the study of questions of reality for these moduli spaces. The main example of this in the literature is the work of Allcock, Carlson and Toledo [2], [5] on the moduli space of cubic surfaces. In the first article they construct a period map from this moduli space to a ball quotient of dimension four and the question of reality for this period map is studied in the second article. One of the components of the moduli space of smooth cubic surfaces was previously studied by Yoshida [45] using the period map.

For the moduli space of smooth plane quartic curves there is also a period map due to Kondo [26]. It maps this moduli space to a ball quotient of dimension six. We will study the question of reality for this period map in Chapter 5. In particular we prove that each of the six components of the moduli space of smooth real plane quartic curves is isomorphic to a real ball quotient. In order to formulate this more precisely we introduce some notation on Gaussian lattices. Let $\mathcal{G} = \mathbb{Z}[i]$ be the Gaussian integers and let $\Lambda_{1,6}$ be the Gaussian lattice $\mathcal{G}^7$ equipped with the Hermitian form $h(\cdot, \cdot)$ defined by the matrix:

$$H = \begin{pmatrix} -2 & 1+i \\ 1-i & -2 \end{pmatrix}^{\oplus 3} \oplus (2). \tag{1.2}$$

We denote the group of unitary transformations of this lattice by $\Gamma = U(\Lambda)$. The lattice $\Lambda_{1,6}$ has hyperbolic signature $(1,6)$ and determines a complex ball of dimension six by the expression:

$$\mathbb{B}_6 = \mathbb{P}\{z \in \Lambda_{1,6} \otimes_{\mathcal{G}} \mathbb{C}\ ;\ h(z,z) > 0\}. \tag{1.3}$$

A root is an element $r \in \Lambda_{1,6}$ such that $h(r,r) = -2$ and for every root r we define its root mirror to be the hypersurface $H_r = \{z \in \mathbb{B}_6\ ;\ h(r,z) = 0\}$. We denote by $\mathbb{B}_6^\circ$ the complement in $\mathbb{B}_6$ of all root mirrors.

Theorem 1.0.2. *There are six projective classes of anti-unitary involutions: $[\chi_j]$ with $j = 1, \ldots, 6$ of the lattice $\Lambda_{1,6}$ up to conjugation by $P\Gamma$. Each of them determines a real ball $\mathbb{B}_6^{\chi_j} \subset \mathbb{B}_6$ and there are isomorphisms of real orbifolds:*

$$\mathcal{Q}_j^{\mathbb{R}} \longrightarrow P\Gamma^{\chi_j} \backslash \left(\mathbb{B}_6^{\chi_j}\right)^\circ. \tag{1.4}$$

The group $P\Gamma^{\chi_j}$ is the stabilizer of the real ball $\mathbb{B}_6^{\chi_j}$ in $P\Gamma$. It is an arithmetic subgroup of $PO(1,6)$ for each $j = 1, \ldots, 6$.

In fact we obtain more information on the groups $P\Gamma^{\chi_j}$ for $j = 1, \ldots, 6$. They are subgroups of finite index of hyperbolic Coxeter groups of finite co-volume and we determine the Coxeter diagrams for these latter groups using Vinberg's algorithm. For the group $P\Gamma^{\chi_1}$ that corresponds to the component $\mathcal{Q}_1^{\mathbb{R}}$ of maximal quartic curves we obtain a more explicit description: it is the semi-direct product of a hyperbolic Coxeter group of finite co-volume by its group of diagram automorphisms. The fundamental domain of this group is a hyperbolic Coxeter polytope C_6 whose Coxeter diagram is shown in Figure 1.3. Its group of diagram automorphisms is the symmetric group S_4. The locus of fixed points in C_6 of this group is a hyperbolic line segment. It corresponds to a one-parameter family of smooth real quartic curves that consist of four ovals with S_4-symmetry and we determine this family explicitly.

Chapter 4. So far we have only discussed smooth plane quartic curves. These are curves of genus three as implied by the title of this thesis. There is however another class of genus three curves: the hyperelliptic ones. A smooth hyperelliptic curve is determined by a set of distinct unordered points on the projective line. The moduli spaces of such point sets were studied by Deligne and Mostow [15]. They are isomorphic to complex ball quotients if and only if the number of points equals $4, 5, 6, 8$ or 12. The question of reality for the case of 5 points has been studied by Yoshida and Apéry [6] and for the case of 6 points by Yoshida [44] and Allcock, Carlson and Toledo [3], [4].

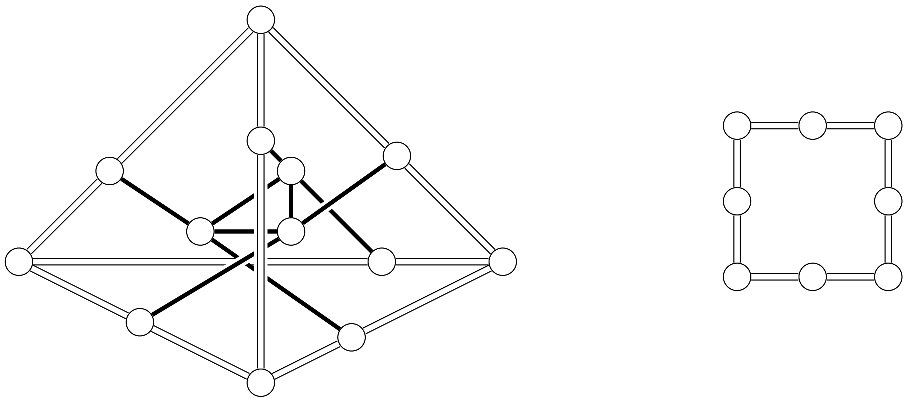


Figure 1.3: The Coxeter diagram of the hyperbolic Coxeter polytope C_6 (left) and the wall that corresponds to C_5 (right). The nodes represent the walls and a double edge connecting two nodes means their walls meet at an angle of $\pi/4$, a thick edge means they are parallel and no edge means they are orthogonal.

A smooth hyperelliptic curve of genus three is determined by eight points on the complex projective line. The question of reality in this case has been studied by Chu [12]. He obtains a result similar to Theorem 1.0.2 using a period map of Deligne-Mostow that maps the moduli space of eight unordered points on $\mathbb{P}^1(\mathbb{C})$ to a complex ball quotient of dimension five. We review this work in Chapter 4 and supplement it by giving an explicit description of the group that corresponds to the real ball quotient of the moduli space of eight points on $\mathbb{P}^1(\mathbb{R})$. This group is the semi-direct product of a hyperbolic Coxeter group of finite co-volume by its group of diagram automorphisms. The Coxeter diagram of its fundamental domain C_5 is shown in Figure 1.3 and its group of diagram automorphisms is the dihedral group D_8. There is a unique central point in C_5 that is invariant under D_8 and has equal distance to all the mirrors. This point corresponds to the configuration of eight-roots of unity on the unit circle in the complex plane.

Chapter 3. The complex ball quotients that describe the two moduli spaces of smooth genus three curves (hyperelliptic and non-hyperelliptic) arise from lattices of hyperbolic signature that are defined over the Gaussian integers. We study such Gaussian lattices in Chapter 3 with a special emphasis on these two examples. The study of questions of reality for moduli spaces then corresponds to the study of anti-unitary involutions on these lattices. This is complicated by the fact that the Hermitian forms on these Gaussian lattices are not unimodular as can be seen from Equation 1.2.

Chapter 2. Finally, or firstly, we discuss the necessary preliminaries and notations in Chapter 2.

Chapter 2

Preliminaries

2.1 Introduction

In this preliminary chapter we review some topics and introduce notations that will be used throughout this thesis. For the most part the subjects treated here are well known and we give references to the literature for proofs. We will also prove some preliminary results and lemma's on examples that appear in later chapters.

2.2 Lattices

A lattice is a pair $(L, (\cdot,\cdot))$ with L a free $\mathbb{Z}$-module of finite rank r and $(\cdot,\cdot)$ a non-degenerate, symmetric bilinear form on L taking values in $\mathbb{Z}$. This bilinear form extends naturally to a bilinear form $(\cdot,\cdot)_{\mathbb{Q}}$ on the rational vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and its signature (r_+, r_-) is called the signature of L. The dual of L is the group $L^\vee = \mathrm{Hom}(L, \mathbb{Z})$ and the lattice L is naturally embedded in $L^\vee$ by the assignment $x \mapsto (x, \cdot)$. The group $L^\vee$ is naturally embedded in the vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ by the identification:

$$L^\vee = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q}\ ;\ (x, y)_{\mathbb{Q}} \in \mathbb{Z} \text{ for all } y \in L\}.$$

Note that the induced bilinear form on $L^\vee$ need not be integer valued but by a small abuse of language we still call $L^\vee$ a lattice. An isomorphism between lattices L_1 and L_2 is a group isomorphism $\phi : L_1 \to L_2$ that preserves the bilinear forms of L_1 and L_2. If $\{e_1, \ldots, e_r\} \subset L$ is a basis for L then the matrix

$$B = \begin{pmatrix} (e_1, e_1) & \cdots & (e_1, e_r) \\ \vdots & \ddots & \vdots \\ (e_r, e_1) & \cdots & (e_r, e_r) \end{pmatrix}$$

is called the Gram matrix. Its determinant $d(L)$ is an invariant called the discriminant of the lattice. A lattice is called unimodular if $d(L) = \pm 1$ or equivalently if $L^\vee = L$. A lattice L is called even if $(x,x) \in 2\mathbb{Z}$ for all $x \in L$, otherwise it is called odd. We denote the automorphism group of a lattice L by $O(L)$. An important class of automorphisms of a lattice is given by its reflections. Suppose $r \in L$ is primitive; we define the reflection s_r in r by the formula:

$$s_r(x) = x - 2\frac{(r,x)}{(r,r)}r. \tag{2.1}$$

This reflection is an automorphism of the lattice L if and only if $2(r,x) \in (r,r)\mathbb{Z}$ for all $x \in L$. In that case we call r a root of L. Since conjugation by an element of $O(L)$ of a reflection is again a reflection, the reflections in roots generate a normal subgroup $W(L) \triangleleft O(L)$.

Let L be an even lattice. The quotient $A_L = L^\vee/L$ is called the discriminant group of L. It is a finite abelian group of order $d(L)$. We denote the minimal number of generators of A_L by $l(A_L)$. If $A_L \cong (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \in \mathbb{N}$ then L is called 2-elementary.

Proposition 2.2.1 (Nikulin [34], Thm. 3.6.2)**.** *An even 2-elementary lattice is determined up to isomorphism by the invariants (r_+, r_-, a, δ). The invariant δ is defined by:*

$$\delta = \begin{cases} 0 & \text{if } (x,x)_\mathbb{Q} \in \mathbb{Z} \text{ for all } x \in L^\vee \\ 1 & \text{else} \end{cases}$$

The discriminant quadratic form q_L on A_L takes values in $\mathbb{Q}/2\mathbb{Z}$ and is defined by the expression:

$$q_L(x+L) \equiv (x,x)_\mathbb{Q} \bmod 2\mathbb{Z} \quad \text{for } x \in L^\vee.$$

The group of automorphisms of A_L that preserve the discriminant quadratic form q_L is denoted by $O(A_L)$ and there is a natural homomorphism: $O(L) \to O(A_L)$. If $\phi_L \in O(L)$ then we denote by $q(\phi_L) \in O(A_L)$ the induced automorphism of A_L.

Theorem 2.2.2 (Nikulin, [34], Thm. 3.6.3)**.** *Let L be an even, indefinite 2-elementary lattice. Then the natural homomorphism $O(L) \to O(A_L)$ is surjective.*

Proposition 2.2.3 (Nikulin, [34], Prop. 1.6.1)**.** *Let L be an even unimodular lattice and M a primitive sublattice of L with orthogonal complement $M^\perp = N$. There is a natural isomorphism $\gamma : A_M \to A_N$ for which $q_N \circ \gamma = -q_M$. Let $\phi_M \in O(M)$ and $\phi_N \in O(N)$. The automorphism (ϕ_M, ϕ_N) of $M \oplus N$ extends to L if and only if $q(\phi_N) \circ \gamma = \gamma \circ q(\phi_M)$.*

Theorem 2.2.4 (Nikulin, [34], Thm. 1.14.4)**.** *Let M be an even lattice of signature (t_+, t_-) and let L be an even unimodular lattice of signature (l_+, l_-). There is a unique primitive embedding of M into L provided the following hold:*

1. $t_+ < s_+$
2. $t_- < s_-$
3. $l(A_M) \leq \text{rank}(L) - \text{rank}(M) - 2$

Notations for lattices

We denote by $L(n)$ the lattice L where the bilinear form is scaled by a factor $n \in \mathbb{Z}$ and we write U for the even unimodular hyperbolic lattice of rank 2 with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Furthermore we denote by A_i, D_j, E_k with $i, j \in \mathbb{N}$, $j \geq 4$ and $k = 6, 7, 8$ the lattices associated to the negative definite Cartan matrices of this type. For example:

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \ A_1 \oplus A_1(2) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}, \ D_4 = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}.$$

Determining if two lattices are isomorphic can be quite difficult. In the following lemma we describe some isomorphic lattices that we will encounter frequently when studying Gaussian lattices.

Lemma 2.2.5. *There are isomorphisms of hyperbolic lattices:*

$$(4) \oplus A_1 \cong (2) \oplus A_1(2) \tag{2.2}$$

$$U(2) \oplus A_1 \cong (2) \oplus A_1^2 \tag{2.3}$$

$$(2) \oplus A_1(2) \oplus D_4(2) \cong (2) \oplus A_1^2 \oplus A_1(2)^3 \tag{2.4}$$

Proof. For the first isomorphism we explicitly determine a base change:

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}^t \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}.$$

For the second isomorphism we calculate the invariants (r_+, r_-, a, δ) of Proposition 2.2.1. They are easily seen to be $(1, 2, 3, 1)$ for both lattices so that the lattices are isomorphic. The third isomorphism is the least obvious. We also determine an explicit base change:

$$B^t \left((2) \oplus A_1^2 \oplus A_1(2)^3\right) B = (2) \oplus A_1(2) \oplus D_4(2)$$

where B is the unimodular matrix:

$$B = \begin{pmatrix} 3 & 2 & 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

□

2.3 Involutions in Coxeter groups

The classification of conjugacy classes of involutions in a Coxeter group was done by Richardson [35] and Springer [40]. Before this the classification of conjugacy classes of elements of finite Coxeter groups was obtained by Carter [10]. In this section we give a brief review of these results.

Definition 2.3.1. A Coxeter system is a pair (W, S) with W a group presented by a (finite) set of generators $S = \{s_1, \dots, s_r\}$ subject to relations

$$(s_i s_j)^{m_{ij}} = 1 \quad \text{with} \quad 1 \leq i, j \leq r$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji}$ are integers ≥ 2. We also allow $m_{ij} = \infty$ in which case there is no relation between s_i and s_j. These relations are encoded by the Coxeter graph of (W, S). This is a graph with r nodes labeled by the generators. Nodes i and j are not connected if $m_{ij} = 2$ and are connected by $m_{ij} - 2$ edges otherwise. If $m_{ij} = \infty$ we connect the vertices by a thick edge.

For a Coxeter system (W, S) we define an action of the group W on the real vector space V with basis $\{e_s\}_{s \in S}$. First we define a symmetric bilinear form B on V by the expression:

$$B(e_i, e_j) = 2 \cos \left(\frac{\pi}{m_{ij}} \right).$$

Then for each $s_i \in S$ the reflection: $s_i(x) = x - B(e_i, x)e_i$ preserves this form B. In this way we obtain a homomorphism $W \to GL(V)$ called the geometric realization of W. For each subset $I \subseteq S$ we can form the standard parabolic subgroup $W_I < W$ generated by the elements $\{s_i; i \in I\}$ acting on the subspace V_I generated by $\{e_i\}_{i \in I}$. We say that W_I (or also I) satisfies the (-1)-condition if there is a $w_I \in W_I$ such that $w_I \cdot x = -x$ for all $x \in V_I$. The element w_I necessarily equals the longest element of (W_I, S_I). This implies in particular that W_I is finite. Let $I, J \subseteq S$, we say that I and J are W-equivalent if there is a $w \in W$ that maps $\{e_i\}_{i \in I}$ to $\{e_j\}_{j \in J}$. Now we can formulate the main theorem of [35]:

Theorem 2.3.2 (Richardson)**.** *Let (W, S) be a Coxeter system and let $\mathcal{J}$ be the set of subsets of S that satisfy the (-1)-condition. Then:*

1. *If $c \in W$ is an involution, then c is conjugate in W to w_I for some $I \in \mathcal{J}$.*

2. *Let $I, J \in \mathcal{J}$. The involutions w_I and w_J are conjugate in W if and only if I and J are W-equivalent.*

This theorem reduces the problem of finding all conjugacy classes of involutions in W to finding all W-equivalent subsets in S satisfying the (-1)-condition. First we determine which subsets $I \subseteq S$ satisfy the (-1)-condition, then we present an algorithm that determines when two subsets $I, J \subseteq S$ are W-equivalent. If (W_I, S_I) is irreducible and satisfies the (-1)-condition then it is of one of the following types:

$$A_1, B_n, D_{2n}, E_7, E_8, F_4, G_2, H_3, H_4, I_2(2p) \tag{2.5}$$

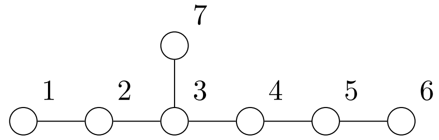
with $n, p \in \mathbb{N}$ and $p \geq 4$. If (W_I, S_I) is reducible and satisfies the (-1)-condition then W_I is the direct product of irreducible, finite standard parabolic subgroups (W_i, S_i) from (2.5). The Coxeter diagrams of the (W_i, S_i) occur as disjoint subdiagrams of the types in the list of the diagram of (W, S). The element w_I is the product of the w_{I_i} which act as -1 on the V_{I_i}. Now let $K \subseteq S$ be of finite type and let w_K be the longest element of (W_K, S_K). The element $\tau_K = -w_K$ defines a diagram involution of the Coxeter diagram of (W_K, S_K) which is non-trivial if and only if $w_K \neq -1$. If $I, J \subseteq K$ are such that $\tau_K I = J$ then I and J are W-equivalent. To see this, observe that $w_K w_I \cdot I = w_K \cdot (-I) = \tau_K I = J$. Now we define the notion of elementary equivalence.

Definition 2.3.3. We say that two subsets $I, J \subseteq S$ are elementary equivalent, denoted by $I \vdash J$, if $\tau_K I = J$ with $K = I \cup \{\alpha\} = J \cup \{\beta\}$ for some $\alpha, \beta \in S$.

It is proved in [35] that I and J are W-equivalent if and only if they are related by a chain of elementary equivalences: $I = I_1 \vdash I_2 \vdash \ldots \vdash I_n = J$. This provides a practical algorithm to determine all the conjugation classes of involutions in a given Coxeter group (W, S) using its Coxeter diagram:

1. Make a list of all the subdiagrams of the Coxeter diagram of (W, S) that satisfy the (-1)-condition. These are exactly the disjoint unions of diagrams in the list (2.5). Every involution in W is conjugate to w_K with K a subdiagram in this list.
2. Find out which subdiagrams of a given type are W-equivalent by using chains of elementary equivalences.

Example 2.3.4 (E_7)**.** We use the procedure described above to determine all conjugation classes of involutions in the Weyl group of type E_7. This result will be used many times later on. Since W_7 contains the element -1 the conjugation classes of involutions come in pairs $\{u, -u\}$. We label the vertices of the Coxeter diagram as follows.



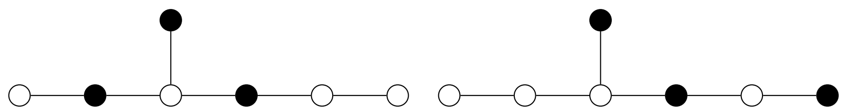


Figure 2.1: The involutions A_1^3 (left) and $A_1^{3\prime}$ (right).

It turns out that all involutions of a given type are equivalent with the exception of type A_1^3: here there are two non-equivalent involutions as seen in Figure 2.1. The types of involutions that occur are:

$$\{1, E_7\}\ ,\ \{A_1, D_6\}\ ,\ \{A_1^2, D_4A_1\}\ ,\ \{A_1^3, A_1^4\}\ ,\ \{D_4, A_1^{3\prime}\}. \tag{2.6}$$

For example, consider the two subdiagrams of type A_1 with vertices $\{1\}$ and $\{2\}$. The diagram automorphism $\tau_{\{1,2\}}$ which is of type A_2 exchanges the vertices $\{1\}$ and $\{2\}$, so they are elementary equivalent. One shows in a similar way that all diagrams of type A_1 are equivalent.

2.4 Hyperbolic reflection groups

Most of the results of this section can be found in [42]. Let L be a hyperbolic lattice of hyperbolic signature $(1, n)$. We can associate to L the space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ with isometry group $O(V) \cong O(1, n)$. A model for real hyperbolic n-space $\mathbb{H}_n$ is given by one of the sheets of the two sheeted hyperboloid $\{x \in V\ ;\ (x, x) = 1\}$ in V. Its isometry group is the subgroup $O(V)^+ < O(V)$ of index two of isometries that preserves this sheet. Another model for $\mathbb{H}_n$ which we will use most of the time is the ball defined by:

$$\mathbb{B}_n = \mathbb{P}\{x \in L \otimes_{\mathbb{Z}} \mathbb{R}\ ;\ (x, x) > 0\}$$

whose isometry group is naturally identified with the group $O(\mathbb{B}) \cong PO(1, n)$. The group $O(L)^+ = O(L) \cap O(V)^+$ is a discrete subgroup of $O(V)^+$ and it has finite covolume by a theorem of Siegel [38]. Let $W(L) < O(L)^+$ be the normal subgroup generated by the reflections in roots of negative norm of L. We can write the group $O(L)^+$ as:

$$O(L)^+ = W(L) \rtimes S(C)$$

where $C \subset \mathbb{B}_n$ is a fundamental chamber of $W(L)$ and $S(C)$ is the subgroup of $O(L)^+$ that maps C to itself. The lattice L is called reflective if $W(L)$ has finite index in $O(L)^+$. In this case C is a hyperbolic polytope of finite volume which we assume from now on. We say that $\{r_i\}_{i \in I}$ with $I = 1, \ldots, k$ is a set of simple roots for C if all pairwise inner products are non-negative and C is the polyhedron bounded by the mirrors H_{r_i} so that:

$$C = \mathbb{P}\{x \in L \otimes_{\mathbb{Z}} \mathbb{R}\ ;\ (x, x) > 0\ ,\ (r_i, x) \geq 0 \text{ for } i = 1, \ldots, k\}. \tag{2.7}$$

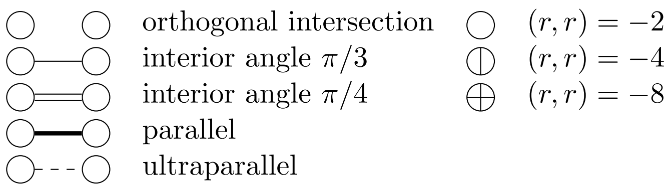


Figure 2.2: Conventions for Coxeter graphs

Since the group $W(L)$ is discrete the root mirrors meet at dihedral angles π/m_{ij} with $m_{ij} \geq 2$ or they are disjoint in $\mathbb{B}_n$. In this last case we say that two root mirrors H_{r_i} and H_{r_j} are parallel if they meet at infinity so that $m_{ij} = \infty$, or ultra-parallel if they do not meet even at infinity. The matrix G with entries $(r_i, r_j)_{i,j \in I}$ is called the Gram matrix of C and the m_{ij} can be calculated from G by the relation:

$$(r_i, r_j)^2 = (r_i, r_i)(r_j, r_j) \cos^2 \left(\frac{\pi}{m_{ij}} \right).$$

The polytope C is described most conveniently by its Coxeter diagram D_I from Definition 2.3.1. In addition to the conventions introduced there we connect two nodes by a dashed edge if their corresponding mirrors are ultraparallel. In the examples that come from Gaussian lattices we will only encounter roots of norm $-2, -4$ and -8 so we also subdivide the corresponding nodes into $0, 2$ and 4 parts respectively. These conventions are illustrated in Figure 2.2

A Coxeter subdiagram $D_J \subset D_I$ with $J \subset I$ is called elliptic if the corresponding Gram matrix is negative definite of rank $|J|$ and parabolic if it is negative semi-definite of rank $|J| - \#$components of D_J. An elliptic subdiagram is a disjoint union of finite Coxeter diagrams and a parabolic subdiagram the disjoint union of affine Coxeter diagrams. The elliptic subdiagrams of D of rank r correspond to the $(n-r)$-faces of the polyhedron $C \in \mathbb{B}_n$. A parabolic subdiagram of rank $n-1$ corresponds to a cusp of C. By the type of a face or cusp of C we mean the type of the corresponding Coxeter subdiagram.

Vinberg's algorithm

Suppose we are given a hyperbolic lattice L of signature $(1, n)$. Vinberg [42] describes an algorithm to determine a set of simple roots of $W(L)$. If the algorithm terminates these simple roots determine a hyperbolic polyhedron $C \subset \mathbb{B}_n$ of finite volume which is a fundamental chamber for the reflection subgroup $W(L)$. We start by choosing a controlling vector $p \in L$ such that $(p, p) > 0$. This implies that $p \in \mathbb{B}_n$. The idea is to determine a sequence of roots $r_1, r_2, \ldots$ so that the hyperbolic distance of p to the mirrors H_{r_i} is increasing. Since the hyperbolic distance $d(p, H_{r_i})$ is given by:

$$\sinh^2 d(p, H_{r_i}) = \frac{(r_i, p)^2}{(r_i, r_i) \cdot (p, p)} \tag{2.8}$$

the height $h(r_i)$ of a root defined by $h(r_i) = (r_i, p)^2/(r_i, r_i)$ is a measure for this distance. First we determine the roots of height 0. They form a finite root system R and we choose a set of simple roots $r_1, \dots, r_i$ to be our first batch of roots. For the inductive step in the algorithm we consider all roots of height h and assume that all roots of smaller height have been enumerated. A root of height h is accepted if and only if it has non-negative inner product with all previous roots of the sequence. The algorithm terminates if the accute angled polyhedron spanned by the mirrors $H_{r_1}, \dots$ has finite volume. This can be checked using the following criterion also due to Vinberg.

Proposition 2.4.1. *A Coxeter polyhedron $C \subset \mathbb{B}_n$ has finite volume if and only if every elliptic subdiagram of rank $n-1$ can be extended in exactly two ways to an elliptic subdiagram of rank n or to a parabolic subdiagram of rank $n-1$. Furthermore there should be at least 1 elliptic subdiagram of rank $n-1$.*

Since an elliptic subdiagram of rank $n-1$ corresponds to an edge of the polyhedron C the geometrical content of this criterion is that every edge connects either two actual vertices, two cusps or a vertex and a cusp. The following example is due to Vinberg, see [41] §4.

Example 2.4.2. Consider the hyperbolic lattice $\mathbb{Z}_{1,n}(2) = (2) \oplus A_1^n$ with its standard orthogonal basis $\{e_0, \dots, e_n\}$ where $2 \le n \le 9$. The possible root norms are -2 and -4. We take as controlling vector $p = e_0$ with $(p,p) = 2$. The height 0 root system is of type B_n and a basis of simple roots is given by:

$$r_1 = e_1 - e_2\ ,\ \dots, r_n = e_{n-1} - e_n\ ,\ r_n = e_n.$$

The next root accepted by Vinberg's algorithm is the root $r_{n+1} = e_0 - e_1 - e_2$ of height 1 for $n = 2$ and the root $r_{n+1} = e_0 - e_1 - e_2 - e_3$ of height 1/2 for $3 \le n \le 8$. This root indeed satisfies $(r_{n+1}, r_i) > 0$ for $i = 1, \dots, n$. The resulting Coxeter polyhedron is a simplex and has finite volume so the algorithm terminates. In all the cases there is a single cusp of type $\widetilde{A}_1$ for $n = 2$ and of type $\widetilde{B}_2$ for $n = 3, \dots, 8$. The Coxeter diagrams are shown in Figure 2.3.

2.5 Real plane algebraic curves

In this section we study the topology of real algebraic curves. For a treatment of curves suitable for geometric invariant theory, which we use in later chapters, we refer to [21] Chapter 4. A real plane algebraic curve C of degree d is the zero locus in the projective plane $\mathbb{P}^2(\mathbb{R})$ of a homogeneous polynomial $f(x,y,z)$ of degree d with real coefficients. If the curve C is smooth its set of real points $C(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ is a compact submanifold of dimension one. It is well known that in this case the connected components of $C(\mathbb{R})$, which we call ovals, are smoothly embedded circles in $\mathbb{P}^2(\mathbb{R})$. There are two possibilities for

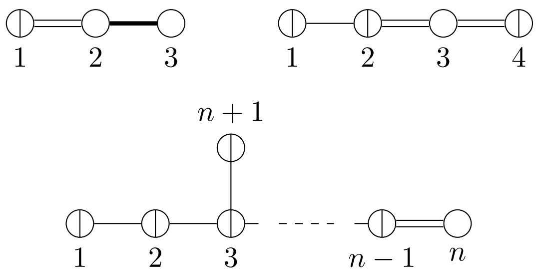


Figure 2.3: The Coxeter diagrams of the groups $O(\mathbb{Z}_{1,n}(2))^+$ for $n = 2, \ldots, 8$.

an oval $\gamma \subset \mathbb{P}^2(\mathbb{R})$: either it bounds a disc in which case we call γ even or it is isotopic to a projective line in $\mathbb{P}^2(\mathbb{R})$ and γ is called odd. If the degree of C is even then all its ovals are even. In this case if $C(\mathbb{R}) \neq \emptyset$ the complement of the curve in $\mathbb{P}^2(\mathbb{R})$ consists of two halves where $f < 0$ and $f > 0$ respectively. One of these halves corresponds to the inside of the curve and is orientable, the other half corresponds to the outside of the curve and is non-orientable. We choose the sign of f so that the half where $f > 0$ is orientable. If $C(\mathbb{R}) = \emptyset$ then its complement is the whole of $\mathbb{P}^2(\mathbb{R})$ which is non-orientable so that we choose the sign of f to be negative.

For a smooth real plane algebraic curve C the possibilities for the number of ovals are completely determined by the following classical theorem due to Harnack.

Theorem 2.5.1 (Harnack [20]). *For any natural number d and any integer n satisfying the inequalities:*

$$\frac{1-(-1)^d}{2} \leq n \leq \frac{(d-1)(d-2)}{2} + 1$$

there exists a smooth real plane algebraic curve C of degree d consisting of n ovals.

The right hand inequality in Harnack's theorem is also called Harnack's inequality. A curve for which this inequality is an equality is called a maximal curve, or M-curve. The determination of the possible relative positions of the ovals of a smooth real plane algebraic curve of degree d is the subject of the first part of Hilbert's 16th problem. It is still open for $d > 7$.

The space of all real plane algebraic curve of degree d is the projective space $P_{d,3}(\mathbb{R}) = \mathbb{P}\,\mathrm{Sym}^d(\mathbb{R}^3)^\vee$ of dimension $N = \frac{d^2+3d}{2}$. The sublocus $\Delta(\mathbb{R}) \subset P_{d,3}(\mathbb{R})$ that corresponds to singular curves is of codimension 1. Two nonsingular curves C and C' are said to be connected by a rigid isotopy if they are in the same connected component of $P_{d,3}(\mathbb{R}) - \Delta(\mathbb{R})$. A rigid isotopy preserves the relative positions of the ovals of a curve but the converse need not be true for curves of degree $d \geq 5$.

Example 2.5.2. For a smooth real plane algebraic curve C of degree four the determination of the relative position of the ovals is classical and goes back to Cayley and Zeuthen. There are six cases: the number of ovals $n(C)$ is $0, 1, 2, 3$ or 4 and if $n(C)$=2 the ovals can be nested or not. In the other cases the ovals cannot be nested otherwise we would get a contradiction with Bezout's theorem. These six cases are shown in Figure 1.1. This classification corresponds with the classification by rigid isotopy, so the space $P_{4,3}(\mathbb{R})-\Delta(\mathbb{R})$ has 6 connected components.

2.6 Complex del Pezzo surfaces

In this section we review the theory of del Pezzo surfaces that will be used throughout this text. A good references for this section is [32]. Two other references on the subject worth mentioning are [16] and [17].

Definition 2.6.1. A del Pezzo surface Y is a smooth, complex projective surface whose anticanonical system $|-K_Y|$ is ample. The degree of Y is the self-intersection number: $d = K_Y \cdot K_Y$ of the canonical class in the Picard group $\mathrm{Pic}(Y)$ of Y. It is an integer with $1 \leq d \leq 9$.

Most del Pezzo surfaces can be realized as the blowup of a configuration of points in the projective plane. The precise statement is given by the following theorem.

Theorem 2.6.2. *A del Pezzo surface of degree d is isomorphic to either:*

1. *The blowup $Y = \mathrm{Bl}_B \mathbb{P}^2$ of the projective plane in a set*

   $$B = \{P_1, \ldots, P_r\} \subset \mathbb{P}^2(\mathbb{C})$$

   of $r = 9 - d$ points in general position ($1 \leq d \leq 9$). A point set is in general position if no 3 points are collinear, no 6 are on a conic and no 8 are on a cubic which is singular at one of these points.

2. *The smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$ in which case $d = 8$.*

From now on we only consider del Pezzo surfaces of the first kind. Exhibiting a del Pezzo surface as a blowup $\pi : Y \to \mathbb{P}^2$ fixes a basis of the Picard group $\mathrm{Pic}(Y)$. This basis consists of the classes $E_i = \pi^{-1}(P_i)$ with $1 \leq i \leq r$ of the exceptional curves over the blown up points and the class E_0 of the strict transform of a general line in $\mathbb{P}^2$. The anticanonical class expressed in this basis of $\mathrm{Pic}(Y)$ is given by:

$$-K_Y = 3E_0 - E_1 - \ldots - E_r.$$

It is represented by the strict transform of a cubic in $\mathbb{P}^2$ through the points $B = \{P_1, \ldots, P_r\}$. We also write $\mathrm{Pic}^0(Y)$ for the orthogonal complement of

$-K_Y$ in Pic(Y). From the description of a del Pezzo surface as a blowup of $\mathbb{P}^2$ it follows that the Picard group Pic(Y) is isomorphic to the hyperbolic lattice $\mathbb{Z}_{1,r}$ of rank $r+1$ and signature $(1,r)$. It has a basis $\{e_0, \ldots, e_r\}$ with inner product defined by the relations:

$$\begin{cases} e_0 \cdot e_0 = 1 & \\ e_i \cdot e_i = -1 & \text{for } 1 \leq i \leq r \\ e_i \cdot e_j = 0 & \text{for } i \neq j. \end{cases}$$

An isomorphism $\phi : \mathbb{Z}_{1,r} \to \mathrm{Pic}(Y)$ is called a *marking* of the del Pezzo surface Y if it maps the element $k = -3e_0 + e_0 + \ldots + e_r$ to the canonical class K_Y of Pic(Y). An isomorphism $(Y, \phi) \cong (Y', \phi')$ of marked del Pezzo surfaces is an isomorphism $F : Y \to Y'$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}_{1,r} & \xrightarrow{\phi} & \mathrm{Pic}(Y) \\ & \searrow^{\phi'} & \downarrow F_* \\ & & \mathrm{Pic}(Y') \end{array}$$

Exhibiting a del Pezzo surface as a blowup $\pi : Y \to \mathbb{P}^2$ is equivalent to adding a marking to Y since from the marking ϕ we recover the blowup map by blowing down the exceptional curves $\phi(e_i)$ for $1 \leq i \leq r$. This determines a set $B = \{P_1, \ldots, P_r\}$ of r points in general postition in $\mathbb{P}^2$. If two marked del Pezzo surfaces are isomorphic then the corresponding point sets B and B' are related by an element of PGL$(3, \mathbb{C})$. As a consequence the space

$$\widetilde{\mathcal{DP}}_d = \left((\mathbb{P}^2)^r - \Delta\right) / \, \mathrm{PGL}(3, \mathbb{C}) \tag{2.9}$$

is a moduli space for marked del Pezzo surfaces of degree $d = 9 - r$. Here Δ denotes the set of configurations of r points in $\mathbb{P}^2$ *not* in general position in the sense of Theorem 2.6.2. For an r-tuple of points in $\mathbb{P}^2$ in general position with $r \geq 4$ there is a unique element of PGL$(3, \mathbb{C})$ that maps the points to the configuration of points represented by the columns of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & x_1 & \ldots & x_{r-4} \\ 0 & 1 & 0 & 1 & y_1 & \ldots & y_{r-4} \\ 1 & 1 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

This implies that $\widetilde{\mathcal{DP}}_d$ is isomorphic to an open subset of $(\mathbb{A}^2)^{r-4}$ and is actually a fine moduli space for marked del Pezzo surfaces of degree d.

The Cremona action of the Weyl group

The orthogonal group $O(\mathbb{Z}_{1,r})^+$ is a hyperbolic Coxeter group with the Coxeter diagram shown in Figure 2.3. The stabilizer W_r of $k = -3e_0 + e_1 + \ldots + e_r$

in this group is the subgroup generated by the reflections in the long roots of $\mathbb{Z}_{1,r}$. It is a finite Coxeter group of type:

$$A_1A_2, A_4, D_5, E_6, E_7, E_8 \tag{2.10}$$

for $r = 3, \ldots, 8$. A set of generators $S = \{s_1, \ldots, s_r\}$ for W_r is given by the reflections in the simple roots:

$$\alpha_1 = e_1 - e_2 \ , \ldots, \ \alpha_{r-1} = e_{r-1} - e_r \ , \ \alpha_r = e_0 - e_1 - e_2 - e_3.$$

These root span a lattice Q_r that is precisely the orthogonal complement of $k^{\perp}$ in $\mathbb{Z}_{1,r}$. The group W_r acts on marked del Pezzo surfaes by composing with the marking: $w \cdot (X, \phi) = (X, \phi \circ w^{-1})$ for $w \in W_r$. This action is simply transitive so that the orbit space $\mathcal{DP}_d = W_r \backslash \widetilde{\mathcal{DP}}_d$ is a coarse moduli space for del Pezzo surfaces of degree d.

We now describe this action on the set of blown up points in $\mathbb{P}^2$. Suppose (X, ϕ) is a marked del Pezzo surface and $\pi : X \to \mathbb{P}^2$ is the corresponding blowing up map with $B \subseteq \mathbb{P}^2$ the set of blown up points. If ϕ' is another marking of X then $\phi' = \phi \circ w$ for some element $w \in W_r$. The element w defines a birational transformation $\rho(w)$ of $\mathbb{P}^2$ in the following way: first blow up $\mathbb{P}^2$ in the points of B. Then blow down the exceptional curves $\phi'(e_i) = \phi(w \cdot e_i)$ for $1 \leq i \leq r$. This determines a new set of points B' and blowup map $\pi' : X \to \mathbb{P}^2$ corresponding to ϕ' such that the following diagram commutes.

$$\begin{array}{ccc} X & & \\ \downarrow \pi & \searrow \pi' & \\ \mathbb{P}^2 & \overset{\rho(w)}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

In this way we obtain a homomorphism of the Weyl group W_r to the group of birational transformations of $\mathbb{P}^2$:

$$\rho : W_r \to \mathrm{Bir}(\mathbb{P}^2)$$

We can calculate this representation on the set S of simple reflections. The element $\rho(s_i)$ with $1 \leq i \leq r-1$ corresponds to the transposition of the points P_i and P_{i+1}. The element s_r gives a more interesting transformation. It acts on $\mathbb{Z}_{1,r}$ by:

$$\begin{aligned} e_1 &\mapsto e_0 - e_2 - e_3 \\ e_2 &\mapsto e_0 - e_1 - e_3 \\ e_3 &\mapsto e_0 - e_1 - e_2 \\ e_i &\mapsto e_i \quad 4 \leq i \leq r. \end{aligned}$$

Geometrically this means that $\rho(s_r)$ is obtained by first blowing up P_1, P_2 and P_3 and then blowing down the strict transforms of the lines connecting them. This birational transformation $\rho(s_r)$ is called the standard Cremona transformation based in P_1, P_2 and P_3. A simple calculation shows that

$s_7(2e_0 - e_1 - e_2 - e_3) = e_0$ so that the image of a conic through P_1, P_2, P_3 under the standard Cremona transformation is a line. If we assume that these points are

$$P_1 = (1:0:0)\ ,\ P_2 = (0:1:0)\ ,\ P_3 = (0:0:1).$$

then $\rho(s_r)$ is given by $(x:y:z) \mapsto (yz:xz:xy)$. To summarise: the group W_r acts on $(\mathbb{P}^2)^r - \Delta$ by permuting the points and by standard Cremona transformations centered in triples of distinct points.

Del Pezzo surfaces of degree two

Suppose that Y is a del Pezzo surface of degree two so that it is isomorphic to the blowup of the projective plane $\mathbb{P}^2$ in 7 points. The anticanonical system of Y defines a morphism:

$$|-K_Y| : Y \to \mathbb{P}^2.$$

It is a double cover of $\mathbb{P}^2$ branched along a smooth quartic $C \subset \mathbb{P}^2$. Conversely a smooth quartic $C = \{f(x,y,z) = 0\}$ determines a del Pezzo surface Y of degree two by the formula:

$$Y = \{w^2 = f(x,y,z)\} \subset \mathbb{P}(2,1,1,1). \tag{2.11}$$

Consequently every del Pezzo surface Y has a special involution that corresponds to the deck transformation of the double cover $Y \to \mathbb{P}^2$. In terms of Equation 2.11 this involution is given by:

$$\rho_Y : [w:x:y:z] \mapsto [-w:x:y:z].$$

If we choose a marking $\phi : \mathrm{Pic}(Y) \to \mathbb{Z}_{1,7}$ then this involution acts on the lattice $\mathbb{Z}_{1,7}$ by:

$$\rho : x \mapsto -x + (x \cdot k)k. \tag{2.12}$$

It fixes the element k and acts as -1 on $k^\perp$ so that it corresponds to the centralizer $-1 \in W(E_7)$. An element $e \in \mathbb{Z}_{1,7}$ that satisfies $e \cdot e = -1$ and $e \cdot k = -1$ is called *exceptional*. The set $\mathcal{E}$ of exceptional elements forms a single $W(E_7)$-orbit and consists of the 56 elements:

1. e_i with $1 \le i \le 7$, the class of the exceptional divisor E_i.

2. $l_{ij} = e_0 - e_i - e_j$, the class of the strict transform of the line L_{ij} through P_i and P_j.

3. $c_{ij} = -k - l_{ij} = 2e_0 - e_1 - \ldots - \hat{e}_i - \ldots - \hat{e}_j - \ldots - e_7$, the class of the strict transform of the conic C_{ij} through 5 of the 7 points.

4. $k_i = -k - e_i = 3e_0 - e_1 - \ldots - 2e_i - \ldots - e_7$, the class of the strict transform of the cubic K_i through 6 points with a node at a seventh point.

The elements of $\mathcal{E}$ come in 28 pairs (e_i, k_i), (l_{ij}, c_{ij}) whose elements are interchanged by the involution ρ. The geometric meaning of this is as follows. An exceptional element $E \in \mathrm{Pic}(Y)$ corresponds to a line on the del Pezzo surface Y and there are 56 of these. The two elements of a pair $(E, \rho_Y(E))$ are mapped to a single bitangent of the quartic curve by the anticanonical map. This accounts for all 28 bitangents of a smooth plane quartic curve.

The Geiser involution

We can also describe the involution ρ_Y explicitly as a birational involution of $\mathbb{P}^2$ which is called the Geiser involution. Let $B = \{P_1, \ldots, P_7\}$ be the 7 blown up points in $\mathbb{P}^2(\mathbb{C})$. Suppose the cubics $\{C_1, C_2, C_3\}$ form a basis for the net of cubics through B. We define the rational map:

$$\begin{aligned} \psi : \mathbb{P}^2(\mathbb{C}) &\dashrightarrow \mathbb{P}^2(\mathbb{C}) \\ [x : y : z] &\mapsto [C_1(x, y, z) : C_2(x, y, z) : C_3(x, y, z)]. \end{aligned}$$

The locus of indeterminacy where C_1, C_2 and C_3 vanish simultaneously is precisely the set B. The cubic curve $V(C)$ defined as the zero locus of $C = aC_1 + bC_2 + cC_3$ with $a, b, c \in \mathbb{C}$ is mapped by ψ to the projective line $aC_1 + bC_2 + cC_3 = 0$.

For a general point $Q \in \mathbb{P}^2(\mathbb{C}) \setminus B$ there is a pencil of cubics through $B \cup \{Q\}$. Let C, C' be a basis for this pencil. The point $\psi(Q)$ is the unique intersection point of the two lines $l = \pi(C)$ and $l' = \pi(C')$. We see that for the inverse image we have:

$$\psi^{-1}\psi(Q) = \psi^{-1}(l \cap l') = (C \cap C') \setminus B.$$

The set $(C \cap C') \setminus B$ consists of two points $\{Q, Q'\}$ with Q' the nineth intersection point of C and C'. This shows that the map ψ is generically $2:1$. The Geiser involution maps Q to Q'. It is not defined on the cubics K_i that pass through B with a double point at P_i. After we blow up the points B, the birational involution ψ lifts to the automorphism ρ_Y of Y.

2.7 $K3$-surfaces

A main ingredient in Kondo's construction of a period map for the moduli space of smooth real plane quartic curves is the period map for $K3$-surfaces. In this section we briefly review the relevant results and refer to [8] Chapter 8 for a detailed exposition and proofs. We also list some results on real $K3$ surfaces.

Definition 2.7.1. A smooth compact complex surface is called a $K3$-surface if it is simply connected and its canonical class K_X is trivial.

Example 2.7.2. The most relevant example for us is that of a smooth quartic surface X in $\mathbb{P}^3$. By the adjunction formula the canonical class K_X is zero in $\mathrm{Pic}(X)$. By the Lefschetz theorem for hyperplane section there are isomorphisms:

$$\pi_1(X) \xrightarrow{\cong} \pi_1(\mathbb{P}^3) \xrightarrow{\cong} \{1\}$$

so that X is simply connected and indeed a $K3$ surface.

Let X be a $K3$ surface, since K_X is trivial there is a non-vanishing holomorphic 2-form $\omega_X \in H^{2,0}(X)$ unique up to scalar multiplication. The Hodge numbers of X are given by the Hodge diamond:

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}\ .$$

There is also a Hodge decomposition of the second cohomology of X, which takes the form:

$$H^2(X,\mathbb{C}) = \mathbb{C}\cdot\omega_X \oplus H^{1,1}(X) \oplus \mathbb{C}\cdot\overline{\omega}_X.$$

The second integral cohomology group $H^2(X,\mathbb{Z})$ of X is equipped with a symmetric bilinear form induced by the cup-product. This turns $H^2(X,\mathbb{Z})$ into a lattice isomorphic to the $K3$-lattice:

$$L = U^3 \oplus E_8^2.$$

It is an even unimodular lattice of rank 22 and signature $(3,19)$. An isomorphism of lattices: $\phi : H^2(X,\mathbb{Z}) \to L$ is called a marking of X. By the Hodge-Riemann bilinear relations the extension of the bilinear form to $H^2(X,\mathbb{C})$ satisfies:

$$(\omega_X,\omega_X) = (\overline{\omega}_X,\overline{\omega}_X) = 0 \quad , \quad (\omega_X,\overline{\omega}_X) \in \mathbb{R}_{>0} \quad , \quad (\omega_X,\gamma) = 0$$

for all $\gamma \in H^{1,1}(X)$. This allows us to define a period map by mapping a marked $K3$-surface (X,ϕ) to the period point $\phi(H^{2,0}(X))$ in the period domain:

$$\Omega = \mathbb{P}\{z \in L\otimes_{\mathbb{Z}}\mathbb{C}\ ;\ (z,z) = 0\ ,\ (z,\bar{z}) \in \mathbb{R}_{>0}\} \tag{2.13}$$

The following two theorems imply that this period domain is actually a (coarse) moduli space for marked $K3$ surfaces.

Theorem 2.7.3 (Weak Torelli theorem)**.** *Two $K3$-surfaces X and X' are isomorphic if and only if there are markings for them such that they map to the same period point in Ω.*

Theorem 2.7.4 (Surjectivity of the period map)**.** *Every point $[x] \in \Omega$ occurs as the period point of some marked $K3$-surface (X, ϕ).*

Real $K3$ surfaces

A $K3$ surface X is called real if it is equipped with an anti-holomorphic involution χ_X. We will also call such an involution a real form of X. The real points of X which we denote by $X(\mathbb{R})$ are the fixed points of the real form.

Theorem 2.7.5. *Let χ be an involution on the $K3$ lattice L. There exists a marked $K3$ surface (X, ϕ) such that $\chi_X = \phi^{-1} \circ \chi \circ \phi$ induces a real form on X if and only if the lattice of fixed points L^χ has hyperbolic signature.*

Proof. See [39] Chapter VIII Theorem 2.3. □

Suppose (X, χ_X) is a real $K3$ surface. By choosing a marking we obtain an involution χ of the $K3$ lattice L. By Theorem 2.7.5 the fixed point lattice of this involution L^χ is of hyperbolic signature. Since the $K3$ lattice is an even unimodular lattice, the lattice L^χ is even and 2-elementary. According to Proposition 2.2.1 the isomorphism type of L^χ is determined by three invariants (r, a, δ) where $r = r_+ + r_- = 1 + r_-$. It is clear that these invariants do not depend on the marking of X. The following theorem originally due to Kharlamov [23] shows that they determine the topological type of the real point set $X(\mathbb{R})$. We will write S_g for a real orientable surface of genus g and kS for the disjoint union of k copies of a real surface S.

Theorem 2.7.6 (Nikulin [34] Thm. 3.10.6)**.** *Let (X, χ_X) be a real $K3$ surface. Then:*

$$X(\mathbb{R}) = \begin{cases} \emptyset & \text{if } (r, a, \delta) = (10, 10, 0) \\ 2S_1 & \text{if } (r, a, \delta) = (10, 8, 0) \\ S_g \sqcup kS_0 & \text{otherwise} \end{cases}$$

where $g = \frac{1}{2}(22 - r - a)$ and $k = \frac{1}{2}(r - a)$.

Chapter 3

Gaussian lattices

3.1 Introduction

This chapter is in a sense the technical heart of the first part of this thesis. We study Gaussian lattices of hyperbolic signature and show how these give rise to arithmetic complex ball quotients. Anti-unitary involutions of the Gaussian lattice then correspond to real forms of these ball quotients. The main examples are the two Gaussian lattices $\Lambda_{1,5}$ and $\Lambda_{1,6}$ whose ball quotients correspond to the moduli spaces of smooth binary octavics and smooth quartic curves. These are the subject of Chapters 4 and 5. An excellent reference on the topic of Gaussian lattices is [1]. It also contains many examples of lattices over the Eisenstein and Hurwitz integers.

3.2 Gaussian lattices

A Gaussian lattice is a pair (Λ, ρ) with Λ a lattice and $\rho \in O(\Lambda)$ an automorphism of order four such that the powers ρ, ρ^2 and ρ^3 act without nonzero fixed points. Such a lattice Λ can be considered as a module over the ring of Gaussian integers $\mathcal{G} = \mathbb{Z}[i]$ by assigning $(a + ib)x = ax + b\rho(x)$ for all $x \in \Lambda$ and $a, b \in \mathbb{Z}$. The expression

$$h(x, y) = (x, y) + i(\rho(x), y)$$

defines a $\mathcal{G}$-valued non-degenerate Hermitian form on Λ which is linear in its second argument and anti-linear in its first argument. Conversely suppose that Λ is a free $\mathcal{G}$-module of finite rank equipped with a $\mathcal{G}$-valued Hermitian form $h(\cdot, \cdot)$. We define a symmetric bilinear form on the underlying $\mathbb{Z}$-lattice of Λ by taking the real part of the Hermitian form: $(x, y) = \operatorname{Re} h(x, y)$. Multiplication by i defines an automorphism ρ of order 4 so the pair (Λ, ρ) is a Gaussian lattice. It is easily checked that these two constructions are inverse to each other. Another way of defining a Gaussian lattice is by prescribing a Hermitian Gaussian matrix. Such a matrix H satisfies $\overline{H}^t = H$ and defines a Hermitian form on $\mathcal{G}^n$ by the formula $h(x,y) = \bar{x}^t H y$. The dual of a Gaussian lattice Λ is the lattice $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathcal{G})$. It is naturally embedded in the vector space $\Lambda \otimes_{\mathcal{G}} \mathbb{Q}(i)$ by the identification:

$$\Lambda^\vee = \{x \in \Lambda \otimes_{\mathcal{G}} \mathbb{Q}(i) \ ; \ h(x,y) \in \mathcal{G} \text{ for all } y \in \Lambda\}.$$

From now on we only consider non-degenerate Gaussian lattices that satisfy the condition $h(x,y) \in (1+i)\mathcal{G}$ for al $x, y \in \Lambda$. This is equivalent to $\Lambda \subset (1+i)\Lambda^\vee$ and implies that the underlying $\mathbb{Z}$-lattice of Λ is even.

Lemma 3.2.1. *The group $U(\Lambda)$ of unitary transformations of a Gaussian lattice Λ is equal to the group:*

$$\Gamma = \{\gamma \in O(\Lambda) \ ; \ \gamma \circ \rho = \rho \circ \gamma\}$$

of orthogonal transformations of the underlying $\mathbb{Z}$-lattice of Λ that commute with ρ.

Proof. If $\gamma \in U(\Lambda)$ then by definition $h(\gamma x, \gamma y) = h(x,y)$ for all $x, y \in \Lambda$. Using the definition of the Hermitian form h this is equivalent to:

$$(\gamma x, \gamma y) + i(\rho\gamma x, \gamma y) = (x,y) + i(\rho x, y).$$

By considering the real part of this equality we see that $\gamma \in O(\Lambda)$. Combining this with the equality of the imaginary parts of the equation we obtain $(\rho\gamma x, \gamma y) = (\gamma\rho x, \gamma y)$ for all $x, y \in \Lambda$. This is equivalent to: $\rho \circ \gamma = \gamma \circ \rho$. This proves the inclusion $U(\Lambda) \subset \Gamma$. For the other inclusion we can reverse the argument. □

A root $r \in \Lambda$ is a primitive element of norm -2. For every root r we define a complex reflection $s_{r,i}$ of order 4 (a tetraflection) by:

$$s_{r,i}(x) = x - (1-i)\frac{h(r,x)}{h(r,r)}r. \tag{3.1}$$

It is a unitary transformation of Λ that maps $r \mapsto ir$ and fixes pointwise the mirror $H_r = \{x \in \Lambda \ ; \ h(r,x) = 0\}$. The tetraflection $s_{r,i}$ and the mirror H_r only depend on the orbit of r under the group of units $\mathcal{G}^* = \{1, i, -1, -i\}$ of $\mathcal{G}$. We call such an orbit a projective root and denote it by $[r]$. If the group generated by tretraflections in the roots has finite index in $\Gamma = U(\Lambda)$ we say that the lattice Λ is tetraflective.

Example 3.2.2 (The Gaussian lattice Λ_2)**.** The D_4 lattice is given by:

$$D_4 = \left\{x \in \mathbb{Z}^4 \ ; \ \sum x_i \equiv 0 \bmod 2\right\}$$

with the symmetric bilinear form induced by the standard form $(x, y) = \sum x_i y_i$ of $\mathbb{Z}^4$ scaled by a factor -1 so that it is negative definite. We choose a non-standard basis for this lattice given by the roots $\{\beta_i\}$ with the Gram matrix B shown below.

$$\begin{aligned}\beta_1 &= e_3 - e_1\\ \beta_2 &= -e_1 - e_3\\ \beta_3 &= e_1 - e_2\\ \beta_4 &= e_3 - e_4\end{aligned} \qquad B = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ -1 & 1 & 0 & -2 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix ρ defines an automorphism of order 4 without fixed points which turns the lattice D_4 into a Gaussian lattice which we will call Λ_2. A basis for Λ_2 is given by the roots $\{\beta_1, \beta_3\}$ and the Gram matrix H with respect to this basis is given by:

$$H = \begin{pmatrix} -2 & 1+i \\ 1-i & -2 \end{pmatrix}.$$

A small calculation shows that there are 6 projective roots which are the $\mathcal{G}^*$-orbits of the roots:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ 1-i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \right\}$$

The group generated by the tetraflections in these roots is the complex reflection group G_8 of order 96 in the Shephard-Todd classification [37]. A basis for the dual lattice $\Lambda_2^\vee$ is given by $\left\{\frac{1}{1+i}\beta_1, \frac{1}{1+i}\beta_3\right\}$ so that $(1+i)\Lambda_2^\vee = \Lambda_2$.

Example 3.2.3. Consider the Gaussian lattice $\Lambda_{1,1}$ with basis $\{e_1, e_2\}$ and Hermitian form defined by the matrix:

$$H = \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}.$$

It is easy to verify that $(1+i)\Lambda_{1,1}^\vee = \Lambda_{1,1}$. A basis $\{\beta_1, \dots, \beta_4\}$ for the underlying $\mathbb{Z}$-lattice and its Gram matrix B are shown below.

$$\begin{aligned}\beta_1 &= e_2\\ \beta_2 &= ie_1\\ \beta_3 &= e_1 - ie_1\\ \beta_4 &= e_2 - ie_2\end{aligned} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

We conclude that the underlying $\mathbb{Z}$-lattice is isomorphic to $U \oplus U(2)$.

Using these two examples of Gaussian lattices we can construct many more by forming direct sums. We are especially interested in the Gaussian lattices of hyperbolic signature since these occur in the study of certain moduli problems. For example the Gaussian lattice $\Lambda_{1,1} \oplus \Lambda_2 \oplus \Lambda_2$ plays an important role in the study of the moduli space $\mathcal{M}_{0,8}$ of 8 points on the projective line. Yoshida and Matsumoto [33] prove that the unitary group of this lattice is generated by 7 tetraflections so that it is in particular tetraflective. We will come back to this lattice in Chapter 4.

3.3 Anti-unitary involutions of Gaussian lattices

Let Λ be a Gaussian lattice of rank n and signature (n_+, n_-). An anti-unitary involution of Λ is an involution χ of the underlying $\mathbb{Z}$-lattice that satisfies:

$$h(\chi(x), \chi(y)) = \overline{h(x, y)}.$$

Equivalently it is an involution that anti-commutes with ρ so that: $\chi \circ \rho = -\rho \circ \chi$. An anti-unitary involution χ naturally extends to the $\mathbb{Q}(i)$-vectorspace $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathcal{G}} \mathbb{Q}(i)$ which can be regarded as a $\mathbb{Q}$-vectorspace of dimension $2n$ and signature $(2n_+, 2n_-)$. The fixed point subspace $\Lambda_{\mathbb{Q}}^{\chi}$ is a $\mathbb{Q}$-vectorspace of dimension n and signature (n_+, n_-). Consider the fixed point lattice $\Lambda^{\chi} = \Lambda \cap \Lambda_{\mathbb{Q}}^{\chi}$. The Hermitian form restricted to Λ^{χ} takes on real values in $(1+i)\mathcal{G}$ so that it is in fact even-valued. This implies that $\Lambda^{\chi}(\frac{1}{2})$ in an integral lattice.

Proposition 3.3.1. *Let Λ be the Gaussian lattice defined by a Hermitian matrix H, so that in particular $\Lambda \cong \mathcal{G}^n$. Every anti-unitary involution of Λ is of the form $\chi = M \circ \mathrm{conj}$ where conj is standard complex conjugation on $\Lambda \cong \mathcal{G}^n$. The matrix M has coefficients in $\mathcal{G}$ and satisfies $\overline{M}M = I$ and: $\overline{M}^t HM = \overline{H}$.*

Proof. Suppose that χ is a anti-unitary involution of Λ. Since every anti-unitary involution on the vector space $\Lambda \otimes_{\mathcal{G}} \mathbb{Q}(i)$ is conjugate to standard complex conjugation there is a matrix N such that $N \circ \chi \circ N^{-1} = \mathrm{conj}$. We can rewrite this as $\chi = M \circ \mathrm{conj}$ where $M = N^{-1}\overline{N}$ and M has coefficients in $\mathcal{G}$. It is clear that $\overline{M}M = I$. Finally we can rewrite the equality $h(\chi(x), \chi(y)) = \overline{h(x, y)}$ as:

$$x^t \overline{M}^t HM\overline{y} = x^t \overline{H} y.$$

This holds for all $x, y \in \Lambda$ so that the last equality of the proposition follows. □

Let χ be an anti-unitary involution of the lattice Λ and let $[\chi]$ be its projective equivalence class. The elements of $[\chi]$ are the involutions $i^k\chi$ with $k = 0, 1, 2, 3$. By conjugation with the scalar i we see that the the two involutions $\{\chi, -\chi\}$ and also $\{i\chi, -i\chi\}$ are conjugate in Γ. The anti-unitary involutions χ and $i\chi$ need not be Γ-conjugate, so in particular their fixed point lattices need not be isomorphic. This can already be seen in the simplest case of anti-unitary involutions on $\mathcal{G}$. The fixed points lattice of the anti-unitary involution conj and $i \circ \mathrm{conj}$ are $\mathbb{Z}$ and $(1+i)\mathbb{Z}$ respectively.

We now present some computational lemma's on anti-unitary involutions of Gaussian lattices of small rank. These will be very useful later on and will be referenced to throughout this text.

χ	Λ	Λ^χ	B^χ
ψ_1	$\mathcal{G}$	A_1	(1)
$i\psi_1$	$\mathcal{G}$	$A_1(2)$	$(1+i)$
ψ_2	Λ_2	A_1^2	$\begin{pmatrix}1+i & 0\\ 1 & 1\end{pmatrix}$
ψ_2	$\Lambda_{1,1}$	$U(2)$	$\begin{pmatrix}1+i & 0\\ 0 & 1\end{pmatrix}$
ψ_2'	Λ_2	$A_1 \oplus A_1(2)$	$\begin{pmatrix}1 & 1+i\\ 1 & 1-i\end{pmatrix}$
ψ_2'	$\Lambda_{1,1}$	$A_1(2) \oplus (2)$	$\begin{pmatrix}1-i & 1\\ 1+i & 1\end{pmatrix}$
ψ_4	Λ_2^2	$D_4(2)$	$\begin{pmatrix}-1 & i & -i & -i\\ 0 & i & 0 & -1-i\\ -i & 1 & -1 & -1\\ 0 & -i & 0 & -1+i\end{pmatrix}$

Table 3.1: Some anti-unitary transformations of Gaussian lattices of small rank.

Lemma 3.3.2. *Let ψ_1, ψ_2, ψ_2' and ψ_4 be the transformations obtained by composing the following matrices with complex conjugation.*

$$M_1 = (1)\ ,\ M_2 = \begin{pmatrix} i & 0\\ 0 & 1\end{pmatrix}\ ,\ M_2' = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\ ,\ M_4 = \begin{pmatrix} 0 & 0 & i & 0\\ 0 & 0 & 0 & 1\\ i & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

They define anti-unitary involutions χ on certain Gaussian lattices Λ shown in Table 3.1. The fixed point lattices Λ^χ are also computed along with a matrix B^χ such that the columns of this matrix form a $\mathbb{Z}$-basis for the fixed point lattice Λ^χ.

Proof. Using the conditions on M_i from Proposition 3.3.1 it is a straightforward calculation to prove that the ψ_i are anti-unitary involutions. Furthermore we need to check that the columns of B^{ψ_i} form a basis for the fixed point lattice and that $\Lambda^{\psi_i} = \overline{B^{\psi_i}}^t \Lambda B^{\psi_i}$. For example the fixed point lattice $(\Lambda_2^2)^{\psi_2}$ is given by the subset

$$\{(z_1, z_2, i\bar{z}_1, \bar{z}_2)\ ;\ z_1, z_2 \in \mathcal{G}\} \subset \Lambda_2^2$$

and it is not difficult to check that the columns of B^{ψ_2} indeed form a $\mathbb{Z}$-basis. The verification for the other lattices proceeds similarly. □

Lemma 3.3.3. *The anti-unitary involution ψ_2 (rep. ψ_2') is conjugate to $i\psi_2$ (resp. $i\psi_2'$) in $U(\Lambda_2)$ as well as in $U(\Lambda_{1,1})$. The anti-unitary involution ψ_4 is conjugate to $i\psi_4$ in $U(\Lambda_2^2)$.*

Proof. The matrix $N = \left(\begin{smallmatrix} 0 & i \\ 1 & 0 \end{smallmatrix}\right)$ satisfies $\overline{N}^t\Lambda_2 N = \Lambda_2$ and $\overline{N}^t\Lambda_{1,1}N = \Lambda_{1,1}$ so it is contained in $U(\Lambda_2)$ and $U(\Lambda_{1,1})$. It also satisfies $N\psi_2\overline{N}^{-1} = i\psi_2$ and $N\psi_2'\overline{N}^{-1} = i\psi_2'$. Similarly conjugation by the matrix $\left(\begin{smallmatrix} 0 & N \\ N & 0 \end{smallmatrix}\right) \in U(\Lambda_2^2)$ maps ψ_4 to $i\psi_4$. □

3.4 Ball quotients from hyperbolic lattices

Let Λ be a Gaussian lattice of hyperbolic signature $(1, n)$ with $n \geq 2$ such that $\Lambda \subset (1+i)\Lambda^\vee$. We can associate to Λ a complex ball:

$$\mathbb{B} = \mathbb{P}\{x \in \Lambda \otimes_{\mathcal{G}} \mathbb{C} \ ; \ h(x,x) > 0\}.$$

The group $P\Gamma = PU(\Lambda)$ acts properly discontinuously on $\mathbb{B}$. The ball quotient $P\Gamma\backslash\mathbb{B}$ is a quasi-projective variety of finite hyperbolic volume by the theorem of Baily-Borel [7]. Recall that a root $r \in \Lambda$ is an element of norm -2. We denote by $\mathcal{H} \subset \mathbb{B}$ the union of all the root mirrors H_r and write $\mathbb{B}^\circ = \mathbb{B} \setminus \mathcal{H}$. The divisor $\mathcal{H}$ is a hyperplane arrangement and the mirrors are its irreducible components. In all the examples we consider later on the space $P\Gamma\backslash\mathbb{B}^\circ$ is a moduli space for certain smooth objects. The image of $\mathcal{H}$ in this space is called the discriminant and parametrizes certain singular objects. The following lemma describes how two mirrors in $\mathcal{H}$ can intersect.

Lemma 3.4.1. *Let r_1, r_2 be two roots in Λ such that $H_{r_1} \cap H_{r_2} \neq \emptyset$. The projective classes $[r_1]$ and $[r_2]$ are either identical, orthogonal or they span a Gaussian lattice of type Λ_2.*

Proof. Since the images of H_{r_1} and H_{r_2} meet in $\mathbb{B}$ there is a vector $x \in \Lambda$ with $h(x,x) > 0$ orthogonal to both r_1 and r_2. This implies that r_1 and r_2 span a negative definite space so that the Hermitian matrix:

$$\begin{pmatrix} -2 & h(r_1,r_2) \\ h(r_2,r_1) & -2 \end{pmatrix}$$

is negative definite. This is equivalent to $|h(r_1,r_2)|^2 < 4$ and since $h(r_1,r_2) \in (1+i)\mathcal{G}$ we see that either $h(r_1,r_2) = 0$ or $h(r_1,r_2) = \pm 1 \pm i$. In the second case we can assume that $h(r_1,r_2) = 1+i$ by multiplying r_1 and r_2 by suitable units in $\mathcal{G}^*$. □

Let $\mathbb{B}^\chi$ be the fixed point set in $\mathbb{B}$ of the real form $[\chi]$. Since the fixed point lattice Λ^χ is of hyperbolic signature this is a real ball given by:

$$\mathbb{B}^\chi = \mathbb{P}\{x \in \Lambda^\chi \otimes_{\mathbb{Z}} \mathbb{R} \ ; \ h(x,x) > 0\}.$$

Note that the lattice $\Lambda^{i\chi}$ defines the same real ball. The isomorphism type of the unordered pair $(\Lambda^{\chi}, \Lambda^{i\chi})$ is an invariant of the $P\Gamma$-conjugacy class of $[\chi]$ as shown by the following lemma. This invariant will prove very useful to distinguish between classes up to $P\Gamma$-conjugacy.

Lemma 3.4.2. *If the projective classes $[\chi]$ and $[\chi']$ of two anti-unitary involutions χ and χ' of Λ are conjugate in $P\Gamma$ then the isomorphism classes of the pairs of lattices $(\Lambda^{\chi}, \Lambda^{i\chi})$ and $(\Lambda^{\chi'}, \Lambda^{i\chi'})$ are equal.*

Proof. Suppose $[\chi]$ and $[\chi']$ are conjugate in $P\Gamma$. Then there is a $g \in \Gamma$ such that $[g\chi g^{-1}] = [\chi']$. This implies that $g\chi g^{-1} = \lambda\chi'$ for some unit $\lambda \in \mathcal{G}^*$ so that the anti-unitary involutions χ and $\lambda\chi'$ are conjugate in Γ. From this we deduce that $\Lambda^{\chi} \cong \Lambda^{\lambda\chi'}$. Since $g \in \Gamma$ commutes with multiplication by i the involutions $i\chi$ and $i\lambda\chi'$ are also conjugate in Γ and we get $\Lambda^{i\chi} \cong \Lambda^{i\lambda\chi'}$. □

Proposition 3.4.3. *Let $P\Gamma^{\chi}$ be the stabilizer of $\mathbb{B}^{\chi}$ in $P\Gamma$. Then:*

$$P\Gamma^{\chi} = \{[g] \in P\Gamma \ ; \ [g] \circ [\chi] = [\chi] \circ [g]\}$$

Proof. The following statements are equivalent:

$$\begin{aligned}
&[g] \in P\Gamma^{\chi},\\
&[gx] \in \mathbb{B}^{\chi} \text{ for all } [x] \in \mathbb{B}^{\chi},\\
&[\chi(gx)] = [g(\chi x)] \text{ for all } [x] \in \mathbb{B}^{\chi},\\
&[\chi(gz)] = [g(\chi z)] \text{ for all } [z] = [x + iy] \ , \ [x], [y] \in \mathbb{B}^{\chi},\\
&[g \circ \chi] = [\chi \circ g].
\end{aligned}$$

□

From Proposition 3.4.3 we see that for every element $[g] \in P\Gamma^{\chi}$ precisely one of the following holds:

I. There is a $g \in [g]$ such that: $g\chi g^{-1} = \chi$ so that: $g\Lambda^{\chi} = \Lambda^{\chi}$.

II. There is a $g \in [g]$ such that $g\chi g^{-1} = i\chi$ so that: $g\Lambda^{\chi} = \Lambda^{i\chi}$.

We use Chu's convention from [12] and say that $[g] \in P\Gamma^{\chi}$ is of type I respectively of type II. The elements of type I form a subgroup of $P\Gamma^{\chi}$ which we denote by $P\Gamma^{\chi}_I$. If there exists an element of type II then this subgroup is of index 2, otherwise every element of $P\Gamma^{\chi}$ is of type I.

Every element $[g] \in P\Gamma^{\chi}$ of type I determines a unique element in $PO(\Lambda^{\chi})$ so there is a natural embedding: $P\Gamma^{\chi} \hookrightarrow PO(\Lambda^{\chi})$. In general not every element $[g] \in PO(\Lambda^{\chi})$ extends to the group $P\Gamma$. Let B be a matrix whose columns represent a basis for the lattice Λ^{χ} in Λ. Then we have:

$$P\Gamma^{\chi}_I = \{[M] \in PO(\Lambda^{\chi}) \ ; \ BMB^{-1} \in \mathcal{G}^{n+1 \times n+1}\} \tag{3.2}$$

so that $P\Gamma^{\chi}_I$ is the subgroup of $PO(\Lambda^{\chi})$ consisting of all elements that extend to unitary transformations of the Gaussian lattice Λ.

Theorem 3.4.4. *The groups $P\Gamma^\chi$ and $PO(\Lambda^\chi)$ are commensurable.*

Proof. We have seen that the intersection of the two groups is given by:

$$P\Gamma^\chi \cap PO(\Lambda^\chi) = P\Gamma_I^\chi$$

and has at most index 2 in $P\Gamma^\chi$. We now prove that this intersection is a congruence subgroup of $PO(\Lambda^\chi)$ so that in particular it has finite index. Recall that the adjoint matrix B^{adj} has coefficients in $\mathcal{G}$ and satisfies $(\det B)B^{-1} = B^{\mathrm{adj}}$. If we write $M = 1 + X$ then by Equation 3.2 we have $[M] \in P\Gamma_I^\chi$ if and only if $\det B$ divides BXB^{adj}. This is certainly the case if $\det B$ divides X so if $M \equiv 1 \bmod (\det B)$. This implies that $P\Gamma_I^\chi$ contains the principal congruence subgroup:

$$\{[M] \in PO(\Lambda^\chi)\ ;\ M \equiv 1 \bmod (\det B)\}.$$

□

In the examples we encounter the lattice Λ^χ is reflective so that the reflections generate a finite index subgroup in $PO(\Lambda^\chi)$. By the results of Section 2.4 the group $PO(\Lambda^\chi)$ is of the form $W(C) \rtimes S(C)$ where $C \subset \mathbb{B}^\chi$ is a Coxeter polytope of finite volume, $W(C)$ its reflection group and $S(C)$ a group of automorphisms of C. The polytope C can be determined by Vinberg's algorithm. We will see that in many cases the reflection subgroup of the group $P\Gamma_I^\chi$ is also of finite index. This can be determined by applying Vinberg's algorithm with the condition that in every step we only accept roots r such that the reflection $s_r \in PO(\Lambda^\chi)$ satisfies Equation 3.2. This is equivalent to the condition:

$$\frac{2r}{h(r,r)} \in \Lambda^\vee. \tag{3.3}$$

We finish this section by describing how a root mirror $H_r \in \mathcal{H}$ can meet the real ball $\mathbb{B}^\chi$. This intersection can be of codimension one or two as shown by the following lemma.

Lemma 3.4.5. *Suppose $r \in \Lambda$ is a root such that $\mathbb{B}^\chi \cap H_r \neq \emptyset$. Then $\mathbb{B}^\chi \cap H_r$ is equal to $\mathbb{B}^\chi \cap L$ with $L^\perp$ a lattice in Λ^χ of type $A_1, A_1(2), A_1 \oplus A_1(2)$ or $A_1(2)^2$.*

Proof. If $x \in H_r \cap \mathbb{B}^\chi$ then x is fixed by both s_r and $s_{\chi r}$ so the intersection $H_r \cap H_{\chi r}$ is non-empty and we are in the situation of Lemma 3.4.1. Suppose that $\chi[r] = [r]$. If $\chi r = \pm r$ then either r or ir is a root of Λ^χ. Both have lenght -2 so they span a root system of type A_1. If $\chi r = \pm ir$ then one of $(1 \pm i)r$ is a root of Λ^χ. Both have norm -4 so they span a root system of type $A_1(2)$. If $\chi[r] \neq [r]$ then the roots r and χr span a rank two Gaussian lattice that is either $(-2) \oplus (-2)$ or Λ_2 according to Lemma 3.4.1. The involution χ acts on these lattices as the anti-unitary involution ψ_2'. The fixed point lattice for $(-2) \oplus (-2)$ is $A_1(2)^2$ as follows from a straightforward computation. For Λ_2 we get the fixed point lattice $A_1 \oplus A_1(2)$ as follows from Lemma 3.3.2. □

3.5 Examples

The Gaussian lattice $\Lambda_{1,2}$

The lattice $\Lambda_{1,2} = \Lambda_2 \oplus (2)$ of signature $(1, 2)$ is related to the moduli space $\mathcal{M}(321^3)$ of eight-tuples of points on $\mathbb{P}^1$ such that there are unique points of multiplicity 3 and 2 and three distinct points of multiplicity 1. We study the anti-unitary involutions of this lattice in some detail. Using Table 3.1 we can immediately write down two anti-unitary involutions of $\Lambda_{1,2}$, namely $\psi_2 \oplus \psi_1$ and $\psi_2' \oplus \psi_1$. We will prove that their projective classes are distinct modulo conjugation in $P\Gamma = PU(\Lambda_{1,2})$. There is however a another anti-unitary involution of $\Lambda_{1,2}$ given by $\psi_3 = M_3 \circ \mathrm{conj}$ where M_3 is the complicated matrix:

$$M_3 = \begin{pmatrix} -2+i & 2-2i & -2-2i \\ 2 & -1 & 2i \\ 1+3i & -2-2i & -3+2i \end{pmatrix}.$$

This anti-unitary involution takes on a much simpler form if we change to a different basis for $\Lambda_{1,2}$ as shown by the following lemma.

Lemma 3.5.1. *The Gaussian lattices $\Lambda_2 \oplus (2)$ and $(-2) \oplus \Lambda_{1,1}$ are isomorphic. The anti-unitary involution ψ_3 of $\Lambda_2 \oplus (2)$ maps to the anti-unitary involution $\psi_1 \oplus \psi_2$ of $(-2) \oplus \Lambda_{1,1}$ under this isomorphism.*

Proof. The underlying $\mathbb{Z}$-lattices of the Gaussian lattices $\Lambda_2 \oplus (2)$ and $(-2) \oplus \Lambda_{1,1}$ are $D_4 \oplus (2)$ and $U \oplus U(2) \oplus A_1$. Both are even 2-elementary lattices and the invariants (r_+, r_-, l, δ) of Theorem 2.2.1 are easily seen to be $(1, 2, 3, 1)$ for both lattices hence they are isomorphic. An explicit base change is given by: $\overline{B}^t(\Lambda_2 \oplus (2))B = (-2) \oplus \Lambda_{1,1}$ for the unimodular matrix

$$B = \begin{pmatrix} 1+i & i & 0 \\ 1-i & 0 & 1 \\ 1 & 1 & i \end{pmatrix}.$$

The final statement follows from the equality $B(\psi_1 \oplus \psi_2)\overline{B}^{-1} = \psi_3$. □

Proposition 3.5.2. *The projective classes of the three anti-unitary involutions χ given by $\psi_2 \oplus (2)$, $\psi_2' \oplus (2)$ and ψ_3 of $\Lambda_{1,2}$ are distinct modulo conjugation in $P\Gamma$. The groups $P\Gamma^\chi$ of these involutions are hyperbolic Coxeter groups and their Coxeter diagrams are shown in Table 3.2.*

Proof. We will use Lemma 3.4.2 to show that the projective classes of the three anti-unitary involutions are not $P\Gamma$-conjugate. For this we need to calculate the fixed point lattices of χ and $i\chi$ for all three anti-unitary involutions. These can be read of from Table 3.1 for the anti-unitary involutions $\psi_2 \oplus \psi_1$ and

$\psi_2' \oplus \psi_1$. For ψ_3 we use Lemma 3.5.1 combined with Table 3.1. We also use Lemma 2.2.5 to simplify the lattices. For example:

$$\begin{aligned}\Lambda_{1,2}^{i(\psi_2\oplus\psi_1)} &\cong (4) \oplus A_1^2 \\ &\cong (2) \oplus A_1 \oplus A_1(2)\end{aligned}$$

where the first isomorphism follows from Table 3.1 and Lemma 3.3.3 and the second follows from Lemma 2.2.5. The results are listed in Table 3.2. The lattices $(2) \oplus A_1 \oplus A_1(2)$ and $U(2) \oplus A_1(2)$ in this table are not isomorphic: if we scale them by a factor $\frac{1}{2}$ then one is even while the other is not. This proves that the $P\Gamma$-conjugation classes of $\psi_2 \oplus \psi_1$ and ψ_3 are distinct. We can distinguish the fixed point lattices of $\psi_2' \oplus \psi_1$ from the previous two by calculating their discriminants.

To prove that the group $P\Gamma^\chi$ is a hyperbolic Coxeter group we use Vinberg's algorithm with the condition that we only accept roots that satisfy Equation 3.3. A vector $x = (x_1, x_2, x_3) \in \Lambda_{1,2} \otimes_{\mathcal{G}} \mathbb{Q}$ is contained in $\Lambda_{1,2}^\vee$ if and only if $x_1, x_2 \in \frac{1}{1+i}\mathcal{G}$ and $x_3 \in \frac{1}{2}\mathcal{G}$ so that we can rewrite Equation 3.3 as:

$$\frac{2(1+i)x_i}{h(x,x)} \in \mathcal{G} \quad \text{for } i = 1,2\ , \quad \frac{4x_3}{h(x,x)} \in \mathcal{G}. \tag{3.4}$$

These equations are satisfied if $h(x,x) = -2$, so for short roots. If $h(x,x) = -4$ the equations are equivalent to x_1 and x_2 being divisible by $1+i$. We demonstrate Vinberg's algorithm with conditions for the anti-unitary involution $\psi_2' \oplus \psi_1$ since this is the most complicated case. For the other anti-unitary involutions the execution of the algorithm is similar. The fixed point lattice $\Lambda_{1,2}^{\psi_2'\oplus\psi_1}$ is isomorphic to $(2) \oplus A_1 \oplus A_1(2)$ and a basis in $\Lambda_{1,2}$ is given by the columns $\{e_0, e_1, e_2\}$ of the matrix:

$$B = \begin{pmatrix} 0 & 1 & 1+i \\ 0 & 1 & 1-i \\ 1 & 0 & 0 \end{pmatrix}.$$

Equations 3.4 can be checked by writing a root on this basis. For example the root $e_0 - e_1 - e_2$ of norm -4 and height $\frac{1}{2}$ is not accepted. The output of Vinberg's algorithm with the condition on the roots is shown in Figure 3.1. The resulting Coxeter diagram has a diagram automorphism whose matrix with respect to the basis $\{e_0, e_1, e_2\}$ is:

$$T = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 0 & -2 \\ -1 & -1 & -1 \end{pmatrix}.$$

A small calculation shows that the matrix $BTB^{-1} \notin \mathcal{G}^{3\times 3}$ so that from Equation 3.2 we see that $T \notin P\Gamma^{\psi_2'\oplus\psi_1}$.

□

	e_0	e_1	e_2
p	1	0	0
height 0			
r_1	0	0	1
r_2	0	1	0
height 1			
r_3	1	0	-1
height 2			
r_4	2	-2	-1

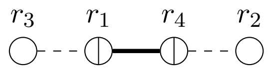


Figure 3.1: Vinberg's algorithm with conditions on the roots for the lattice $(2) \oplus A_1 \oplus A_1(2)$.

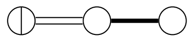
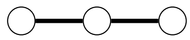
χ	$\Lambda_{1,2}^{\chi}$	$\Lambda_{1,2}^{i\chi}$	$P\Gamma^{\chi}$
$\psi_2 \oplus \psi_1$	$(2) \oplus A_1^2$	$(2) \oplus A_1 \oplus A_1(2)$	
ψ_3	$(2) \oplus A_1^2$	$U(2) \oplus A_1(2)$	
$\psi_2' \oplus \psi_1$	$(2) \oplus A_1 \oplus A_1(2)$	$(2) \oplus A_1(2)^2$	

Table 3.2: The three classes of anti-unitary involutions of the lattice $\Lambda_{1,2}$.

We will see in Section 4.5 that the moduli space $\mathcal{M}(321^3)$ has three connected components so the three projective classes actually form a complete set of representatives for $P\Gamma$-conjugation classes of anti-unitary involutions in $\Lambda_{1,2}$.

The Gaussian lattice $\Lambda_{1,6}$

The lattice $\Lambda_{1,6} = \Lambda_2^3 \oplus (2)$ is related to the moduli space of plane quartic curves which is the topic of Chapter 5. In this section we collect some useful properties of this lattice that will be used in that chapter. We start by introducing a very convenient basis.

Lemma 3.5.3. *There is a basis $\{e_1, \ldots, e_7\}$ for $\Lambda_{1,6}$ so that the basis vectors are enumerated by the vertices of the Coxeter diagram of type E_7 as in Example 2.3.4. By this we mean that the basis satisfies:*

$$h(e_i, e_j) = \begin{cases} -2 & \text{if } i = j \\ 1 + \operatorname{sign}(j-i)i & \text{if } i,j \text{ connected} \\ 0 & \text{else.} \end{cases}$$

Proof. An example of such a basis is given by the column vectors of the matrix:

$$B_{E_7} = \begin{pmatrix} 1 & -1-i & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1-i & 1 & -1-i & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1-i & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

□

The tetraflections $s_{e_i,i} \in U(\Lambda_{1,6})$ with $i = 1, \ldots, 7$ satisfy the commutation and braid relations of the Artin group $A(E_7)$ of type E_7 so that they induce a representation: $A(E_7) \to U(\Lambda_{1,6})$ by tetraflections. If this homomorphism is surjective then the lattice $\Lambda_{1,6}$ is tetraflective. We suspect this is the case but will not need it in what follows.

Proposition 3.5.4. *Let V be the $\mathbb{F}_2$ vectorspace defined by*

$$V = \Lambda_{1,6}/(1+i)\Lambda_{1,6} \cong (\mathbb{F}_2)^7.$$

with the invariant quadratic form $q(x) \equiv \frac{1}{2}h(x,x) \bmod 2$. Reduction modulo $(1+i)$ induces a surjective homomorphism:

$$U(\Lambda_{1,6}) \to O(V,q) \cong W(E_7)^+.$$

where we denote by $W(E_7)^+$ the Weyl group of type E_7 divided out by its center $\{\pm 1\}$. This group is generated by the images of the tetraflections $s_{e_i,i}$ with $i = 1, \ldots, 7$.

Proof. The tetraflections $s_{e_i,i}$ with $i = 1, \ldots, 7$ act as reflections on the vectorspace V since their squares act as the identity. This defines a representation of the Weyl group $W(E_7)$ on V. The matrices of these tetraflections modulo $(1+i)$ are identical to the matrices of the simple generating reflections of $W(E_7)$ modulo 2. These act naturally on the $\mathbb{F}_2$-vectorspace $V' = Q/2Q$ where Q is the root lattice of type E_7. This space is equipped with the invariant quadratic form defined by $q'(x) \equiv \frac{1}{2}(x,x) \bmod 2$ where $(\cdot,\cdot)$ is the natural bilinear form on Q defined by the Gram matrix of type E_7. We conclude that the representation spaces (V,q) and (V',q') for $W(E_7)$ are isomorphic. The proposition now follows from [9], §4 Ex. 3 in which it is stated that there is an exact sequence:

$$1 \to \{\pm 1\} \to W(E_7) \to O(V,q) \to 1.$$

□

Let $U(\Lambda_{1,6})^a$ be the set of anti-unitary transformations of $\Lambda_{1,6}$. Reduction modulo $(1+i)$ also induces a map

$$U(\Lambda_{1,6})^a \to O(V,q) \cong W(E_7)^+$$

since complex conjugation induces the identity map on V. The projective class of an anti-unitary involution $[\chi]$ maps to an involution u of $W(E_7)^+$ under this map. Its image does not depend on the choice of representative for the class $[\chi]$ since multiplication by i acts as the identity on V. This implies that the conjugation class of the involution u in $W(E_7)^+$ is an invariant of the $P\Gamma$-conjugation class of $[\chi]$. The conjugation classes of involutions of $W(E_7)$ were determined in Example 2.3.4. There are ten conjugation classes that come in five pairs $\{u, -u\}$. Since both u and $-u$ map to the same involution $\overline{u} \in W(E_7)^+$ each pair determines a unique conjugation class in $W(E_7)^+$. We will use this to prove the following theorem.

Theorem 3.5.5. *The transformations χ_i with $i = 1, \ldots, 6$ defined by:*

$$\chi_1 = \psi_2^3 \oplus \psi_1 \qquad \chi_2 = \psi_2^2 \oplus \psi_2' \oplus \psi_1 \qquad \chi_3 = \psi_2 \oplus (\psi_2')^2 \oplus \psi_1$$
$$\chi_4 = (\psi_2')^3 \oplus \psi_1 \qquad \chi_5 = \psi_4 \oplus \psi_2 \oplus \psi_1 \qquad \chi_6 = \psi_4 \oplus \psi_3$$

are anti-unitary involutions of the lattice $\Lambda_2^3 \oplus (2)$. Their projective classes are distinct modulo conjugation by $P\Gamma$.

Proof. According to Lemma 3.3.2 and the previous example it is clear that the χ_i are anti-unitary involutions of the lattice $\Lambda_2^3 \oplus (2)$. By reducing the χ_i modulo $(1+i)$ they map to involutions $\overline{u}_i$ in $W(E_7)^+$. To distinguish them we calculate the dimensions of the fixed point spaces in V and compare them to those of the involutions in $W(E_7)^+$. From this we conclude that $\overline{u}_1, \overline{u}_2$ and $\overline{u}_4$ are of type $(1, E_7), (A_1, D_6)$ and (A_1^3, A_1^4) respectively. It is clear that $\overline{u}_5 = \overline{u}_6$. We used the computer algebra package SAGE to determine that both are of type $(D_4, A_1^{3\prime})$ and that $\overline{u}_3$ is of type (A_1^2, D_4A_1). All of this is summarized in Table 3.3.

This method is insufficient to distinghuish the classes of χ_5 and χ_6. For this we determine the fixed point lattice $\Lambda_{1,6}^{\chi_i}$ and $\Lambda_{1,6}^{i\chi_i}$ for $i = 5, 6$ and use Lemma 3.4.2. The lattices $\Lambda_{1,6}^{\chi_5}$ and $\Lambda_{1,6}^{\chi_6}$ are both isomorphic to $(2) \oplus A_1^2 \oplus D_4(2)$. The lattice $\Lambda_{1,6}^{i\chi_5}$ is isomorphic to:

$$(2) \oplus A_1 \oplus A_1(2) \oplus D_4(2) \cong (2) \oplus A_1^3 \oplus A_1(2)^3$$

where we used Lemma 2.2.5. The fixed point lattice $\Lambda_{1,6}^{i\chi_6}$ is isomorphic to $U(2) \oplus A_1(2) \oplus D_4(2)$. After scaling by a factor $\frac{1}{2}$ we see that $\Lambda_{1,6}^{i\chi_5}$ is odd while the $\Lambda_{1,6}^{i\chi_6}$ is even so that they are not isomorphic. Consequently the $P\Gamma$-conjugacy classes of the $[\chi_5]$ and $[\chi_6]$ are distinct. □

χ_i	Type of $\overline{u}_i$	$\dim V^{\overline{u}_i}$
χ_1	$(1, E_7)$	7
χ_2	(A_1, D_6)	6
χ_3	(A_1^2, D_4A_1)	5
χ_4	(A_1^3, A_1^4)	4
χ_5	$(D_4, A_1^{3\prime})$	5
χ_6	$(D_4, A_1^{3\prime})$	5

Table 3.3: The six projective classes of anti-unitary involutions of $\Lambda_{1,6}$ and the type of the involution they induce in $W(E_7)^+$ by reducing modulo $1+i$.

χ_i	$\Lambda_{1,6}^{\chi_i}$	$d(\Lambda_{1,6}^{\chi_i})$	$\Lambda_{1,6}^{i\chi_i}$	$d(\Lambda_{1,6}^{i\chi_i})$
χ_1	$(2) \oplus A_1^6$	2^7	$(2) \oplus A_1^5 \oplus A_1(2)$	2^8
χ_2	$(2) \oplus A_1^5 \oplus A_1(2)$	2^8	$(2) \oplus A_1^4 \oplus A_1(2)^2$	2^9
χ_3	$(2) \oplus A_1^4 \oplus A_1(2)^2$	2^9	$(2) \oplus A_1^3 \oplus A_1(2)^3$	2^{10}
χ_4	$(2) \oplus A_1^3 \oplus A_1(2)^3$	2^{10}	$(2) \oplus A_1^2 \oplus A_1(2)^4$	2^{11}
χ_5	$(2) \oplus A_1^2 \oplus D_4(2)$	2^9	$(2) \oplus A_1^3 \oplus A_1(2)^3$	2^{10}
χ_6	$(2) \oplus A_1^2 \oplus D_4(2)$	2^9	$U(2) \oplus A_1(2) \oplus D_4(2)$	2^{10}

Table 3.4: The fixed point lattices for χ_j and $i\chi_j$ for $j = 1, \ldots, 6$ and their discriminants.

Remark 3.5.6. The question remains whether the list of anti-unitary involutions from Theorem 3.5.5 is complete. This is in fact the case as we will see in Chapter 5, Proposition 5.6.1. It is a consequence of the fact that the moduli space of smooth real quartics consists of six connected components.

Theorem 3.5.7. *The hyperbolic lattices $\Lambda_{1,6}^{\chi}$ for $\chi = \chi_j, i\chi_j$ where $j = 1, \ldots, 6$ from Table 3.4 are all reflective and the hyperbolic Coxeter diagrams for the groups $PO(\Lambda_{1,6}^{\chi})$ are shown in Figure 3.2. By projecting the diagrams for χ_j with $j = 1, \ldots, 5$ onto a wall defined by a short root we recover Chu's diagrams [12].*

Proof. We observe from Table 3.4 that there are seven distinct hyperbolic lattices. To prove that they are reflective we apply Vinberg's algorithm. We demonstrate this for the hyperbolic lattice $(2) \oplus A_1^2 \oplus D_4(2)$ corresponding to the anti-unitary involutions χ_5 and χ_6. Let $\{e_0, e_1, e_2\}$ be an orthonormal

	e_0	e_1	e_2	u_1	u_2	u_3	u_4
p	1	0	0	0	0	0	0
height 0							
r_1	0	1	−1	0	0	0	0
r_2	0	0	1	0	0	0	0
r_3	0	0	0	1	−1	−1	−1
r_4	0	0	0	0	0	0	2
r_5	0	0	0	0	0	1	−1
r_6	0	0	0	0	1	−1	0
height$\frac{1}{2}$							
r_7	1	−1	0	−1	−1	0	0
height1							
r_8	1	−1	−1	0	0	0	0

Table 3.5: Vinberg's algorithm for the hyperbolic lattice $(2) \oplus A_1^2 \oplus D_4(2)$. This lattice corresponds to the two anti-unitary involutions χ_5 and χ_6.

basis for $(2) \oplus A_1^2$. Recall that the root lattice $D_4(2)$ is given by:

$$D_4(2) = \{(u_1, u_2, u_3, u_4) \in \mathbb{Z}^4 \; ; \; \sum_{i=0}^{4} u_i \equiv 0 \pmod 2\}.$$

It contains root of norm -4 and -8 and both form a root system of type D_4. Together these roots form a root system of type F_4. If we choose the controlling vector e_0 the height 0 root system is of type B_2F_4 spanned by the roots $\{r_1, \ldots, r_6\}$ from Table 3.5. This table also shows how the algorithm proceeds. The resulting Coxeter diagram is shown in Figure 3.2. The Coxeter diagrams for the other six hyperbolic lattices can be computed similarly and are also shown in this figure. □

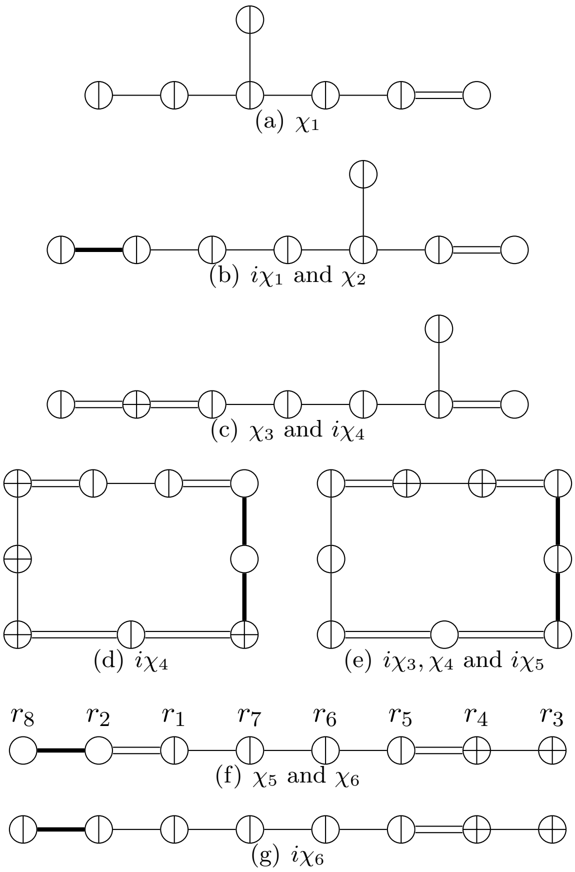


Figure 3.2: The Coxeter diagram of the groups $PO(\Lambda_{1,6}^{\chi})$ for $\chi = \chi_j, i\chi_j$ with $j = 1, \ldots, 6$.

Chapter 4

Hyperelliptic curves of genus three

4.1 Introduction

A smooth complex curve C of genus 3 is hyperelliptic if it admits a degree 2 cover of the projective line $\mathbb{P}^1(\mathbb{C})$. By the Riemann-Hurwitz formula such a double cover is ramified over eight distinct points in $\mathbb{P}(\mathbb{C})^1$. Conversely this eight-tuple of distinct points determines a complex curve C of genus 3 by the formula:

$$C = \{w^2 = f(x, y)\}$$

where $f(x, y)$ is a smooth complex binary octavic whose zeroes on the projective line are the eight points. The eight-tuple of distinct points on $\mathbb{P}(\mathbb{C})^1$, or rather its $\mathrm{Aut}(\mathbb{P}(\mathbb{C})^1)$-orbit determines the curve C up to isomorphism. Consequently the moduli space of smooth hyperelliptic curves of genus 3 is isomorphic to the moduli space $\mathcal{M}(1^8)$ of eight distinct unordered points on the projective line. This moduli space has a description as an arithmetic complex ball quotient due to the work of Deligne and Mostow [15]. It has since been studied by many authors. Chu [12] has shown that the five components of the moduli space $\mathcal{M}^{\mathbb{R}}(1^8)$ of smooth real binary octavics are isomorphic to arithmetic real ball quotients. In this chapter we review this work and prove some new results on the maximal real component of this moduli space where all zeroes of the real binary octavic are real. We also study the moduli spaces of certain stable real binary octavics.

4.2 Configurations of eight points on the projective line

A complex binary octavic is a homogeneous polynomial in two variables of degree eight with complex coefficients. The zero set of a binary octavic $p(x, y)$

consists of eight points in $\mathbb{P}(\mathbb{C})^1$. Conversely an eight-tuple of unordered points determines a binary octavic up to scalar multiplication by the formula:

$$p(x,y) = \prod_{i=1}^{8} (\beta_i x - \alpha_i y)$$

where the points are given up to permutation by: $p_i = [\alpha_i : \beta_i]$. The space of binary octavics up to scalar multiplication is the projective space $P_{8,2} = \mathbb{P}\,\mathrm{Sym}^8(\mathbb{C}^2)^\vee$ of dimension eight. The locus of singular octavics $\Delta \subset P_{8,2}$ consists of all octavics such that two (or more) points coincide, is of codimension one. The moduli space of smooth binary octavics or equivalently of 8 distinct unordered points on $\mathbb{P}(\mathbb{C})^1$ is given by:

$$\mathcal{M}(1^8) = \mathrm{PGL}_2(\mathbb{C})\backslash \left(P_{8,2}(\mathbb{C}) - \Delta(\mathbb{C})\right).$$

There is a period map $\mathrm{Per} : \mathcal{M}(1^8) \to \Gamma\backslash\mathbb{B}_5$ that maps this moduli space to a ball quotient of dimension five. The construction follows from the work of Deligne and Mostow in [15]. There is also a construction of this ball quotient using periods of $K3$ surfaces by Kondo [27]. We will not review these constructions but we will give a description of the ball quotient in terms of the Gaussian lattice $\Lambda_{1,5} = \Lambda_2^2 \oplus \Lambda_{1,1}$ due to Matsumoto and Yoshida in [33]. This lattice determines a complex ball by the expression:

$$\mathbb{B}_5 = \mathbb{P}\{x \in \Lambda_{1,5} \otimes_{\mathcal{G}} \mathbb{C} \ ; \ h(x,x) > 0\}.$$

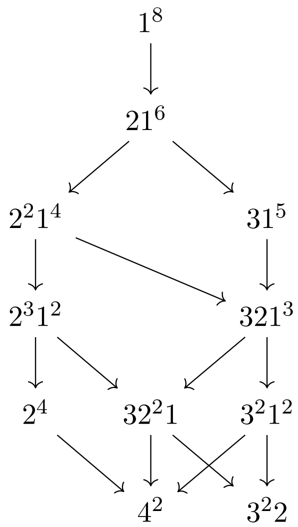
The discriminant locus $\mathcal{H} \subset \mathbb{B}$ is the collection of all root mirrors H_r. We will also write $\mathbb{B}^\circ = \mathbb{B} \setminus \mathcal{H}$ and Γ for the unitary group $U(\Lambda_{1,5})$.

Theorem 4.2.1 (Deligne-Mostow)**.** *The period map induces an isomorphism:*

$$\mathrm{Per} : \mathcal{M}(1^8) \to P\Gamma\backslash\mathbb{B}_5^\circ$$

This isomorphism can be extended to include certain singular octavics. A binary octavic called semi-stable if at most four of its zeroes coincide and stable if at most three coincide. We will denote the moduli space of stable octavics by $\mathcal{M}_s$ and use the notation $\mathcal{M}(4^{n_4}3^{n_3}2^{n_2}1^{n_1})$ for the locus of octavics with n_i points of multiplicity i. The strictly semi-stable octavics correspond to a single point in the GIT-compactification $\overline{\mathcal{M}}_s$ of $\mathcal{M}_s$. The extension of the period isomorphism to this GIT-compactification maps the non-smooth stable octavics to $P\Gamma\backslash\mathcal{H}$ and the strictly semi-stable octavics to the unique cusp of $P\Gamma\backslash\mathbb{B}$.

For a smooth point $x \in \mathcal{H}$ there is a unique root $r \in \Lambda_{1,5}$ such that the mirror H_r contains x. There is a single $P\Gamma$-orbit of roots and for every root the lattice $r^\perp$ is isomorphic to the hyperbolic Gaussian lattice $\Lambda_2^2 \oplus (2)$. A smooth point of $\mathcal{H}$ corresponds to an octavic whose eight-tuple of zeroes has precisely two points that coincide. By the above discussion we see that moduli



Locus in $\mathcal{M}_s$	Gaussian lattice
1^8	$\Lambda_2^2 \oplus \Lambda_{1,1}$
21^6	$\Lambda_2^2 \oplus (2)$
2^21^4	$\Lambda_2 \oplus (-2) \oplus (2)$
31^5	$\Lambda_2 \oplus \Lambda_{1,1}$
2^31^2	$(-2)^2 \oplus (2)$
321^3	$(-2) \oplus \Lambda_{1,1}$
2^4	$U(2)$
32^21	$(-2) \oplus (2)$
3^21^2	$\Lambda_{1,1}$

Figure 4.1: The loci of binary octavics and their corresponding hyperbolic Gaussian lattices

space of such octavics $\mathcal{M}(21^6)$ is the ball quotient corresponding to the lattice $\Lambda_2^2 \oplus (2)$. We can iterate this construction: in the lattice $\Lambda_2^2 \oplus (2)$ there are two orbits of roots which should correspond to the loci $\mathcal{M}(31^5)$ and $\mathcal{M}(2^21^4)$. We can calculate $r^\perp$ in each case and compare this to Table 1 of [27] to determine which root corresponds to which type of octavic . The results are listed in Figure 4.1.

4.3 Real binary octavics

The moduli space of real binary octavics is studied by Chu in [12]. We review the main results in this section. A real binary octavic is an element of the space $P_{8,2}(\mathbb{R})$ so that it has real coefficients. The eight points constituting the zero set of such an octavic need not be real: a zero is either real or part of a pair of complex conjugate zeroes. There are five possibilities: the number of complex conjugate pairs of zeroes can be $0, 1, 2, 3$ and 4. Each of these possibilities determines a component in the moduli space of smooth real binary octavics:

$$\mathcal{M}^{\mathbb{R}}(1^8) = \mathrm{PGL}_2(\mathbb{R}) \backslash \left(P_{8,2}(\mathbb{R}) - \Delta(\mathbb{R})\right).$$

We denote the five components by $\mathcal{M}_j^{\mathbb{R}}(1^8)$ where j is the number of complex conjugate pairs of zeroes of a representative octavic . One of the main results of Chu is that each of these components admits a real hyperbolic structure.

Theorem 4.3.1 (Chu)**.** *There are isomorphisms of real analytic orbifolds:*

$$\mathcal{M}_j^{\mathbb{R}}(1^8) \to P\Gamma^{\chi_j} \backslash (\mathbb{B}_5^{\chi_j})^\circ \qquad j = 0, \ldots, 4$$

j	χ_j	$\Lambda_{1,5}^{\chi_j}$	Type of $u \in S_8$
0	ψ_2^3	$(2) \oplus A_1^5$	(1)
1	$\psi_2' \oplus \psi_2^2$	$(2) \oplus A_1^4 \oplus A_1(2)$	(2)
2	$(\psi_2')^2 \oplus \psi_2$	$(2) \oplus A_1^3 \oplus A_1(2)^2$	$(2)^2$
3	$(\psi_2')^3$	$(2) \oplus A_1^2 \oplus A_1(2)^3$	$(2)^3$
4	$\psi_4 \oplus \psi_2$	$U(2) \oplus D_4(2)$	$(2)^4$

Table 4.1: The real forms and their fixed point lattices.

for certain anti-unitary involutions χ_j of $\Lambda_{1,5}$. The group $P\Gamma^{\chi_j}$ is the stabilizer of the real ball $\mathbb{B}_5^{\chi_j}$ in $P\Gamma$. It is commensurable with the orthogonal group $PO(\Lambda_{1,5}^{\chi_j})$ which is a hyperbolic Coxeter group of finite co-volume.

We can express the anti-unitary involutions χ_j in the elementary ones from Lemma 3.3.2. For χ_4 this gives a simpler expression for the fixed point lattice compared to Chu's.

Proposition 4.3.2. *The real forms χ_j and their fixed point lattices given in Table 4.1 correspond to the ones defined in Appendix B of [12].*

Proof. The real forms χ_0, χ_1 and χ_2 are in fact identical to the ones given in [12]. For χ_3 and χ_4 we see that the fixed point lattices are isomorphic to L_3 and L_4 given in B.3 of [12]. This is immediate for L_3 and for L_4 we need to perform a change of basis: $B^t L_4 B = D_4(2) \oplus U(2)$ where $\det(B) = 1$ and:

$$L_4 = \begin{pmatrix} -4 & -4 & 2 & 0 & 0 & -4 \\ -4 & -12 & 6 & 0 & 0 & -8 \\ 2 & 6 & -4 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 & 2 & -2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -4 & -8 & 4 & -2 & 0 & -8 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

□

Remark 4.3.3. Note that according to Lemma 3.3.3 the anti-unitary involutions χ_j and $i\chi_j$ are conjugate in $\Gamma = U(\Lambda_{1,5})$ for $j = 0, \ldots, 4$.

As in Proposition 3.5.4 we can reduce the lattice $\Lambda_{1,5}$ modulo $(1+i)$. Let V_6 be the $\mathbb{F}_2$-vectorspace $\Lambda_{1,5}/(1+i)\Lambda_{1,5}$ equipped with the quadratic form $q(x) \equiv \frac{1}{2}h(x,x) \bmod 2$. There is a natural map $U(\Lambda_{1,5})^a \to O(V_6, q) \cong S_8$ where $U(\Lambda_{1,5})^a$ is the collection of anti-unitary transformations of the Gaussian lattice $\Lambda_{1,5}$. The image of an anti-unitary involution of $\Lambda_{1,5}$ under this map is an involution of S_8. It is well known that there are five conjugation classes of involutions in S_8 distinguished by their cycle type which is $(2)^j$ with

$j = 0, \dots, 4$. This type is a full invariant for the $P\Gamma$-conjugation class of $[\chi]$. The group S_8 also has a geometric meaning: its generating transpositions $(i\ i+1)$ act by Dehn twists on the space of ordered eight-tuples of points. As a result the number of transpositions of an involution $u \in S_8$ belonging to $\chi \in U(\Lambda_{1,5})^a$ corresponds to the number of complex conjugate pairs of points of a smooth octavic fixed by χ.

4.4 The maximal real component

We now study the component $\mathcal{M}_0^{\mathbb{R}}(1^8)$ consisting of eight real points in more detail. This component is called maximal since it corresponds to real binary octavics with the maximum number of real points. The corresponding fixed point lattice $\Lambda_{1,5}^{\chi_0} = (2) \oplus A_1^5$ is reflective and the group $PO(\Lambda_{1,5}^{\chi_0})$ is a Coxeter group; its Coxeter diagram can be found in Figure 2.3. We see from the diagram that there are two possible root norms: either a root has norm -2 or -4. Recall that $P\Gamma_I^{\chi_0}$ consists of all elements of $PO(\Lambda_{1,5}^{\chi_0})$ that are induced from $U(\Lambda_{1,5})$. A reflection s_r is in $P\Gamma_I^{\chi_0}$ if and only if it satisfies Equation 3.3. Since the lattice $\Lambda_{1,5}$ satisfies $\Lambda_{1,5} = (1+i)\Lambda_{1,5}^\vee$ this is equivalent to:

$$2r \in \frac{h(r,r)}{1+i}\Lambda_{1,5}$$

If r has norm -2 then this condition is satisfied. If r has norm -4 it is equivalent to $r \in (1+i)\Lambda_{1,5}$ so that $r = (1+i)r'$ with r' a primitive vector of norm -2. This means that r is primitive in $\Lambda_{1,5}^{\chi_0}$ but not in $\Lambda_{1,5}$. Consider the matrix B_0 shown below whose columns e_i with $i = 0 \dots, 5$ form a basis for $\Lambda_{1,5}^{\chi_0}$. The condition can be checked by writing a root r of norm -4 on this basis.

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 1+i & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1+i & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 \\ 1+i & 0 & 0 & 0 & -1-i & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Theorem 4.4.1. *The group $P\Gamma^{\chi_0}$ is isomorphic to the semi-direct product:*

$$W(C_5) \rtimes \mathrm{Aut}(C_5)$$

where $C_5 \subset \mathbb{B}_5^{\chi_0}$ is a hyperbolic Coxeter polytope with finite volume whose Coxeter diagram is shown in Figure 4.2. Its automorphism group $\mathrm{Aut}(C_5)$ is isomorphic to a dihedral group D_8 of order 16.

Proof. We apply Vinberg's algorithm with the condition that in each step of the algorithm we only accept roots such that the reflection s_r is induced from

	e_0	e_1	e_2	e_3	e_4	e_5
p	1	0	0	0	0	0
height 0						
r_1	0	1	0	−1	0	0
r_2	0	0	0	1	0	0
r_3	0	0	1	0	−1	0
r_4	0	0	0	0	1	0
r_5	0	0	0	0	0	1
height $\frac{1}{2}$						
r_6	1	−1	0	−1	0	−1
r_7	1	0	−1	0	−1	−1
height 1						
r_8	1	−1	−1	0	0	0

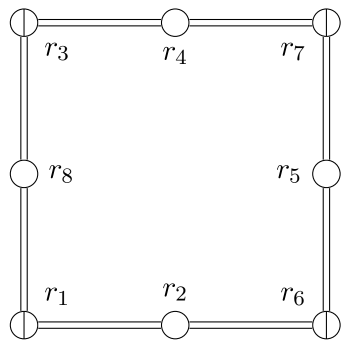
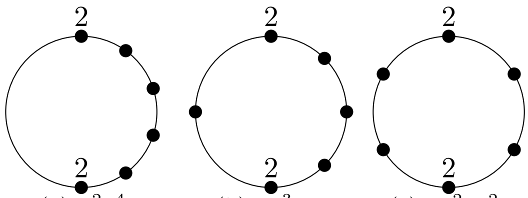


Figure 4.2: The hyperbolic Coxeter polytope C_5

$U(\Lambda_{1,5})$. This implies that r has norm -2 or it has norm -4 and is divisible by $1+i$. For example we see directly from the matrix B_0 that the only height 0 vectors of norm -4 are $e_1 - e_3$ and $e_2 - e_4$. The algorithm proceeds as in Figure 4.2 and terminates. The result is the hyperbolic Coxeter polytope C_5. The group $P\Gamma^{\chi_0}$ is the semi-direct product of $W(C_5)$ by the subgroup of $\mathrm{Aut}(C_5)$ that is contained in $P\Gamma^{\chi_0}$. The two involutions $t_1, t_2 \in PO(\Lambda_{1,5}^{\chi_0})$ shown below are contained in the subgroup $P\Gamma_I^{\chi_0}$; we can verify that the matrices $B_0 \cdot t_1 \cdot B_0^{-1}$ and $B_0 \cdot t_2 \cdot B_0^{-1}$ have coefficients in $\mathcal{G}$ so that Equation 3.2 is satisfied. Together the elements t_1 and t_2 generate a dihedral group D_4 of order 8. To get the full automorphism group of the polytope C_5 we also need to add the element t_3 shown below. It can be verified by straightforward computation that the element $g = B_0 \cdot \frac{\sqrt{2}}{1+i} t_3 \cdot B_0^{-1}$ is an element of $P\Gamma^{\chi_0}$ of type II; simply check that it has coefficients in $\mathcal{G}$ and satisfies $g\chi_0 g^{-1} = i\chi_0$.

$$t_1 = s_{e_4-e_5} \cdot s_{e_0-e_1-e_2-e_3}$$
$$t_2 = s_{e_1-e_2} \cdot s_{e_3-e_4}$$
$$t_3 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

The factor $1/\sqrt{2}$ in the expression for t_3 comes from the fact that t_2 exhanges long and short roots. The rotation $t_1 t_3$ has order 8 and induces a cyclic permutation of the nodes of the Coxeter diagram of C_5. We can conclude that



(a) $2^2 1^4$ (b) $21^3 21$ (c) $21^2 21^2$

Figure 4.3: The three configuration of zeroes on $\mathbb{P}^1(\mathbb{R})$ of maximal real stable octavics of type $2^2 1^4$.

the group generated by t_1, t_3 is a dihedral group of order 16 and is in fact the full automorphism group $\mathrm{Aut}(C_5)$. □

There is a unique central point κ of C_5 that is invariant under the action of $\mathrm{Aut}(C_5)$. It is also the unique point that has equal distance to all the walls of C_5. In terms of the basis $\{e_0, \ldots, e_5\}$ for $\Lambda_{1,5}^{\chi_0} \otimes_{\mathbb{Z}} \mathbb{R}$ this point is given by:

$$\kappa = (-3 - 2\sqrt{2}, 1 + \sqrt{2}, 1 + \sqrt{2}, 1, 1, 1).$$

The point κ corresponds to a binary octavic $p(x, y)$ whose symmetry group is D_8. To see this it is convenient to map $\mathbb{P}^1(\mathbb{R})$ conformally to the unit circle in $\mathbb{C}$ by a Möbius transformation. This maps the zeroes of such an octavic $p(x, y)$ to the eight roots of unity. These form a regular 8-gon so the symmetry group of the corresponding octavic is indeed dihedral of order 16.

The orbits under the group $\mathrm{Aut}(C_5) \cong D_8$ of the faces of the polytope $C_5 \subset \mathbb{B}_5^{\chi_0}$ correspond to the different loci in the moduli space $\mathcal{M}_{0,s}^{\mathbb{R}}$ of maximal real stable octavics. The *real type* of such a locus in $\mathcal{M}_{0,s}^{\mathbb{R}}$ is determined up to cyclic permutation and orientation reversal by an ordered r-tuple of integers $(n_1, \ldots, n_r)$ such that $\sum_{i=1}^{r} n_i = 8$. It corresponds to a real stable octavic of the form:

$$f(x, y) = \prod_{i=1}^{r} (x - \alpha_i y)^{n_i} \ , \ \alpha_1 < \ldots < \alpha_r \ , \ \alpha_1, \ldots, \alpha_r \in \mathbb{R}.$$

In visualizing the real type of an octavic $f(x, y)$ it is convenient to represent $\mathbb{P}^1(\mathbb{R})$ as a circle and the zeroes of $f(x, y)$ as points on the circle with multiplicity. Examples of this are shown in Figure 4.3. The representatives for some types are shown in Figure 4.3. All information on the faces of C_5 and the corresponding octavics are listed in Table 4.2.

4.5 The moduli space $\mathcal{M}^{\mathbb{R}}(321^3)$

The moduli space of real stable octavics of type 321^3 consists of three connected components. We see in Figure 4.3 that the two maximal real types

k	Type	Octavic	$\#D_8$-orbits	Representatives
0	$\widetilde{B}_2^2$	$[4^2]$	1	$[41^4, 421^2, 4121, 42^2, 431, 4^2]$
	$B_2^2A_1$	3^22	1	3^22
1	B_2^2	3^21^2	2	$3^21^2, 3131$
	$B_2A_1^2$	32^21	2	$32^21, 3212$
	A_1^4	2^4	1	2^4
2	B_2A_1	321^3	2	$321^3, 31^221$
	A_1^3	2^31^2	2	$2^31^2, 21212$
3	B_2	31^5	1	31^5
	A_1^2	2^21^4	3	$2^21^4, 21^321, 21^221^2$
4	A_1	21^6	1	21^6

Table 4.2: The k-faces of the polytope C_5 and their orbits under $\mathrm{Aut}(C_5) \cong D_8$ along with the real types of their representatives. The unique cusp of type $\widetilde{B}_2^2$ corresponds to the union of all strictly semi-stable orbits.

321^3 and 31^221 account for two connected components. The third component consists of octavics of type 321^3 such that two points of multiplicity one form a pair of complex conjugate points. We denote this locus by $321(11^c)$. These three components correspond to the three anti-unitary involutions:

$$\psi_2 \oplus \psi_1 \ , \ \psi_3 \ , \ \psi_2' \oplus \psi_1$$

of the Gaussian lattice $\Lambda_{1,2} = \Lambda_2 \oplus (2)$ that we found in Section 3.5.

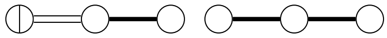
Proposition 4.5.1. *The correspondence between the three real types of stable octavics of type* 321^3 *and the three anti-unitary involutions of* $\Lambda_{1,2}$ *is as follows:*

$$\begin{aligned} 321^3 &\longleftrightarrow \psi_2 \oplus \psi_1 \\ 31^221 &\longleftrightarrow \psi_3 \\ 321(11^c) &\longleftrightarrow \psi_2' \oplus \psi_1. \end{aligned}$$

Proof. Just like we did for the lattice $\Lambda_{1,5}$ we can reduce the Gaussian lattice $\Lambda_{1,2}$ modulo $(1+i)$ to obtain the three dimensional $\mathbb{F}_2$ vectorspace:

$$V_3 = \Lambda_{1,2}/(1+i)\Lambda_{1,2} \cong \mathbb{F}_2^3$$

with the quadratic form $q(x) = \frac{1}{2}h(x,x) \bmod 2$. There is a natural map $U(\Lambda_{1,2})^a \to O(V_3, q) \cong S_3$. It is easy to check from the defining matrices that the anti-unitary involutions $\psi_2 \oplus \psi_1$ and ψ_3 map to the identity in S_3 so that they correspond to maximal real stable octavics. Recall from Table 3.2 that the Coxeter diagram of the groups $P\Gamma^\chi$ for these real forms are given by:



(a) $\psi_2 \oplus \psi_1$ (b) ψ_3

Only the diagram corresponding to $\psi_2 \oplus \psi_1$ has an edge with two actual vertices (and no cusp). This edge corresponds to the real type 3212 which cannot degenerate to a semi-stable real type: it can only degenerate to the stable real type 332 or to an unstable real type with a point of multiplicity ≥ 5. The real type 321^3 can degenerate to 3212 while 31^221 cannot. We conclude that the anti-unitary involution $\psi_2 \oplus \psi_1$ corresponds to the real type 321^3 and ψ_3 corresponds to the real type 31^221. The anti-unitary involution $\psi_2' \oplus \psi_1$ reduces modulo $1+i$ to the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O(V_3, q)$$

so it maps to a transposition in S_3. As a consequence the anti-unitary involution $\psi_2' \oplus \psi_1$ corresponds to real stable octavics of type $321(11^c)$ with two complex conjugate points of multiplicity one. □

Chapter 5

Real plane quartic curves and real hyperbolic geometry

5.1 Introduction

In this chapter we study a real period map for the moduli space of smooth real plane quartic curves in the spirit of Allcock, Carlson and Toledo [5]. This moduli space has six connected components and we prove that each is isomorphic to an arithmetic real ball quotient. For the component corresponding to maximal quartics we obtain much more information by studying a certain real hyperbolic Coxeter polytope. Our starting point is Kondo's construction in [26] of a period map for the moduli space of smooth complex plane quartic curves using the period map for $K3$ surfaces.

5.2 Plane quartic curves

The set of complex points of a plane quartic curve is the zero locus in the projective plane $\mathbb{P}^2(\mathbb{C})$ of a homogeneous polynomial $f(x, y, z)$ of degree four with coefficients in $\mathbb{C}$. The space of all such polynomials up to scalar multiplication is the projective space $P_{4,3}(\mathbb{C}) = \mathbb{P}\,\mathrm{Sym}^4(\mathbb{C}^3)^\vee$ of dimension 14. We denote the locus of codimension one that defines singular quartics by $\Delta(\mathbb{C}) \subset P_{4,3}(\mathbb{C})$. The moduli space of smooth quartic curves is the space:

$$\mathcal{Q} = PGL_3(\mathbb{C})\backslash\left(P_{4,3}(\mathbb{C}) - \Delta(\mathbb{C})\right).$$

We briefly recall some terminology from Mumford's geometric invariant theory of quartic curves. A complex quartic curve is called stable if it has at worst ordinary nodes and cusps as singularities and semi-stable if it has at worst tacnodes as singularities or is a smooth conic of multiplicity 2.

A complex plane quartic is real if it is invariant under complex conjugation of $\mathbb{P}^2(\mathbb{C})$ or equivalently if its defining polynomial has real coefficients. The moduli space of smooth real plane quartics is the space:

$$\mathcal{Q}^{\mathbb{R}} = PGL_3(\mathbb{R})\backslash \left(P_{4,3}(\mathbb{R}) - \Delta(\mathbb{R})\right).$$

As we have seen in Example 2.5.2 the space $P_{4,3}(\mathbb{R}) - \Delta(\mathbb{R})$ consists of six connected components. Since the group $PGL_3(\mathbb{R})$ is connected this implies that the moduli space $\mathcal{Q}^{\mathbb{R}}$ also consists of six connected components which we denote by $\mathcal{Q}_i^{\mathbb{R}}$ for $i = 1, \ldots, 6$. These components are distinguished by the topological type of $C(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ of a representative curve C. The real point set $C(\mathbb{R})$ can consist of $0, 1, 2, 3$ or 4 ovals and in the case of two ovals they can be nested or not. The correspondence is shown in Table 5.1.

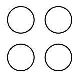
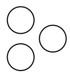
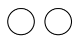
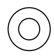
$\mathcal{Q}_1^{\mathbb{R}}$	$\mathcal{Q}_2^{\mathbb{R}}$	$\mathcal{Q}_3^{\mathbb{R}}$	$\mathcal{Q}_4^{\mathbb{R}}$	$\mathcal{Q}_5^{\mathbb{R}}$	$\mathcal{Q}_6^{\mathbb{R}}$
					$\emptyset$

Table 5.1: The topological types of representative curves $C(\mathbb{R})$ for the six components of $\mathcal{Q}_i^{\mathbb{R}} \subset \mathcal{Q}^{\mathbb{R}}$ for $i = 1, \ldots, 6$.

5.3 Kondo's period map

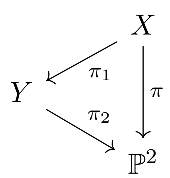
In this section we review Kondo's construction of a period map for complex plane quartic curves in [26]. Let C be a smooth quartic curve in $\mathbb{P}^2$ defined by a homogeneous polynomial $f(x, y, z)$ of degree four. We define the surface X to be the four-fold cyclic cover of $\mathbb{P}^2$ ramified over C so that:

$$X = \{w^4 = f(x, y, z)\} \subset \mathbb{P}^3.$$

The surface X is a $K3$-surface of degree four with an action of the group of covering transformations of the cover $\pi : X \to \mathbb{P}^2$. This group is cyclic of order four and a generator is given by the transformation:

$$\rho_X \cdot [w : x : y : z] = [iw : x : y : z].$$

The involution $\tau_X = \rho_X^2$ also acts on X and the quotient surface $Y = X/\tau_X$ is a double cover of $\mathbb{P}^2$ ramified over the quartic C. It is a del Pezzo surface of degree two. The situation is summarized by the following commutative diagram.



The cohomology group $H^2(X,\mathbb{Z})$ is isomorphic to the $K3$ lattice $L = E_8^2 \oplus U^3$ and a choice of isomorphism $\phi : H^2(X,\mathbb{Z}) \to L$ is called a marking of X. We fix a marking and let ρ and τ denote the automorphisms of L induced by ρ_X and τ_X. Kondo [26] proves that the eigenlattices of τ for the eigenvalues $+1$ and -1 are isomorphic to:

$$L_+ \cong A_1^7 \oplus (2) \quad , \quad L_- \cong D_4^3 \oplus (2)^2. \tag{5.1}$$

Remark 5.3.1. The expression for L_- in Equation 5.1 is different from the lattice $U(2)^2 \oplus D_8 \oplus A_1^2$ given by Kondo. Since the lattice L_- is even and 2-elementary its isomorphism type is determined by the invariants (r_+, r_-, a, δ) from Theorem 2.2.1. These invariants are $(2, 12, 8, 1)$ for both lattices so that the lattices are isomorphic. For the lattice $U \oplus U(2) \oplus D_4^2 \oplus A_1^2$ the invariants also take these values so that it is isomorphic to the previous two lattices.

For applications later on it is convenient to have a more explicit desciption of the involution τ. This is provided by the following lemma.

Lemma 5.3.2. *Let $L = U^3 \oplus E_8^2$ be the $K3$ lattice. The involution τ is conjugate in $O(L)$ to the involution given by:*

$$-I_2 \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \oplus u \oplus u \tag{5.2}$$

where $u \in O(E_8)$ is an involution of type D_4A_1.

Proof. Since the involution u is of type D_4A_1, its negative $-u$ is of type A_1^3. This implies that the eigenlattice for the eigenvalue 1 of u in E_8 is isomorphic to A_1^3. The ± 1 eigenlattices in L of the involution in Equation 5.2 are then given by:

$$\begin{gathered} U(2) \oplus A_1^6 \cong (2) \oplus A_1^7 \\ U \oplus U(2) \oplus D_4^2 \oplus A_1^2 \cong D_4^3 \oplus (2)^2 \end{gathered} \tag{5.3}$$

The lattice $(2) \oplus A_1^7$ has a unique embedding into the $K3$ lattice L up to automorphisms in $O(L)$ by Theorem 2.2.4. This implies that the involution of Equation 5.2 is conjugate to τ in $O(L)$. □

The map π_1 induces a primitive embedding of lattices $\pi_1^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ and the image is precisely the lattice $\phi^{-1}(L_+)$. It is the Picard group of the del Pezzo surface Y scaled by a factor two which comes from the fact that the map π_1 is of degree 2.

The powers ρ, ρ^2 and ρ^3 act on the lattice L_- without fixed points. This action turns L_- into a Gaussian lattice of signature $(1, 6)$ isomorphic to the Gaussian lattice $\Lambda_{1,6} = \Lambda_2^3 \oplus (2)$. From now on we identify L_- considered as a Gaussian lattice with $\Lambda_{1,6}$ and write L_- for the underlying $\mathbb{Z}$-lattice. If

$\gamma \in \pi_1^* \operatorname{Pic}(Y)$ then $(\omega, \gamma) = 0$ for all $\omega \in H^{2,0}(X, \mathbb{C})$ so that the the complex ball

$$\mathbb{B} = \mathbb{P}\{x \in \Lambda_{1,6} \otimes_{\mathcal{G}} \mathbb{C} \ ; \ h(x,x) > 0\}$$

is a period domain for smooth plane quartic curves. Let $\Gamma = U(\Lambda_{1,6})$ be the unitary group of the Gaussian lattice $\Lambda_{1,6}$. Equivalently it is the group of orthogonal transformations of the lattice L_- that commute with ρ. The period map $\operatorname{Per} : \mathcal{Q} \to P\Gamma \backslash \mathbb{B}$ is injective by the Torelli theorem for $K3$ surfaces but not surjective. Its image misses certain divisors in $\mathbb{B}$ which we now describe. An element $r \in \Lambda_{1,6}$ is called a root if $h(r,r) = -2$ and for every root we define the mirror $H_r = \{z \in \mathbb{B} \ ; \ h(r,z) = 0\}$. We denote by $\mathcal{H} \subset \mathbb{B}$ the union of all the root mirrors H_r and write $\mathbb{B}^\circ = \mathbb{B} \setminus \mathcal{H}$.

Theorem 5.3.3 (Kondo)**.** *The period map defines an isomorphism of orbifolds:*

$$\operatorname{Per} : \mathcal{Q} \to P\Gamma \backslash \mathbb{B}^\circ.$$

Proof. The proof consists of constructing an inverse map of the period map. We give a brief sketch of the main arguments used in [26]. Let $z \in \mathbb{B}^\circ$. There is a $K3$ surface X together with a marking $\phi : H^2(X, \mathbb{Z}) \to L$ such that the period point of X is z. This $K3$ surface X has an automorphism ρ_X of order four such that its action on $H^2(X, \mathbb{Z})$ corresponds to the action of ρ on L. The quotient surface $Y = X/\langle \tau_X \rangle$ with $\tau_X = \rho_X^2$ is a del Pezzo surface of degree two. Its anticanonical map: $|K_Y| : Y \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ ramified over a smooth plane quartic curve C. The inverse period map associates to the $P\Gamma$-orbit of $z \in \mathbb{B}^\circ$ the isomorphism class of this quartic curve C. □

Furthermore Kondo proves in [26] Lemma 3.3 that there are two Γ-orbits of roots in $\Lambda_{1,6}$. This determines a decomposition $\mathcal{H} = \mathcal{H}_n \cup \mathcal{H}_h$ where:

$$\begin{aligned} \mathcal{H}_n &= \left\{ H_r \in \mathcal{H} \ ; \ H_r \cap \Lambda_{1,6} \cong \Lambda_2^2 \oplus \left(\begin{smallmatrix} -2 & 0 \\ 0 & 2 \end{smallmatrix} \right) \right\} \\ \mathcal{H}_h &= \left\{ H_r \in \mathcal{H} \ ; \ H_r \cap \Lambda_{1,6} \cong \Lambda_2^2 \oplus \Lambda_{1,1} \right\}. \end{aligned} \tag{5.4}$$

A smooth point of a mirror $H_r \in \mathcal{H}_n$ corresponds to a plane quartic curve with a node and a smooth point of a mirror $H_r \in \mathcal{H}_h$ corresponds to a smooth hyperelliptic curve of genus three. This last case we studied in Chapter 4.

5.4 The lattices L_+ and L_-

The main result of this section is Lemma 5.4.4 which states that an anti-unitary involution of the Gaussian lattice $\Lambda_{1,6}$ can be lifted to an involution of the $K3$ lattice such that its fixed point lattice is of hyperbolic signature. This will be an important ingredient in the proof of one of our main results: the real analogue of Kondo's period map for real quartic curves in Section 5.5. We start with a detailed analysis of the lattices L_+ and L_-.

The lattice $L_+ \cong (2) \oplus A_1^7$ has an orthogonal basis $\{e_0, \ldots, e_7\}$ that satisfies $(e_0, e_0) = 2$ and $(e_i, e_i) = -2$ for $i = 1, \ldots, 7$. According to Kondo the automorphism ρ acts on L_+ by fixing the element $k = -3e_0 + e_1 + \ldots + e_7$ and acting as -1 on its orthogonal complement $k^\perp$ in L_+. This special element k satisfies $(k, k) = 4$ and represents the canonical class of the del Pezzo surface Y as we have seen in Section 2.6. The orthogonal complement $k^\perp$ is isomorphic to the root lattice $E_7(2)$. By the results of Section 2.4 there is an isomorphism of groups:

$$O(L_+) \cong O(L_+)^+ \rtimes \mathbb{Z}/2\mathbb{Z}$$

where the second factor is generated by $-1 \in O(L_+)$. The group $O(L_+)^+$ is a hyperbolic Coxeter group as we have seen in Example 2.4.2 and its Coxeter diagram shown is Figure 5.1.

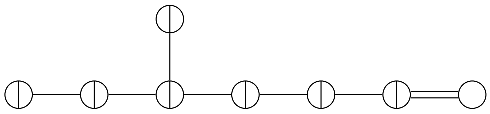


Figure 5.1: The Coxeter diagram of the group $O(L_+)^+$.

From this diagram we see that the reflections in the long negative simple roots of L_+ form a subgroup $W(E_7) < O(L_+)^+$ of type E_7. It is precisely the stabilizer of the element $k \in L_+$. Recall from Section 2.2 that the discriminant group of a lattice L is defined by $A_L = L^\vee / L$. Since the dual lattice $L_+^\vee$ can be naturally identified with the lattice $\frac{1}{2} L_+$ we have:

$$A_{L_+} = \frac{1}{2} L_+ / L_+ \cong (\mathbb{Z}/2\mathbb{Z})^8.$$

Proposition 5.4.1. *The natural map $O(L_+) \to O(A_{L_+})$ maps the subgroup $W(E_7) < O(L_+)^+$ isomorphically onto $O(A_{L_+})$.*

Proof. The bilinear form on $L_+ \cong (2) \oplus A_1^7$ is even valued so that a reflection s_r in a short root r of norm ± 2 satisfies:

$$s_r(x) = x \pm (r, x) r \equiv x \bmod L_+$$

for $x \in \frac{1}{2} L_+$. This implies that these reflections are contained in the kernel of the map $O(L_+) \to O(A_{L_+})$. A consequence is that the image of this map is generated by the subgroup $W(E_7) < O(L_+)^+$ of reflections in negative simple long roots. According to Kondo [26] Lemma 2.2 the group $O(A_{L_+})$ is isomorphic to $W(E_7)^+ \times \mathbb{Z}/2\mathbb{Z} \cong W(E_7)$. Since the natural map $O(L_+) \to O(A_{L_+})$ is surjective by Theorem 2.2.2 the proposition follows. □

The $K3$ lattice L is an even unimodular lattice and the primitive sublattices L_+ and L_- satisfy: $L_-^\perp = L_+$. According to Proposition 2.2.3 there is a natural isomorphism $O(A_{L_-}) \cong O(A_{L_+})$ which allows us to identify these groups. In particular we have: $O(A_{L_-}) \cong W(E_7)^+ \times \mathbb{Z}/2\mathbb{Z}$. We prefer to consider L_- as the Gaussian lattice $\Lambda_{1,6}$ so that:

$$\begin{aligned} A_{\Lambda_{1,6}} &= \Lambda_{1,6}^\vee / \Lambda_{1,6} \\ &\cong \left(\frac{1}{1+i}\mathcal{G}/\mathcal{G}\right)^6 \times \frac{1}{2}\mathcal{G}/\mathcal{G} \end{aligned} \tag{5.5}$$

Remark 5.4.2. Note that $\frac{1}{1+i}\mathcal{G}/\mathcal{G} \cong \mathbb{Z}/2\mathbb{Z}$ and that $\frac{1}{2}\mathcal{G}/\mathcal{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The generators of this last group are $\frac{1}{2}$ and $\frac{i}{2}$ and they are exchanged by multiplication by i.

Proposition 5.4.3. *The composition of homomorphisms:*

$$U(\Lambda_{1,6}) \to O(A_{\Lambda_{1,6}}) \cong W(E_7)^+ \times \mathbb{Z}/2\mathbb{Z}$$

is given by reduction modulo $1+i$ on the first factor and the second factor is generated by the image of the central element $\rho \in U(\Lambda_{1,6})$.

Proof. Let $A'_{\Lambda_{1,6}}$ be the subset of $A_{\Lambda_{1,6}}$ where the discriminant quadratic form takes values in $\mathbb{Z}/2\mathbb{Z}$. The Gaussian lattice $\Lambda_{1,6}$ satisfies $\Lambda_{1,6} \subset (1+i)\Lambda_{1,6}^\vee$ so that the following equalities hold:

$$\begin{aligned} A'_{\Lambda_{1,6}} &= \{x \in \Lambda_{1,6}^\vee/\Lambda_{1,6}\ ;\ h(x,x) \in \mathbb{Z}\} \\ &= \frac{1}{1+i}\Lambda_{1,6}/\Lambda_{1,6}. \end{aligned}$$

By writing: $h(\frac{1}{1+i}x, \frac{1}{1+i}x) = \frac{1}{2}h(x,x)$ for $x \in \Lambda_{1,6}$ we see that the $\mathbb{F}_2$-vectorspace $A'_{\Lambda_{1,6}}$ with its induced quadratic form $q_{\Lambda_{1,6}}$ is isomorphic to the quadratic space (V,q) from Proposition 3.5.4. According to this proposition there is an isomorphism: $O(A'_{\Lambda_{1,6}}) \cong W(E_7)^+$ and the composition of natural maps:

$$U(\Lambda_{1,6}) \to O(A_{L_-}) \to O(A'_{L_-}) \cong W(E_7)^+$$

corresponds to mapping an element $g \in U(\Lambda_{1,6})$ to its reduction $\overline{g}$ modulo $(1+i)$. The automorphism $\rho \in U(\Lambda_{1,6})$ corresponds to multiplication by i and by definition commutes with every element in $U(\Lambda_{1,6})$. It maps to the identity in $O(A'_{\Lambda_{1,6}})$ but acts as a non-trivial involution in $O(A_{\Lambda_{1,6}})$ by Remark 5.4.2. This implies that $O(A_{\Lambda_{1,6}})$ is isomorphic to the direct product of $O(A'_{\Lambda_{1,6}})$ with the subgroup $\mathbb{Z}/2\mathbb{Z} \lhd O(A_{\Lambda_{1,6}})$ generated by ρ. ☐

Lemma 5.4.4. *Let $\chi_- \in U(\Lambda_{1,6})^a$ be an anti-unitary involution of $\Lambda_{1,6}$. There is a unique $\chi \in O(L)$ that restricts to χ_- on L_- so that the fixed point lattice L^χ is of hyperbolic signature.*

Proof. Since complex conjugation on $\Lambda_{1,6}$ induces the identity on $O(A_{\Lambda_{1,6}})$ the statement of Proposition 5.4.3 is also true for the composition of homomorphisms:

$$U(\Lambda_{1,6})^a \to O(A_{\Lambda_{1,6}}) \cong W(E_7)^+ \times \mathbb{Z}/2\mathbb{Z}.$$

Consider the image of the anti-unitary transformation $\chi_- \in U(\Lambda_{1,6})^a$ under this composition. This image is of the form $(\bar{u}, \pm 1)$ where the involution $\bar{u} \in W(E_7)^+$ is obtained by reducing $\chi_- \in U(\Lambda_{1,6})^a$ modulo $(1+i)$. Observe that if the anti-unitary involution χ_- maps to $(\bar{u}, 1)$ then $i\chi$ maps to $(\bar{u}, -1)$. The involution

$$\chi_+ = (\pm u, -1) \in W(E_7) \times \mathbb{Z}/2\mathbb{Z} < O(L_+) \tag{5.6}$$

maps to $(\bar{u}, \pm 1) \in O(A_{L_+})$ by Proposition 5.4.1. Since χ_+ maps $k \mapsto -k$ and $(k,k) = 4$ the lattice of fixed points $L_+^{\chi_+}$ is negative definite. By Proposition 2.2.3 there is a unique involution $\chi \in O(L)$ that restricts to $\chi_- \in U(\Lambda_{1,6})$ and $\chi_+ \in O(L_+)$ respectively. Since $\Lambda_{1,6}^{\chi_-}$ is of hyperbolic signature and $L_+^{\chi_+}$ is negative definite the fixed point lattice L^χ is of hyperbolic signature. □

Proposition 5.4.5. *Consider the* 12 *anti-unitary involutions* χ_j *and* $i\chi_j$ *for* $j = 1, \ldots, 6$ *from Theorem 3.5.5. For each of them the corresponding involution* $\chi_+ \in O(L_+)$ *is of the form* $(u, -1) \in W(E_7) \times \mathbb{Z}/2\mathbb{Z}$*. The conjugation classes of the involutions* $u \in W(E_7)$ *are shown in Table 5.2.*

j	1	2	3	4	5	6
χ_j	1	A_1	A_1^2	A_1^3	D_4	D_4
$i\chi_j$	E_7	D_6	D_4A_1	A_1^4	$A_1^{3\prime}$	$A_1^{3\prime}$

Table 5.2: The conjugation classes in $W(E_7)$ of the 12 anti-unitary involutions $\chi_j, i\chi_j \in U(\Lambda_{1,6})^a$ for $j = 1, \ldots, 6$.

Proof. This follows from Table 3.3 and the proof of Lemma 5.4.4. □

5.5 Periods of real quartic curves

Let $C = \{f(x,y,z) = 0\} \subset \mathbb{P}^2$ be a smooth real plane quartic curve. This means that C is invariant under complex conjugation of $\mathbb{P}^2(\mathbb{C})$ or equivalently that the polynomial f has real coefficients. The $K3$ surface X that corresponds to C is also defined by an equation with real coefficients. Complex conjugation on $\mathbb{P}^3(\mathbb{C})$ induces an anti-holomorphic involution χ_X on X.

Remark 5.5.1. There are two anti-holomorphic involutions on the $K3$ surface $X = \{w^4 = f(x,y,z)\}$. Since we chose the sign of $f(x,y,z)$ to be positive on the interior of the curve $C(\mathbb{R})$ in Section 2.5 the anti-holomorphic involution χ_X is determined without ambiguity.

By fixing a marking $\phi : H^2(X, \mathbb{Z}) \to L$ of the $K3$ surface X we associate to χ_X the involution:

$$\chi = \phi \circ \chi_X^* \circ \phi^{-1}$$

of the $K3$ lattice L. Since the involution χ commutes with τ it preserves the ± 1-eigenlattices of the involution τ. We denote by χ_- (resp. χ_+) the induced involution on L_- (resp. L_+). It is clear that χ and ρ satisfy the relation:

$$\rho \circ \chi = \tau \circ \chi \circ \rho$$

so that on the eigenlattice L_- where τ acts as -1 they anti-commute and on L_+ they commute. This implies that χ_- is an anti-unitary involution of the Gaussian lattice $\Lambda_{1,6}$.

By the results of Section 5.3.3 on Kondo's period map we can associate to a smooth real plane quartic curve C a period point $[x] \in \mathbb{B}^\circ$ and the real form $[\chi_-]$ of $\mathbb{B}$ we just defined fixes $[x]$. The following lemma shows that the $P\Gamma$-conjugation class of $[\chi_-]$ does not change if we vary C in its connected component of $\mathcal{Q}^{\mathbb{R}}$.

Lemma 5.5.2. *If two smooth real plane quartic curves C and C' are real isomorphic then the projective classes $[\chi_-]$ and $[\chi'_-]$ of their corresponding anti-unitary involutions in $\Lambda_{1,6}$ are conjugate in $P\Gamma$.*

Proof. Since C and C' are real plane curves a real isomorphism $C \to C'$ is induced from an element in $PGL(3, \mathbb{R})$. We can lift this element to $PGL(4, \mathbb{R})$ so that it induces an isomorphism $\alpha_C : X \to X'$ that commutes with the covering transformations ρ_X and $\rho_{C'}$ of X and X'. Since the real forms χ_X and χ'_X of X and X' are both induced by complex conjugation on $\mathbb{P}^3$ they satisfy $\chi'_X = \alpha_C \circ \chi_X \circ \alpha_C^{-1}$. By fixing markings of the $K3$ surfaces X and X' we obtain induced orthogonal transformations χ, χ' and α of the $K3$-lattice L and they satisfy: $\chi' = \alpha \circ \chi \circ \alpha^{-1}$. Since α commutes with ρ the restriction α_- of α to L_- is contained in Γ. This proves the lemma. □

Let $\mathbb{B}^{\chi_-}$ be the fixed point set in $\mathbb{B}$ of the real form $[\chi_-]$. The fixed point lattice $\Lambda_{1,6}^{\chi_-}$ has hyperbolic signature $(1, 6)$ so that $\mathbb{B}^{\chi_-}$ is the real ball:

$$\mathbb{B}^{\chi_-} = \mathbb{P}\{x \in \Lambda_{1,6}^{\chi_-} \otimes_{\mathbb{Z}} \mathbb{R} \ ; \ h(x, x) > 0\}.$$

As before we denote by $P\Gamma^{\chi_-}$ the stabilizer of $\mathbb{B}^{\chi_-}$ in the ball $\mathbb{B}$. Since the period point of a smooth real quartic curve C is fixed by $[\chi_-]$ it lands in the real ball quotient: $P\Gamma^{\chi_-} \backslash (\mathbb{B}^{\chi_-})^\circ$. This gives rise to a real period map. More precisely we have the following real analogue of Theorem 5.3.3.

Theorem 5.5.3. *The real period map* $\mathrm{Per}^{\mathbb{R}}$ *that maps a smooth real plane quartic to its period point in* $P\Gamma\backslash\mathbb{B}^{\circ}$ *defines an isomorphism of real orbifolds:*

$$\mathrm{Per}^{\mathbb{R}} : \mathcal{Q}^{\mathbb{R}} \to \coprod_{[\chi_-]} P\Gamma^{\chi_-}\backslash\left(\mathbb{B}^{\chi_-}\right)^{\circ} \tag{5.7}$$

where $[\chi_-]$ *varies over the* $P\Gamma$*-conjugacy classes of projective classes of anti-unitary involutions of* $\Lambda_{1,6}$*.*

Proof. We construct an inverse to the real period map. Let $z \in \mathbb{B}^{\circ}$ be such that $\chi_-(z) = z$ for a certain anti-unitary involution of $\Lambda_{1,6}$. From the proof of 5.3.3 we see that there is a marked $K3$ surface X that corresponds to z. According to Lemma 5.4.4 the involution χ_- lifts to an involution $\chi \in O(L)$ such that for its restriction χ_+ to L_+ the fixed point lattice $L_+^{\chi_+}$ is negative definite. Since $\Lambda_{1,6}^{\chi_-}$ is of hyperbolic signature the lattice L^{χ} is also of hyperbolic signature. According to Theorem 2.7.5 this implies that the marked $K3$ surface X is real. Its real form χ_X commutes with τ_X so that it induces a real form on χ_Y on the del Pezzo surface $Y = X/\langle\tau_X\rangle$. The anticanonical system $|-K_Y| : Y \to \mathbb{P}^2$ is the double cover of $\mathbb{P}^2$ ramified over a smooth real plane quartic curve C. The inverse of the real period map associates to the $P\Gamma^{\chi_-}$ orbit of $z \in (\mathbb{B}^{\chi_-})^{\circ}$ the real isomorphism class of the real quartic curve C. □

5.6 The six components of $\mathcal{Q}^{\mathbb{R}}$

In this section we complete our description of the real period map $\mathrm{Per}^{\mathbb{R}}$ by connecting the six connected components of the moduli space $\mathcal{Q}^{\mathbb{R}}$ of smooth real plane quartic curves to the six projective classes of anti-unitary involutions of the Gaussian lattice $\Lambda_{1,6}$ from Theorem 3.5.5. We first prove that these six anti-unitary involutions are in fact all of them.

Proposition 5.6.1. *There are six projective classes of anti-unitary involutions of the Gaussian lattice* $\Lambda_{1,6}$ *up to conjugation by* $P\Gamma$*.*

Proof. Since $\mathcal{Q}^{\mathbb{R}}$ consists of six connected components and the real period map $\mathrm{Per}^{\mathbb{R}}$ is surjective the number of projective classes is at most six. In Theorem 3.5.5 we found six projective classes of anti-unitary involutions up to conjugation by $P\Gamma$ so these six are all of them.

□

The following corollary follows from the proof of Theorem 5.5.3.

Corollary 5.6.2. *Suppose* $z \in \mathbb{B}^{\circ}$ *is a real period point so that it is fixed by an anti-unitary involution* $\chi \in U(\Lambda_{1,6})^a$*. By the real period map we associate to* $z \in \mathbb{B}^{\circ}$ *a real del Pezzo surface* Y *of degree two together with a marking:*

$$H^2(Y,\mathbb{Z}) \to L_+\left(\tfrac{1}{2}\right)$$

such that the induced involution of the real form of Y on $L_+(\frac{1}{2})$ is given by χ_+.

We review some results of [43] on real del Pezzo surfaces of degree two. Other references on this subject are Kollár [25] and Russo [36]. A real del Pezzo surface Y of degree two is the double cover of the projective plane $\mathbb{P}^2$ ramified over a smooth real plane quartic curve $C \subset \mathbb{P}^2$ so that:

$$Y = \{w^2 = f(x, y, z)\}.$$

Recall from section 2.5 that we choose the sign of f so that $f > 0$ on the orientable interior part of $C(\mathbb{R})$. By using the deck transformation ρ_Y of the cover we see that there are two real forms of Y:

$$\begin{aligned} \chi_Y^+ &: [w : x : y : z] \mapsto [\bar{w} : \bar{x} : \bar{y} : \bar{z}] \\ \chi_Y^- &: [w : x : y : z] \mapsto [-\bar{w} : \bar{x} : \bar{y} : \bar{z}]. \end{aligned} \tag{5.8}$$

These real forms satisfy: $\chi_Y^- = \rho_Y \circ \chi_Y^+$ and we denote the real point sets of χ_Y^+ and χ_Y^- by $Y^+(\mathbb{R})$ and $Y^-(\mathbb{R})$ respectively. Note that $Y^+(\mathbb{R})$ is an orientable surface while $Y^-(\mathbb{R})$ is non-orientable. Suppose that $H^2(Y, \mathbb{Z}) \to L_+(\frac{1}{2})$ is a marking of Y. The deck transformation ρ_Y induces the involution:

$$\rho = (-1, 1) \in W(E_7) \times \mathbb{Z}/2\mathbb{Z}.$$

in $O(L_+(\frac{1}{2}))$. This implies that the two real forms $\chi_Y^\pm$ form a pair:

$$(\chi_Y^+, \chi_Y^-) \longleftrightarrow (\pm u, -1) \in W(E_7) \times \mathbb{Z}/2\mathbb{Z}.$$

In [43] Wall determines the correspondence between the conjugation classes of the $u \in W(E_7)$ and the topological type of $Y(\mathbb{R})$. The results are shown in Table 5.3. We use the notation kX for the disjoint union and $\#kX$ for the connected sum of k copies of a real surface X. From this table we see that except for the classes of D_4 and $A_1^{3\prime}$ the conjugation class of $u \in W(E_7)$ determines the topological type of the real plane quartic curve $C(\mathbb{R})$.

Theorem 5.6.3. *The correspondence between the six projective classes of anti-unitary involutions of the lattice $\Lambda_{1,6}$ up to conjugation by $P\Gamma$ and the real components of $\mathcal{Q}^{\mathbb{R}}$ is given by:*

$$\mathcal{Q}_j^{\mathbb{R}} \longleftrightarrow \chi_j \qquad j = 1, \ldots, 6.$$

The index j on the left is given by Table 5.3 and the index j on the right by Table 5.2.

Proof. For $j = 1, 2, 3, 4$ the statement follows by comparing Table 5.3 and Table 5.2. Unfortunately this does not work for the projective classes anti-unitary involutions χ_5 and χ_6 since both correspond to the involutions $D_4 \in$

j	$C(\mathbb{R})$	$u \in W(E_7)$	$Y(\mathbb{R})$
1	○○ ○○	1	$\#8\mathbb{P}^2(\mathbb{R})$
		E_7	$4S_0$
2	○○○	A_1	$\#6\mathbb{P}^2(\mathbb{R})$
		D_6	$3S_0$
3	○○	A_1^2	$\#4\mathbb{P}^2(\mathbb{R})$
		D_4A_1	$2S_0$
4	○	A_1^3	$\#2\mathbb{P}^2(\mathbb{R})$
		A_1^4	S_0
5	◎	D_4	$S_0 \sqcup \#2\mathbb{P}^2(\mathbb{R})$
		$A_1^{3\prime}$	S_1
6	$\emptyset$	D_4	$2\mathbb{P}^2(\mathbb{R})$
		$A_1^{3\prime}$	$\emptyset$

Table 5.3: The real topological types of real del Pezzo surfaces of degree two and their corresponding involutions in the Weyl group $W(E_7)$.

$W(E_7)$. To distinguish these two we will prove that the anti-unitary involution $i\chi_6$ extends to an involution of the $K3$ lattice whose real $K3$ surface X has no real points. This implies that the projective class of χ_6 corresponding to the component $\mathcal{Q}_6^{\mathbb{R}}$ of smooth real quartic curves with no real points.

For this let $L = U^3 \oplus E_8^2$ be the $K3$ lattice and consider the involution:

$$\chi = -I_2 \oplus \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & I_8 \\ I_8 & 0 \end{pmatrix} \in O(L). \tag{5.9}$$

It is clear from the expression for χ that the fixed point lattice L^χ is isomorphic to $U(2) \oplus E_8(2)$. The invariants (r, a, δ) of this lattice are given by $(10, 10, 0)$ so that $X(\mathbb{R}) = \emptyset$ according to Theorem 2.7.6. Using the explicit embedding of L_+ and L_- into the $K3$ lattice L from Lemma 5.3.2 it is easily seen that:

$$L_-^{\chi-} \cong U(2) \oplus D(4) \oplus A_1(2) \quad , \quad L_+^{\chi+} \cong A_1(2)^3. \tag{5.10}$$

By consulting Table 3.4 we now deduce that χ is conjugate to $i\chi_6$ in $P\Gamma$.

□

5.7 The geometry of maximal quartics

We now study the component $\mathcal{Q}_1^{\mathbb{R}} \cong P\Gamma^{\chi_1}\backslash\mathbb{B}_6^{\chi_1}$ that corresponds to M-quartics in more detail. An M-quartic is a smooth real plane quartic curve C such that its set of real points $C(\mathbb{R})$ consists of four ovals. Much of the geometry of such quartics is encoded by a hyperbolic polytope $C_6 \subset \mathbb{B}_6^{\chi_1}$.

Theorem 5.7.1. *The group $P\Gamma^{\chi_1}$ is isomorphic to the semi-direct product:*

$$W(C_6) \rtimes \mathrm{Aut}(C_6)$$

where $C_6 \subset \mathbb{B}_6^{\chi_1}$ is a hyperbolic Coxeter polytope whose Coxeter diagram is shown in Figure 5.7. Its automorphism group $\mathrm{Aut}(C_6)$ is isomorphic to the symmetric group S_4.

Proof. Recall that an element $[g] \in P\Gamma^{\chi}$ is of type II if and only if there is a $g \in [g]$ such that $g\Lambda_{1,6}^{\chi} = \Lambda_{1,6}^{i\chi}$. We see from Table 3.4 that the lattice $\Lambda_{1,6}^{\chi}$ is not isomorphic to the lattice $\Lambda_{1,6}^{i\chi}$ so that the group $P\Gamma^{\chi}$ does not contain elements of type II. Therefore the group $P\Gamma^{\chi_1}$ consists of all element of $PO(\Lambda_{6,1}^{\chi_1})$ that are induced from $U(\Lambda_{1,6})$. The lattice $\Lambda_{1,6}^{\chi_1}$ is isomorphic to $(2) \oplus A_1^6$. A basis $\{e_0, \ldots, e_6\}$ in $\Lambda_{1,6}$ is given by the columns of the matrix:

$$B_1 = \begin{pmatrix} 0 & 1+i & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is a reflective lattice and the group $PO(\Lambda_{1,6}^{\chi_1})$ is a Coxeter group whose Coxeter diagram can be found in Figure 3.2. A reflection $s_r \in PO(\Lambda_{1,6}^{\chi_1})$ is induced from $U(\Lambda_{1,6})$ if and only if the root r satisfies Equation 3.3. Note that a vector $r = (r_1, \ldots, r_7) \in \Lambda_{1,6} \otimes_{\mathcal{G}} \mathbb{Q}$ is contained in $\Lambda_{1,6}^{\vee}$ if and only if $r_i \in \frac{1}{1+i}\mathcal{G}$ for $i = 1, \ldots, 6$ and $r_7 \in \frac{1}{2}\mathcal{G}$ so that we can rewrite this equation as:

$$\frac{2(1+i)r_i}{h(r,r)} \in \mathcal{G} \quad \text{for } i = 1, \ldots, 6 \quad , \quad \frac{4r_7}{h(r_7, r_7)} \in \mathcal{G}.$$

These equations are automatically satisfied if $h(r,r) = -2$ and if $h(r,r) = -4$ they are equivalent to: $(1+i)$ divides r_i for $i = 1, \ldots, 6$. This can be checked from the matrix B_1. Now we run Vinberg's algorithm with this condition and the result is the hyperbolic Coxeter polytope C_6 shown in Figure 5.7. The vertices r_1, r_3, r_5 and r_{13} of norm -4 roots form a tetrahedron. Every symmetry of this tetrahedron extends to the whole Coxeter diagram. Consequently the symmetry group of the Coxeter diagram is the symmetry group of a tetrahedron which is isomorphic to S_4. Consider the two elements $s, t \in PO(\Lambda_{1,6}^{\chi_1})$ defined by:

$$\begin{aligned} s &= s_{e_4-e_6} \cdot s_{e_3-e_5} \cdot s_{e_1-e_3} \cdot s_{e_2-e_4} \\ t &= s_{e_0-e_1-e_3-e_4} \cdot s_{e_0-e_1-e_5-e_6}. \end{aligned} \tag{5.11}$$

The element s has order three and corresponds to the rotation of the tetrahedron that fixes r_{13} and cyclically permutes $(r_1 r_5 r_3)$. The element t has order two and corresponds to the reflection of the tetrahedron that interchanges r_1 and r_{13} and fixes r_3 and r_5. Together these transformations generate S_4. We can check that both are contained in $P\Gamma^{\chi_1}$ by using Equation 3.2. □

We see from the Coxeter diagram of the polytope C_6 that there are three orbits of roots under the automorphism group $\mathrm{Aut}(C_6) \cong S_4$. The orbit of a root r corresponding to a grey node of norm -2 satisfies $r^\perp \cong \Lambda_2^2 \oplus \Lambda_{1,1}$. According to Equation 5.4 the mirror of such a root is of hyperelliptic type. This means that the smooth points of such a mirror correspond to a smooth hyperelliptic genus three curves. The Coxeter diagram of the wall that corresponds to the hyperelliptic root r_{11} is the subdiagram consisting of the nodes belonging to the roots:

$$\{r_1, r_2, r_3, r_5, r_6, r_7, r_9, r_{13}\}.$$

It is isomorphic to the Coxeter diagram of Figure 4.2. This is also the case for the other two hyperelliptic roots so they correspond to the maximal real component of real hyperelliptic genus three curves of Section 4.4.

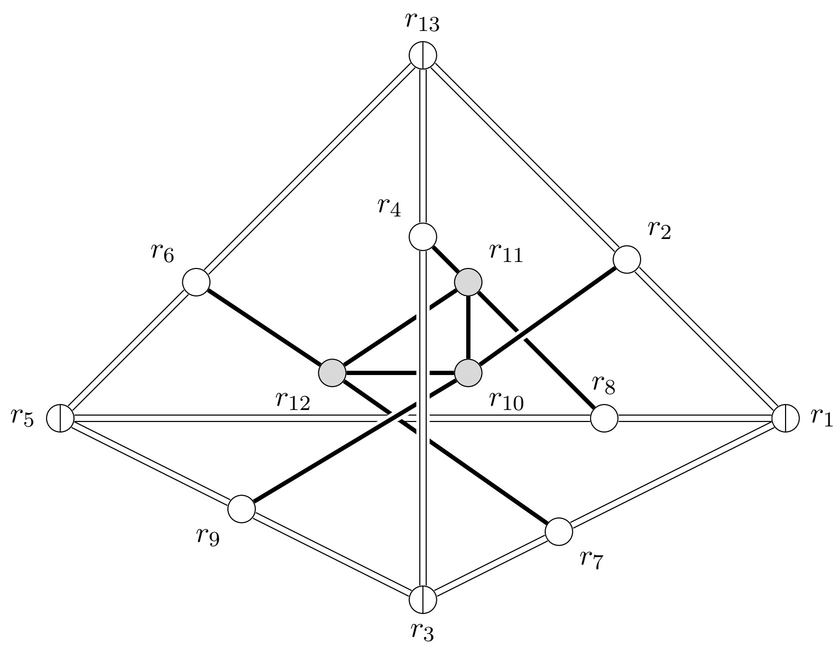
The other two orbits of roots satisfy $r^\perp \cong \Lambda_2^2 \oplus \left(\begin{smallmatrix} -2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ so that their mirrors are of nodal type. For a white root of norm -2 the orthogonal complement $r^\perp$ in the lattice $\Lambda_{1,6}^{\chi_1}$ is isomorphic to $(2) \oplus A_1^5$. The smooth points of such a mirror correspond to quartic curves with a nodal singularity such that the tangents at the node are real. Locally such a node is described by the equation $x^2 - y^2 = 0$. This happens when two ovals touch each other; since there are four ovals this can happen in $\binom{4}{2} = 6$ ways; hence there are six mirrors of this type.

For a nodal root of norm -4 the orthogonal complement is given by $r^\perp \cong (2) \oplus A_1^4 \oplus A_1(2)$ in $\Lambda_{1,6}^{\chi_1}$. The smooth points of such a mirror correspond to quartic curves with a nodal singularity such the tangents at the node are complex conjugate. Locally this is described by $x^2 + y^2 = 0$. It happens when an oval shrinks to a point which can occur for each of the four ovals; hence there are four mirrors of this type.

The complex ball quotient $P\Gamma \backslash \mathbb{B}_6^\circ$ has a single cusp that corresponds to the unique minimal semi-stable orbit of quartics consisting of a conic of multiplicity two. The real ball quotient $P\Gamma^{\chi_1} \backslash \mathbb{B}_6^{\chi_1}$ however has two cusps. This can be seen from the Coxeter diagram of the polytope C_6 since there are two $\mathrm{Aut}(C_6)$-orbits of cusps both of type $\widetilde{B_2}^2 \times \widetilde{A_1}$. The geometric meaning of this

	e_0	e_1	e_2	e_3	e_4	e_5	e_6
p	1	0	0	0	0	0	0
height 0							
r_1	0	1	−1	0	0	0	0
r_2	0	0	1	0	0	0	0
r_3	0	0	0	1	−1	0	0
r_4	0	0	0	0	1	0	0
r_5	0	0	0	0	0	1	−1
r_6	0	0	0	0	0	0	1
height 1							
r_7	1	−1	0	−1	0	0	0
r_8	1	−1	0	0	0	−1	0
r_9	1	0	0	−1	0	−1	0
r_{10}	1	−1	−1	0	0	0	0
r_{11}	1	0	0	−1	−1	0	0
r_{12}	1	0	0	0	0	−1	−1
height 2							
r_{13}	2	−1	−1	−1	−1	−1	−1

(a) Vinberg's algorithm



(b) The Coxeter diagram of the reflection part of the group $P\Gamma^{\chi_1}$

is that there are two real minimal semi-stable orbits: a double conic can have real points or not. Examples are given by the two conics:

$$C_0 = \{(x^2 + y^2 + z^2 + 1)^2 = 0\} \quad , \quad C_1 = \{(x^2 + y^2 + z^2 - 1)^2 = 0\}.$$

A point $[x] \in C_6$ that is invariant under the action of $\mathrm{Aut}(C_6) \cong S_4$ corresponds to an M-quartic whose automorphism group is isomorphic to S_4. These points are described by the following lemma.

Lemma 5.7.2. *A point* $[x] \in C_6$ *with* $x = (x_0, \ldots, x_6) \in \Lambda_{1,6}^{\chi_1} \otimes_{\mathbb{Z}} \mathbb{R}$ *is invariant under* $\mathrm{Aut}(C_6) \cong S_4$ *if and only if it lies on the hyperbolic line segment:*

$$L = \mathbb{P}\{(-2b - a, b, a, b, a, b, a) \ ; \ a, b \in \mathbb{R}\} \subset C_6.$$

The line segment L *has fixed distances* d_1, d_2 *and* d_3 *to mirrors of type* ○,● *and* ⦶ *respectively and these distances satisfy:*

$$[\sinh^2 d_1 : \sinh^2 d_2 : \sinh^2 d_3] = [a^2 : b^2 : (a - b)^2/2].$$

Proof. The group $\mathrm{Aut}(C_6)$ is generated by the two elements s and t from Equation 5.11. A small computation shows that a point $x \in \Lambda_{1,6}^{\chi_1} \otimes_{\mathbb{Z}} \mathbb{R}$ is invariant under these two generators if and only if it is of the form

$$x = (-2b - a, b, a, b, a, b, a).$$

The second statement of the Lemma follows from the formula for hyperbolic distance (Equation 2.8) and the equalities:

$$(x, r_i) = \begin{cases} -a & i = 2, 4, 6, 7, 8, 9 \\ -b & i = 10, 11, 12 \\ a - b & i = 1, 3, 5, 13. \end{cases} \tag{5.12}$$

□

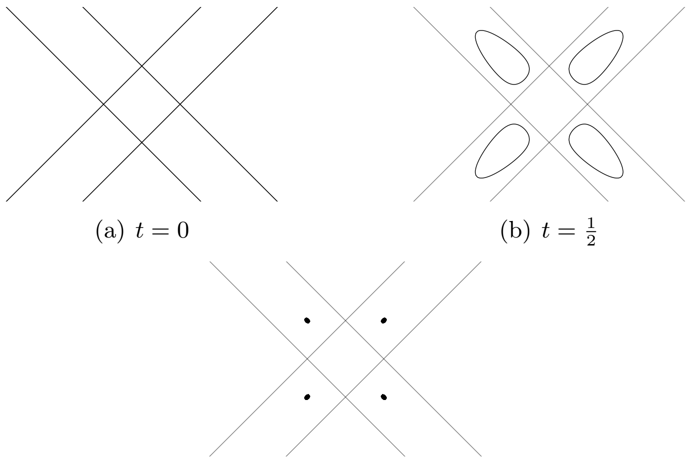
The line segment L connects the vertex $L_0 = (-2, 1, 0, 1, 0, 1, 0) \in C_6$ of type A_1^6 to the point $L_1 = (-3, 1, 1, 1, 1, 1, 1)$. A consequence of the real period map of Theorem 5.5.3 is that there is a unique one-parameter family of smooth plane quartics with automorphism group S_4 that corresponds to the line segment $L \subset C_6$. It is described by the following proposition.

Proposition 5.7.3. *The one-parameter family of quartic curves* C_t *by:*

$$C_t = \prod(\pm x \pm y + z) + t(x^4 + y^4 + z^4) \quad , \quad 0 \leq t \leq 1.$$

corresponds to the line segment L *under the real period map.*

Proof. This family is invariant under permutations of the coordinates (x, y, z) and the transformations: $(x, y, z) \mapsto (\pm x, \pm y, z)$. Together these generate a group $S_3 \rtimes V_4 \cong S_4$. The curve C_0 is a degenerate quartic that consists of four lines and has six real nodes corresponding to the intersection points of the lines. For $0 < t < 1$ the curve C_t is an M-quartic. The quartic C_1 has no real points except for four isolated nodes. □



(a) $t = 0$ (b) $t = \frac{1}{2}$ (c) $t = 1$

Figure 5.2: The one-parameter family of quartic curves C_t

5.8 Concluding remarks

It would be interesting to also describe the Weyl chambers of the other five components of the moduli space of smooth real plane quartic curves. A similar question can be asked for the other components of the moduli space of smooth real binary octavics. For the component that corresponds to binary octavics with six points real and one pair of complex conjugate points we managed to compute by hand the Coxeter diagram of this chamber. The result was already much more complicated then the diagram of Figure 4.2. This leads us to believe that the Coxeter diagrams of the remaining five components of the moduli space of smooth real plane quartics will be even more complicated. Computing them would require implementing our version of Vinberg's algorithm in a computer and the result will likely be a very complicated diagram.

Chapter 6

Moduli of real pointed quartic curves

6.1 Introduction

In this chapter we describe a natural open stratum in the moduli space of smooth real plane quartic curves equipped with a point and determine its connected components. This stratum consists of real isomorphism classes of pairs (C, p) with $p \in C(\mathbb{R})$ such that the tangent line at p intersects the curve in two other distinct points. It can be described very explicitly in terms of real tori defined by involutions in the Weyl group of type E_7. This allows us to prove that is has 20 connected components and we find representative curves for each one.

6.2 Moduli of del Pezzo pairs

We start by studying del Pezzo surfaces obtained by blowing up r points on a fixed plane singular cubic. The strict transform of this cubic is a singular anti-canonical curve on the del Pezzo surface. This is the situation studied by Looijenga for del Pezzo surfaces of degree two in [29] and for general del Pezzo surfaces in [30]. In this text we restrict ourselves to the case of del Pezzo surfaces of degree two.

Definition 6.2.1. A del Pezzo pair of degree two is a pair (Y, Z) consisting of a del Pezzo surface Y of degree two and a singular anti-canonical curve $Z \subset Y$. We denote the moduli space of del Pezzo pairs of degree two by $\mathcal{DPP}_2$. By adding a marking to the del Pezzo surface Y we obtain a marked del Pezzo pair (Y, Z, ϕ) with $\phi : \mathbb{Z}_{1,7} \to \mathrm{Pic}(Y)$ a marking of Y. The moduli space of marked del Pezzo pairs is denoted by $\widetilde{\mathcal{DPP}}_2$.

The smooth points of an irreducible plane cubic admit a group law. For smooth cubics this is well known. A similar construction for the group law can be applied to singular cubics as follows. Let Z be a irreducible plane cubic curve and let $\mathcal{O}$ be an inflection point of Z. The map:

$$\begin{aligned} Z^{\text{ns}}(\mathbb{C}) &\to \text{Pic}^0(Z) \\ P &\mapsto [P] - [\mathcal{O}] \end{aligned}$$

is a bijection and defines a group law on $Z^{\text{ns}}(\mathbb{C})$. For a nodal cubic it is well known that there is an isomorphism of groups: $\text{Pic}^0(Z) \cong \mathbb{C}^*$. It is unique up to multiplication by an element of $\text{Aut}(\mathbb{C}^*) \cong \{\pm 1\}$. Similarly for a cuspidal cubic we have an isomorphism $\text{Pic}^0(Z) \cong \mathbb{C}$ unique up to multiplication by an element of $\text{Aut}(\mathbb{C}) \cong \mathbb{C}^*$. A useful property of the group law is the following.

Proposition 6.2.2. *Let Z be a plane cubic curve and let $P_1, \dots P_{3d}$ be points on Z^{ns}. Then $\sum_{i=1}^{3d} P_i = 0$ if and only if $\{P_1, \dots, P_{3d}\} = C^{ns} \cap D$ for some plane curve D of degree d. In particular three points of Z^{ns} add up to zero if and only if they are colinear.*

Proof. The condition $\sum_{i=1}^{3d} P_i = 0$ is equivalent to $\sum_{i=1}^{3d} ([P_i] - [\mathcal{O}]) = 0$ in $\text{Pic}^0(C)$. This implies that the divisor $\sum_{i=1}^{3d} (P_i - \mathcal{O})$ is principal of the form $\text{div}(f/g^d)$ with g the equation of the flex line at $\mathcal{O}$ and f a homogeneous polynomial of degree d which defines the curve D. □

Suppose that (Y, Z, ϕ) is a marked del Pezzo pair and $\pi : Y \to \mathbb{P}^2$ is the corresponding blowup map. The anti-canonical curve $Z \subset Y$ is the strict transform of a plane singular cubic Z_B through the set of 7 blown up points $B = \{P_1, \dots, P_7\}$. Since the points are in general position the cubic Z_B is irreducible, otherwise there would be 3 points on a line or 6 on a conic. In Table 6.1 we distinguish four cases according to the type (nodal or cuspidal) of Z_B and the location of the points B. We also list the Kodaira type of the curve Z in each of these cases.

These four types of $Z \subset Y$ each define a stratum in the moduli space of del Pezzo pairs. The stratum of type I_1 where Z_B is a nodal cubic and $B \subset Z_B^{\text{ns}}$ is generic and defines an open subset $\mathcal{DPP}_2^\circ \subset \mathcal{DPP}_2$. Assume for now that Z is of type I_1 and identify Z with Z_B so that we can make use of the group law on the singular cubic Z_B. By composing the marking $\phi : \mathbb{Z}_{1,7} \to \text{Pic}(Y)$ with the restriction homomorphism $\text{Pic}(Y) \to \text{Pic}(Z)$ we obtain a map that assigns $e_i \mapsto [P_i]$ for $1 \le i \le r$ and $e_0 \mapsto 3[\mathcal{O}]$ where $[\mathcal{O}]$ is an inflection point of Z. Restricting this map to the root lattice $Q < \mathbb{Z}_{1,7}$ induces a homomorphism $\chi \in \text{Hom}(Q, \text{Pic}^0(Z))$ characterized by the relations:

$$\begin{aligned} \chi(e_i - e_{i+1}) &= [P_i] - [P_{i+1}] \\ \chi(e_0 - e_1 - e_2 - e_3) &= 3[\mathcal{O}] - [P_1] - [P_2] - [P_3]. \end{aligned} \tag{6.1}$$

Proposition 6.2.3. *No root lies in the kernel of $\chi : Q \to \text{Pic}^0(Z)$.*

Z	Z_B
I_1	Nodal cubic and the points B are contained in the smooth locus Z_B^{ns} of Z. In this case $\pi\|_Z$ is an isomorphism and we can identify Z with Z_B.
I_2	Nodal cubic and the node coincides with a blown up point. In this case Z consists of two smooth rational curves intersecting transversally in two distinct points.
II	Cuspidal cubic and $B \subset Z^{\mathrm{ns}}$. In this case $Z \cong Z_B$.
III	Cuspidal cubic and the cusp of Z_B coincides with a blown up point. In this case Z consists of two smooth rational curves intersecting with multiplicity 2 in a single point.

Table 6.1: Strata in the moduli space of del Pezzo pairs (Y, Z) of degree two according to the Kodaira type of Z.

Proof. From the construction of χ and Proposition 6.2.2 we see that:

$$\chi(e_i - e_j) = 0 \Leftrightarrow P_i = P_j$$
$$\chi(e_0 - e_i - e_j - e_k) = 0 \Leftrightarrow P_i, P_j, P_k \text{ colinear}$$
$$\chi(2e_0 - e_1 - \ldots - \hat{e}_i - \ldots - e_7) = 0 \Leftrightarrow P_1, \ldots, \hat{P}_i, \ldots, P_7 \text{ conconic}$$

so that the points being in general position is equivalent to $\chi(\alpha) \neq 0$ for all roots $\alpha \in R$. □

After fixing an isomorphism $\mathrm{Pic}^0(Z) \cong \mathbb{C}^*$ we can identify the space:

$$\mathrm{Hom}(Q, \mathrm{Pic}^0(Z))$$

with the complex torus $\mathbb{T} = \mathrm{Hom}\,(Q, \mathbb{C}^*)$. This identification is not canonical but is unique up to multiplication by an element of $\mathrm{Aut}(\mathbb{C}^*) \cong \{\pm 1\}$ which acts on $\mathbb{T}$. The Weyl group W of type E_7 acts on $\mathbb{T}$ by its natural action on Q and we denote the complement of the toric mirrors for this action by $\mathbb{T}^\circ$.

Theorem 6.2.4 (Looijenga)**.** *Let (Y, Z, ϕ) be a marked del Pezzo pair of degree two with Z a nodal anti-canonical curve. The association:*

$$(Y, Z, \phi) \mapsto \big(\chi : Q \to \mathrm{Pic}^0(Z)\big)$$

extends to an isomorphism of orbifolds:

$$\widetilde{\mathcal{DPP}}_2^\circ \to \{\pm 1\}\backslash\mathbb{T}^\circ. \tag{6.2}$$

The left hand side is the open stratum of the moduli space of marked del Pezzo pairs of degree two with Z of type I_1. Similarly we have an isomorphism of orbifolds:

$$\mathcal{DPP}_2^\circ \to W\backslash\mathbb{T}^\circ$$

Proof. Let χ be an element of $\mathbb{T}^\circ = \mathrm{Hom}(Q, \mathbb{C}^*)^\circ$. We construct an inverse to the map of Equation 6.2 by constructing seven points on a fixed nodal cubic Z. Fix an isomorphism $\mathbb{C}^* \to Z^{\mathrm{ns}}$ by choosing one of the three inflection points $\mathcal{O}$ on Z as a unit element. The group law then satisfies $t_i t_j t_k = 1$ if and only if the corresponding points P_i, P_j, P_k on Z^{ns} are colinear. Since the seven points should satisfy (6.1) they must also satisfy the equality:

$$P_i = \chi(e_i - e_0/3) \tag{6.3}$$

where we consider χ as an element of $\mathrm{Hom}(Q \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{C}^*)$. This determines the seven points uniquely up to addition of an inflection point of Z^{ns} (or equivalently multiplication by a third root of unity of $\mathbb{C}^*$). Blowing up these seven points determines a marked del Pezzo surface Y and the pullback of Z under the blowup map defines a nodal anti-canonical curve on Y isomorphic to Z. □

To conclude this section we obtain explicit descriptions of the standard Cremona transformation centered in three points on a plane nodal cubic Z and of the Geiser involution in terms of the coordinate $t \in \mathbb{C}^* \cong Z^{\mathrm{ns}}(\mathbb{C})$. The Cremona map $\rho(s_7)$ centered in the points P_1, P_2, P_3 of Z with coordinate t maps Z to another nodal cubic Z' which can be mapped back to Z with new coordinate t' by an element of $\mathrm{PGL}(3, \mathbb{C})$. If $t_i, t_j, t_k, t_1, t_2, t_3 \in Z$ are distinct points lying on a conic, then t'_i, t'_j, t'_k lie on a line by the properties of the standard Cremona transformation so that:

$$1 = t_i t_j t_k t_1 t_2 t_3 = t'_i t'_j t'_k.$$

Similarly, the standard Cremona transformation maps the line L_{12} to t'_3, so that for a point t_i on L_{12}:

$$1 = t_i t_1 t_2 = t'_3 t'^{-1}_i.$$

From these formulas we compute:

$$t' = \begin{cases} t(t_1 t_2 t_3)^{-2/3} & t = t_1, t_2, t_3 \\ t(t_1 t_2 t_3)^{1/3} & t \text{ general} \end{cases} \tag{6.4}$$

which determines t' up to multiplication by a third root of unity. These formulas can also be derived by computing the action of $s_7 \in W$ on Equation 6.3. From Proposition 6.2.2 we see that if the seven points $\{P_1, \ldots, P_7\}$ lie on a nodal cubic then the Geiser involution is given by:

$$\gamma : t \mapsto t^{-1}(t_1 \ldots t_7)^{-1} \tag{6.5}$$

6.3 Strata of smooth pointed quartic curves

We have seen that the moduli space $\mathcal{DP}_2$ of del Pezzo surfaces of degree two and the moduli space $\mathcal{Q}$ of plane quartic curves are isomorphic. In this section we relate the moduli space $\mathcal{DPP}_2$ of del Pezzo pairs of degree two and its strata to the moduli space of smooth pointed plane quartics $\mathcal{Q}_1$. We first define this latter space $\mathcal{Q}_1$.

Definition 6.3.1. Let k be the field of real or complex numbers. A pointed plane quartic curve is a pair (C,p) with C a plane quartic curve and $p \in C(k)$. The space Γ of smooth pointed quartics curve is defined by:

$$\Gamma(k) = \{(C,p)\ ;\ p \in C(k)\} \subset P_{4,3}(k) \times \mathbb{P}^2.$$

The group $\mathrm{PGL}(3,k)$ acts on Γ and the quotient:

$$\mathcal{Q}_1 = \mathrm{PGL}(3,k)\backslash\Gamma(k)$$

is a (coarse) moduli space for smooth pointed plane quartics.

To a pointed quartic (C,p) we can associate a del Pezzo pair (Y,Z) in the following way. The del Pezzo surface Y of degree two defined by:

$$Y = \{w^2 = f(x,y,z)\} \subset \mathbb{P}(2,1,1,1) \tag{6.6}$$

in weighted projective space. The morphism defined by the anti-canonical map $|-K_Y|$ is realized by the projection map $\psi : Y \to \mathbb{P}^2$ given by:

$$[w : x : y : z] \mapsto [x : y : z].$$

Every anti-canonical curve on Y is the pullback under ψ of a line in $\mathbb{P}^2$. We define $Z = \psi^{-1}T_pC$ to be the pullback of the tangent line to C at p. It is a singular anti-canonical curve on Y of arithmetic genus 1. Its Kodaira type is determined by the type of the intersection divisor $D = (C \cdot T_pC)$ defined below.

Definition 6.3.2. Let $D = \sum_{i=1}^{k} d_i(p_i)$ be a divisor on a curve C with the p_i distinct and ordered in such a way that $d_1 \geq \ldots \geq d_r$. The type of D is the r-tuple $\underline{d} = (d_1, \ldots, d_r)$.

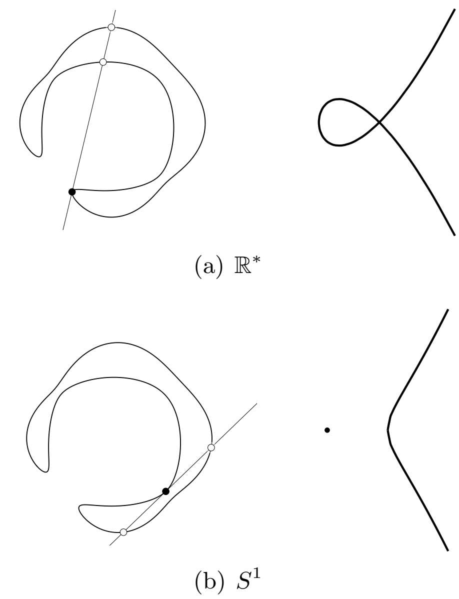
There are four possibilities for the type of D corresponding to the types for Z in Table 6.1. Similarly we obtain four strata in the space Γ. The strata Γ^{bit} and Γ^{flex} where the point p is respectively a bitangent and an inflection point have codimension one and the stratum Γ^{hflex} where p is a hyperflex has codimension two in the space Γ.

Stratum	D	Z	Codim.
Γ°	$(2,1,1)$	I_1	0
Γ^{bit}	$(2,2)$	I_2	1
Γ^{flex}	$(3,1)$	II	1
Γ^{hflex}	(4)	III	2

Table 6.2: Strata in the space of pointed quartics

6.4 Moduli of real del Pezzo pairs of degree two

Let (C, p) be a smooth real pointed plane quartic curve. By the results of Section 5.6 and the previous section we can associate to (C, p) a real del Pezzo pair (Y, Z) with real form χ_Y^- such that $Y^-(\mathbb{R})$ in nonorientable. The real form restricts to Z which is a real curve of arithmetic genus 1 on Y and $Z^{\text{ns}}(\mathbb{R}) \neq \emptyset$. If (C, p) is in the open stratum Γ° then the tangent line T_pC intersects C in two other distinct points which can both be real or a pair of complex conjugate points. In both cases the curve Z is of type I_1 (it has a single node). Since $Z^{\text{ns}}(\mathbb{R}) \neq \emptyset$ there are two possibilities for the real form induced by χ_Y^- on $Z^{\text{ns}}(\mathbb{C}) \cong \mathbb{C}^*$. Either it maps: $t \mapsto \bar{t}$ and $Z^{\text{ns}}(\mathbb{R}) \cong \mathbb{R}^*$ or $t \mapsto \bar{t}^{-1}$ and $Z^{\text{ns}}(\mathbb{R}) \cong S^1$. An example of both is given in figure 6.1.



(a) $\mathbb{R}^*$

(b) S^1

Figure 6.1: The two possibilities for $Z^{\text{ns}}(\mathbb{R})$ for a quartic curve with one component.

Theorem 6.4.1. *The map* $(C,p) \mapsto \big(\chi : \operatorname{Pic}^0 Y \to \operatorname{Pic}^0 Z\big)$ *extends to an isomorphism:*

$$(\mathcal{Q}_1^\circ)^{\mathbb{R}} \cong (W\backslash\mathbb{T}^\circ)(\mathbb{R})$$

where $\mathbb{T}^\circ$ *denotes the complement in* $\mathbb{T} = \operatorname{Hom}(Q, \mathbb{C}^*)$ *of the mirrors of the action of the Weyl group* W *of type* E_7*.*

Proof. A lot of work has already been done in the proof of Theorem 6.2.4. We need to show that the element $\chi : Q \to \mathbb{C}^*$ we associate to (C,p) is a real point of $W\backslash\mathbb{T}^\circ$. By definition this means that $w \cdot \chi = \bar{\chi}$ for some element $w \in W$. The real structure χ_Y^- of Equation 5.6 acts on $\operatorname{Pic}^0(Y) \cong Q$ as an involution $u \in W$. Since $Y^-(\mathbb{R})$ is nonorientable we see from Table 5.3 that this involution is of type $1, A_1, A_1^2, A_1^3$ or D_4. The action of the restriction of χ_Y^- to $\operatorname{Pic}^0(Y) \cong \mathbb{C}^*$ is one of $t \mapsto \bar{t}^{\pm 1}$ so the element χ satisfies $u \cdot \chi = \bar{\chi}^{\pm 1}$. Since the Weyl group W of type E_7 contains -1 we can rewrite this as $\pm u \cdot \chi = \bar{\chi}$ so that χ is indeed a real element of $W\backslash\mathbb{T}^\circ$.

Conversely, let χ be a real point of $(W\backslash\mathbb{T}^\circ)(\mathbb{R})$. By Proposition 6.5.6 we can assume that $u \cdot \chi = \bar{\chi}^{\pm 1}$ with $u \in W$ an involution of type $1, A_1, A_1^2, A_1^3$ or D_4. As in the proof of Theorem 6.2.4 we fix a real nodal cubic Z in $\mathbb{P}^2$ and an isomorphism $Z^{\mathrm{ns}}(\mathbb{C}) \cong \mathbb{C}^*$ by choosing a real inflection point. The real form of Z is then equivalent to one of $t \mapsto \bar{t}^{\pm 1}$. As in Equation 6.2 the element χ determines seven points in $\mathbb{C}^*$ by the formula $t_i = \chi(e_i - \frac{e_0}{3})$ which we interpret as points on $Z^{\mathrm{ns}}(\mathbb{C})$. Since χ is real these points satisfy:

$$u \cdot (t_1, \ldots, t_7) = (\bar{t}_1^{\pm 1}, \ldots, \bar{t}_7^{\pm 1}).$$

where the involution $u \in W$ acts by the Cremona action of the Weyl group as a birational involution of $\mathbb{P}^2$. This involution lifts an anti-holomorphic involution of the del Pezzo surface Y obtained by blowing up the seven points. These two construction are inverse to each other.

□

6.5 Reflection groups and real tori

In this section we study the connected components of the space $(W\backslash\mathbb{T}^\circ)(\mathbb{R})$ where $\mathbb{T}$ is the complex torus $\mathbb{T} = \operatorname{Hom}(Q, \mathbb{C}^*)$ for Q a root lattice of type ADE. For type E_7 this space has 20 connected components which we describe explicitly as quotients of certain real subtori of $\mathbb{T}$.

Reflection groups and root systems

We start by recalling some facts about reflection groups and root systems. Our main reference is [9] Chapter VI. Let V be a real, finite dimensional vector space of dimension n with an inner product $(\cdot,\cdot)$. For every nonzero $\alpha \in V$

we define the reflection $s_\alpha \in O(V)$ by:

$$s_\alpha(x) = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha$$

for all $x \in V$. The mirror H_α is defined as the fixed point locus of the reflection s_α. A *root system* $R \subset V$ is a finite set of nonzero vectors called *roots* that satisfy the following properties:

R1 The $\mathbb{R}$-span of R is V.

R2 If $\alpha \in R$ then $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$.

R3 $s_\alpha R = R$ for all $\alpha \in R$.

R4 $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

A system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subseteq R$ is a basis for V such that every root is an integral linear combination $\sum_{i=1}^r c_i \alpha_i$ of simple roots of the same sign. From now on we assume we have fixed a system of simple roots $\Delta \subset R$. For every root $\alpha \in R$ we define the coroot $\alpha^\vee$ by:

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

The set of coroots $R^\vee$ is again a root system (the coroot system) with corresponding coroot lattice $Q^\vee = \mathbb{Z}R^\vee$. A root system R is called *irreducible* if it is non-empty and cannot be decomposed as an orthogonal direct sum $R = R_1 \oplus R_2$ of two non-empty root systems R_1 and R_2. Let R be an irreducible root system. We define the *highest root* $\tilde{\alpha}$ of R with respect to Δ as the unique root such that $\sum_{i=1}^r c_i$ is maximal. We also define $\alpha_0 = -\tilde{\alpha}$. The Weyl group W is the group generated by the reflections s_α with $\alpha \in R$ or equivalently by the simple reflection s_α with $\alpha \in \Delta$. It is a finite group and acts simply transitively on the connected components of $V \setminus \cup H_\alpha$ which are called *chambers*. The fundamental chamber $\mathcal{C}$ is defined by:

$$\mathcal{C} = \{x \in V \mid (\alpha_i, x) > 0 \text{ for } 1 \leq i \leq r\}.$$

Its closure $\bar{\mathcal{C}}$ is a fundamental domain for the action of W on V.

The *affine Weyl group* W_a is the group generated by the affine reflections $s_{\alpha,k}$ with $\alpha \in R$ and $k \in \mathbb{Z}$ defined by:

$$s_{\alpha,k}(x) = x - (\alpha, x)\alpha^\vee + k\alpha^\vee.$$

The mirror of $s_{\alpha,k}$ is the affine hyperplane $H_{\alpha,k} = \{x \in V; (\alpha, x) = k\}$. The affine Weyl group W_a is the semidirect product of W by the coroot lattice: $W_a = Q^\vee \rtimes W$. This allows us to write $s_{\alpha,k} = t(k\alpha^\vee)s_\alpha$ where $t(k\alpha^\vee)$ denotes translation over $k\alpha^\vee$ in V. The group W_a acts simply transitively on the connected components of the space $V^\circ = V \setminus \cup H_{\alpha,k}$ which are called *alcoves*. The fundamental alcove $\mathcal{A}$ is the simplex given by:

$$\mathcal{A} = \{x \in V \mid (\tilde{\alpha}, x) < 1,\ (\alpha_i, x) > 0 \text{ for } 1 \leq i \leq r\}$$

and its closure $\bar{\mathcal{A}}$ is a fundamental domain for the action of W_a on V. The $r+1$ closed facets $\bar{\mathcal{A}}_i$ of $\bar{\mathcal{A}}$ are given by:

$$\bar{\mathcal{A}}_i = \begin{cases} H_{\alpha_i} \cap \bar{\mathcal{A}} & \text{if } 1 \leq i \leq r \\ H_{\tilde{\alpha},1} \cap \bar{\mathcal{A}} & \text{if } i = 0 \end{cases}$$

A reducible root system R can be decomposed into a direct sum of irreducible root systems $\{R_i\}_{i \in I}$ for some finite index set I. The Weyl group $W(R)$ of R is the direct product of the Weyl groups $\{W(R_i)\}_{i \in I}$. This decomposition is unique up to permutation of the factors. A fundamental domain for the action of $W(R)$ on V is now the direct product of the fundamental chambers of the factors. Similarly for the affine Weyl group $W_a(R) = Q^\vee \rtimes W(R)$ a fundamental domain on V is the product of the fundamental alcoves of the factors.

We want to determine the stabilizer $\mathrm{Stab}_{W_a}(x)$ of an $x \in V$ in the affine Weyl group. Since all points in the orbit $W_a \cdot x$ have conjugate stabilizers, we can assume that $x \in \bar{\mathcal{A}}$. The stabilizer $\mathrm{Stab}_{W_a}(x)$ is the group generated by the reflections in the mirrors $H_{\alpha,k}$ that contain x. It is a Weyl group with root system $R(x)$ and system of simple roots $\Delta(x)$ given by:

$$R(x) = \{\alpha \in R\ ;\ (\alpha, x) \in \mathbb{Z}\}\ ,\ \Delta(x) = \left\{\alpha_i\ ; 0 \leq i \leq r,\ x \in \bar{\mathcal{A}}_i\right\}. \qquad (6.7)$$

These root systems can be reducible, even if the root system R is irreducible.

The extended affine Weyl group

The coweight lattice $P^\vee$ is defined by:

$$P^\vee = \{\mathbb{Z} \in V; (\mathbb{Z}, \alpha) \in \mathbb{Z} \quad \forall \alpha \in R\}$$

and contains $Q^\vee$ as a subgroup of finite index. It has a basis $\{\varpi_1^\vee, \ldots, \varpi_r^\vee\}$ dual to the basis of simple roots of R, so that $(\alpha_i, \varpi_j^\vee) = \delta_{ij}$. The *extended affine Weyl group* W'_a is defined as the semidirect prodct $P^\vee \rtimes W$ with $P^\vee$ acting on V by translations. We will prove that W'_a is the extension of W_a by a finite group of automorphisms of the fundamental alcove.

Let $n_i = (\tilde{\alpha}, \varpi_i^\vee)$ be the coefficient of α_i in the highest root $\tilde{\alpha}$. For notation it is convenient to define $\varpi_0^\vee = 0 \in P^\vee$ and $n_0 = 1$. The fundamental alcove $\mathcal{A}$ is the open n-simplex with vertices $\{\varpi_i^\vee/n_i\}_{i=0}^r$. Let J be the set of indices $0 \leq i \leq r$ such that $n_i = 1$. The vertices $\varpi_i^\vee$ with $i \in J$ or equivalently: $R(\varpi_i^\vee/n_i) \cong R$ are called *special*. Put $R_0 = R$ and let w_0 be the longest element of W with respect to the basis of simple roots in equation $\{\alpha_i\}_{i=1}^r$.

We also define for every $i \in J\setminus\{0\}$ the root system R_i generated by the simple roots:

$$\{\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_r\}. \tag{6.8}$$

Let w_i be the longest element of the Weyl group $W(R_i)$ with respect to the basis of simple roots (6.8). For every $i \in J$ we now define the following element of the extended affine Weyl group:

$$\gamma_i = t(\varpi_i^\vee)w_i w_0.$$

Observe that $\gamma_i(0) = \varpi_i^\vee$ and $\gamma_0 = w_0^2 = 1$. Proposition 6 from [9] §2 VI states that we have equality:

$$\{w \in W_a'; w(\mathcal{A}) = \mathcal{A}\} = \{\gamma_i\}_{i\in J} \tag{6.9}$$

and we can identify the group (6.9) with the finite Abelian group $P^\vee/Q^\vee$ by assigning to γ_i the class of ϖ_i mod $Q^\vee$ where $i \in J$. We see that the group $P^\vee/Q^\vee$ acts simply transitively on the special points. Since the affine Weyl group W_a acts simply transitively on the alcoves it follows from the above that we have an isomorphism:

$$W_a \rtimes P^\vee/Q^\vee \xrightarrow{\sim} P^\vee \rtimes W = W_a' \tag{6.10}$$

by assigning $(t(\mathbb{Z})w, \gamma_i) \mapsto t(\mathbb{Z} + w\varpi_i^\vee)ww_iw_0$. The extended affine Weyl group acts transivitely connected components of V°, but the action need not be free. The action of $P^\vee/Q^\vee$ on the fundamental alcove $\mathcal{A}$ can have fixed points. Also W_a' is in general not a Coxeter group.

Lemma 6.5.1. *Let $x \in \bar{\mathcal{A}}$, then:*

$$\mathrm{Stab}_{W_a'}(x) = \mathrm{Stab}_{W_a}(x) \rtimes \mathrm{Stab}_{P^\vee/Q^\vee}(x)$$

Proof. Let $t(\mathbb{Z})w \in W_a$ and $\gamma \in P^\vee/Q^\vee$ be such that $t(\mathbb{Z})w\gamma(x) = x$. Define $y := \gamma(x) \in \bar{\mathcal{A}}$. Now $t(\mathbb{Z})w(y) = x$ with $x, y \in \bar{\mathcal{A}}$, and because $\bar{\mathcal{A}}$ is a strict fundamental domain for the action of W_a we can conclude $x = y$, so $\gamma(x) = x$. This also implies that $t(\mathbb{Z})w(x) = x$. □

The centralizer of an involution in a reflection group

Let (W, S) be a finite Coxeter group and let $u \in W$ be an involution. We want to determine the centralizer $C_W(u)$ of u in W. By the classification of involutions in Coxeter groups there is a subset $I \subseteq S$ such that u is conjugate in W to the involution w_I: the unique longest element -1 in the parabolic subgroup W_I. Felder en Veselov in [19] observe the following:

Proposition 6.5.2. *If W_I is a parabolic subgroup of W that satisfies the (-1) condition then $C_W(w_I) = N_W(W_I)$.*

Proof. The element u is the unique longest element of W_I, so that $wuw^{-1} = u$ for all $w \in N_W(W_I)$ and $N_W(W_I) < C_W(u)$. For the other inclusion let $w \in C_W(u)$. We need to prove that $ws_{\alpha_i}w^{-1} \in W_u^-$, or equivalently: $w \cdot \alpha_i \in R_u^-$ for all $i \in I$. The element w preserves the eigenspace decomposition of V so that $w \cdot \alpha_i \in R \cap V_u^- = R_u^-$. □

This allows us to use the classification of normalizers of parabolic subgroups of reflection groups by Howlett [22]. Note that a lot of the results of this section also appear in [28]. Let $u = w_I$ and decompose V into ± 1-eigenspaces for u: $V = V_u^+ + V_u^-$ where $V_u^- = V_I$. This defines two orthogonal root systems and corresponding Coxeter groups:

$$R_u^{\pm} = R \cap V_u^{\pm} \quad , \quad W_u^{\pm} = W(R_u^{\pm}).$$

Observe that $W_u^- = W_I$ and that the eigenspace V_u^- is spanned by the roots of R_u^-. The group $W_u = W_u^- \times W_u^+$ is generated by all reflections that commute with u and is contained in the centralizer $C_W(u)$ of u. This centralizer also contains a non-reflection part G_u which now describe. Consider the elements:

$$\rho_{\pm} = \frac{1}{2} \sum_{\alpha \in R_u^{\pm}(+)} \alpha$$

where the sum runs over all positive roots of $R_u^{\pm}$. The set:

$$R_u^c = \{\alpha \in R \ ; \ (\alpha, \rho_+) = (\alpha, \rho_-) = 0\}$$

is a root system which can be written as an orthogonal disjoint union of subroot systems $R_u^c = R_1 \cup R_2$. These factors are isomorphic root systems and are exchanged by the involution u. The Weyl group $W(R_u^c)$ is a product:

$$W(R_u^c) = W(R_1) \times W(R_2)$$

whose factors are exchanged by conjugation with u. The group G_u consists of all elements of $W(R_u^c)$ that commute with u and is the diagonal of this product:

$$G_u = \{(w, uwu) \ ; \ w \in W(R_1)\}.$$

This group is generated by pairs of commuting reflections $s_\alpha s_{u \cdot \alpha}$ with $\alpha \in R_1$ and is isomorphic to $W(R_1)$. Now we can formulate the main theorem of this section.

Theorem 6.5.3. *The centralizer of an involution $u \in W$ splits as a semidirect product:*

$$\begin{aligned} C_W(u) &\cong W_u \rtimes G_u \\ &\cong W_u^- \rtimes G_u^+ \end{aligned}$$

where G_u^+ is the reflection group defined by $G_u^+ = \{w \in W \ ; \ wI = I\}$ which contains W_u^+ as a normal subgroup.

Root tori and their invariants

If the type of a root system R occurs in Equation 2.10 we say it is of del Pezzo type. Such root systems are products of root systems of type ADE so that all roots have the same length. This implies that we can identify the root (resp weight) and the coroot (resp coweight) lattices R and $R^\vee$ (resp P and $P^\vee$). To simplify notation we use this identification from now on. To a root system R of del Pezzo type we associate the complex torus $\mathbb{T} = \mathbb{C}^* \otimes P = \mathrm{Hom}(Q, \mathbb{C}^*)$. It has a natural action of the Weyl group W.

Theorem 6.5.4. *The quotient $W\backslash\mathbb{T}$ is an affine toric variety and the algebra of W-invariants of $\mathbb{Z}[Q]$ is the semi-group algebra given by*

$$\mathbb{Z}[Q]^W \cong \mathbb{Z}[P_+ \cap Q] \tag{6.11}$$

where $P_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\varpi_i$ is the lattice cone spanned by the fundamental weights.

Proof. For the proof of the isomorphism of Equation 6.11 we refer to [31] Section 6.3.5. The coordinate ring of $W\backslash\mathbb{T}$ is $\mathbb{C}[\mathbb{T}]^W \cong \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[P_+ \cap Q]$. It is the complexification of a semi-group algebra and its spectrum is by definition an affine toric variety. □

This theorem is a generalisation of classical exponential invariant theory for root systems as described in [9] VI §3. The main theorem of that section states that the algebra of W-invariants of $\mathbb{Z}[P]$ is a polynomial algebra:

$$\mathbb{Z}[P]^W \cong \mathbb{Z}[P_+].$$

This is a toric analogue of a well known theorem of Chevalley. The algebra $\mathbb{C}[P]$ is the coordinate ring of the algebraic torus $T = \mathbb{C}^* \otimes Q$. Since the W-invariants form a polynomial algebra we can rephrase the theorem as $W\backslash T \cong \mathbb{C}^n$. The torus T is a finite cover of $\mathbb{T}$ where the group of deck transformations is isomorphic to P/Q and there is an isomorphism of orbifolds:

$$W\backslash\mathbb{T} \cong (P/Q)\backslash\mathbb{C}^n.$$

The action of W on $\mathbb{T}$ on the complement of the mirrors $\mathbb{T}^\circ$ is not free in general: the group P/Q can have fixed points in $\mathbb{T}^\circ$. The stabilizers are described by the following lemma.

Lemma 6.5.5. *For $t \in \mathbb{T}$ the stabilizer $\mathrm{Stab}_W(t)$ is the extension of a reflection group $\mathrm{Stab}^r_W(t)$ by a finite group of diagram automorphisms.*

Proof. Consider the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 1$$

where $\exp : z \mapsto e^{2\pi i z}$. By tensoring from the right with P we obtain another exact sequence:

$$0 \longrightarrow P \longrightarrow V_{\mathbb{C}} \xrightarrow{\exp} \mathbb{T} \longrightarrow 1$$

where $V_{\mathbb{C}} = \mathbb{C} \otimes V$ is the complexification of V. From the sequence we read off that $W \backslash \mathbb{T} \cong W'_a \backslash V_{\mathbb{C}}$ where the extended affine group $W'_a = P \rtimes W$ acts on $V_{\mathbb{C}}$ by the formula:

$$t(\lambda) w \cdot (x + iy) = (w \cdot x + \mathbb{Z}) + i(w \cdot y)$$

for $\mathbb{Z} \in Q$ and $w \in W$. Write $z = \log t$, by Lemma 6.5.1 the group $\mathrm{Stab}_W(t)$ is isomorphic to:

$$\mathrm{Stab}_{W'_a}(z) \cong \mathrm{Stab}_{W_a}(z) \rtimes \mathrm{Stab}_{P/Q}(z).$$

The group $\mathrm{Stab}_{W_a}(z)$ is a Weyl group generated by the reflection in the mirrors that contain z. □

Real root tori and their connected components

Complex conjugation on $\mathbb{C}^*$ defines a real form on the complex torus $\mathbb{T} = \mathbb{C}^* \otimes P$. This in turn defines a real form on the quotient $W \backslash \mathbb{T}$. Let $q : \mathbb{T} \to W \backslash \mathbb{T}$ be the quotient map. The real points of $W \backslash \mathbb{T}$ are the points $q(t)$ such that t and $\bar{t}$ are in the same W-orbit so that:

$$q(t) \in (W \backslash \mathbb{T})(\mathbb{R}) \Longleftrightarrow w \cdot t = \bar{t} \quad \text{for some } w \in W.$$

We will prove in Proposition 6.5.6 that we can assume that w is an involution in W. Every involution in W defines a real form on $\mathbb{T}$ by composing with complex conjugation. The real points of such a real form are given by:

$$\mathbb{T}_u(\mathbb{R}) = \{t \in \mathbb{T} \; ; \; u \cdot t = \bar{t}\}.$$

The following proposition is a slight modification of a result due to Tits ([28], Proposition 2.2) to the present situation. The proof is similar to the one given there.

Proposition 6.5.6 (Tits). *The following equality holds:*

$$q^{-1}(W \backslash \mathbb{T})(\mathbb{R}) = \bigcup_{u \in W; u^2 = 1} \mathbb{T}_u(\mathbb{R}).$$

Proof. Let $t \in \mathbb{T}$ be such that $w \cdot t = \bar{t}$ for some $w \in W$. We will prove that there is a w' in the reflection part $\mathrm{Stab}^r_W(t)$ of the stabilizer $\mathrm{Stab}_W(t)$ such that $u = ww'$ is an involution in $\mathrm{Stab}_W(t)$. The reflection part of the stabilizer is a finite reflection group which acts on the tangent space $T_t\mathbb{T}$ through its complexified reflection representation. Since $\mathrm{Stab}_W(t)$ is also the stabilizer of $\bar{t}$ we see that:

$$w\,\mathrm{Stab}_W(t)w^{-1} = \mathrm{Stab}_W(w \cdot t) = \mathrm{Stab}_W(\bar{t}) = \mathrm{Stab}_W(t).$$

This implies that w permutes the chambers of $\mathrm{Stab}^r_W(t)$ so that we can find a $w' \in \mathrm{Stab}^r_W(t)$ such that $u = ww'$ leaves a chamber invariant. Since $\mathrm{Stab}^r_W(t)$ acts simply transitively on its chambers it follows that $u^2 = 1$.

□

The group W permutes the real tori $\mathbb{T}_u$ according to:

$$w \cdot \mathbb{T}_u = \mathbb{T}_{wuw^{-1}}$$

so W-equivalent real tori correspond to conjugate involutions. Furthermore the stabilizer of a real torus $\mathbb{T}_u(\mathbb{R})$ in W is precisely the centralizer $C_W(u)$ of u in W. We want to study the real tori $\mathbb{T}_u$ and especially their connected components in more detail. The involution u acts naturally on the weight lattice P and there exists a so called *normal* basis for P such that:

$$\begin{aligned} u &= I_{n_1} \oplus (-I_{n_2}) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{n_3} \\ P &= P_{1,u} \oplus P_{2,u} \oplus P_{3,u}. \end{aligned} \tag{6.12}$$

This is described in detail in [11]. The decomposition of Equation 6.12 is not unique but the triple (n_1, n_2, n_3) which we call the *type* of the involution $u \in W$ is an invariant of the involution. A choice of normal basis determines an isomorphism:

$$\mathbb{T}_u(\mathbb{R}) \cong (\mathbb{R}^*)^{n_1} \times (S^1)^{n_2} \times (\mathbb{C}^*)^{n_3} \tag{6.13}$$

with $n_1, n_2, n_3 \in \mathbb{N}$ and $n_1 + n_2 + 2n_3 = n$. This product consists of factors of *split* ($\mathbb{R}^*$), *compact* (S^1) and *complex* ($\mathbb{C}^*$) type. To determine the numbers n_i we have the following lemma from [11].

Lemma 6.5.7. *There are isomorphisms of abelian groups:*

$$\frac{\ker(u-1)}{\mathrm{im}(u+1)} \cong (\mathbb{Z}/2\mathbb{Z})^{n_1} \quad , \quad \frac{\ker(u+1)}{\mathrm{im}(u-1)} \cong (\mathbb{Z}/2\mathbb{Z})^{n_2}$$

where the first of these can be identified with the component group $\pi_0(\mathbb{T}_u(\mathbb{R}))$.

Proof. We construct the first of these isomorphisms. After choosing a normal basis for P we can use the lattice decomposition (6.12) to see that the lattice $\ker(u-1)$ is isomorphic to $P_{1,u} \oplus P_{3,u}$. Similarly the lattice $\mathrm{im}(u+1)$ is isomorphic to $2P_{1,u} \oplus P_{3,u}$ and the quotient of these lattices is $P_{1,u}/2P_{1,u}$. □

To determine the number of connected components of $C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$ we need to compute the number of orbits under the action of $C_W(u)$ on the connected components of $\mathbb{T}_u(\mathbb{R})$. The following lemma shows that in fact only W_u^+ acts non-trivially on the components.

Lemma 6.5.8. *The group* $W_u^- \rtimes G_u$ *is contained in the kernel of the action of* $C_W(u)$ *on* $\ker(u-1)/\operatorname{im}(u+1)$.

Proof. Suppose $x \in \ker(u-1)$. In particular $x \in V_u^+$ so that $w \cdot x = x$ for all $w \in W_u^-$. The group is G_u is generated by products of commuting reflections $s_\alpha s_{u\cdot\alpha}$ where $\alpha \in R_1$. Such an element acts trivially on the class of x in $\ker(u-1)/\operatorname{im}(u+1)$ since:

$$s_\alpha s_{u\cdot\alpha} \cdot x - x = (u+1)\left((\alpha, x)\alpha\right) \in \operatorname{im}(u+1).$$

□

Connected components of $W\backslash\mathbb{T}_1$

For the trivial involution $u = 1$ the real torus $\mathbb{T}_1$ is maximally split and we have:

$$\mathbb{T}_u(\mathbb{R}) \cong (\mathbb{R}^*)^n \quad , \quad \frac{\ker u - 1}{\operatorname{im} u + 1} \cong \frac{P}{2P} \quad , \quad C_W(u) = W.$$

The decomposition of this real torus into connected components is described by:

$$\mathbb{T}_1(\mathbb{R}) = \bigsqcup_{[\varpi] \in P/2P} \mathbb{T}_1^{\varpi} \quad \text{where} \quad \mathbb{T}_1^{\varpi} = \exp\left(\frac{1}{2}\varpi + iV\right).$$

We can use the basis of fundamental weights $\{\varpi_1, \ldots, \varpi_n\}$ of P to identify $\mathbb{T}$ with $(\mathbb{C}^*)^n$ through the isomorphism:

$$\mathbb{C}^* \otimes P \to (\mathbb{C}^*)^n$$
$$\sum_{i=1}^n t_i \otimes \varpi_i \mapsto (t_1, \ldots, t_n).$$

In this way we can also identify the component group $P/2P$ of $\mathbb{T}_1$ with the subgroup $\{-1, 1\}^n \subset (\mathbb{R}^*)^n$.

Remark 6.5.9. An element of $P/2P$ can be represented by a colouring of the Coxeter diagram of W where the ith node is coloured white if the corresponding coefficient of ϖ_i is 1 and coloured black if it is -1. To determine the action of W on two-coloured Coxeter diagrams first observe that a simple reflection for W acts on the fundamental weights as:

$$s_i \cdot \varpi_j = \begin{cases} \varpi_i & i \neq j \\ -\varpi_i + \sum_{k \in I_j} \varpi_j & i = j \end{cases} \tag{6.14}$$

where the sum runs over the set I_j of neighbouring vertices of the jth vertex of the Coxeter diagram. Now the generator s_i only acts nontrivially if the ith node v_i is black. In this case the action of s_i changes the colour of all neighbouring vertices of v_i but leaves v_i unchanged.

It is often convenient to use the group $\frac{1}{2}P/P$ for representing the connected components of $\mathbb{T}_u(\mathbb{R})$ instead of the group $P/2P$. The reason for this is that there are bijections of orbit spaces:

$$W\Big\backslash\left(\frac{1}{2}P/P\right) \cong (P \rtimes W)\Big\backslash\frac{1}{2}P \cong (P/Q)\Big\backslash\left(\frac{1}{2}P\cap\bar{\mathcal{A}}\right). \qquad (6.15)$$

We can count the points in the intersection $\frac{1}{2}P\cap\bar{\mathcal{A}}$ and the group P/Q is typically small. Its action is easily determined for Weyl groups of type ADE. We do this for root systems of type A_n in Example 6.5.10.

Example 6.5.10 (A_n)**.** We will use the above method to describe the orbit space $W\backslash(P/2P)$ for type A_n. This will be used frequently in the next section. Representatives for the orbits are given by:

$$\begin{cases}\{0,\varpi_1,\ldots,\varpi_{n/2}\} & \text{n even}\\ \{0,\varpi_1,\ldots,\varpi_{(n+1)/2}\} & \text{n odd.}\end{cases}$$

Proof. For a root system of type A_n all the roots have coefficient 1 in the highest root so the fundamental alcove is the convex hull of the fundamental weights. From this we determine:

$$\bar{\mathcal{A}}\cap\frac{1}{2}P=\left\{0,\frac{\varpi_i}{2},\frac{\varpi_i+\varpi_j}{2}\right\}_{1\le i\neq j\le n}.$$

The group P/Q is cyclic of order $n+1$ and is generated by γ_1 which acts as the permutation $(01\ldots n)$ on the indices of the fundamental weights $\{\varpi_i\}$. A small calculation shows that:

$$\gamma_1\left(\frac{\varpi_i}{2}\right)=\frac{\varpi_{i+1}+\varpi_1}{2}$$

where we use the notation $\varpi_0=0$ and the indices are considered mod $n+1$. A typical γ_1-orbit (for which $n+1\neq 2i$) is of the form:

$$\begin{aligned}\frac{\varpi_i}{2}\mapsto\frac{\varpi_{i+1}+\varpi_1}{2}\mapsto\ldots\mapsto\frac{\varpi_n+\varpi_{n-i}}{2}\mapsto\frac{\varpi_{n-i+1}}{2}\\ \mapsto\frac{\varpi_{n-i+2}+\varpi_1}{2}\mapsto\ldots\mapsto\frac{\varpi_n+\varpi_{i-1}}{2}\mapsto\frac{\varpi_i}{2}.\end{aligned}$$

A representative is given by $\varpi_i/2$. If n is even then all orbits are of this form and there are $n/2$ orbits. If n is odd then there is one additional orbit with $n+1=2i$ given by:

$$\frac{\varpi_{(n+1)/2}}{2}\mapsto\frac{\varpi_{i+1}+\varpi_1}{2}\mapsto\ldots\mapsto\frac{\varpi_n+\varpi_{n-i}}{2}\mapsto\frac{\varpi_{(n+1)/2}}{2}.$$

A representative is given by $\varpi_{(n+1)/2}$. □

For non-trivial involutions u it is more complicated to determine the connected components of:

$$C_W(u)\backslash\mathbb{T}_u(\mathbb{R}).$$

After choosing a normal basis for P in which u takes a normal form we can identify the component group $\pi_0(\mathbb{T}_u(\mathbb{R}))$ with $P_{1,u}/2P_{1,u}$. However since there is no canonical choice for $P_{1,u}$ we have to compute:

$$W_u^+\backslash (P_{1,u}/2P_{1,u})$$

case by case. The real torus $\mathbb{T}_u(\mathbb{R})$ can be written as the disjoint union of its connected components in the following way:

$$\mathbb{T}_u(\mathbb{R}) = \bigsqcup_{[\varpi]\in\frac{1}{2}P_{1,u}/P_{1,u}} \mathbb{T}_u^{\varpi} \quad \text{where} \quad \mathbb{T}_u^{\varpi} = \exp\left(\frac{1}{2}\varpi + iV_u^+ + V_u^-\right). \tag{6.16}$$

Connected components of real tori of type E_7

In this section we determine all connected components of the space:

$$C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$$

where $u \in W$ is an involution in the Weyl group W of type E_7 and $\mathbb{T}_u$ is the corresponding real torus. The results are listed in Table 6.3. In the first column are the pairs of conjugation classes of involutions W we determined in Example 2.3.4. The total number of connected components equals 20. Recall that the nodes of the E_7 diagram are numbered as in the diagram below.

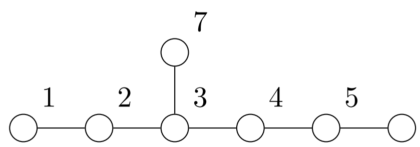


Figure 6.2: The labelling of the nodes of the Coxeter graph of type E_7.

If u is an involution with $n_1 = 0$ then the real torus $\mathbb{T}_u(\mathbb{R})$ is connected. Consequently the quotient $C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$ is also connected. Now suppose that $n_1 = 1$ so that $P_{1,u}/2P_{1,u} \cong \mathbb{Z}/2\mathbb{Z}$. Since $\{0\}$ is a single W_u^+-orbit, so is $\{\varpi\}$ with $\varpi \in P$ a generator for $P_{1,u}/2P_{1,u}$ and there are two connected components. From Table 6.3 we see that this is the case for A_1^3 and D_4 which we represent by $I = \{s_2, s_4, s_7\}$ and $I = \{s_2, s_3, s_4, s_7\}$ respectively. In both cases the fundamental weight ϖ_6 is a generator for $P_{1,u}/2P_{1,u}$. For $n_1 > 2$ the situation becomes more complicated and we have to determine the action of W_u^+ on the generators of $P_{1,u}/2P_{1,u}$. These cases are $u = 1, A_1, A_1^2$ or $A_1^{3\prime}$ and we treat them below.

$u \in W$	n_1	n_2	n_3	#components	representatives
1	7	0	0	4	$\{0, \varpi_5, \varpi_6, \varpi_7\}$
E_7	0	7	0	1	$\{0\}$
A_1	5	0	1	3	$\{0, \varpi_3, \varpi_4\}$
D_6	0	5	1	1	$\{0\}$
A_1^2	3	0	2	3	$\{0, \varpi_4, \varpi_5\}$
D_4A_1	0	3	2	1	$\{0\}$
A_1^3	1	0	3	2	$\{0, \varpi_6\}$
A_1^4	0	1	3	1	$\{0\}$
D_4	1	2	2	2	$\{0, \varpi_6\}$
$A_1^{3\prime}$	2	1	2	2	$\{0, \varpi_1\}$

Table 6.3: The number of components of $C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$ for all conjugation classes of involutions $u \in W$. We also list the corresponding representatives in $W_u^+ \backslash (P_{1,u}/2P_{1,u})$.

1 The involution $u = 1$ is of type $(7, 0, 0)$ so that we can use Formula 6.15. The closure of the fundamental alcove $\overline{\mathcal{A}}$ intersected with lattice of half weights:

$$\bar{\mathcal{A}} \cap \frac{1}{2}P = \mathrm{Conv}\left(0, \frac{\varpi_1}{2}, \frac{\varpi_2}{3}, \frac{\varpi_3}{4}, \frac{\varpi_4}{3}, \frac{\varpi_5}{2}, \varpi_6, \frac{\varpi_7}{2}\right) \cap \frac{1}{2}P$$

consists of the six elements $\{0, \varpi_1/2, \varpi_5/2, \varpi_6, \varpi_6/2, \varpi_7/2\}$. The group P/Q is of order two and acts on this set by γ_6 which interchanges $0 \leftrightarrow \varpi_6$ and $\varpi_1/2 \leftrightarrow \varpi_5/2$. We conclude that there are four orbits in $W\backslash (P/2P)$ represented by $\{0, \varpi_5, \varpi_6, \varpi_7\}$.

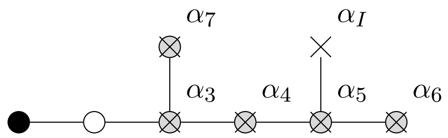
A_1 The involution A_1 is of type $(5, 0, 1)$. As a representative we pick $I = \{s_1\}$. Let S_u be the matrix of u with respect to the basis of fundamental weights for P and let B_u be a matrix whose columns represent a normal basis in the sense of Equation 6.12. The normal basis for P is not uniquely determined but we fix the choice below.

$$S_u = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \oplus I_5 \quad , \quad B_u = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \oplus I_5.$$

A basis for $P_{1,u}$ and system of simple roots for W_u^+ are then given by respectively:

$$P_{1,u} = \mathbb{Z}\{\varpi_7, \varpi_3, \varpi_4, \varpi_5, \varpi_6\} \ , \ \Delta(D_6) = \{\alpha_7, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_I\}$$

where $\alpha_I = e_0 - e_3 - e_4 - e_5$. The lattice $P_{1,u}$ is a weight lattice of type A_5 and the group W_u^+ which acts on $P_{1,u}$ is of type D_6. All this is shown in the picture below. The black nodes represent the set I of W_u^-, the crossed nodes the root system of W_u^+ and the grey nodes the fundamental weights of $P_{1,u}$.



We need to determine the action of W_u^+ on $P_{1,u}$. First we observe that the parabolic subgroup $W(A_5)$ of W_u^+ generated by the reflections represented by grey nodes in the diagram is of type A_5 and acts on $P_{1,u}$ in the usual way. We see from example 6.5.10 that there are four orbits for $W(A_5)\backslash(P_{1,u}/2P_{1,u})$ represented by $\{0, \varpi_7, \varpi_3, \varpi_4\}$. The remaining generating reflection s_I acts on the basis for $P_{1,u}$ as

$$s_{\alpha_I}(\varpi_7, \varpi_3, \varpi_4, \varpi_5, \varpi_6) = (\varpi_7 + \varpi_5, \varpi_3 + 2\varpi_5, \varpi_4 + \varpi_5, \varpi_5, \varpi_6).$$

A small calculation using Equation 6.14 shows that

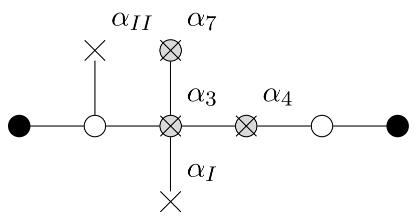
$$\begin{aligned} s_4 s_3 s_7 s_I(\varpi_7) &= s_4 s_3 s_7(\varpi_7 + \varpi_5) \\ &= s_4 s_3(\varpi_7 + \varpi_3 + \varpi_5) \\ &= s_4(\varpi_3 + \varpi_4 + \varpi_5) \\ &= \varpi_4 \end{aligned}$$

so that the reflection s_I exchanges the orbits $\varpi_7 \leftrightarrow \varpi_4$. This leaves three orbits for $W(D_6)\backslash(P_{1,u}/2P_{1,u})$ represented by $\{0, \varpi_3, \varpi_4\}$.

A_1^2 The involution A_1^2 is of type $(3, 0, 2)$. As a representative we choose $I = \{s_1, s_6\}$. As a basis for the lattice $P_{1,u}$ we can choose $\mathbb{Z}\{\varpi_3, \varpi_4, \varpi_7\}$ which is of type A_3. The group W_u^+ is of type D_4A_1 with simple system:

$$\Delta(D_4A_1) = \{\alpha_3, \alpha_4, \alpha_7, \alpha_I, \alpha_{II}\}$$

where $\alpha_I = e_0 - e_3 - e_6 - e_7$ and $\alpha_{II} = -e_0 + e_3 + e_4 + e_5$. The corresponding diagram is given below.



The parabolic subgroup $W(A_3)$ of W_u^+ of type A_3 generated by the reflection represented by the grey nodes of the diagram acts on $P_{1,u}$

in the usual way. We can represent the orbits of $W(A_3)\backslash(P_{1,u}/2P_{1,u})$ by $\{0, \varpi_3, \varpi_4\}$. The reflection $s_{\alpha_{II}}$ acts trivially on these orbits. The reflection s_{α_I} acts as:

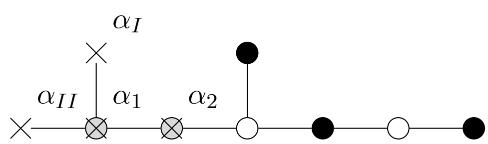
$$s_{\alpha_I}(\varpi_7, \varpi_3, \varpi_4) = (\varpi_7 + \varpi_3, \varpi_3, \varpi_4).$$

This action is identical to that of $s_7 \in W(A_3)$ so the number of orbits of $W(D_4A_1)\backslash(P_{1,u}/2P_{1,u})$ remains three with representatives $\{0, \varpi_3, \varpi_4\}$.

$A_1^{3\prime}$ The involution $A_1^{3\prime}$ is of type $(2,1,2)$ and is represented by $I = \{s_4, s_6, s_7\}$. The group W_u^+ is of type D_4 with simple system

$$\Delta(D_4) = \{\alpha_1, \alpha_2, \alpha_I, \alpha_{II}\}$$

where $\alpha_I = e_0 - e_1 - e_4 - e_5$ and $\alpha_{II} = e_0 - e_1 - e_6 - e_7$. As a basis for $P_{1,u}$ we can choose $\mathbb{Z}\{\varpi_1, \varpi_2\}$. The diagram is shown below.



The parabolic subgroup $W(A_2)$ of W_u^+ generated by the reflections represented by the grey nodes in the diagram acts on $P_{1,u}$ in the usual way. The space $W(A_2)\backslash(P_{1,u}/2P_{1,u})$ consists of a single orbit represented by $\{\varpi_1\}$. The reflections s_{α_I} and $s_{\alpha_{II}}$ both act as s_2 on $P_{1,u}$. So there are two W_u^+ orbits in $W(D_4)\backslash(P_{1,u}/2P_{1,u})$ represented by $\{0, \varpi_1\}$.

The complement of the mirrors

In this section we prove that for a root system of type ADE satisfying certain assumptions the connected components of the space $(W\backslash\mathbb{T}^\circ)(\mathbb{R})$ are of the form:

$$\mathrm{Stab}_W(\mathbb{T}_u^{\varpi})\backslash(\mathbb{T}_u^{\varpi})^\circ \quad \text{where} \quad [\varpi] \in P_{1,u}/2P_{1,u}. \tag{6.17}$$

This implies that removing the mirrors from $\mathbb{T}_u(\mathbb{R})$ does not add new components to the quotient $C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$. In particular the number of connected components of $C_W(u)\backslash\mathbb{T}_u^\circ(\mathbb{R})$ for involutions u in $W(E_7)$ are the same as the numbers in Table 6.3.

Definition 6.5.11. Let $q : \mathbb{T} \to W\backslash\mathbb{T}$ be the quotient map. The discriminant $D_{\mathbb{T}}$ is the set of critical values of q. It consists of union of the q-images of the toric mirrors and the q-image of the set:

$$\mathbb{T}_{P/Q} = \bigcup_i \exp(V_{\mathbb{C}}^{\gamma_i})$$

where we denote by $V_{\mathbb{C}}^{\gamma_i}$ the fixed points in $V_{\mathbb{C}}$ of the generator γ_i for P/Q.

Lemma 6.5.12. *The q-images of the real tori $\mathbb{T}_u(\mathbb{R})$ are disjoint in:*

$$(W\backslash\mathbb{T})(\mathbb{R}) - D_{\mathbb{T}}(\mathbb{R}).$$

Proof. Suppose that $t \in \mathbb{T}_{u_1} \cap \mathbb{T}_{u_2}$ for involutions $u_1, u_2 \in W$. This implies that $u_1 \cdot t = u_2 \cdot t = \bar{t}$ so that in particular $u_1 u_2 \cdot t = t$. But then $q(t) \in D_{\mathbb{T}}(\mathbb{R})$. □

Since we are interested in connected components it suffices to consider the part of $D_{\mathbb{T}}(\mathbb{R})$ of codimension 1 in $(W\backslash\mathbb{T})(\mathbb{R})$. This motivates the following definition.

Definition 6.5.13. The *real discriminant* $D_{\mathbb{T},\mathbb{R}}$ of $(W\backslash\mathbb{T})(\mathbb{R})$ is the closure of the nonsingular part of $D_{\mathbb{T}}(\mathbb{R})$. The difference is a $D_{\mathbb{T}}(\mathbb{R}) - D_{\mathbb{T},\mathbb{R}}$ has codimension ≥ 2 in $(W\backslash\mathbb{T})(\mathbb{R})$.

Proposition 6.5.14. *If we assume that $\mathbb{T}_{P/Q} \cap \mathbb{T}_u$ has codimension ≥ 2 for all involutions $u \in W$ then:*

$$q^{-1} D_{\mathbb{T},\mathbb{R}} = \bigcup (\mathbb{T}_u \cap H_s)$$

where the union runs over all involutions $u \in W$ and reflections $s \in W$ that commute with u.

Proof. Under the assumption of the proposition the set $\mathbb{T}_{P/Q}$ does not contribute to the real discriminant. In this case an element $t \in \mathbb{T}_u$ is mapped to a nonsingular point of $D_{\mathbb{T}}(\mathbb{R})$ by q if and only if there is a unique reflection $s \in W$ that fixes t. Since the reflection usu also fixes t, we must have that s commutes with u. □

The assumption is satisfied for type E_7. In that case P/Q is generated by the involution γ_6. The locus of fixed points V^{γ_6} has dimension four so that the codimension of $\mathbb{T}_{P/Q} \cap \mathbb{T}_u \geq 3$. In order to prove that the space of equation (6.17) is connected it is sufficient to prove that the space:

$$\mathrm{Stab}_W(\mathbb{T}_u^{\varpi})\backslash\mathbb{T}_u^{\varpi} - D_{\mathbb{T}_u^{\varpi},\mathbb{R}}$$

is connected. We prove the slightly stronger result that the quotient

$$\mathrm{Stab}_{W_u}(\mathbb{T}_u^{\varpi})\backslash\mathbb{T}_u^{\varpi} - D_{\mathbb{T}_u^{\varpi},\mathbb{R}}$$

by the smaller group $\mathrm{Stab}_{W_u}(\mathbb{T}_u^{\varpi})$ is connected. We start with a lemma on the decomposition $V = V_u^+ \oplus V_u^-$ into ± 1-eigenspaces for u on the weight lattice P. Denote by P_u^+ and P_u^- the orthogonal projections of P into V_u^+ and V_u^- respectively.

Lemma 6.5.15. *The lattice P_u^- is equal to the weight lattice $P(W_u^-)$ of W_u^-. If we also assume that $-1 \in W$ then $P_u^+ = P(W_u^+)$ and R_u^+ spans V_u^+.*

Proof. The simple system $\{\alpha_i\}_{i\in I}$ for the root system R_u^- is a basis for V_u^-. The dual basis is given by $\{\varpi_i^-\}_{i\in I}$ where $\varpi_i^- = \mathrm{Proj}_{V_u^-}(\varpi_i)$. We have:

$$\begin{aligned} P(W_u^-) &= \{\mathbb{Z} \in V_u^- \ ; \ (\mathbb{Z}, \alpha_i) \in \mathbb{Z} \quad \forall i \in I\} \\ &= \mathbb{Z}\left\{\varpi_i^-\right\}_{i\in I} \\ &= P_u^-. \end{aligned}$$

If $-1 \in W$ then for every involution $u \in W$ its opposite $-u$ is also an involution in W. Furthermore $V_u^\pm = V_{-u}^\mp$ and $W_u^\pm = W_{-u}^\mp$ so that we have equalities

$$P(W_u^+) = P(W_{-u}^-) = \mathrm{Proj}_{V_{-u}^-}(P) = \mathrm{Proj}_{V_u^+}(P).$$

Similarly we have $\mathbb{R}R_u^+ = \mathbb{R}R_{-u}^- = V_{-u}^- = V_u^+$ so that R_u^+ spans V_u^+. □

Theorem 6.5.16. *Assume that $-1 \in W$ and that $\mathbb{T}_{P/Q} \cap \mathbb{T}_u$ has codimension ≥ 2 for all involutions $u \in W$. Let $\mathcal{A}_u^-$ be the fundamental alcove for the action of the affine Weyl group $Q_u^- \rtimes W_u^-$ on V_u^- and let $\mathcal{C}_u^{\varpi+}$ be the fundamental chamber of the action of the Weyl group $\mathrm{Stab}_{W_u^+}\left(\frac{1}{2}\varpi\right)$ on the affine space $\frac{1}{2}\varpi + iV_u^+$. Then there is an isomorphism of orbifolds:*

$$\mathrm{Stab}_{W_u}(\mathbb{T}_u^\varpi)\backslash\mathbb{T}_u^\varpi - D_{\mathbb{T}_u^\varpi,\mathbb{R}} \cong \mathrm{Stab}_{P_u^+/Q_u^+}\left(\frac{1}{2}\varpi\right)\backslash\mathcal{C}_u^{\varpi+} \times \left(P_u^-/Q_u^-\right)\backslash\mathcal{A}_u^-.$$

Proof. Similar to the decomposition $V = V_u^+ \oplus V_u^-$ there is a decomposition:

$$\begin{aligned} \mathbb{T}_u^\varpi &= \exp\left(\frac{1}{2}\varpi + iV_u^+ + V_u^-\right) \\ &\cong \exp\left(\frac{1}{2}\varpi + iV_u^+\right) \times \exp\left(V_u^-\right) \end{aligned}$$

where $[\varpi] \in P_{1,u}/2P_{1,u}$ (so that in particular $\varpi \in V_u^+$). The stabilizer of $\mathbb{T}_u^\varpi$ in W_u also splits into a product:

$$\begin{aligned} \mathrm{Stab}_{W_u}\left(\mathbb{T}_u^\varpi\right) &\cong \mathrm{Stab}_{W_u^+}\exp\left(\frac{1}{2}\varpi + iV_u^+\right) \times \mathrm{Stab}_{W_u^-}\exp\left(V_u^-\right) \\ &\cong \mathrm{Stab}_{P_u^+ \rtimes W_u^+}\left(\frac{1}{2}\varpi\right) \times \mathrm{Stab}_{P_u^- \rtimes W_u^-}\left(V_u^-\right). \end{aligned}$$

The result now follows from applying Lemma 6.5.1 to these factors and taking the quotient. □

Remark 6.5.17. While the roots of the Weyl group W_u^+ span the vector space V_u^+ the same need not be true for the Weyl group $\mathrm{Stab}_{W_u^+}(\frac{1}{2}\varpi)$, even if the group W contains -1. An example is given by the Weyl group of type E_7 and the trivial involution $u = 1$. In that case the Weyl group $\mathrm{Stab}_W(\frac{1}{2}\varpi_6)$ is of type E_6 and has rank six while V is of dimension seven. A fundamental domain for this action is the product of a Weyl chamber of type E_6 and its orthogonal complement in V which is a copy of $\mathbb{R}$. From this discussion we see that the chamber $\mathcal{C}_u^{\varpi+}$ is not a Weyl chamber in the traditional sense but the product of a Weyl chamber and an affine space. In particular it is connected, as is the alcove $\mathcal{A}_u^-$ so that the following corollary is immediate.

Corollary 6.5.18. *Under the assumptions of Theorem 6.5.16 the space*

$$\mathrm{Stab}_{W_u}(\mathbb{T}_u^{\varpi})\backslash\mathbb{T}_u^{\varpi} - D_{\mathbb{T}_u^{\varpi},\mathbb{R}}$$

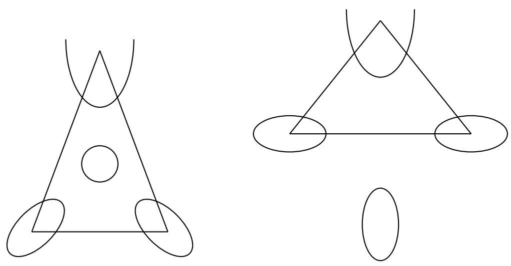
is connected.

6.6 The Geometry of the 20 components

In this section we relate the 20 connected components of the space $(W\backslash\mathbb{T}^{\circ})(\mathbb{R})$ for a root system of type E_7 to the components of the moduli space $(\mathcal{Q}_1^{\circ})^{\mathbb{R}}$. For each of these components we find a representative pair (C, p) with $p \in C(\mathbb{R})$. The results are listed in the tables of Section 6.7.

Theorem 6.6.1. *The curves in the tables of Section 6.7 represent the* 20 *different components of* $(\mathcal{Q}_1^{\circ})^{\mathbb{R}}$.

Proof. It is clear that for the curves in the left columns the associated del Pezzo pair (Y, Z) satisfies $Z^{\mathrm{ns}}(\mathbb{R}) \cong \mathbb{R}^*$ so that they belong to the space $\mathrm{Stab}_W(\mathbb{T}_u)\backslash\mathbb{T}_u^{\circ}$ for u of type $1, A_1, A_1^2, A_1^3$ or D_4. Similarly the curves in the right column satisfy $Z^{\mathrm{ns}}(\mathbb{R}) \cong S^1$ and belong to $\mathrm{Stab}_W(\mathbb{T}_{-u})\backslash\mathbb{T}_{-u}^{\circ}$ (so $-u$ is of type E_7, D_6, D_4A_1, A_1^4 or $A_1^{3\prime}$). Just check from the pictures whether $Z^{\mathrm{ns}}(\mathbb{R})$ has one or two components. With the exception of the two M-curves labeled ϖ_6 and ϖ_7 for all of the curves in the table the topological types of the pairs $(C(\mathbb{R}), T_pC(\mathbb{R}))$ are distinct. It is not possible to deform one of them into the other without passing through one of the strata $(\mathcal{Q}_1^{\mathrm{flex}})^{\mathbb{R}}$, $(\mathcal{Q}_1^{\mathrm{bit}})^{\mathbb{R}}$ or $(\mathcal{Q}_1^{\mathrm{hflex}})^{\mathbb{R}}$ so that they lie indeed in different components of $(\mathcal{Q}_1^{\circ})^{\mathbb{R}}$. We need to prove that the M-curves labeled ϖ_6 and ϖ_7 are not in the same component. For this consider the affine quartics obtained by placing the tangent line T_pC at infinity for these two curves. They are shown in the following figure.



ϖ_6 ϖ_7

The triangle drawn in the picture forms an obstruction to deforming one into the other: it is not possible to move the central oval of the curve ϖ_6 out of the triangle without contradicting Bezout's theorem (a line intersects C in four points). This is in agreement with Table 15 in Appendix 1 of [13] where certain affine M-quartics are classified. $\square$

To obtain an alternative description of the components of $(W\backslash\mathbb{T}^\circ)(\mathbb{R})$ we explicitly use the construction from the proof of Theorem 6.4.1 to associate to $\chi \in \mathbb{T}_u^\varpi(\mathbb{R})$ seven points in general position on the nonsingular locus of a real plane nodal cubic $Z \subseteq \mathbb{P}^2$. As before we identify $Z^{\text{ns}}(\mathbb{C}) \cong \mathbb{C}^*$ so that the points are defined by the formula:

$$\chi \mapsto (P_1, \ldots, P_7) \quad \text{with} \quad P_i = \chi\left(e_i - \frac{e_0}{3}\right)$$

up to addition of an inflection point of Z. We start with the components corresponding to M-quartics.

If χ is an element of the compact torus $\mathbb{T}_{-1} \cong \text{Hom}(Q, S^1)$ then this construction determines seven points on the real point set $Z(\mathbb{R})$ of a real plane nodal cubic with $Z^{\text{ns}}(\mathbb{R}) \cong S^1$.

If χ is an element of the split torus $\mathbb{T}_1 \cong \text{Hom}(Q, \mathbb{R}^*)$ then the construction determines seven points on the real point set of a real plane nodal cubic with $Z^{\text{ns}}(\mathbb{R}) \cong \mathbb{R}^*$. If we choose the unique real inflection point of Z as the unit element for the group law on Z^{ns} then these seven points are real. The Weyl group W acts on $(\mathbb{R}^*)^7$ by permuting the coordinates and by Cremona transformations in triples of points. For a seven-tuple $t = (t_1, \ldots, t_7) \in (\mathbb{R}^*)^7$ let m_+ denote the number of positive coordinates and m_- the number of negative coordinates. The permutation orbit of t is uniquely determined by the pair (m_+, m_-). From Formula 6.4 we see that if we perform a Cremona transformation in t_i, t_j, t_k the sign of these points remains unchanged and the remaining points change sign if and only if one or three of the three points are negative. This describes the action of W on the pairs (m_+, m_-) and there are four orbits. These correspond to the four components of:

$$W\backslash\mathbb{T}_1^\circ = \bigsqcup_{[\varpi]\in W\backslash(P/2P)} \text{Stab}_W(\mathbb{T}_1^\varpi)\big\backslash(\mathbb{T}_1^\varpi)^\circ$$

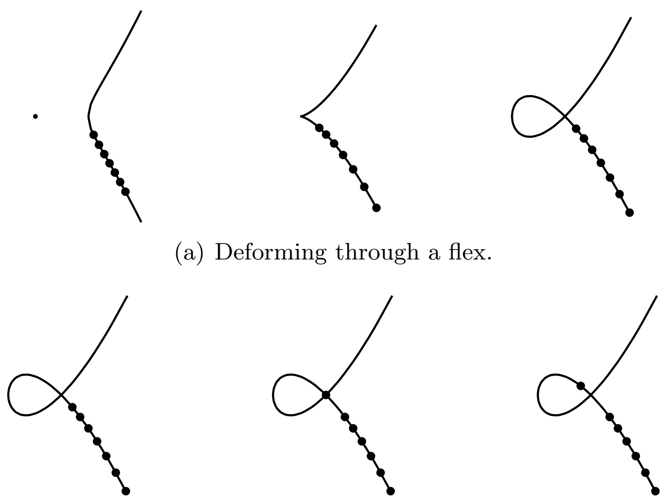
where $\varpi \in \{0, \varpi_5, \varpi_6, \varpi_7\}$. The precise correspondence is shown in Table 6.4. The stabilizers for the components in the table are calculated using Formula 6.7 and Lemma 6.5.1.

To determine which picture from the tables in Section 6.7 for $u = 1$ belongs to which of the components from Table 6.4 we determine the adjacency relations between the five components corresponding to pointed M-quartics in $(\mathcal{Q}_1^\circ)^\mathbb{R}$. Two components are adjacent if their corresponding pointed quartics (C, p) can be deformed into each other by moving through the stata $(\mathcal{Q}_1^{\text{bit}})^\mathbb{R}$

representative	$\mathrm{Stab}_W(\mathbb{T}_1^{\varpi})$	$W \cdot (m_+, m_-)$
$[0]$	$W(E_7)$	$\{(7,0)\}$
$[\varpi_6]$	$W(E_6) \rtimes \mathbb{Z}/2\mathbb{Z}$	$\{(6,1),(2,5)\}$
$[\varpi_5]$	$W(D_6A_1)$	$\{(5,2),(3,4),(1,6)\}$
$[\varpi_7]$	$W(A_7) \rtimes \mathbb{Z}/2\mathbb{Z}$	$\{(4,3),(0,7)\}$

Table 6.4: Connected components for $W\backslash\mathbb{T}_1(\mathbb{R})$. The first column lists the representatives for $W\backslash(P/2P)$.

or $(\mathcal{Q}_1^{\mathrm{flex}})^{\mathbb{R}}$ of codimension one. The effect of these two deformations is shown in figure 6.3.



(a) Deforming through a flex.

(b) Deforming through a bitangent.

Figure 6.3: The two ways of passing through a codimension one stratum in the moduli space $(\mathcal{Q}_1^{\circ})^{\mathbb{R}}$.

Proposition 6.6.2. *The adjacency graph for the five components of real pointed M-curves in* $(\mathcal{Q}_1^{\circ})^{\mathbb{R}}$ *is given by:*

$$(7) \text{ ——— } (7,0) \text{ ——— } (6,1) \text{ ——— } (5,2) \text{ ——— } (4,3)$$

where we label components of $W\backslash\mathbb{T}_1^{\circ}$ *by a representative for the corresponding orbit* $W\cdot(m_+,m_-)$ *and the component* $W\backslash\mathbb{T}_{-1}^{\circ}$ *by* (7).

Proof. A curve in the component coresponding to $W\backslash\mathbb{T}_1^{\circ}$ can only be deformed to one in the component of $W\backslash\mathbb{T}_{-1}^{\circ}$ by deforming through a flex point. This is a transition from (7) to $(7,0)$. By repeatedly deforming through a bitangent one moves through the components

$$(7,0) \leftrightarrow (6,1) \leftrightarrow (5,2) \leftrightarrow (4,3).$$

This proves the proposition and shows that the pictures corresponding to the components are indeed the ones shown in the table for $u = 1$. □

For $\chi \in \mathbb{T}_u(\mathbb{R})$ with u of type A_1^i with $i = 1, 2, 3$ we can do a similar analysis. In this case the construction associates to χ: $7 - 2i$ real points and i pairs of complex conjugate points for a suitable representative u (not involving the reflection s_7). For example the involution $u = s_6 s_4$ of type A_1^2 acts as

$$s_6 s_4 \cdot (P_1, P_2, P_3, P_4, P_5, P_6, P_7) = (P_1, P_2, P_3, P_5, P_4, P_7, P_6)$$

on the P_i so that $\chi \in \mathbb{T}_{s_6 s_4}$ produces seven points in $Z^{\mathrm{ns}}(\mathbb{C}) \cong \mathbb{C}^*$ with P_1, P_2, P_3 real points and (P_4, P_5) and (P_6, P_7) complex conjugate pairs. The centralizer $C_W(u)$ is more complicated in this case. It acts on the points by permutations preserving the real points and conjugate pairs and Cremona transformations centered in a triples of real points or a real point and a pair of conjugate points. The orbits are calculated in Table 6.5 and confirm the numbers we computed earlier in Table 6.3.

u	representative	$C_W(u) \cdot (m_+, m_-)$
A_1	$[0]$	$\{(5,0)\}$
	$[\varpi_4]$	$\{(4,1),(2,3),(0,5)\}$
	$[\varpi_3]$	$\{(3,2),(1,4)\}$
A_1^2	$[0]$	$\{(3,0)\}$
	$[\varpi_4]$	$\{(2,1),(0,3)\}$
	$[\varpi_3]$	$\{(1,2)\}$
A_1^3	$[0]$	$\{(1,0)\}$
	$[\varpi_6]$	$\{(0,1)\}$

Table 6.5: Connected components of $C_W(u)\backslash\mathbb{T}_u(\mathbb{R})$ for u of type A_1^i. The second column shows the representatives for $C_W(u)\backslash(P_{1,u}/2P_{1,u})$ from Table 6.3.

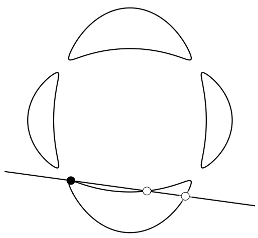
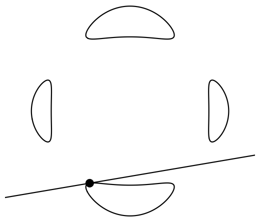
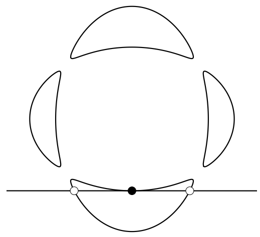
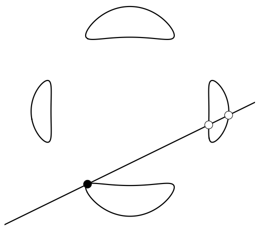
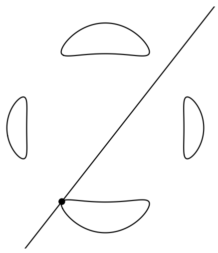
Recall that the Geiser involution centered in the seven points on $Z^{\mathrm{ns}}(\mathbb{R}) \subset \mathbb{P}^2$ lifts to an involution of Y whose fixed points correspond to the quartic curve C. The unique node of Z corresponds to the point $p \in C(\mathbb{R})$ of the pair (C, p). The remaining two fixed points on Y (corresponding to the remaining two points of $T_pC \cap C$) can be calculated using Equation 6.5 for the Geiser involution restricted to Z. They are the solutions to the equation:

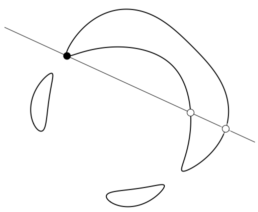
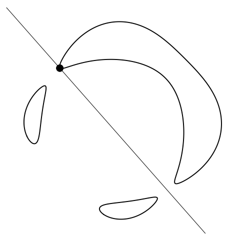
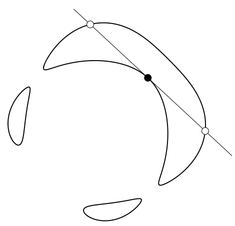
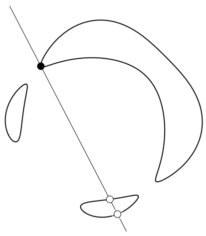
$$t^2 = \frac{1}{t_1 \cdot \ldots \cdot t_7}.$$

If m_- is even then these points are real and if m_- is odd they are complex conjugate. This is in agreement with the data in the tables.

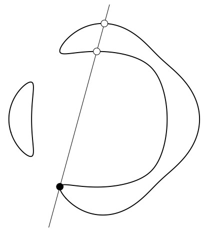
For $\chi \in \mathbb{T}_u$ with u of type D_4 or A_3' the situation is different. In this case u acts as a nontrivial Cremona transformation on the points. In fact it acts as a de Jonquiéres involution of order three centred in 5 of the points (for the definition we refer to [36]). The curve $C(\mathbb{R})$ consists of two nested ovals and only the outer oval can contain an inflection point, otherwise we would again get a contradiction with Bezout's theorem. This implies that the component with p on the outer oval is the unit component of $\mathbb{T}_u(\mathbb{R})$ for u of type D_4 and A_3'.

6.7 Tables

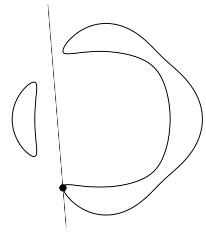
1		E_7
 (a) 0	 (b) ϖ_6	 (c) 0
 (d) ϖ_5	 (e) ϖ_7	

A_1		D_6
 (f) 0	 (g) ϖ_4	 (h) 0
 (i) ϖ_3		

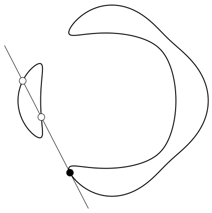
A_1^2	D_4A_1



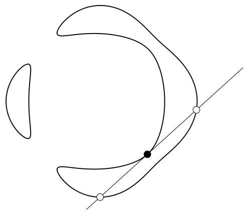
(j) 0



(k) ϖ_4

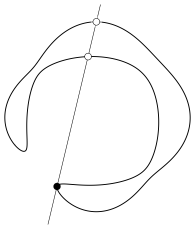


(m) ϖ_3

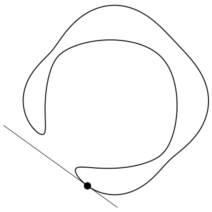


(l) 0

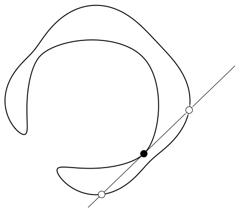
A_1^3	A_1^4



(n) 0

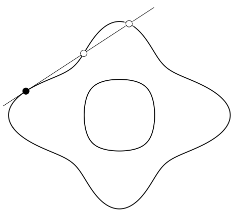


(o) ϖ_6

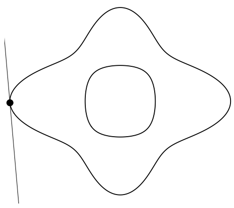


(p) 0

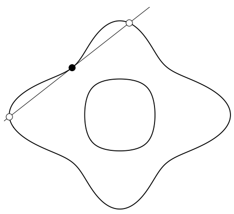
D_4	$A_1^{3\prime}$



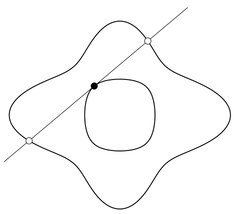
(q) 0



(r) ϖ_6



(s) 0



(t) ϖ_1

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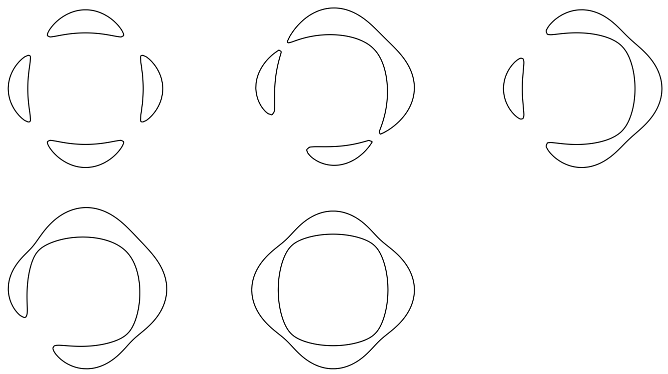
Samenvatting

De hoofdrolspelers van dit proefschrift zijn de vlakke reële krommen van graad vier. Dit soort krommen wordt volledig bepaald door vergelijkingen van de vorm:

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + fx^3 + gx^2y + \dots o = 0$$

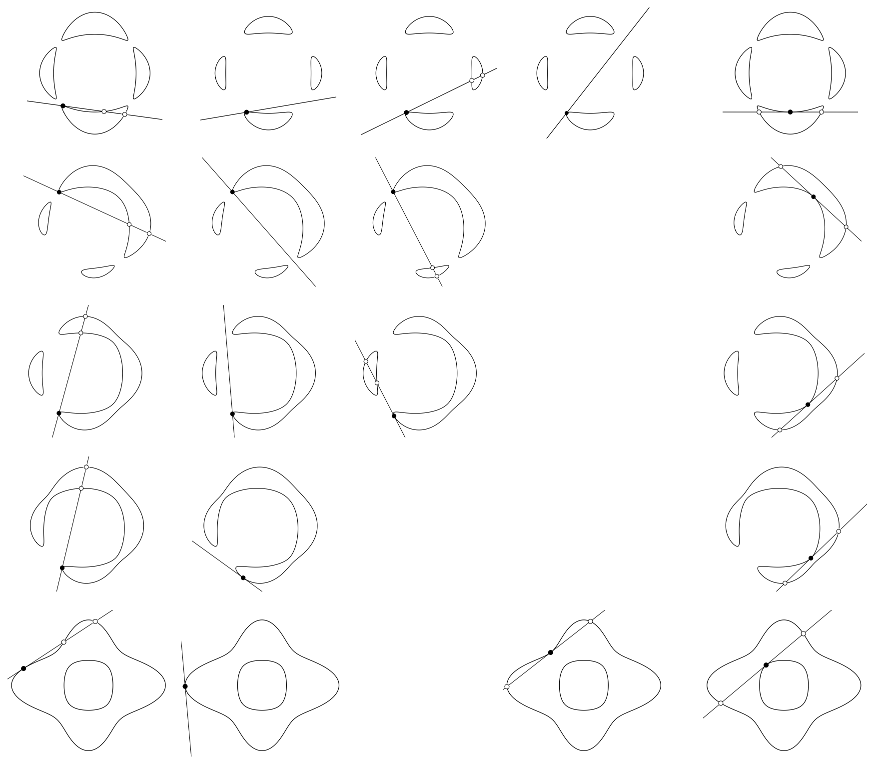
waar de 15 coëfficienten $a, b, c, \dots, o$ reële getallen zijn. Deze coëfficienten zijn vrij te kiezen zolang de resulterende kromme maar glad is. Dit betekent dat deze zichzelf niet doorsnijdt en geen scherpe randen heeft. Door de coëfficienten en het coördinatensysteem geschikt te kiezen kunnen alle zes de verschijningsvormen uit onderstaande figuur gemaakt worden en geen andere.



De kromme rechtsonder heeft geen reële punten en wordt bijvoorbeeld gegeven door de vergelijking $x^4 + y^4 = -1$.

Dit proefschrift bestaat uit twee delen. In het eerste deel bestuderen we de meetkunde van de ruimte van gladde vlakke reële krommen van graad vier met behulp van een periodenafbeelding. Deze afbeelding slaat een brug tussen de meetkundige wereld en de arithmetische wereld van de balquotiënten. Het hoofdresultaat van dit deel is dat voor elk van de zes soorten gladde vlakke reële krommen van graad vier de ruimte van dit soort krommen volledig beschreven wordt door een zesdimensionaal hyperbolisch polytoop van eindig volume. Voor krommen die bestaan uit vier ovalen (zoals linksboven in het bovenstaande plaatje) wordt het Coxeterdiagram van dit polytoop gegeven door de figuur op de kaft van dit proefschrift.

In het tweede deel van het proefschrift bestuderen we de ruimte van gladde vlakke reële krommen van graad vier met een punt zodanig dat de raaklijn in dit punt de kromme snijdt in twee verschillende punten. Deze twee punten zijn reëel of vormen een complex geconjugeerd paar. We bewijzen dat deze ruimte bestaat uit 20 componenten die we expliciet beschrijven met behulp van een soort van periodenafbeelding. Voorbeelden van de 20 soorten gepunteerde krommen zijn te zien in onderstaande figuur.



Acknowledgements

Now that my thesis is complete it is my pleasure to thank some people that have contributed to it directly or indirectly.

Op de eerste plaats wil ik mijn begeleider Gert Heckman bedanken. Gert, onze wekelijkse bijeenkomsten vormden voor mij het inspirerende hoogtepunt van de werkweek. Veel dank voor alle mooie wiskunde en het vertrouwen dat je vanaf het begin in me had. Ik wil Eduard Looijenga bedanken voor het mogelijk maken van mijn bezoek aan Tsinghua University. Mijn verblijf was zowel productief als plezierig. Hiervoor wil ik ook Elisabeth Giljam hartelijk bedanken. Many thanks also to Dali Shen for showing me around the campus. I would like to thank Professor Kharlamov for his interest in my work and giving me the opportunity to speak at the university of Strasbourg. Many thanks also to the members of the manuscript committee for their critical reading of this thesis. To all of my fellow Ph.D. students in Nijmegen: thank you for the great atmosphere during these last four years. I really enjoyed all the lively lunches, coffee breaks, dinners, hamburger party's, all-you-can-eat sushi evenings, operas and especially the holiday trips. Ik wil Bert en Pieter bedanken dat ze mijn paranimfen willen zijn. Ook ben ik Rutger veel dank verschuldigd voor zijn geweldige hulp bij het maken van de kaft van dit proefschrift. Verder wil ik mijn familie bedanken omdat ze altijd voor me klaarstaan. Pap, Mam en Ilse: bedankt. Ten slotte wil ik dit proefschrift opdragen aan Heleen. Zij was vanaf het begin betrokken bij de totstandkoming ervan en heeft alle ups en downs meebeleefd. Ook was zij de eerste die het zeer kritisch van kaft tot kaft gelezen heeft. Heleen: bedankt dat je samen met mij door het leven wilt gaan.

Curriculum vitae

Sander Rieken was born on July 19, 1985 in Roermond. He grew up in the small village of Vlodrop and attended Lyceum Schöndeln in Roermond. After graduating in 2003 he started his studies at the Radboud University Nijmegen majoring in Physics. Two years later he switched to Mathematics and obtained his Master of Science degree *cum laude* under supervision of Joseph Steenbrink in 2010. That same year he started his Ph.D. research under supervision of Gert Heckman. In March 2014 he was given the opportunity to visit Tsinghua University in Beijing for a month hosted by Eduard Looijenga.