SYMPLECTIC GEOMETRY

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Dedicated to the memory of my teacher
Hans Duistermaat (1942-2010)

Contents

Preface

These are lecture notes for a course on symplectic geometry in the Dutch Mastermath program. There are several books on symplectic geometry, but I still took the trouble of writing up lecture notes. The reason is that this one semester course was aiming for students at the beginning of their masters. So I took quite some time reviewing basic facts about manifolds, algebraic topoplogy, de Rham theory and Lie groups.

There was of course not enough time to treat these subjects in a complete way, but I tried to explain the basic theorems with references to the literature, in order that they could be applied for the development of the symplectic geometry. For example, the fundamental proof by Moser of the equivalence under diffeomorphisms of normalized volume forms on compact connected manifolds uses de Rham theory. Moser's argument can be easily adapted to give a proof of the Darboux theorem. In order to explain the monodromy calculation for the spherical pendulum by Duistermaat as a Picard–Lefschetz formula I have to use Poincaré duality (admittedly just for a two dimensional torus).

My original plan was to discuss more examples from classical mechanics, like the integrable motion of rigid bodies, and the Lagrange points as relative equilibria. If time permits I plan to write additional parts of the text in the future in this direction.

I profited greatly from lecture notes of Hans Duistermaat on symplectic geometry, written for a Spring School in 2004. Because of the sudden decease of Hans in the spring of 2010 I took the freedom to put his unpublished notes on my website as well.

A final word of thanks for the students in the class is in place. For their questions during and after the class, and for their comments on the text. It was a great pleasure working with them.

1 Symplectic Linear Algebra

1.1 Symplectic Vector Spaces

Let V be a real finite dimensional vector space, and let $V^* = \mathrm{Hom}(V, \mathbb{R})$ denote the dual vector space. A bilinear form b on V is a bilinear map

$$b : V \times V \to \mathbb{R}$$

in the sense that the maps $u \mapsto b(u, v)$ for fixed v and $v \mapsto b(u, v)$ for fixed u are both linear maps. In turn a bilinear form b determines and is determined by a linear map $b : V \to V^*$ by $b(u)v = b(u, v)$ for all u, v. The bilinear form b is called nondegenerate if and only if the associated linear map $b : V \hookrightarrow V^*$ is an injection, so explicitly if $b(u, v) = 0$ for all v then necessarily $u = 0$, or equivalently if and only if $b : V \to V^*$ is a linear isomorphism (because V is finite dimensional). This latter condition is equivalent to $b^* : V \to V^*$ being a linear isomorphism, or explicitly if $b(u, v) = 0$ for all $u \in V$ then necessarily $v = 0$.

There are two cases of nondegenerate bilinear forms b on V of interest to us: $b = g$ is a Euclidean form or $b = \omega$ is symplectic form. A Euclidean form $b = g$ is symmetric (in the sense that $g(v, u) = g(u, v)$ for all u, v) and positive definite ($g(u, u) > 0$ for all nonzero u), and (V, g) is called a Euclidean vector space. By definition a symplectic form $b = \omega$ is antisymmetric (in the sense that $\omega(v, u) = -\omega(u, v)$ for all u, v) and nondegenerate ($\omega : V \to V^*$ is a linear isomorphism), and (V, ω) is called a symplectic vector space.

Suppose $b : V \times V \to \mathbb{R}$ is a nondegenerate bilinear form on V, which is either symmetric or antisymmetric. For $U < V$ a linear subspace put

$$U^b = \{v \in V; b(u, v) = 0 \ \forall u \in U\} = \{u \in V; b(u, v) = 0 \ \forall v \in U\}$$

for the orthogonal complement of U with respect to b. Since the form b is nondegenerate it is clear that $\dim U + \dim U^b = \dim V$. It is obvious that $U < U^{bb}$ and therefore we get $U = U^{bb}$. For $U_1 < U_2 < V$ linear subspaces we have $V > U_1^b > U_2^b$. The restriction of b to U is nondegenerate if and only if $U \cap U^b = \{0\}$, in which case $V = U \oplus U^b$ is the direct sum of two subspaces U and U^b with the restriction of b to both subspaces nondegenerate.

The standard example of a Euclidean vector space is the Cartesian vector space $\mathbb{R}^m$ with Euclidean form $g(x, y) = \sum x_j y_j$. If (V, g) is a Euclidean vector space, then there exists an orthonormal basis $(e_1, \cdots, e_m)$ in V, which means that $g(e_j, e_k) = \delta_{jk}$. Indeed pick a nonzero vector v in V and put $e_1 = v/g(v, v)^{1/2}$ such that $g(e_1, e_1) = 1$. The orthogonal complement $(\mathbb{R}v)^g$ has codimension one, and by induction there exists an orthonormal basis

$(e_2, \cdots, e_m)$ in $(\mathbb{R}v)^g$, which extends to an orthonormal basis $(e_1, \cdots, e_m)$ in V. In case we denote $v = \sum x_j e_j$ for the coordinates relative to this basis then (V, g) gets identified with the Cartesian vector space $\mathbb{R}^m$ with its standard Euclidean form.

Example 1.1. *A basic example of a symplectic vector space is $V = U \oplus U^*$ with standard symplectic form*

$$\omega((x, \xi), (y, \eta)) = \xi(y) - \eta(x)$$

for $x, y \in U$ and $\xi, \eta \in U^*$. Choose a basis $(e_1, \cdots, e_n)$ in U and let $(f_1, \cdots, f_n)$ be the dual basis in U^*, defined by $f_j(e_k) = \delta_{jk}$. The basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ of V satisfies

$$\omega(e_j, e_k) = \omega(f_j, f_k) = 0 \ , \ \omega(f_j, e_k) = -\omega(e_j, f_k) = \delta_{jk}$$

and is called a symplectic basis for V. Written out in coordinates relative to such a symplectic basis we get

$$\omega((x, \xi), (y, \eta)) = \sum (\xi_j y_j - \eta_j x_j)$$

with $x = \sum x_j e_j, y = \sum y_j e_j, \xi = \sum \xi_j f_j, \eta = \sum \eta_j f_j$.

In the context of classical mechanics one usually uses position coordinates $q = \sum q_j e_j \in U$ and momentum coordinates $p = \sum p_j f_j \in U^*$, and so the standard symplectic form becomes

$$\omega((q, p), (q', p')) = \sum (p_j q'_j - p'_j q_j)$$

in canonical coordinates $(q, p), (q', p') \in U \times U^*$.

Lemma 1.2. *In a symplectic vector space (V, ω) one can choose a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ characterized by*

$$\omega(e_j, e_k) = \omega(f_j, f_k) = 0 \ , \ \omega(f_j, e_k) = -\omega(e_j, f_k) = \delta_{jk} \ .$$

In particular symplectic vector spaces have even dimension.

Proof. The proof is similar to the proof of the existence of an orthonormal basis in a Euclidean vector space. Let (V, ω) be a symplectic vector space, and choose a nonzero vector e_1 in V. Since ω is nondegenerate one can choose f_1 in V with $\omega(f_1, e_1) = 1$. By the antisymmetry of ω it is clear that $\omega(e_1, e_1) = \omega(f_1, f_1) = 0$. The restriction of ω to the plane $U = \mathbb{R}e_1 + \mathbb{R}f_1$ is clearly nondegenerate. Hence $V = U \oplus U^\omega$ and the restriction of ω to U^ω is also nondegenerate. By induction on the dimension we can choose a symplectic basis $(e_2, \cdots, e_n, f_2, \cdots, f_n)$ for U^ω, which together with (e_1, f_1) extends to a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ of V. $\square$

Let (V, ω) be a symplectic vector space. A linear subspace $U < V$ is called isotropic if $U < U^\omega$, that is $\omega(u,v) = 0$ for all $u, v \in U$. A linear subspace $U < V$ is called coisotropic if $U^\omega < U$, and so U is isotropic (coisotropic) if and only if U^ω is coisotropic (isotropic). If the linear subspace $U < V$ satisfies $U = U^\omega$, so U is both isotropic and coisotropic, then U is called a Lagrangian subspace. The dimension of a Lagrangian subspace is half the dimension of the total vector space.

1.2 Hermitian Forms

A complex structure on a finite dimensional real vector space V is a linear map $J : V \to V$ with $J^2 = -1$. Multiplication of a complex scalar $z = x + \mathrm{i}y$ on $v \in V$ by $zv = xv + yJv$ turns V into a complex vector space, and so (V, J) is considered as a complex vector space. A map

$$h : V \times V \to \mathbb{C}$$

is called a Hermitian form if $u \mapsto h(u,v)$ is complex linear for all v, and $h(v,u) = \overline{h(u,v)}$ (and so $v \mapsto h(u,v)$ is complex antilinear for all u), and finally $h(u,u) > 0$ for all nonzero u. The triple (V, J, h) is called a Hermitian vector space.

If we denote

$$h = g + \mathrm{i}\omega \ , \ g = \Re(h) \ , \ \omega = \Im(h)$$

then the real part g is a Euclidean form and the imaginary part ω a symplectic form on V. For example the antisymmetry of ω follows from

$$\omega(v,u) = \Im(h(v,u)) = \Im(\overline{h(u,v)}) = -\Im(h(u,v)) = -\omega(u,v)$$

for all u, v in V. The relation between the Euclidean form g, the symplectic form ω and the complex structure J is given by

$$g(u,v) = \omega(Ju, v) \ , \ \omega(u,v) = g(u, Jv)$$

for all u, v in V.

Definition 1.3. *Let V be a finite dimensional real vector space equiped with a Euclidean form g, a symplectic form ω and a complex structure J. These three structures are called compatible if $h = g + \mathrm{i}\omega$ is a Hermitian form on (V, J), or equivalently if*

$$g(u,v) = \omega(Ju, v) \ , \ \omega(u,v) = g(u, Jv)$$

for all u, v in V.

Note that any two of a compatible triple (g, ω, J) on V determine the third. Observe that for a compatible triple the operator J is orthogonal with respect to g, and given this orthogonality of J the two equations in the above definition become equivalent.

Lemma 1.4. *Given a symplectic vector space (V, ω) there exist a compatible Euclidean form g and complex structure J on V.*

Proof. Choose an arbitrary Euclidean form G on V, and let $A \in \mathrm{Aut}(V)$ be defined by $\omega(u, v) = G(u, Av)$ for all u, v. Because G is symmetric and ω antisymmetric we get $G(Au, v) + G(u, Av) = 0$. In other words, if A^* is the adjoint of A with respect to G then $A^* = -A$, so A is skewadjoint with respect to G. The product A^*A is selfadjoint with respect to G, and positive definite. Hence the square root $\sqrt{A^*A} \in \mathrm{Aut}(V)$ exists uniquely as a positive definite selfadjoint operator. Just take the eigenspace decomposition of A^*A, and the square root per eigenvalue. Any linear operator on V commuting with A^*A also commutes with its square root $\sqrt{A^*A}$.

Since $A^*A = -A^2$ commutes with A it is clear that $\sqrt{A^*A}$ also commutes with A. Define $J \in \mathrm{Aut}(V)$ by the polar decomposition $A = \sqrt{A^*A}J = J\sqrt{A^*A}$, and put

$$g(u, v) = \omega(Ju, v) = G(Ju, Av)$$

for all u, v. Then $g(u, v) = G(Ju, \sqrt{A^*A}Jv)$ is a Euclidean form on V. Moreover $J^2 = A^2(A^*A)^{-1} = -1$, and so J is a complex structure on V. Since $J^* = A^*(\sqrt{A^*A})^{-1} = \sqrt{A^*A}A^{-1} = J^{-1} = -J$ and $g(u, v) = \omega(Ju, v)$ we also get $\omega(u, v) = G(u, Av) = -G(J^2u, Av) = G(Ju, JAv) = g(u, Jv)$, and so the three structures g, ω, J are compatible. $\square$

An alternative proof goes as follows. By Lemma 1.2 we can choose a symplectic basis $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ in (V, ω). In canonical coordinates $q = \sum q_j e_j$, $p = \sum p_j f_j$ we take as complex coordinates $z = \sum z_j e_j$ with $z_j = q_j + \mathrm{i}p_j$ (so $Je_j = f_j$). Then the standard Hermitian form

$$h(z, z') = \sum z_j \overline{z'_j}$$

has the standard symplectic form $\omega((q, p), (q', p')) = \sum(p_j q'_j - p'_j q_j)$ as imaginary part, as required. However the first proof has an advantage in later use.

1.3 Exterior Algebra

Let V be a real vector space of dimension m. Denote by $\Lambda^p V^*$ the vector space of antisymmetric multilinear forms of degree p (for short p-forms) on

V, so $\alpha \in \Lambda^p V^*$ means

$$\alpha : V^p = V \times \cdots \times V \to \mathbb{R}$$

is linear in each argument while keeping the other arguments fixed, and

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_p) = -\alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_p)$$

for any pair of indices $1 \leq i < j \leq p$. For $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$ the exterior product $\alpha \wedge \beta \in \Lambda^{p+q} V^*$ is defined by

$$(\alpha \wedge \beta)(v_1, \cdots, v_{p+q}) = \sum \epsilon(\sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(p)})\beta(v_{\sigma(p+1)}, \cdots, v_{\sigma(p+q)})$$

with the sum taken over left cosets of permutations $\sigma \in \mathfrak{S}_{p+q}$ modulo the subgroup $\mathfrak{S}_p \times \mathfrak{S}_q$, and $\epsilon(\sigma)$ denotes the sign of the permutation σ. Note that terms with equivalent permutations are equal, so the expression is well defined. The exterior product is associative, that is $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$. It is anticommutative in the sense that $\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta$ for $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$. With this product

$$\Lambda V^* := \bigoplus_{p \geq 0} \Lambda^p V^*$$

becomes a graded associative algebra. Here $\Lambda^0 V^* = \mathbb{R}$ by convention. Clearly $\Lambda^1 V^* = V^*$ generates ΛV^* as an associative algebra. Note that the even degree part of ΛV^* is a commutative subalgebra of ΛV^*.

Let e_j be a basis of V and ϵ_k the dual basis of V^*, so $\epsilon_k(e_j) = \delta_{jk}$. For strictly increasing functions $J, K : \{1, \cdots, p\} \to \{1, \cdots, m\}$ let us write

$$e_J = (e_{J(1)}, \cdots, e_{J(p)}) \in V^p \ , \ \epsilon_K = \epsilon_{K(1)} \wedge \cdots \wedge \epsilon_{K(p)} \in \Lambda^p V^*$$

and so $\epsilon_K(e_J) = \delta_{JK}$. Therefore, for $\alpha \in \Lambda^p V^*$, we get $\alpha = \sum_K \alpha(e_K)\epsilon_K$, by applying both sides to e_J and observing that α is determined by the numbers $\alpha(e_J)$. Conversely, if $\sum_K a_K \epsilon_K = 0$, then application to e_J yields $a_J = 0$. Hence the ϵ_K form a basis of $\Lambda^p V^*$, and so

$$\dim \Lambda^p V^* = \binom{m}{p} \ , \ \dim \Lambda V^* = 2^m \ .$$

In particular $\Lambda^p V^* = 0$ for $p > m$, and the space $\Lambda^m V^*$ of volume forms on V is one dimensional.

Let ω be a symplectic form on V. In the above terminology $\omega \in \Lambda^2 V^*$ is a nondegenerate 2-form . If $(e_1, \cdots, e_n, f_1, \cdots, f_n)$ is a symplectic basis of V, and $(\epsilon_1, \cdots, \epsilon_n, \phi_1, \cdots, \phi_n)$ the dual basis of V^*, then

$$\omega = \sum_1^n \phi_j \wedge \epsilon_j$$

by Lemma 1.2. Hence the n-th exterior power

$$\omega^n = \omega \wedge \cdots \wedge \omega = n! \phi_1 \wedge \epsilon_1 \wedge \cdots \wedge \phi_n \wedge \epsilon_n$$

of ω is a nonzero volume form on V. This implies that all intermediate powers $\omega^k \in \Lambda^{2k} V^*$ are also nonzero, $0 \leq k \leq n$.

The volume form $\omega^n / n!$ is called the Liouville form associated with the symplectic form ω on V. The Liouville measure of the unit parallellepiped spanned by a symplectic basis of V is equal to 1.

1.4 The Word "Symplectic"

The planes through the origin in $V = \mathbb{R}^m$ form the Grassmannian $\mathrm{G}(2, m)$ of lines in the projective space $\mathbb{P}(V)$. The Plücker embedding, sending a plane $L = \mathbb{R}u + \mathbb{R}v$ in V to the element $[u \wedge v]$ in $\mathbb{P}(\Lambda^2 V)$, maps the Grassmannian $\mathrm{G}(2, m)$ into the projective space $\mathbb{P}(\Lambda^2 V)$. The image is a smooth projective variety of dimension $2m - 4$, given as an intersection of quadrics. Indeed, if $u = (x_1, \cdots, x_m)$ and $v = (y_1, \cdots, y_m)$ then $u \wedge v$ has Plücker coordinates $p_{jk} = x_j y_k - x_k y_j$ for $j < k$, and the Plücker relations are given by

$$p_{ij} p_{kl} + p_{ik} p_{lj} + p_{il} p_{jk} = 0$$

for all $i < j < k < l$. Indeed the expression

$$\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \begin{vmatrix} x_k & y_k \\ x_l & y_l \end{vmatrix} + \begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} \begin{vmatrix} x_l & y_l \\ x_j & y_j \end{vmatrix} + \begin{vmatrix} x_i & y_i \\ x_l & y_l \end{vmatrix} \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} =$$

$$(x_i y_j - y_i x_j) \begin{vmatrix} x_k & y_k \\ x_l & y_l \end{vmatrix} + (x_i y_k - y_i x_k) \begin{vmatrix} x_l & y_l \\ x_j & y_j \end{vmatrix} + (x_i y_l - y_i x_l) \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} =$$

$$x_i \left(y_j \begin{vmatrix} x_k & y_k \\ x_l & y_l \end{vmatrix} + y_k \begin{vmatrix} x_l & y_l \\ x_j & y_j \end{vmatrix} + y_l \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} \right) +$$

$$-y_i \left(x_j \begin{vmatrix} x_k & y_k \\ x_l & y_l \end{vmatrix} + x_k \begin{vmatrix} x_l & y_l \\ x_j & y_j \end{vmatrix} + x_l \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} \right) =$$

$$x_i \begin{vmatrix} x_j & y_j & y_j \\ x_k & y_k & y_k \\ x_l & y_l & y_l \end{vmatrix} - y_i \begin{vmatrix} x_j & x_j & y_j \\ x_k & x_k & y_k \\ x_l & x_l & y_l \end{vmatrix}$$

vanishes because both latter matrices have two equal columns. It can be shown [16] that the quadratic Plücker relations are sufficient to describe the image of $\mathrm{G}(2,m)$ inside $\mathbb{P}(\Lambda^2 V)$. For example the Grassmannian $\mathrm{G}(2,4)$ embedds in $\mathbb{P}^5 = \mathbb{P}(\Lambda^2\mathbb{R}^4)$ as a smooth quadric fourfold Q.

Since $(\Lambda^p V)^* = \Lambda^p(V^*)$ a plane $L = \mathbb{R}u + \mathbb{R}v$ in a symplectic vector space (V,ω) is isotropic if and only if its image $[u \wedge v] \in \mathbb{P}(\Lambda^2 V)$ under the Plücker embedding lies on the hyperplane H defined by the linear form ω on $\Lambda^2 V$. For example, for a four dimensional symplectic vector space (V,ω) the image of the Lagrangian Grassmannian of Lagrangian planes in $\mathrm{G}(2,4)$ under the Plücker embedding in $\mathbb{P}^5 = \mathbb{P}(\Lambda^2 V)$ is a smooth threefold $Q \cap H$, so a smooth quadric in $\mathbb{P}^4$.

Definition 1.5. *A line complex of degree d is a smooth hypersurface of the Grassmannian $\mathrm{G}(2,m)$, which is obtained as the intersection under the Plücker embedding $\mathrm{G}(2,m) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{R}^m)$ with a hypersurface of degree d.*

The basic example of a linear line complex is the space of isotropic planes in a symplectic vector space (V,ω). A well studied example in 19th century algebraic geometry is the case of a quadric line complex in $\mathrm{G}(2,4)$ and its beautiful relation with Kummer surfaces [16].

For a symplectic vector space (V,ω) the subgroup of $\mathrm{GL}(V)$ leaving ω invariant acts transitively on the associated linear line complex, and was called the linear complex group. In his book “The Classical Groups” of 1936 Hermann Weyl wrote [58]

> The name “complex group” formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word “complex” in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective “symplectic”.

The word “com-plex” means “plaited together” and the Greek transcription became “sym-plectic”. Ever since the group $\mathrm{Sp}(V,\omega)$ of all linear transformations of V leaving the form ω invariant is called the symplectic group, (V,ω) is called a symplectic vector space, and ω is called a symplectic form.

1.5 Exercises

Exercise 1.1. *Show that for a Hermitian vector space (V,J,h) the complex structure J is both unitary and skewadjoint, that is $h(Ju,Jv) = h(u,v)$ and $h(Ju,v) + h(u,Jv) = 0$ for all $u,v \in V$.*

Exercise 1.2. *Suppose V is a finite dimensional real vector space with a Euclidean form g, a symplectic form ω and a complex structure J. Show that J is orthogonal with respect to g if and only if J is skew adjoint with respect to g, and under this assumption $g(u,v) = \omega(Ju,v)$ holds for all u,v if and only if $\omega(u,v) = g(u,Jv)$ holds for all u,v.*

Exercise 1.3. *Let V be a finite dimensional real vector space with compatible Euclidean form g, symplectic form ω and complex structure J. Let $h = g + \mathrm{i}\omega$ be the associated Hermitian form. Show that*

$$\mathrm{U}(V,J,h) = \mathrm{O}(V,g) \cap \mathrm{Sp}(V,\omega) = \mathrm{O}(V,g) \cap \mathrm{GL}(V,J) = \mathrm{Sp}(V,\omega) \cap \mathrm{GL}(V,J)$$

with $\mathrm{O}(V,g)$ the orthogonal group of (V,g), $\mathrm{Sp}(V,\omega)$ the symplectic group of (V,ω), $\mathrm{GL}(V,J)$ the complex general linear group of (V,J) and $\mathrm{U}(V,J,h)$ the unitary group of (V,J,h).

Exercise 1.4. *Show that the Lagrangian Grassmannian of all Lagrangian subspaces in $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic form $\omega = \sum (p_j q_j' - p_j' q_j)$ is isomorphic to $\mathrm{U}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{R})$. Hint: Identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$ via $(x,y) \simeq (x + \mathrm{i}y = z)$, and let $h(z,z') = \sum z_j \overline{z_j'}$ be the standard Hermitian form with $\omega = \Im(h)$. Observe that a Lagrangian subspace is obtained as the real span of an orthonormal (with respect to h) basis of $\mathbb{C}^n$.*

Exercise 1.5. *Show that $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ for $\alpha \in \Lambda^p V^*$, $\beta \in \Lambda^q V^*$ and $\gamma \in \Lambda^r V^*$.*

Exercise 1.6. *Let V be a vector space of dimension m and v a vector in V. The linear operator $\mathrm{i}_v : \Lambda^p V^* \to \Lambda^{p-1} V^*$ is defined by*

$$\mathrm{i}_v \,\alpha(v_2, \cdots, v_p) = \alpha(v, v_2, \cdots, v_p)$$

for $\alpha \in \Lambda^p V^*$ and vectors $v_2, \cdots, v_p \in V$. The $(p-1)$-form $\mathrm{i}_v \,\alpha$ is called the inner product or the contraction of the p-form α with the vector v. Show that i_v is an antiderivation of ΛV^* in the sense that

$$\mathrm{i}_v(\alpha \wedge \beta) = (\mathrm{i}_v \,\alpha) \wedge \beta + (-1)^p \alpha \wedge (\mathrm{i}_v \,\beta)$$

for $\alpha \in \Lambda^p V^*$ and $\beta \in \Lambda^q V^*$.

Given a nonzero volume form μ on V show that the linear map

$$V \to \Lambda^{m-1} V^* \;,\; v \mapsto \mathrm{i}_v \,\mu$$

is a bijection. Given a symplectic form ω on V show that the linear map

$$V \to \Lambda^1 V^* = V^* \;,\; v \mapsto \mathrm{i}_v \,\omega$$

is a bijection. Hint: For reasons of dimension it suffices to prove injectivety of both linear maps.

Exercise 1.7. *For* $\alpha \in \Lambda^2 V^*$ *an antisymmetric bilinear form on* V *the rank* $\mathrm{rk}(\alpha)$ *is defined as the dimension of* $V/\operatorname{Ker}(\alpha)$ *with* $\operatorname{Ker}(\alpha) = V^\alpha$. *Show that* $\mathrm{rk}(\alpha) \in 2\mathbb{N}$ *is even. Show that* $\mathrm{rk}(\alpha) \leq 2$ *if and only if* $\alpha \wedge \alpha = 0$. *This is a set of quadratic equations on* $\alpha \in \Lambda^2 V^*$, *and the abstract form of the Plücker relations.*

Exercise 1.8. *Let* (V, ω) *be the Cartesian space* $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ *with basis* (e_1, e_2, f_1, f_2) *and standard symplectic form* $\omega = f_1 \wedge e_1 + f_2 \wedge e_2$. *Show that the vector space* $\Lambda^2 V$ *carries a nondegenerate symmetric bilinear form* $(\cdot, \cdot)$ *defined by*

$$\alpha \wedge \beta = (\alpha, \beta)\omega^2/2$$

with signature $(3, 3)$. *Show that the map* $\mathrm{Aut}(\mathbb{R}^4) \ni A \mapsto \Lambda^2 A \in \mathrm{Aut}(\Lambda^2 \mathbb{R}^4)$ *induces a natural homomorphism*

$$\mathrm{Sp}(4, \mathbb{R}) \to \mathrm{O}(2, 3, \mathbb{R}) ,$$

which is called the spin homomorphism for the so called anti-de Sitter group $\mathrm{O}(2, 3, \mathbb{R})$.

2 Calculus on Manifolds

2.1 Vector Fields and Flows

Throughout these notes we assume that M is a smooth manifold. A smooth vector field v on M assigns to each point $x \in M$ a vector v_x in the tangent space T_xM of M at x varying smoothly with x. Given a smooth vector field v on M there exists for each x in M a unique solution curve

$$\gamma = \gamma_x : I_x \to M \ , \ \frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t} = v_{\gamma(t)} \ , \ \gamma(0) = x$$

defined on a maximal open interval I_x around 0 in $\mathbb{R}$. In local coordinates this is just the existence and uniqueness theorem for a first order system of ordinary differential equations [7]. If $s \in I_x$ and $t \in I_{\gamma_x(s)}$ then it is clear that

$$\gamma_x(t+s) = \gamma_{\gamma_x(s)}(t) \ ,$$

and so $(t+s) \in I_x$. We also write $\phi_t(x) = \gamma_x(t)$. The map $\phi_t : D_t \to M$ is smooth with domain

$$D_t = \{x \in M; t \in I_x\} \ ,$$

and is called the flow after time t. The flow satisfies the group property that

$$\phi_{t+s}(x) = \phi_t(\phi_s(x))$$

for $x \in D_s$ and $\phi_s(x) \in D_t$.

If for $x \in M$ one has $s := \sup I_x < \infty$ then for each compact subset K of M there exists an $\epsilon = \epsilon_K > 0$ such that $\gamma_x(t) \notin K$ for all $t \in (s-\epsilon, s)$. In other words, the solution $\gamma_x(t)$ runs out of every compact subset of M in finite time. In particular if M is compact we have $\sup I_x = +\infty$ and likewise $\inf I_x = -\infty$ for all x in M.

The vector field v on M is called complete if $I_x = \mathbb{R}$ for all x in M, or equivalently if $D_t = M$ for all $t \in \mathbb{R}$. The group property of flows then implies that $t \mapsto \phi_t$ is a homomorphism of the additive group $(\mathbb{R}, +)$ to the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M. For this reason $t \mapsto \phi_t$ is also called a one-parameter group of diffeomorphisms. Smooth vector fields on compact manifolds are always complete.

If the vector field v_t depends smoothly on the time parameter $t \in \mathbb{R}$ then the solution curve $t \mapsto \gamma_x(t)$ through $x = \gamma_x(0)$ exists likewise for a maximal open interval I_x around 0 in $\mathbb{R}$. In case of a complete vector field v_t (so $I_x = \mathbb{R}$ for all x in M) this leads to a one-parameter family $t \mapsto \phi_t$ of diffeomorphisms of M with $\phi_0 = \mathrm{Id}$, but the group property need no longer hold. A one parameter family $t \mapsto \phi_t$ of diffeomorphisms of M with $\phi_0 = \mathrm{Id}$ is also called an isotopy.

2.2 Lie Derivatives

Let $\Omega^p(M)$ denote the space of smooth p-forms on the manifold M. So a smooth p-form α on M assigns to each point x of M an antisymmetric multilinear form α_x of degree p on the tangent space T_xM varying smoothly with x. Note that $\Omega^0(M) = \mathcal{F}(M)$ is the space of smooth real valued functions on M. With respect to the exterior product the space

$$\Omega(M) = \bigoplus_{p\geq 0} \Omega^p(M)$$

becomes a graded associative algebra. The exterior product is anticommutative in the sense that $\beta\wedge\alpha = (-1)^{pq}\alpha\wedge\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. If M and N are smooth manifolds, and $\phi : M \to N$ is a smooth map, then the pullback of a smooth p-form α on N under ϕ is a smooth p-form $\phi^*\alpha$ on M defined by

$$(\phi^*\alpha)_x(v_{1x}, \cdots, v_{px}) = \alpha_{\phi(x)}((T_x\phi)v_{1x}, \cdots, (T_x\phi)v_{px})$$

with $T_x\phi : T_xM \to T_{\phi(x)}N$ the tangent map of ϕ at x and $v_{1x}, \cdots, v_{px}$ in T_xM. Note that the pullback ϕ^* is a linear operator from $\Omega^p(N)$ to $\Omega^p(M)$, and the word "pullback" reminds one of the fact that the direction of the arrow $\phi^* : \Omega^p(N) \to \Omega^p(M)$ is reversed compared to the arrow $\phi : M \to N$. It also helps to remember that for $\psi : L \to M$ and $\phi : M \to N$ the order in the composition formula $(\phi\circ\psi)^* = \psi^*\circ\phi^*$ is reversed. The pullback $\phi^* : \Omega(N) \to \Omega(M)$ becomes a homomorphism of algebras by the rule

$$\phi^*(\alpha\wedge\beta) = (\phi^*\alpha)\wedge(\phi^*\beta)$$

for $\alpha, \beta \in \Omega(N)$. In particular if $\phi : M \to N$ is a diffeomorphism then $\phi^* : \Omega(N) \to \Omega(M)$ is an isomorphism of algebras, and an automorphism of $\Omega(M)$ in case $N = M$.

The exterior derivative $\mathrm{d} : \Omega^p(M) \to \Omega^{p+1}(M)$ is a linear map, satisfying $\mathrm{d}\,\mathrm{d} = 0$ and is an antiderivation of $\Omega(M)$ by the rule

$$\mathrm{d}(\alpha\wedge\beta) = (\mathrm{d}\,\alpha)\wedge\beta + (-1)^p\alpha\wedge(\mathrm{d}\,\beta)$$

for α a smooth p-form and β a smooth q-form. For $f \in \mathcal{F}(M)$ the smooth 1-form $\mathrm{d}\,f$ is given by

$$\mathrm{d}\,f(v) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^* f\Big\}_{t=0}$$

for v a smooth vector field on M and $t \mapsto \phi_t$ the corresponding one-parameter group of diffeomorphisms. The function $\mathrm{d}\,f(v)$ is called the derivative of f in the direction of v. The exterior derivative behaves naturally under smooth maps, in the sense that

$$\phi^*(\mathrm{d}\,\alpha) = \mathrm{d}(\phi^*\alpha)$$

for $\phi : M \to N$ a smooth map and α a smooth p-form on N. The Poincaré lemma says that the equation $\mathrm{d}\,\alpha = 0$ for $\alpha \in \Omega^p(M)$ implies that there is a neighborhood U around each point of M on which $\alpha = \mathrm{d}\,\beta$ for some $\beta \in \Omega^{p-1}(U)$.

Let $\mathcal{X}(M)$ denote the vector space of smooth vector fields on M. For $v \in \mathcal{X}(M)$ and $\alpha \in \Omega^p(M)$ the contraction $\mathrm{i}_v\,\alpha \in \Omega^{p-1}(M)$ of the p-form α with the vector field v is defined by

$$(\mathrm{i}_v\,\alpha)(v_2, \cdots, v_p) = \alpha(v, v_2, \cdots, v_p)$$

for $v_2, \cdots, v_p \in \mathcal{X}(M)$. The $(p-1)$-form $\mathrm{i}_v\,\alpha$ is also called the inner product of α with v, which is the reason for the notation $\mathrm{i}_v\,\alpha$. The contraction operator i_v has somewhat similar properties as the exterior derivative d. Indeed $\mathrm{i}_v : \Omega^p(M) \to \Omega^{p-1}(M)$ is a linear map, satisfying $\mathrm{i}_v\,\mathrm{i}_v = 0$ and is an antiderivation of $\Omega(M)$ by the rule

$$\mathrm{i}_v(\alpha \wedge \beta) = (\mathrm{i}_v\,\alpha) \wedge \beta + (-1)^p \alpha \wedge (\mathrm{i}_v\,\beta)$$

for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$. The operator i_v behaves naturally under a diffeomorphism $\phi : M \to N$, in the sense that

$$\phi^*(\mathrm{i}_v\,\alpha) = \mathrm{i}_{\phi^* v}(\phi^*\alpha)$$

for $v \in \mathcal{X}(N)$ and $\alpha \in \Omega^p(N)$. Here the pullback $\phi^* v \in \mathcal{X}(M)$ of $v \in \mathcal{X}(N)$ under a diffeomorphism $\phi : M \to N$ is defined by

$$(\phi^* v)_x = (T_x\phi)^{-1} v_{\phi(x)}$$

for $x \in M$.

For $v \in \mathcal{X}(M)$ the Lie derivative $\mathcal{L}_v\alpha$ of $\alpha \in \Omega^p(M)$ in the direction of the vector field v is the element of $\Omega^p(M)$ defined by

$$\mathcal{L}_v\alpha = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^*\alpha\Big\}_{t=0}$$

with the derivative taken in the vector space $\Omega^p(M)$. The Lie derivative $\mathcal{L}_v : \Omega^p(M) \to \Omega^p(M)$ is a linear operator, commuting with the exterior derivative, and becomes a derivation of $\Omega(M)$ by the rule

$$\mathcal{L}_v(\alpha \wedge \beta) = (\mathcal{L}_v\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_v\beta)$$

for $\alpha, \beta \in \Omega(M)$. In particular $\mathcal{L}_v$ acts as a derivation of the commutative subalgebra $\mathcal{F}(M)$ of $\Omega(M)$. Note that for $v \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$

$$\mathcal{L}_v f = \mathrm{d}\, f(v)$$

is just the derivative of f in the direction of v.

Theorem 2.1. *For $v \in \mathcal{X}(M)$ and $\alpha \in \Omega^p(M)$ we have*

$$\mathcal{L}_v \alpha = \mathrm{i}_v(\mathrm{d}\,\alpha) + \mathrm{d}(\mathrm{i}_v\,\alpha)$$

and $\mathcal{L}_v = \mathrm{i}_v\,\mathrm{d} + \mathrm{d}\,\mathrm{i}_v$ is called the Cartan formula.

Proof. First check that the Cartan formula is correct on $\Omega^0(M) = \mathcal{F}(M)$. Subsequently check that both sides of the Cartan formula commute with the exterior derivative d, and that both sides are derivations of the associative algebra $(\Omega(M), \wedge)$. Finally observe that for $U \hookrightarrow M$ a coordinate chart the algebra $\Omega(U)$ is generated by $\mathcal{F}(U)$ and $\mathrm{d}\,\mathcal{F}(U)$. Hence the Cartan formula, which is local in nature, holds on $\Omega(M)$. $\square$

For $v \in \mathcal{X}(M)$ the Lie derivative $\mathcal{L}_v$ is also defined as a linear operator on $\mathcal{X}(M)$ by

$$\mathcal{L}_v w = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^* w\Big\}_{t=0}$$

for $w \in \mathcal{X}(M)$ and $t \mapsto \phi_t$ the flow of v at time t. It is customary to write $[v, w] = \mathcal{L}_v w$ and call it the Lie bracket of the vector fields v, w. For $v, w \in \mathcal{X}(M)$ and $\alpha \in \Omega^p(M)$ we have

$$\mathcal{L}_v(\mathrm{i}_w\,\alpha) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^*(\mathrm{i}_w\,\alpha)\Big\}_{t=0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\mathrm{i}_{\phi_t^* w}(\phi_t^*\alpha)\Big\}_{t=0} = \mathrm{i}_{[v,w]}\,\alpha + \mathrm{i}_w(\mathcal{L}_v\alpha)$$

by the chain rule. If we substitute $\alpha = \mathrm{d}\,f$ for $f \in \mathcal{F}(M)$ then the Cartan formula implies

$$\mathcal{L}_{[v,w]} f = [\mathcal{L}_v, \mathcal{L}_w] f$$

in which the bracket on the right hand side denotes the commutator bracket $[A, B] = AB - BA$ of linear operators A, B on the vector space $\mathcal{F}(M)$.

A Lie algebra $\mathfrak{g}$ is a vector space with a bilinear operation

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\ ,\ (X, Y) \mapsto [X, Y]$$

which is antisymmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. If $\mathcal{A}$ is an associative algebra then it is easy to check that the space $\mathrm{Der}(\mathcal{A})$ of derivations of $\mathcal{A}$ is a Lie algebra with respect to the commutator product. The conclusion from the previous paragraph is that the map

$$\mathcal{X}(M) \to \mathrm{Der}(\mathcal{F}(M)) \ , \ v \mapsto \mathcal{L}_v$$

is an injective homomorphism of Lie algebras. In particular this shows that $\mathcal{X}(M)$ is a Lie algebra with respect to the Lie bracket of vector fields. It is customary in differential geometry to identify a smooth vector field v with the derivation $\mathcal{L}_v$ of the commutative algebra $\mathcal{F}(M)$. Elements of a Lie algebra are commonly denoted by capital letters $X, Y, \cdots$ and some authors use these capitals also for smooth vector fields on M. This explains the notation $\mathcal{X}(M)$ for the Lie algebra of smooth vector fields on M.

2.3 Singular Homology

For $p \in \mathbb{N}$ the standard p-simplex Δ^p in $\mathbb{R}^{p+1}$ is defined as

$$\Delta^p = \{x = (x_0, \cdots, x_p) \in \mathbb{R}^{p+1}; x_j \geq 0 \ \forall \ j, \sum x_j = 1\}$$

and so Δ^p is just the convex hull of the standard basis $(e_0, \cdots, e_p)$ of $\mathbb{R}^{p+1}$. The "boundary" of Δ^p consists of $(p+1)$ codimension-one faces, obtained by putting one of the coordinates equal to 0. For $0 \leq j \leq p$ denote by

$$\rho_j^p : \Delta^{p-1} \to \Delta^p \ , \ (x_0, \cdots, x_{p-1}) \mapsto (x_0, \cdots, x_{j-1}, 0, x_j, \cdots, x_{p-1})$$

the inclusion of the standard $(p-1)$-simplex as the j-th boundary face of the standard p-simplex.

Now let M be a smooth manifold. A smooth singular p-simplex σ in M is a continuous map $\sigma : \Delta^p \to M$ which has a smooth extension to an open neighborhood of Δ^p in the hyperplane $V^p = \{x \in \mathbb{R}^{p+1}; \sum x_j = 1\}$. Note that ρ_j^p is in fact a smooth singular $(p-1)$-simplex in V^p, and likewise is Δ^p via the identity map a smooth singular p-simplex in V^p. A smooth p-chain c in M is a formal finite real linear combination $c = \sum c_\sigma \sigma$ of smooth singular p-simplices, and the vector space of smooth p-chains in M is denoted by $C_p(M)$. A smooth map $\phi : M \to N$ of manifolds induces a "pushforward" linear map

$$\phi_* : C_p(M) \to C_p(N) \ , \ \phi_*(\sum c_\sigma \sigma) = \sum c_\sigma \phi \circ \sigma \ .$$

The name pushforward reminds one of the property $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ when acting on smooth p-chains.

Working with chains rather than singular simplices makes it possible to introduce a boundary operator $\partial = \partial_p : C_p(M) \to C_{p-1}(M)$ for $p \geq 1$ as a linear operator, defined by

$$\partial(\sigma) = \sum_{j=0}^{p} (-1)^j \sigma \circ \rho_j^p = \sum_{j=0}^{p} (-1)^j \sigma_* \rho_j^p = \sigma_*(\sum_{j=0}^{p} (-1)^j \rho_j^p)$$

for σ a smooth singular p-simplex. The boundary operator behaves naturally under smooth maps, in the sense that

$$\phi_*(\partial c) = \partial(\phi_* c)$$

for $\phi : M \to N$ a smooth map and $c \in C_p(M)$. It is convenient to put $C_p(M) = 0$ for $p \in \mathbb{Z}, p < 0$ and likewise $\partial_p = 0$ for $p \leq 0$. The choice with the signs gives rise to a fundamental property of boundary operators.

Theorem 2.2. *We have $\partial_p \partial_{p+1} = 0$ for all $p \in \mathbb{Z}$.*

Proof. It is clear that for all $p \geq 1$ and $0 \leq j \leq k \leq p$

$$\rho_j^{p+1} \circ \rho_k^p = \rho_{k+1}^{p+1} \circ \rho_j^p$$

since both sides embed Δ^{p-1} as a codimension-two face inside Δ^{p+1} by putting 0 for the j-th and $(k+1)$-th coordinate. For σ a smooth singular $(p+1)$-simplex we get

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\Big\{\sum_{j=0}^{p+1} (-1)^j \sigma_* \rho_j^{p+1}\Big\} = \sigma_*\Big\{\sum_{j=0}^{p+1} (-1)^j \partial(\rho_j^{p+1})\Big\} \\
&= \sigma_*\Big\{\sum_{j=0}^{p+1} \sum_{k=0}^{p} (-1)^{j+k} \rho_j^{p+1} \circ \rho_k^p\Big\} \\
&= \sigma_*\Big\{\sum_{j \leq k} (-1)^{j+k} \rho_j^{p+1} \circ \rho_k^p + \sum_{k<j} (-1)^{j+k} \rho_j^{p+1} \circ \rho_k^p\Big\} \\
&= \sigma_*\Big\{\sum_{j \leq k} (-1)^{j+k} \rho_{k+1}^{p+1} \circ \rho_j^p + \sum_{j<k} (-1)^{j+k} \rho_k^{p+1} \circ \rho_j^p\Big\} \\
&= \sigma_*\Big\{\sum_{j \leq k} (-1)^{j+k} \rho_{k+1}^{p+1} \circ \rho_j^p + \sum_{j \leq k} (-1)^{j+k+1} \rho_{k+1}^{p+1} \circ \rho_j^p\Big\} \\
&= \sigma_*\Big\{\sum_{j \leq k} [(-1)^{j+k} + (-1)^{j+k+1}] \rho_{k+1}^{p+1} \circ \rho_j^p\Big\} = 0
\end{aligned}$$

and $\partial_p \partial_{p+1} = 0$ on all of $C_{p+1}(M)$ by linearity. □

The sequence

$$\cdots \to C_{p+1}(M) \xrightarrow{\partial_{p+1}} C_p(M) \xrightarrow{\partial_p} C_{p-1}(M) \to \cdots$$

is called the smooth chain complex of M, where the word "complex" refers to the fact that the composition of any two arrows is zero. If we introduce the space $Z_p(M)$ of smooth p-cycles and $B_p(M)$ of smooth p-boundaries on M by

$$Z_p(M) = \mathrm{Ker}(\partial_p) \ , \ B_p(M) = \mathrm{Im}(\partial_{p+1})$$

then it is clear from the above theorem that $B_p(M)$ is a linear subspace of $Z_p(M)$. The quotient space

$$H_p(M) = Z_p(M)/B_p(M)$$

is called the smooth singular homology space of M in degree p. For c in $Z_p(M)$ we denote by $[c]$ the corresponding homology class in $H_p(M)$. If $\phi : M \to N$ is a smooth map of manifolds then we have an induced map

$$\phi_* : H_p(M) \to H_p(N)$$

which is called the pushforward of $\phi : M \to N$ in homology.

This all mimics the definition of the continuous singular homology of a topological space M as developed in a standard course on algebraic topology. The only differences are that we work with smooth rather than continuous maps, and with real rather than integral coefficients. Since continuous chains can be uniformly approximated by smooth chains to arbitrary precision we have

$$H_p^{\mathrm{smooth}}(M, \mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H_p^{\mathrm{continuous}}(M, \mathbb{Z})$$

for all p and any smooth manifold M.

2.4 Integration over Singular Chains and Stokes Theorem

Given a smooth manifold M of dimension m a volume form is a smooth m-form μ such that $\mu_x \in \Lambda^m T_x^* M$ is nonzero for all x in M. If μ is a given volume form on M then any smooth m-form on M is of the form $f\mu$ for some $f \in \mathcal{F}(M)$. For example, on $\mathbb{R}^{p+1}$ with coordinates $(x_0, \cdots, x_p)$ we take $\mathrm{d}\,x_0 \wedge \cdots \wedge \mathrm{d}\,x_p$ for the standard Euclidean volume form. On the hyperplane V^p embedded via $\iota : V^p = \{x \in \mathbb{R}^{p+1}; \sum x_j = 1\} \hookrightarrow \mathbb{R}^{p+1}$ we take as standard Euclidean volume form

$$\mu^p = \iota^*(\mathrm{i}_n(\mathrm{d}\,x_0 \wedge \cdots \wedge \mathrm{d}\,x_p))$$

with $n = n^p = (1, \cdots, 1)/\sqrt{p+1}$ the outward unit normal of V^p.

If $\alpha \in \Omega^p(M)$ and σ a smooth p-simplex in M then $\sigma^*\alpha = f\mu^p$ for some smooth function f on Δ^p. In turn we define the integral of α over σ by

$$\int_\sigma \alpha = \int_{\Delta^p} \sigma^*\alpha = \int_{\Delta^p} f(x)d|\mu^p|(x)$$

where the integral on the right hand side is the Riemann integral of f against the Euclidean measure $|\mu^p|$ on V^p. The integral of α over a smooth p-chain $c = \sum c_\sigma \sigma$ in M is defined by linearity: $\int_c \alpha = \sum_\sigma c_\sigma \int_\sigma \alpha$. If $\phi : M \to N$ is a smooth map then

$$\int_c \phi^*\alpha = \int_{\phi_* c} \alpha$$

for $\alpha \in \Omega^p(N)$ and $c \in C_p(M)$.

Perhaps the most important theorem in integration theory on manifolds is the Stokes theorem.

Theorem 2.3. *Let M be a smooth manifold. Then*

$$\int_c \mathrm{d}\,\alpha = \int_{\partial c} \alpha$$

for $c \in C_p(M)$ and $\alpha \in \Omega^{p-1}(M)$.

For $\sigma : \Delta^1 \to M$ a smooth singular 1-simplex and $f \in \mathcal{F}(M)$ we have

$$\begin{aligned}\int_\sigma \mathrm{d}\,f = \int_{\Delta^1} \sigma^*(\mathrm{d}\,f) = \int_{\Delta^1} \mathrm{d}(f \circ \sigma) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{f(\sigma(1-t,t))\Big\}\,\mathrm{d}\,t \\ &= f(\sigma(0,1)) - f(\sigma(1,0)) = \int_{\partial c} f\end{aligned}$$

and so for $p = 1$ the Stokes' theorem boils down to the fundamental theorem of calculus.

The Stokes theorem is a crucial ingredient in the de Rham theory, as discussed in the next section. For a proof we refer to the text book by Warner [56].

2.5 De Rham Theorem

Denote by $\mathrm{d}_p : \Omega^p(M) \to \Omega^{p+1}(M)$ the exterior derivative with subindex p to emphasize its action on p-forms. Since $\mathrm{d}_p \circ \mathrm{d}_{p-1} = 0$ we obtain a cochain (with "co" indicating that indices go up rather than down) complex

$$\cdots \to \Omega^{p-1}(M) \xrightarrow{\mathrm{d}_{p-1}} \Omega^p(M) \xrightarrow{\mathrm{d}_p} \Omega^{p+1}(M) \to \cdots$$

with the exterior derivative as coboundary operator. This complex is called the de Rham complex of M. If we introduce the space $Z^p(M)$ of cocycles (or closed forms) and $B^p(M)$ the space of coboundaries (or exact forms) on M by

$$Z^p(M) = \mathrm{Ker}(\mathrm{d}_p)\ ,\ B^p(M) = \mathrm{Im}(\mathrm{d}_{p-1})$$

then the coboundary property $\mathrm{d}\,\mathrm{d} = 0$ implies that $B^p(M)$ is a linear subspace of $Z^p(M)$. The quotient space

$$H^p_{\mathrm{dR}}(M) = Z^p(M)/B^p(M)$$

is called the de Rham cohomology space in degree p. For $\alpha \in Z^p(M)$ we denote by $[\alpha]$ the corresponding cohomology class in $H^p_{\mathrm{dR}}(M)$.

If z is a p-cycle and γ a p-cocycle on M then the (bilinear in z and γ) integral

$$\int_z \gamma$$

is called the period of γ over z. Now the Stokes' theorem implies that the period integral remains invariant under additions $z \mapsto z + \partial c$ and $\gamma + \mathrm{d}\,\alpha$ for some $c \in C_{p+1}(M)$ and $\alpha \in \Omega^{p-1}(M)$. This means that the period integral defines a period pairing (pairing means bilinear map)

$$\langle \cdot, \cdot \rangle : H_p(M) \times H^p_{\mathrm{dR}}(M) \to \mathbb{R}$$

by

$$\langle [z], [\gamma] \rangle = \int_z \gamma$$

for $[z] \in H_p(M)$ and $[\gamma] \in H^p_{\mathrm{dR}}(M)$. The de Rham theorem, obtained by Georges de Rham in his doctoral thesis of 1931, says that this pairing is nondegenerate [45].

Theorem 2.4. *The period pairing* $\langle \cdot, \cdot \rangle : H_p(M) \times H^p_{\mathrm{dR}}(M) \to \mathbb{R}$ *is nondegenerate.*

For a proof of the de Rham theorem we refer again to the text book of Warner [56].

2.6 Integration on Oriented Manifolds and Poincaré Duality

Let M be a smooth manifold of dimension m. A volume form on M is a nowhere vanishing smooth m-form on M, and M is called orientable if there exists a volume form on M. If both μ and ν are volume forms on M then

$\mu = f\nu$ with f a nowhere vanishing smooth function on M. If $f > 0$ on all of M then $\mu \sim \nu$ are called equivalent volume forms on M. An orientation of M is a choice of an equivalence class of volume forms on an orientable M. For M connected and orientable there are just two orientations of M. Suppose M has a fixed orientation represented by the volume form ν on M. The pair (M, ν) is called an oriented manifold.

For $\mu \in \Omega^m(M)$ a smooth m-form with compact support we wish to define the integral

$$\int_M \mu$$

over the oriented manifold (M, ν).

A smooth singular m-simplex $\sigma : \Delta^m \to M$ is called regular if σ extends to a diffeomorphism on a neighborhood of Δ^m in V^m. A regular m-simplex σ in an oriented manifold (M, ν) is called oriented if $\sigma^*\nu$ and the standard volume form μ^m on V^m are equivalent on that neighborhood.

Lemma 2.5. *Suppose (M, ν) is an oriented smooth manifold of dimension m. Suppose $\sigma, \tau : \Delta^m \to M$ are oriented regular m-simplices. If $\mu \in \Omega^m(M)$ is a smooth m-form with support contained in $\sigma(\Delta^m) \cap \tau(\Delta^m)$ then*

$$\int_\sigma \mu = \int_\tau \mu \ .$$

Proof. As before, let μ^m denote the standard Euclidean volume form on the hyperplane V^m containing Δ^m, with associated Euclidean measure $|\mu^m|$ on V^m. If $\sigma^*\mu = f\mu^m$ and $\tau^*\mu = g\mu^m$ with f and g smooth functions on V^m with support contained in Δ^m then

$$\int_\sigma \mu = \int_{\Delta^m} f(x) d|\mu^m|(x) \ , \ \int_\tau \mu = \int_{\Delta^m} g(y) d|\mu^m|(y)$$

by definition. Putting $\sigma(x) = \tau(y)$ we can apply the Jacobi substitution theorem to the orientation preserving diffeomorphism

$$\sigma^{-1}(\sigma(\Delta^m) \cap \tau(\Delta^m)) \to \tau^{-1}(\sigma(\Delta^m) \cap \tau(\Delta^m)) \ , \ x \mapsto y = \phi(x) \ , \ \phi = \tau^{-1}\sigma$$

yielding

$$\int_{\Delta^m} g(y) d|\mu^m|(y) = \int_{\Delta^m} g(\phi(x)) J\phi(x) d|\mu^m|(x)$$

with $J\phi$ the Jacobian of the map $\phi : \Delta^m \to \Delta^m$. Indeed, $J\phi > 0$ since ϕ is orientation preserving, and therefore we can forget the absolute value around $J\phi$ in the substitution theorem.

On the other hand, we have

$$f\mu^m = \sigma^*\mu = \phi^*(\tau^*\mu) = \phi^*(g\mu^m) = \phi^*(g)\phi^*(\mu^m) = \phi^*(g)J\phi\mu^m$$

which in turn implies that $f = \phi^*(g)J\phi$. Equivalently $f(x) = g(\phi(x))J\phi(x)$ for all $x \in \Delta^m$, which proves the lemma. $\square$

Let us write $\Delta^{m\circ}$ for the interior of Δ^m. For each point $x \in M$ we can choose an oriented regular m-simplex σ with $x \in \sigma(\Delta^{m\circ})$ by working in a chart. Now let $\mu \in \Omega^m(M)$ with compact support $K \subset M$. We can choose a finite number of oriented regular m-simplices $\sigma_j : \Delta^m \to M$, such that K is covered by the open sets $U_j = \sigma_j(\Delta^{m\circ})$. Let $U_0 = M - K$ and let $\{f_j; j \geq 0\}$ be a partition of unity subordinate to the cover $\{U_j; j \geq 0\}$ of M.

Definition 2.6. *The integral of the m-form μ over the oriented manifold (M, ν) is defined by*

$$\int_M \mu = \sum_{j \geq 1} \int_{\sigma_j} f_j \mu \,.$$

The above lemma ensures that this definition is independent of the choices of the open cover $\{U_j; j \geq 0\}$ and the partition of unity $\{f_j; j \geq 0\}$ subordinate to it. The Stokes theorem for integration over oriented manifolds takes the following form.

Theorem 2.7. *Let (M, ν) be an oriented manifold on dimension m, and let D be a compact domain in M with smooth (via an outward directed vector field compatibly) oriented boundary ∂D. Then*

$$\int_D \mathrm{d}\,\alpha = \int_{\partial D} \alpha$$

for $\alpha \in \Omega^{m-1}(M)$.

Suppose M is an oriented compact smooth manifold of dimension m. Under these assumptions we can define the intersection pairing

$$H^p_{\mathrm{dR}}(M) \times H^{m-p}_{\mathrm{dR}}(M) \to \mathbb{R}$$

by

$$\langle [\alpha], [\beta] \rangle = \int_M \alpha \wedge \beta$$

for $\alpha \in Z^p(M)$ and $\beta \in Z^{m-p}(M)$ representatives of the cohomology classes. It follows from the Stokes theorem that the intersection pairing descends from closed differential forms to de Rham cohomology classes.

Theorem 2.8. *For M an oriented compact smooth manifold of dimension m the intersection pairing $\langle\cdot,\cdot\rangle : H^p_{\mathrm{dR}}(M) \times H^{m-p}_{\mathrm{dR}}(M) \to \mathbb{R}$ is nondegenerate.*

Under the above conditions on M the intersection pairing determines a linear isomorphism

$$H^p_{\mathrm{dR}}(M) \cong (H^{m-p}_{\mathrm{dR}}(M))^*$$

which is called Poincaré duality. Since $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$ for M connected we arrive at the following conclusion.

Corollary 2.9. *If M is an oriented connected compact smooth manifold of dimension m then $H^m_{\mathrm{dR}}(M) \cong \mathbb{R}$.*

Theorem 2.10. *If M is an oriented compact smooth manifold, then the de Rham cohomology spaces $H^p_{\mathrm{dR}}(M)$ are all finite dimensional.*

For a proof of Poincaré duality and the finite dimensionality result of de Rham cohomology (using Hodge theory and analysis of elliptic differential operators) we refer once more to the text book of Warner [56].

2.7 Moser Theorem

Suppose (M, ν) is an oriented connected compact manifold, and suppose $\phi : M \to M$ is an orientation preserving diffeomorphism. Then $\mu = \phi^*\nu \sim \nu$ is another volume form representing the same orientation with

$$\int_M \mu = \int_M \nu$$

by the Jacobi substitution theorem, meaning that the (positive) volume of M relative to μ and ν is equal. Moser asked himself the converse question [37].

Question 2.11. *Suppose M is an oriented connected compact manifold, and $\mu \sim \nu$ are equivalent volume forms representing the given orientation. Does the assumption*

$$\int_M \mu = \int_M \nu$$

conversely imply that $\mu = \phi^*\nu$ for an orientation preserving diffeomorphism $\phi : M \to M$?

Moser showed that the answer is yes, with a very elegant argument.

Theorem 2.12. *Suppose M is an oriented connected compact manifold and $\mu \sim \nu$ are equivalent volume forms representing the given orientation. Suppose that the volumes of M relative to μ and ν are equal. Then there exists an orientation preserving diffeomorphism $\phi : M \to M$ with $\mu = \phi^*\nu$.*

Proof. Let $\mu_0 = \mu$ and $\mu_1 = \nu$. If we put $\mu_t = (1-t)\mu_0 + t\mu_1$ for $0 \leq t \leq 1$ then μ_t is a volume form, and the volume of M relative to μ_t is constant. Since $H^m_{\mathrm{dR}}(M) \cong \mathbb{R}$ by Corollary 2.9 and de Rham cohomology classes are distinguished by their periods by the de Rham theorem it follows that de Rham volume form classes are determined by their volume. In turn we find

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mu_t = \nu - \mu = \mathrm{d}\,\lambda$$

for some $\lambda \in \Omega^{m-1}(M)$.

The Moser trick is the search for an isotopy ϕ_t of M such that for all t

$$\phi_t^*\mu_t = \mu$$

and so $\phi = \phi_1$ does the job. Differentiation of the left hand side with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^*\mu_t\Big\} = \phi_t^*\Big\{\mathcal{L}_{v_t}\mu_t + \frac{\mathrm{d}}{\mathrm{d}\,t}\mu_t\Big\}$$

by the chain rule. Here v_t is the time dependent vector field whose solution curves correspond to the isotopy ϕ_t of M. Using the Cartan formula this expression becomes

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^*\mu_t\Big\} = \phi_t^*\Big\{\mathrm{d}(\mathrm{i}_{v_t}\,\mu_t) + \mathrm{d}\,\lambda\Big\} = \phi_t^*\,\mathrm{d}\Big\{\mathrm{i}_{v_t}\,\mu_t + \lambda\Big\}$$

and so will certainly vanish if

$$\mathrm{i}_{v_t}\,\mu_t + \lambda = 0\;.$$

This is called the Moser equation for the time dependent vector field v_t corresponding to the isotopy ϕ_t of M. But this equation determines v_t uniquely as shown in Exercise 1.6. □

Remark 2.13. *The orientation preserving diffeomorphism $\phi : M \to M$ is highly nonunique. At first there is the choice of a smooth path μ_t of volume forms on M with $\mu_0 = \mu$ and $\mu_1 = \nu$. In the proof we chose the linear path but in fact any smooth path of volume forms on M will do as long as the volume of M relative to μ_t is constant. The equation*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mu_t = \mathrm{d}\,\lambda_t$$

has a solution $\lambda_t \in \Omega^{m-1}(M)$ but again nonunique. Indeed the difference of any two choices for λ_t will be in $Z^{m-1}(M)$. Having chosen the path μ_t and the corresponding $(m-1)$-form λ_t the Moser equation

$$\mathrm{i}_{v_t}\,\mu_t + \lambda_t = 0$$

has indeed a unique solution v_t.

In the next chapter we will adapt Moser's argument for symplectic forms rather than volume forms.

2.8 Exercises

Exercise 2.1. *Suppose we have given a smooth vector field v on M with flow $\phi_t : D_t \to M$ after time t. Show that $\sup I_x < \infty$ for some $x \in M$ implies that the solution curve $t \mapsto \phi_t(x)$ runs out of every compact subset of M in finite time.*

Exercise 2.2. *Show that the pullback $\phi^* : \Omega(N) \to \Omega(M)$ of a smooth map $\phi : M \to N$ is a homomorphism of associative algebras.*

Exercise 2.3. *Suppose the smooth vector field v on M has flow $\phi_t : D_t \to M$ after time t. By the previous exercise the pullback $\phi_t^* : \Omega(M) \to \Omega(D_t)$ is a homomorphism. Show that in turn the Lie derivative $\mathcal{L}_v$ is a derivation of $\Omega(M)$.*

Exercise 2.4. *Let $\Omega(M) = \Omega^+(M) \oplus \Omega^-(M)$ denotes the decomposition of differential forms in even and odd degree parts. A linear operator L on $\Omega(M)$ is called odd if $\alpha \in \Omega^\pm(M)$ impies that $L\alpha \in \Omega^\mp(M)$. Show that the square L^2 of an odd antiderivation L of $\Omega(M)$ becomes a derivation. Conclude that the right hand side of the Cartan formula is a derivation of $\Omega(M)$. Check the further details of the proof of Theorem 2.1.*

Exercise 2.5. *Let $\phi : M \to N$ be a diffeomorphism, and let $v \in \mathcal{X}(N)$ and $\phi^* v \in \mathcal{X}(M)$. Show that $\mathcal{L}_{\phi^* v} = \phi^* \mathcal{L}_v (\phi^{-1})^*$ as operators on $\Omega(M)$, and conclude that $\phi^*[u,v] = [\phi^* u, \phi^* v]$ for all $u, v \in \mathcal{X}(N)$. Hint: Use that $\phi^*(\mathrm{i}_v\,\alpha) = \mathrm{i}_{\phi^* v}\,\phi^*\alpha$ for all $\alpha \in \Omega(N)$.*

Exercise 2.6. *Show that for an associative algebra $\mathcal{A}$ the space* $\mathrm{Der}(\mathcal{A})$ *of derivations of $\mathcal{A}$ is a Lie algebra with respect to the commutator bracket.*

Exercise 2.7. *Show that $\mathcal{L}_{[v,w]} = [\mathcal{L}_v, \mathcal{L}_w]$ as operators on $\Omega(M)$ for all $v, w \in \mathcal{X}(M)$.*

Exercise 2.8. *Show that for a connected smooth manifold* $H_0(M) \cong \mathbb{R}$.

Exercise 2.9. *Show that for a connected smooth manifold* $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$.

Exercise 2.10. *Show that a smooth map* $\phi : M \to N$ *induces a well defined pushforward* $\phi_* : H_p(M) \to H_p(N)$ *in smooth singular homology, and a well defined pullback* $\phi^* : H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ *in de Rham cohomology.*

Exercise 2.11. *Suppose* (M, ν) *is an oriented smooth manifold of dimension* m. *Suppose* $\mu \in \Omega^m(M)$ *with compact support* $K \subset M$. *Suppose*

$$\int_M \mu = \sum_{j \geq 0} \int_{\sigma_j} f_j \mu \ , \ \int_M \mu = \sum_{k \geq 0} \int_{\tau_k} g_k \mu$$

is defined using two pairs $\{U_j = \sigma_j(\Delta^{m\circ}), f_j\}$ *and* $\{V_k = \tau_k(\Delta^{m\circ}), g_k\}$ *of open covers of* K *with subordinate partitions of unity. Show that the outcome of the two definitions is indeed the same.*

Exercise 2.12. *Suppose* (M, μ) *is an oriented connected smooth manifold. Suppose* $\phi : M \to M$ *is an involution in the sense that* $\phi^2 = \mathrm{Id}$ *while* $\phi \neq \mathrm{Id}$. *If* ϕ *has no fixed points then the quotient* N *of* M *by the action of the order two group* $\{\mathrm{Id}, \phi\}$ *is again a manifold. Show that* N *is orientable if* ϕ *preserves the orientation, and is not orientable if* ϕ *reverses the orientation. Show that the unit sphere* $\mathbb{S}^m$ *in* $\mathbb{R}^{m+1}$ *is orientable, and conclude that the real projective space* $\mathbb{P}^m(\mathbb{R})$ *is orientable if and only if* m *is odd. Hint: If* $e_x = x$ *is the Euler vector field on* $\mathbb{R}^{m+1}$ *then the restriction* μ *of* $\mathrm{i}_e(dx_0 \wedge \cdots \wedge dx_m)$ *to* $\mathbb{S}^m$ *is a volume form on* $\mathbb{S}^m$ *with* $\phi^*\mu = (-1)^{m+1}\mu$ *for* ϕ *the antipodal map* $x \mapsto -x$ *on* $\mathbb{S}^m$.

Exercise 2.13. *Suppose* M *is an oriented connected compact manifold. Show that the intersection pairing*

$$\langle \cdot, \cdot \rangle : Z^p(M) \times Z^{m-p}(M) \to \mathbb{R}$$

descends to the level of cohomology.

Exercise 2.14. *Suppose* M *is an oriented connected compact manifold of even dimension* $m = 2n$. *Show that for* n *an odd number the intersection form gives a symplectic form on the de Rham cohomology space* $H^p_{\mathrm{dR}}(M)$ *in the middle dimension* $p = n$.

3 Symplectic Manifolds

3.1 Riemannian Manifolds

A Riemannian metric g on a smooth manifold M is a function which assigns to any point x in M a Euclidean form g_x on the tangent space T_xM of M at x varying smoothly with x, in the sense that $x \mapsto g_x(u_x, v_x)$ is a smooth function on M for all $u, v \in \mathcal{X}(M)$. The pair (M, g) of a smooth manifold M with a Riemannian metric g is called a Riemannian manifold. Riemannian metrics exist on each manifold by a partion of unity argument. Suppose (M, g) is a given Riemannian manifold.

The Euclidean form g_x on T_xM gives a linear isomorphism

$$g_x : T_xM \to T_x^*M \ , \ g_x(u_x)v_x = g_x(u_x, v_x) \ , \ u, v \in \mathcal{X}(M)$$

between tangent and cotangent space of M at x. If $f \in \mathcal{F}(M)$ is a smooth function on M then this isomorphism turns the smooth 1-form $\mathrm{d}\, f$ into a smooth vector field $\mathrm{grad}\, f$, which is called the gradient vector field of the function f (relative to the Riemannian metric g). Hence $\mathrm{d}\, f(v) = g(\mathrm{grad}\, f, v)$ for $f \in \mathcal{F}(M)$ and $v \in \mathcal{X}(M)$. The flow $\phi_t : M \to M$ corresponding to the gradient vector field $\mathrm{grad}\, f$ is called the gradient flow of f. Note that

$$\mathcal{L}_{\mathrm{grad}\, f} f = \mathrm{d}\, f(\mathrm{grad}\, f) = g(\mathrm{grad}\, f, \mathrm{grad}\, f) \geq 0$$

which implies that f is nondecreasing along the integral curves of its gradient flow.

If $\gamma : [a, b] \to M$ is a smooth curve in M from $x = \gamma(a)$ to $y = \gamma(b)$ then the distance $\mathrm{L}(\gamma)$ from x to y along γ is defined by

$$\mathrm{L}(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt$$

with $\dot{\gamma}(t) = \mathrm{d}\, \gamma / \mathrm{d}\, t$ the velocity vector of γ at time t. The length $\mathrm{L}(\gamma)$ of γ is invariant under reparametrizations of the curve. The curve γ is called a geodesic if it locally minimizes the distance and is traversed with constant speed. However in the next sections we shall give another definition of geodesics, which is more natural from the symplectic point of view.

3.2 Symplectic Manifolds

A symplectic form ω on a smooth manifold M is a smooth 2-form ω on M that is closed and nondegenerate, so $\omega \in \Omega^2(M)$ with $\mathrm{d}\, \omega = 0$ and ω_x is nondegenerate on T_xM for all $x \in M$. The pair (M, ω) of a smooth manifold M with a symplectic form ω is called a symplectic manifold. A necessary condition for the existence of a symplectic form ω on M is that M should have even dimension $2n$. Moreover $\omega^n/n!$ is a volume form (the so called Liouville form) giving M an orientation. If in addition M is connected and compact then the even dimensional de Rham cohomology spaces $H^{2p}_{\mathrm{dR}}(M)$ should be all nonzero for $0 \le p \le n$. Indeed $[\omega]^n = [\omega^n] \ne 0$ which in turn implies that $[\omega^p] = [\omega]^p \ne 0$ for $0 \le p \le n$. Suppose (M, ω) is a given symplectic manifold.

The flow ϕ_t of a smooth vector field $v \in \mathcal{X}(M)$ leaves the symplectic form ω invariant if and only if

$$0 = \mathcal{L}_v\omega = \mathrm{i}_v(\mathrm{d}\,\omega) + \mathrm{d}(\mathrm{i}_v\,\omega) = \mathrm{d}(\mathrm{i}_v\,\omega)\ ,$$

and so if and only if $\mathrm{i}_v(\omega)$ is closed. Here we have used the Cartan formula in the second identity and the fact that ω is closed in the third identity. By the Poincaré lemma the condition that $\mathrm{i}_v\,\omega$ is closed is locally equivalent to the condition that $\mathrm{i}_v\,\omega$ is equal to the exterior derivative of a smooth function, so

$$\mathrm{i}_v\,\omega = -\,\mathrm{d}\,f$$

for a locally defined smooth function, where the minus sign is a matter of convention. If $H^1_{\mathrm{dR}}(M) = 0$ then there exists a globally defined function $f \in \mathcal{F}(M)$ such that the above relation holds. Moreover, if M is connected then f is uniquely determined up to an additive constant.

Conversely, given $f \in \mathcal{F}(M)$ the fact that for each $x \in M$ the linear map

$$\omega_x : T_xM \to T^*_xM\ ,\ \omega_x(v_x) = (\mathrm{i}_v\,\omega)_x$$

is a bijection implies the existence of a unique vector field $v \in \mathcal{X}(M)$ such that $\mathrm{i}_v\,\omega = -\,\mathrm{d}\,f$. This smooth vector field v on M is called the Hamiltonian vector field v_f on M defined by the function f, and the function f is called the Hamiltonian function of the vector field v_f. The flow of v_f is called the Hamiltonian flow defined by the function f. It is quite remarkable that there are so many smooth vector fields whose flows leave ω invariant. Indeed, for every smooth function f on M the Hamiltonian flow of v_f preserves ω.

Note that

$$\mathcal{L}_{v_f} f = \mathrm{d}\,f(v_f) = -\omega(v_f, v_f) = 0$$

which implies that the Hamiltonian flow of v_f preserves the hypersurfaces on which f is constant. We say that f is a constant of motion for its Hamiltonian flow.

Example 3.1. *Let us take* $M = \mathbb{R}^{2n}$ *with canonical coordinates* (q, p) *equal to* $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ *and standard (translation invariant) symplectic form* $\omega = \sum \mathrm{d}\, p_j \wedge \mathrm{d}\, q_j$*. Hence* $\mathrm{i}_v\, \omega = \mathrm{d}\, q_j$ *if* $v = \partial/\partial p_j$ *and* $\mathrm{i}_v\, \omega = -\, \mathrm{d}\, p_j$ *if* $v = \partial/\partial q_j$*, and so* $v_{q_j} = -\partial/\partial p_j$ *and* $v_{p_j} = \partial/\partial q_j$*. If* f *is a smooth function on* $\mathbb{R}^{2n}$ *then*

$$\mathrm{d}\, f = \sum_j \left(\frac{\partial f}{\partial q_j} \mathrm{d}\, q_j + \frac{\partial f}{\partial p_j} \mathrm{d}\, p_j \right) ,$$

and therefore the Hamiltonian vector field of f *becomes*

$$v_f = \sum_j \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right) .$$

In other words, the integral curves of the Hamiltonian flow of the function f *are solutions of the system of first order differential equations*

$$\dot{q}_j = \frac{\partial f}{\partial p_j} \ , \ \dot{p}_j = -\frac{\partial f}{\partial q_j} \ .$$

This is Hamilton's equation for the Hamiltonian function f *on* M*, which is the reason for the sign convention in the equation*

$$\mathrm{i}_v\, \omega = -\, \mathrm{d}\, f$$

for the relation between the Hamiltonian vector field v *and the Hamiltonian function* f*.*

In the particular case that the Hamiltonian function $f = H$ *is of the classical form*

$$H(q, p) = K(p) + V(q)$$

with kinetic term $K(p) = \sum p_j^2/2m$ *and potential function* $V(q)$ *Hamilton's equation* $m\dot{q} = p, \dot{p} = -\,\mathrm{grad}\, V$ *by elimination of* p *boils down to*

$$F(q) = m\ddot{q} \ ,$$

which is the famous Newton equation for the motion of a point particle in $\mathbb{R}^n$ *with mass* $m > 0$ *in a conservative force field* $F(q) = -\,\mathrm{grad}\, V(q)$*, as formulated by him in the Principia Mathematica of 1687 as the second law [41].*

3.3 Fiber Bundles

Suppose E, B and F are smooth manifolds, and $\pi : E \to B$ is a smooth surjective map.

Definition 3.2. *The quadruple (E, B, F, π) is called a smooth fiber bundle (or smooth fibration) if around each point of B there exists an open neighborhood U and a diffeomorphism*

$$\phi_U : U \times F \to \pi^{-1}(U) \ , \ \pi(\phi_U(x, y)) = x$$

for all $x \in U$ and $y \in F$.

We call E the total space, B the base space, F the fiber space and π the projection map. The diffeomorphism ϕ_U is called a local trivialization over U. The closed submanifold $E_x = \pi^{-1}(x)$ of E is diffeomorphic to F, and called the fiber over $x \in B$. A smooth map $\sigma : B \to E$ with $\pi(\sigma(x)) = x$ is called a smooth section.

Suppose (E, B, F, π) is a smooth fibration. Suppose U and V are both neighborhoods in B with local trivializations ϕ_U and ϕ_V. For $x \in U \cap V$ fixed the equation $\phi_U(x, y) = \phi_V(x, z)$ with $y, z \in F$ has a unique solution

$$(x, z) = \phi_V^{-1}(\phi_U(x, y)) \Longleftrightarrow z = \phi_{VU}(x)(y)$$

with $\phi_{VU}(x)$ a diffeomorphism of F. Note that $\phi_{UU}(x) = \mathrm{Id}$ for all $x \in U$. These diffeomorphisms of F are called the transition maps from ϕ_U to ϕ_V. It is easy to check that

$$\phi_{WV}(x) \circ \phi_{VU}(x) = \phi_{WU}(x)$$

for all $x \in U \cap V \cap W$. This relation is called the cocycle condition.

Conversely, given smooth manifolds B and F with an open cover $\{U\}$ of B, and for each pair U, V from this cover smooth maps $\phi_{VU} : U \cap V \to \mathrm{Diff}(F)$ satisfying the cocycle condition, then we can form a smooth fiber bundle (E, B, F, π) by glueing together the sets $U \times F$ and $V \times F$ by means of ϕ_{VU}. So put

$$E = \bigcup_U (U \times F)/_\sim$$

with $U \times F \ni (x, y) \sim (x, z) \in V \times F$ if $z = \phi_{VU}(x)(y)$. The relation $\sim$ is an equivalence relation due to the cocycle condition.

A Lie group G is both a group and a manifold, and the two structures are compatible in the sense that multiplication $G \times G \to G$ and inversion $G \to G$

are smooth maps. The prototype example of a Lie group is the general linear group $\mathrm{GL}(V)$ of all invertible linear transformations of a finite dimensional vector space V. Indeed, after a choice of basis $(e_1, \cdots, e_m)$ in V we can identify $\mathrm{GL}(V)$ with the group $\mathrm{GL}(m, \mathbb{R})$ of nonsingular real matrices of size $m \times m$, and $\mathrm{GL}(m, \mathbb{R})$ is a Lie group as open subset of $\mathrm{Mat}(m, \mathbb{R}) \cong \mathbb{R}^{m^2}$. Hermann Weyl called the general linear group "Her All Embracing Majesty" in part because of the following result of Élie Cartan [56].

Theorem 3.3. *Any closed subgroup G of $GL(V)$ is itself a Lie group.*

Closed subgroups of $\mathrm{GL}(V)$ are called linear Lie groups, and for most practical purposes linear Lie groups suffice.

If a Lie group G acts smoothly on the fiber F of a fiber bundle (E, B, F, π) such that all transition maps $\phi_{VU}(x) : F \to F$ are obtained from the action of G on F then the Lie group G (together with its action on F) is called a structure group for the fiber bundle. For example, the trivial fiber bundle $E = B \times F$ with π projection on the first factor has the trivial group as a structure group.

Definition 3.4. *A fiber bundle (E, B, F, π) with fiber F a finite dimensional real vector space and structure group the general linear group $\mathrm{GL}(F)$ is called a real vector bundle.*

All fibers of a vector bundle (E, B, F, π) inherit in a natural way the structure of a finite dimensional real vector space. In turn the space $\Gamma(B, E)$ of smooth sections in (E, B, F, π) becomes a real vector space. Constructions of linear algebra give a natural way of making new vector bundles from old ones (all vector bundles having the same base space B). For example if (E_1, B, F_1, π_1) and (E_2, B, F_2, π_2) are two vector bundles then we can form the direct sum bundle $(E_1 \oplus E_2, B, F_1 \oplus F_2, B, \pi_1 \oplus \pi_2)$ and the tensor product bundle $(E_1 \otimes E_2, B, F_1 \otimes F_2, \pi_1 \otimes \pi_2)$. A Euclidean metric on a vector bundle (E, B, F, π) is a smooth section $g \in \Gamma(B, S^2 E^*)$ such that g_x is a Euclidean form on E_x for all $x \in B$. Euclidean metrics on vector bundles always exist by a partition of unity argument. Likewise the space $\Omega^p(M)$ is just the space $\Gamma(M, \Lambda^p T^*(M))$ of smooth sections of the vector bundle $\Lambda^p T^*(M)$ over M.

Suppose (E, B, F, π) is a smooth vector bundle, and $\phi : A \to B$ is a smooth map. Then there exists a smooth vector bundle

$$\phi^* E = \{(x, y) \in A \times E; \phi(x) = \pi(y)\}$$

over the base space A. The projection map is the projection on the first factor. The fiber $(\phi^* E)_x$ is equal to $E_{\phi(x)}$, and its smooth sections are given by $\Gamma(A, \phi^*E) = \{\sigma \circ \phi; \sigma \in \Gamma(B, E)\}$. The vector bundle ϕ^*E is called the pullback of the vector bundle E over B under the smooth map $\phi : A \to B$.

Natural examples of vector bundles are the tangent bundle TN and the cotangent bundle T^*N of a smooth manifold N. If $\iota : S \hookrightarrow N$ is a closed submanifold then the tangent bundle TS is a vector subbundle of $\iota^*(TN)$, and the quotient bundle $NS = \iota^*(TN)/TS$ is called the normal bundle of S in N. In case N has a Riemannian metric g, the normal bundle NS can be viewed as vector subbundle of $\iota^*(TN)$, namely, the fiber N_xS is equal to the orthogonal complement with respect to g_x of T_xS in T_xN.

Definition 3.5. *A principal fiber bundle with structure group a Lie group G is a fiber bundle (E, B, G, π) with fiber the Lie group G and transition maps $\phi_{VU} : U \cap V \to G$ with $\phi_{VU}(x)$ for $x \in U \cap V$ acting on the fiber G by left multiplication.*

Note that a principal fiber bundle (E, B, G, π) gives rise to a natural right action

$$E \times G \to E \ , \ (x, a) \mapsto xa$$

of the structure group on the total space obtained by right multiplication in the fibers. This right action is free in the sense that the stabilizer group $G_x = \{a \in G; xa = x\}$ is trivial for all $x \in E$. The base space B is just the orbit space of G in the total space E.

3.4 Cotangent Bundles

Let N be a smooth manifold of dimension n, and let $M = T^*N$ be the cotangent bundle of N of dimension $m = 2n$. A point of M is a cotangent vector $\xi \in T_x^*N$ for some $x \in N$. Let π denote the projection map from M to N, so $\pi(\xi) = x$ for all $\xi \in T_x^*N$. Hence the tangent map $T_\xi\pi$ at ξ is a linear map from $T_\xi M$ to T_xN, and if we subsequently apply $\xi \in T_x^*N = (T_xN)^*$ we obtain a linear form

$$\theta_\xi = \xi \circ T_\xi\pi = (T_\xi\pi)^*\xi$$

on $T_\xi M$. This defines a smooth 1-form θ on the cotangent bundle $M = T^*N$.

Any smooth 1-form α on N is a smooth section $\alpha : N \to M$ of the cotangent bundle $\pi : M \to N$. It follows that

$$(\alpha^*\theta)_x = \theta_{\alpha(x)} \circ T_x\alpha = \alpha(x) \circ T_{\alpha(x)}\pi \circ T_x\alpha = \alpha(x) \circ T_x(\pi \circ \alpha) = \alpha(x) = \alpha_x$$

where in the first identity we use the definition of pullback, in the second identity use the definition of θ, in the third identity use the chain rule, and in the last identity use $\pi \circ \alpha = \mathrm{Id}$. The equation

$$\alpha^*\theta = \alpha \; \forall \alpha \in \Omega^1(N)$$

says that every 1-form on N is equal to the pullback of θ under the 1-form viewed as a mapping from N to $M = T^*N$. For this reason θ is called the tautological 1-form on the cotangent bundle.

The exterior derivative

$$\omega = \mathrm{d}\,\theta$$

of the tautological 1-form on M is a closed (even exact) 2-form on M. In local coordinates $(x_1, \cdots, x_n)$ on N with corresponding dual coordinates $(\xi_1, \cdots, \xi_n)$ the projection map $\pi : M \to N$ sends $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ to $(x_1, \cdots, x_n)$, and therefore the tautological 1-form θ takes the form

$$\theta = \sum_{j=1}^{n} \xi_j \,\mathrm{d}\, x_j \ .$$

In turn we get

$$\omega = \sum_{j=1}^{n} \mathrm{d}\,\xi_j \wedge \mathrm{d}\, x_j \ ,$$

which shows that $\omega = \sum \mathrm{d}\, p_j \wedge \mathrm{d}\, q_j$ is the standard symplectic form of Example 3.1 if we substitute $x_j = q_j$ and $\xi_j = p_j$. The conclusion is that $\omega = \mathrm{d}\,\theta$ is a symplectic form on $M = T^*N$.

Definition 3.6. *The form $\omega = \mathrm{d}\,\theta$ is called the canonical symplectic form on the cotangent bundle $M = T^*N$.*

Suppose N_1 and N_2 are smooth manifolds of dimension n with cotangent bundles $M_1 = T^*N_1$ and $M_2 = T^*N_2$ and with tautological 1-forms θ_1 and θ_2. A diffeomorphism $\phi : N_1 \to N_2$ induces a diffeomorphism

$$\Phi : M_1 \to M_2 \ , \ \Phi(\xi) = ((T_x\phi)^*)^{-1}\xi$$

for all $\xi \in T_x^*N_1$, which is called the lift of ϕ, and in such a way that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ N_1 & \xrightarrow{\phi} & N_2 \end{array}$$

is commutative. The vertical arrows are the two projection maps.

Lemma 3.7. *The pullback of the tautological* 1*-form* θ_2 *on* M_2 *under the lift* Φ *of a diffeomorphism* $\phi : N_1 \to N_2$ *is equal to the tautological* 1*-form* θ_1 *on* M_1*.*

Proof. We have to show that

$$\Phi^* \theta_2 = \theta_1 \ .$$

Suppose $\phi(x_1) = x_2$ and $(T_{x_1}\phi)^* \xi_2 = \xi_1$ for $\xi_1 \in T^*_{x_1} N_1$ and $\xi_2 \in T^*_{x_2} N_2$, which amounts to saying that $\Phi(\xi_1) = \xi_2$. Then we have

$$\begin{aligned}(\Phi^*\theta_2)_{\xi_1} = (T_{\xi_1}\Phi)^*(\theta_2)_{\xi_2} = (T_{\xi_1}\Phi)^*(T_{\xi_2}\pi_2)^*\xi_2 = (T_{\xi_1}(\pi_2 \circ \Phi))^*\xi_2 = \\ (T_{\xi_1}(\phi \circ \pi_1))^*\xi_2 = (T_{\xi_1}\pi_1)^*(T_{x_1}\phi)^*\xi_2 = (T_{\xi_1}\pi_1)^*\xi_1 = (\theta_1)_{\xi_1}\end{aligned}$$

which proves the lemma. ☐

A diffeomorphism between two symplectic manifolds that preserves the two symplectic forms is called a symplectomorphism (or a canonical transformation). Because the exterior derivative behaves naturally under smooth maps we obtain the following corollary.

Corollary 3.8. *The lift* $\Phi : M_1 \to M_2$ *of a diffeomorphism* $\phi : N_1 \to N_2$ *of manifolds is a symplectomorphism of cotangent bundles, in the sense that*

$$\Phi^* \omega_2 = \omega_1$$

with ω_1 *and* ω_2 *the canonical symplectic forms.*

Definition 3.9. *Let* (M, ω) *be a symplectic manifold. A closed submanifold* $L \hookrightarrow M$ *is called Lagrangian if for each* $x \in L$ *the tangent space* $T_x L$ *is a Lagrangian subspace of* $T_x M$*.*

In other words, a submanifold $\iota : L \hookrightarrow M$ is a Lagrangian submanifold if the dimension of L is half the dimension of M and $\iota^*\omega = 0$.

Example 3.10. *Let* $M = T^*N$ *with canonical symplectic form* $\omega = \mathrm{d}\,\theta$*. If we consider a smooth* 1*-form* α *on* N *as a smooth section* $\alpha : N \to M$ *of the cotangent bundle* $\pi : M \to N$*, then the submanifold* $\alpha : N \hookrightarrow M$ *is Lagrangian if and only if* α *is closed.*

3.5 Geodesic Flow

Suppose (N, g) is a Riemannian manifold, so the Riemannian structure is given by a Euclidean form

$$g_x : T_xN \times T_xN \to \mathbb{R}$$

on the tangent space T_xN varying smoothly with the base point $x \in N$. In turn g_x determines and is determined by the linear isomorphism

$$g_x : T_xN \to T_x^*N \ , \ g_x(u)v = g_x(u, v)$$

for all $u, v \in T_xN$. By abuse of notation we write $g : TN \to T^*N$ for the natural vector bundle isomorphism between the tangent and cotangent bundle. Let $\pi : M = T^*N \to N$ be the cotangent bundle equipped with its canonical symplectic form $\omega = \mathrm{d}\,\theta$. The pullback $g^*\pi : TN \to N$ is the projection map for the tangent bundle. A commutative diagram

$$\begin{array}{ccc} TN & \xrightarrow{g} & M = T^*N \\ {\scriptstyle g^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\mathrm{Id}} & N \end{array}$$

might be helpful to visualize the geometry.

Let f be the smooth function on the cotangent bundle M defined by

$$f(\xi) = g(v, v)/2$$

with $\xi \in \Omega^1(N)$ and $v \in \mathcal{X}(N)$ related by $\xi = g(v)$. Let $\phi_t : M \to M$ be the flow of the Hamiltonian vector field v_f of the function f on M. For simplicity of notation we shall assume that this flow is complete, which is always the case if the manifold N is compact.

Definition 3.11. *The conjugated flow $\psi_t : TN \to TN$ defined by*

$$\psi_t(v) = g^{-1}(\phi_t(g(v))) \ , \ v \in T_xN \ , \ x \in N$$

is called the geodesic flow on the tangent bundle. Moreover the smooth curve $\gamma_v : \mathbb{R} \to N$ defined by

$$\gamma_v(t) = g^*\pi(\psi_t(v)) = \pi(\phi_t(g(v)))$$

is called the geodesic in N through x with initial velocity v.

From the symplectic perspective geodesics are the trajectories for the motion of a free point particle in a Riemannian landscape. The terminology geodesic through $x = \gamma_v(0)$ with initial velocity v is justified by the following result.

Theorem 3.12. *If $\gamma_v : \mathbb{R} \to N$ is the geodesic through $x \in N$ with initial velocity $v \in T_xN$ then*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\gamma_v(t)\Big\}_{t=0} = v\ .$$

Proof. Let $x \in N$ and $v \in T_xN$ with $\xi = g(v) \in T^*_xN$. It is clear that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\gamma_v(t)\Big\}_{t=0} = T_\xi\pi((v_f)_\xi)$$

with the Hamiltonian vector field v_f given by the usual equation

$$(\mathrm{d}\,f)_\xi(r) = \omega_\xi(r, (v_f)_\xi)$$

for all $r \in T_\xi M$. The tangent space $T_\xi M$ admits a short exact sequence

$$0 \to T^*_xN \to T_\xi M \to T_xN \to 0$$

induced by the inclusion map $\iota_x : T^*_xN \hookrightarrow M = T^*N$ and the projection map $\pi : M \to N$. In fact we already tacitly identified $\xi \cong \iota_x(\xi)$ in order to keep the notation transparent.

Choosing a linear section $T_xN \to T_\xi M$ yields a linear isomorphism

$$T_\xi M \cong T_xN \times T^*_xN \ni (u + \eta)$$

with $T_\xi\pi(u + \eta) = u \in T_xN$ the horizontal and $(T_\xi\iota_x)\eta \cong \eta \in T^*_xN$ the vertical component of the tangent vector $(u+\eta) \in T_\xi M$. In these coordinates the symplectic form on $T_\xi M$ is the standard symplectic form

$$\omega_\xi(u + \eta, w + \zeta) = \eta(w) - \zeta(u)$$

with $u, w \in T_xN$ and $\eta, \zeta \in T^*_xN$ as in Example 1.1.

We have to show that the horizontal component of $(v_f)_\xi$ is equal to v. Taking $r = \eta$ in the defining equation for v_f yields

$$(\iota^*_x(\mathrm{d}\,f))_\xi(\eta) = (df)_\xi(T_\xi i_x(\eta) \cong \eta) = \omega_\xi(\eta, (v_f)_\xi) = \eta(T_\xi\pi((v_f)_\xi))$$

for all $\eta \in T^*_xN$. Because $\iota^*_xf(\xi) = g(g^{-1}(\xi), g^{-1}(\xi))/2$ we get

$$\begin{aligned}\mathrm{d}(\iota^*_xf)_\xi(\eta) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{g(g^{-1}(\xi + t\eta), g^{-1}(\xi + t\eta))/2\Big\}_{t=0} \\ &= g(g^{-1}(\eta), g^{-1}(\xi)) = \eta(g^{-1}(\xi)) = \eta(v)\end{aligned}$$

for all $\eta \in T^*_xN$. Since $(\iota^*_x(\mathrm{d}\,f))_\xi(\eta) = \mathrm{d}(\iota^*_xf)_\xi(\eta)$ for all $\eta \in T^*_xN$ we conclude that $T_\xi\pi((v_f)_\xi) = v$. $\square$

This proof might become easier if we write things out in local canonical coordinates $(q,p) = (q_1, \cdots, q_n, p_1, \cdots, p_n)$ on T^*U with $U \subset N$ small and open and $\omega = \sum_j dp_j \wedge dq_j$. If the Riemannian metric is given by

$$ds^2 = \sum_{i,j} g_{ij}(q) dq_i dq_j$$

with g_{ij} a positive definite symmetric matrix depending smoothly on $q \in U$, then our function f as given above takes the form $f(q,p) = \sum_{ij} g_{ij}(q) p_i p_j / 2$. Hence Hamilton's equations become

$$\dot{q}_k = \sum_j g_{kj} p_j \ , \ \dot{p}_k = -\sum_{i,j} \frac{\partial g_{ij}}{\partial q_k} p_i p_j / 2$$

for $k = 1, \cdots, n$. Inversion of the first equation gives $p_k = \sum_j g^{kj} \dot{q}_j$ and substitution in the second equation gives the geodesic equation. This is a second order nonlinear ordinary differential equation in q which has a unique local solution for prescribed initial values $q_k(0), \dot{q}_k(0)$. It is clear that the initial velocity $\dot{q}(0)$ is given by $g(0)(p)$, which amounts to $\dot{\gamma}_v(0) = v$.

Since the Hamiltonian flow of f preserves the level hypersurfaces of f it is clear that geodesics are traversed with constant speed. The property that geodesics locally minimize distance can be derived by variational calculus, and we refer to Chapter 3 of Arnold's book on classical mechanics for a fine exposition [2]. The statement of the theorem remains valid if the kinetic term $f(\xi) = g(v,v)/2$ is replaced by the function $f(\xi) = g(v,v)/2 + V(\pi(\xi))$ with $V \in \mathcal{F}(N)$ a potential function on the configuration space N.

Definition 3.13. *For $x \in N$ fixed the smooth map*

$$\exp_x : T_x N \to N \ , \ \exp_x(v) = \gamma_v(1)$$

is called the exponential map.

For $f(\xi) = g(v,v)/2$ and $t > 0$ the homothety $v \mapsto tv$ implies $f \mapsto t^2 f$ and $\omega \mapsto t\omega$. Hence $v_f \mapsto t v_f$ which in turn implies that $\gamma_{tv}(1) = \gamma_v(t)$. Therefore the above theorem gives the following result.

Corollary 3.14. *The tangent map $T_0(\exp_x) : T_x N \to T_x N$ of $\exp_x$ at the origin of $T_x N$ is equal to* Id.

In turn the inverse function theorem implies that for each $x \in N$ there exists an $\epsilon > 0$ such that

$$\exp_x : \mathbb{B}_\epsilon(x) = \{v \in T_x N; g_x(v,v) < \epsilon^2\} \to N$$

is a diffeomorphism onto its image. These coordinates around x are called geodesic normal coordinates.

Suppose $\iota : S \hookrightarrow N$ is a connected compact submanifold. The normal bundle NS of S in N is viewed as a vector subbundle of $\iota^* TN$. Define the relative exponential map

$$\exp_S : NS \to N$$

by $\exp_S(v) = \exp_x(v)$ if $v \in N_x S$. Under the natural map $S \hookrightarrow NS$ as zero section the tangent map at $x \in S$ of the relative exponential map

$$T_x(\exp_S) : T_x NS \cong T_x S \times N_x S \cong T_x N \to T_x N$$

is equal to Id. Likewise using the inverse function theorem we obtain the tubular neighborhood theorem.

Theorem 3.15. *Let $\iota : S \hookrightarrow N$ be a connected compact submanifold with relative exponential map $\exp_S : NS \to N$. Then there exists an $\epsilon > 0$ such that*

$$\exp_S : \mathbb{B}_\epsilon(S) = \{v \in N_x S; x \in S, g_x(v,v) < \epsilon^2\} \to N$$

is a diffeomorphism onto its image. This image is called a normal geodesic tubular neighborhood of S in N.

The restriction of the projection map $NS \to S$ turns $\mathbb{B}_\epsilon(S) \to S$ into a fiber bundle with fiber an ϵ-ball of dimension equal to the codimension of S in N. In case S is disconnected the same theorem holds by working on each component of S seperately.

3.6 Kähler Manifolds

An almost complex structure on a manifold M is a complex structure

$$J_x : T_x M \to T_x M \ , \ J_x^2 = -\,\mathrm{Id}$$

on each tangent space $T_x M$, depending smoothly on $x \in M$. The pair (M, J) is called an almost complex manifold. If around each point $x \in M$ there are complex coordinates from $\mathbb{C}^n$ such that the transition maps from one chart to another are biholomorphic, then M is called a complex manifold. In turn multiplication by i in the tangent spaces gives rise to a natural almost complex structure on any complex manifold M. There are local conditions for an almost complex structure on M to come from a complex manifold structure on M (the vanishing of the Nijenhuis tensor of J), which were obtained by Newlander and Nirenberg [40]. Examples of almost complex compact manifolds (of dimension 4), which are excluded on topological grounds (the numerology of Chern numbers) to be complex manifolds, were found by Van de Ven [55]. The concept of almost complex structure was introduced by Ehresmann and Hopf around 1945.

The structures of a Riemannian manifold (M, g), a symplectic manifold (M, ω) and an almost complex manifold (M, J) are called a compatible triple if for each $x \in M$ the three structures (g_x, ω_x, J_x) are compatible on T_xM. Two out of three compatible structures determine the third since

$$g_x(u_x, v_x) = \omega_x(J_x u_x, v_x) \ , \ \omega_x(u_x, v_x) = g_x(u_x, J_x v_x)$$

for all $u, v \in \mathcal{X}(M)$ and all $x \in M$. Two out of the three structures (g, ω, J) on M are called compatible if there exist a (unique) third such that all three are compatible.

Lemma 3.16. *Each symplectic manifold (M, ω) has a compatible almost complex structure J.*

Proof. Suppose (M, ω) is a symplectic manifold. By a partition of unity we can choose a Riemannian metric G on M. We just carry out the construction of Lemma 1.4 in the tangent space T_xM at each point $x \in M$. Indeed, let $A \in \Gamma(M, \mathrm{End}(TM))$ be a smooth section with

$$\omega(u, v) = G(u, Av)$$

for all $u, v \in \mathcal{X}(M)$. Writing

$$A = \sqrt{A^*A}J = J\sqrt{A^*A} \ , \ g(u, v) = \omega(Ju, v) = G(Ju, Av)$$

gives the desired compatible triple (g, ω, J). □

Definition 3.17. *Suppose M is a complex manifold with associated (almost) complex structure J. A Kähler structure on M is a Hermitian metric h on the tangent bundle of M whose imaginary part ω is a closed 2-form. For $h = g + \mathrm{i}\omega$ the Kähler condition means that the triple (g, ω, J) of Riemannian metric g, symplectic form ω and complex structure J is compatible. The symplectic form ω is called the Kähler form, and (M, ω) is called a Kähler manifold.*

Examples of compact symplectic manifolds (of dimension 4), which are excluded on topological grounds to be Kähler manifolds, were found (using the structure of the fundamental group) by Thurston [54] and (in the simply connected case) byMcDuff [35]. An example of a simply connected both complex and symplectic compact manifold (in dimension 6), which is not Kähler, was recently found in [3].

Example 3.18. *A lattice L in an Hermitian vector space (V, h) is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis of V. The imaginary part ω of h is a translation invariant symplectic structure on V, and the Kähler manifold $(V/L, h)$ is called a complex torus.*

Example 3.19. *Let $(z_0, \cdots, z_n)$ be coordinates on $\mathbb{C}^{n+1}$ and let*

$$\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n(\mathbb{C})$$

be the natural projection map with corresponding homogeneous coordinates $[z] = (z_0 : \cdots : z_n)$ on $\mathbb{P}^n(\mathbb{C})$. Let $h = g + \mathrm{i}\omega$ be the standard Hermitian form

$$h(z, z') = \sum z_j \overline{z'_j}$$

on $\mathbb{C}^{n+1}$ with g the standard Euclidean form and ω the standard symplectic form on $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$. If $\iota : \mathbb{S}^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ is the inclusion of the unit sphere in $\mathbb{C}^{n+1}$ then the diagram

$$\mathbb{C}^{n+1} \overset{\iota}{\hookleftarrow} \mathbb{S}^{2n+1} \overset{\pi}{\twoheadrightarrow} \mathbb{P}^n(\mathbb{C})$$

induces a canonical symplectic form ω_{FS} on $\mathbb{P}^n(\mathbb{C})$ determined by the relation

$$\pi^*\omega_{\mathrm{FS}} = -\iota^*\omega/(\pi = 3.14\cdots)$$

which is called the Fubini–Study symplectic form on $\mathbb{P}^n(\mathbb{C})$.

Indeed, the closed 2-form $\iota^*\omega$ has a one dimensional kernel at each point of $\mathbb{S}^{2n+1}$, and the orbits of the circle group $\mathrm{U}_1(\mathbb{C})$ on $\mathbb{S}^{2n+1}$ are exactly the leaves of the null foliation. In turn this defines a unique 2-form ω_{FS} on the complex projective space $\mathbb{P}^n(\mathbb{C})$ as the quotient space of $\mathbb{S}^{2n+1}$ by the action of the group $\mathrm{U}_1(\mathbb{C})$. Since the projection map π is a submersion the pullback π^* is injective on differential forms. In particular ω_{FS} is closed since

$$\pi^*(\mathrm{d}\,\omega_{\mathrm{FS}}) = \mathrm{d}(\pi^*\omega_{\mathrm{FS}}) = -\,\mathrm{d}(\iota^*\omega)/\pi = -\iota^*(\mathrm{d}\,\omega)/\pi = 0\ .$$

The above construction is called the symplectic reduction method and will be discussed in greater generality in a later chapter on the moment map. Together with the natural complex structure $(\mathbb{P}^n(\mathbb{C}), \omega_{\mathrm{FS}})$ becomes a compact Kähler manifold.

Since holomorphic submanifolds of Kähler manifolds are again Kähler manifolds (for the restriction of the Kähler form on the ambient space to the submanifold) we arrive at the following conclusion.

Corollary 3.20. *Smooth projective manifolds (that is compact holomorphic submanifolds of a complex projective space) are Kähler manifolds.*

Hence complex algebraic geometry provides us with an overwhelming abundance of compact Kähler manifolds. By a theorem of Chow compact holomorphic submanifolds of projective space are in fact defined by homogeneous polynomial equations. However Kähler manifolds are more general than complex projective manifolds. Examples of compact Kähler manifolds, which can not be embedded in projective space $\mathbb{P}^n(\mathbb{C})$, are complex tori $(V/L, \omega = \Im h)$ of complex dimension at least 2 and L a generic lattice. The complex tori that can be embedded in projective space are called Abelian varieties, whose study is a central subject in algebraic geometry.

We have the following diagram of implications of structures

$$\begin{array}{ccc} \text{Kaehler} & \longrightarrow & \text{symplectic} \\ \Big\downarrow & & \Big\downarrow \\ \text{complex} & \longrightarrow & \text{almost complex} \end{array}$$

and refer to Section 17.3 in the book of Cannas da Silva for a discussion of related examples [4].

A long standing question is whether the unit sphere $\mathbb{S}^6$ in $\mathbb{R}^7$ has a complex structure. It is known that $\mathbb{S}^6$ has an almost complex structure coming from a transitive action of the exceptional simple Lie group of dimension 14 (of exceptional Cartan type G_2) on $\mathbb{S}^6$ with point stabilizer isomorphic to $\mathrm{SU}_3(\mathbb{C}) < \mathrm{SO}_6(\mathbb{R})$. It is known that this almost complex structure is not integrable, and so does not come from an honest complex structure on $\mathbb{S}^6$. The sphere $\mathbb{S}^6$ can not be symplectic because $H^2_{\mathrm{dR}}(\mathbb{S}^6) = 0$. Therefore $\mathbb{S}^6$ can not be Kähler, and in particular can not be a projective manifold. But there is nothing known to prevent $\mathbb{S}^6$ from having some odd complex structure!

3.7 Darboux Theorem

In this section we will adapt the Moser theorem on equivalence of volume forms to the case of symplectic forms. But first we shall explain a moduli count in the spirit of Riemann [46], which "explains" nicely that the Darboux theorem is a reasonable result to expect.

A Riemannian manifold (M, g) is given in local coordinates $(x_1, \cdots, x_m)$ by a Riemannian metric

$$ds^2 = \sum_{i,j} g_{ij}(x)\, \mathrm{d}\, x_i\, \mathrm{d}\, x_j$$

with $g_{ij}(x) = g_{ji}(x)$ freely chosen (under the restriction that the matrix $g_{ij}(x)$ is positive definite for all x) functions of the coordinates. Hence initially there are $m(m+1)/2$ functions to be picked, but after removing the ambiguity of the m coordinates there remains a choice of $m(m-1)/2$ free functions, which are captured by the Riemann curvature tensor. It is not at all true that any two Riemannian manifolds locally look alike, at least if the dimension m is greater than 1.

Similarly, a symplectic manifold (M, ω) is given in local coordinates $(x_1, \cdots, x_m)$ by a (nondegenerate) closed 2-form ω, which by the Poincaré lemma takes the form $\omega = \mathrm{d}\,\theta$, with

$$\theta = \sum_i \theta_i(x)\,\mathrm{d}\,x_i$$

some 1-form, and $\theta_i(x)$ are m freely chosen functions of the coordinates (under the restriction that $\omega = \mathrm{d}\,\theta$ is nondegenerate). After removing the ambiguity of the m coordinates one could hope that no choice remains left. Then any two symplectic manifolds of the same dimension m would locally look alike. That is the statement of the Darboux theorem.

Theorem 3.21. *Let M be a compact manifold, and let ω_0 and ω_1 be two symplectic forms on M. Suppose that both symplectic forms have the same periods, or equivalently by the de Rham theorem that both cohomology classes $[\omega_0]$ and $[\omega_1]$ in $H^2_{\mathrm{dR}}(M)$ are equal. Suppose also that $\omega_t = (1-t)\omega_0 + t\omega_1$ is nondegenerate for all $0 \leq t \leq 1$. Then there exists a diffeomorphism $\phi : M \to M$ with $\omega_0 = \phi^*\omega_1$.*

Proof. The proof is rather similar to the proof of the Moser volume theorem. Indeed $t \mapsto \omega_t = (1-t)\omega_0 + t\omega_1$ for $0 \leq t \leq 1$ gives a line segment of closed forms, and by assumption ω_t is nondegenerate and $[\omega_t]$ is constant in $H^2_{\mathrm{dR}}(M)$. Hence the time derivative of ω_t is exact, and so

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\omega_t = \mathrm{d}\,\lambda$$

for some $\lambda \in \Omega^1(M)$.

The Moser trick searches an isotopy ϕ_t of M with

$$\phi_t^*\omega_t = \omega_0$$

for all $0 \leq t \leq 1$, and so $\phi = \phi_1$ will work. Differentiation of the left hand side with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{\phi_t^*\omega_t\Big\} = \phi_t^*\Big\{\mathcal{L}_{v_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}\,t}\omega_t\Big\} = \phi_t^*\Big\{\mathrm{d}(\mathrm{i}_{v_t}\,\omega_t) + \mathrm{d}\,\lambda\Big\} = \phi_t^*\,\mathrm{d}\Big\{\mathrm{i}_{v_t}\,\omega_t + \lambda\Big\}$$

with v_t the time dependent vector field whose solution curves correspond to the isotopy ϕ_t of M. This expression will certainly vanish if the Moser equation

$$\mathrm{i}_{v_t}\,\omega_t + \lambda = 0$$

holds. But the Moser equation determines the vector field v_t uniquely as shown in Exercise 1.6. □

Corollary 3.22. *Suppose (M, J) is an almost complex compact manifold, and ω_0 and ω_1 are both compatible symplectic forms on M having the same periods on all of $H_2(M)$. Then there exists a diffeomorphism $\phi : M \to M$ with $\omega_0 = \phi^*\omega_1$.*

Proof. The interpolation $\omega_t = (1-t)\omega_0 + t\omega_1$ of closed 2-forms has constant periods on $H_2(M)$, which are compatible with J via the interpolation $h_t = (1-t)h_0 + th_1$ of Hermitian forms. Hence ω_t is symplectic for all $0 \le t \le 1$, and we can apply the previous theorem. □

Corollary 3.23. *Suppose M is a compact complex manifold. If ω_0 and ω_1 are two Kähler forms on M having the same periods on all of $H_2(M)$ then there existst a diffeomorphism $\phi : M \to M$ with $\omega_0 = \phi^*\omega_1$.*

By a minor adaption we also get a clean proof of the Darboux theorem.

Theorem 3.24. *If (M, ω) is a symplectic manifold then each point x in M has a coordinate neighborhood $\iota : U \hookrightarrow M$ with coordinates*

$$(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$$

such that $\iota^*\omega = \sum \mathrm{d}\,\xi_j \wedge \mathrm{d}\,x_j$ is the standard symplectic form on U.

Proof. By the linear Darboux lemma (Lemma 1.2) we can choose local coordinates $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ with $x \leftrightarrow (0, 0)$ such that ω in these coordinates becomes ω_0 and coincides with the translation invariant symplectic form $\omega_1 = \sum \mathrm{d}\,\xi_j \wedge \mathrm{d}\,x_j$ at the origin $(0, 0)$. Say we work on a small ball $\mathbb{B}_\epsilon$ of radius $\epsilon > 0$ around $(0, 0)$. By the Poincaré lemma

$$\omega_1 - \omega_0 = \mathrm{d}\,\lambda$$

for some $\lambda \in \Omega^1(\mathbb{B}_\epsilon)$, and we may assume that λ vanishes at the origin.

Writing $\omega_t = (1-t)\omega_0 + t\omega_1$ we can solve the Moser equation

$$\mathrm{i}_{v_t}\,\omega_t + \lambda = 0$$

for a unique vector field v_t which vanishes at the origin 0 of $\mathbb{B}_\epsilon$ for all $0 \le t \le 1$. The corresponding isotopy ϕ_t is well defined on $\mathbb{B}_\delta$ for some $0 < \delta \le \epsilon$ and leaves the origin fixed. Moreover $\phi_t^*\omega_t = \omega_0$ and in particular $\phi_1^*\omega_1 = \omega_0$. This proves the Darboux theorem by the Moser trick. □

The Darboux theorem was obtained by Gaston Darboux in 1882 [8]. The fact that all symplectic manifolds of the same dimension locally look alike is at first surprising. In this sense symplectic geometry is rather different from Riemannian geometry, where the Riemann curvature tensor locally distinguishes Riemannian manifolds.

The classical proof by Darboux of his theorem goes by induction on the dimension [53]. The proof of the Darboux theorem as given above is due to Moser [37]. This method of proof has been further refined by Weinstein obtaining a standard form of a symplectic manifold in a neighborhood of a closed submanifold rather than just a point [57]. For example this leads to an equivariant form of the Darboux theorem, usually called the Darboux–Weinstein theorem, in a neighborhood of the fixed point locus for the action of a compact Lie group acting canonically on a symplectic manifold [18].

3.8 Exercises

Exercise 3.1. *Given a Riemannian manifold (M, g) show that $\mathcal{L}_u f = g(u, \operatorname{grad} f)$ for $u \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$. Show that the vector field* grad f *gives the direction in which f increases mostly.*

Exercise 3.2. *For a symplectic manifold (M, ω) show that $\mathcal{L}_u f = \omega(u, v_f)$ for $u \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$. Conclude that for all regular values $r \in \mathbb{R}$ of f (meaning that $(\mathrm{d} f)_x$ is nonzero for all $x \in M$ with $f(x) = r$) the level hypersurface $M_r = \{x \in M; f(x) = r\}$ is a smooth submanifold with tangent space $T_x M_r$ equal to the orthogonal complement with respect to ω_x of $\mathbb{R}(v_f)_x$ in $T_x M$. Here TM_r is identified with its image $T\iota_r(TM_r)$ inside TM with $\iota_r : M_r \hookrightarrow M$ the inclusion map.*

Exercise 3.3. *Let (E, B, G, π) be a principal fiber bundle with structure group G and let $\rho : G \to \mathrm{GL}(V)$ be a smooth representation of G on a vector space V. How would you define the associated vector bundle $E \times_\rho V$ with base space B and fiber V?*

Exercise 3.4. *In a standard course of algebraic topology it is shown that $H_p(\mathbb{S}^m)$ is one dimensional for p equal to 0 or m, and zero otherwise. Using this result show that the unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ has a symplectic structure if and only if m is 0 or 2.*

Exercise 3.5. *Let $M = T^*N$ with canonical symplectic form $\omega = \mathrm{d}\,\theta$. Consider a smooth 1-form α on N as a smooth section $\alpha : N \to M$ of the cotangent bundle $\pi : M \to N$. Show that $\alpha : N \hookrightarrow M$ is Lagrangian if and only if $\mathrm{d}\,\alpha = 0$.*

Exercise 3.6. *Let $\pi : T^*N \to N$ be the cotangent bundle with canonical symplectic form $\omega = \mathrm{d}\,\theta$. Show that for β a closed 2-form on N the 2-form $\omega + \pi^*\beta$ is again a symplectic form on $M = T^*N$. For a smooth 1-form α on N we denote*

$$\omega_\alpha = \omega + \pi^*(\mathrm{d}\,\alpha) = \mathrm{d}(\theta + \pi^*\alpha)$$

and call (M, ω_α) a twisted cotangent bundle (with twist α).

Exercise 3.7. *(using some Lie theory) Let σ denote the Euclidean measure on the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4 \cong \mathbb{C}^2$. Show that*

$$\int_{\mathbb{S}^3} \mathrm{d}\,\sigma(x) = \int_0^\pi 4\pi \sin^2\theta \,\mathrm{d}\,\theta = 2\pi^2$$

and conclude that

$$\int_{\mathbb{P}^1(\mathbb{C})} \mathrm{d}\,\sigma_{\mathrm{FS}}(y) = 2\pi^2/(2\pi \times \pi) = 1$$

with σ_{FS} the Fubini–Study measure on $\mathbb{P}^1(\mathbb{C})$.

Exercise 3.8. *(using Lie theory) Let $h(z, z') = z_1\overline{z_1'} + z_2\overline{z_2'}$ be the standard Hermitian metric on $\mathbb{C}^2$. Write $h = g + \mathrm{i}\omega$ for the decomposition of h in real and imaginary part. Show that $\omega(z, \mathrm{i}z) = -h(z, z)$ and conclude from the previous exercise that the Fubini–Study area of $\mathbb{P}^1(\mathbb{C})$ taken with its natural orientation is equal to 1.*

Exercise 3.9. *(using Lie theory) Show that the distance function d_{FS} for the Fubini–Study metric on $\mathbb{P}^n(\mathbb{C})$ is given by*

$$\cos(d_{\mathrm{FS}}([z], [z'])) = \frac{|h(z, z')|}{\sqrt{h(z, z)}\sqrt{h(z', z')}}$$

for $[z], [z'] \in \mathbb{P}^n(\mathbb{C})$.

Exercise 3.10. *(using integration over compact groups) Suppose we have given a symplectic action $K \times M \to M$ of a compact Lie group K on a symplectic manifold (M, ω). Show that there exists a compatible triple (g, ω, J) on M with g a Riemannian metric and J an almost complex structure that* *are both invariant under this action. Check that our proof of the Darboux theorem around a fixed point for this action with this g provides Darboux coordinates in which the action is linearized. Conclude that the fixed point locus M^K is a symplectic submanifold. Hint: Apply the proof of Lemma 3.16 to a Riemannian metric G on M that is invariant under the action. Such a G can be obtained by averaging an arbitrary Riemannian metric over K.*

Exercise 3.11. *(using some algebraic geometry) Let $x = (x_0, \cdots, x_n)$ be a nonzero point of $\mathbb{C}^{n+1}$, and write $[x] = [x_0 : \cdots : x_n]$ for the corresponding point of $\mathbb{P}^n(\mathbb{C})$. For $f \in \mathbb{C}[x_0, \cdots, x_n]$ a homogeneous polynomial of degree $d \geq 1$ the locus*

$$\{[x] \in \mathbb{P}^n(\mathbb{C}); f(x) = 0\}$$

is called a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$. Check that the hypersurface is smooth if the equations

$$x \wedge \mathrm{grad}(f)(x) = 0 \;,\; f(x) = 0$$

do not have a common nonzero solution in $\mathbb{C}^{n+1}$. Show that the Fermat cubic hypersurface in $\mathbb{P}^n(\mathbb{C})$ with equation $f(x) = x_0^3 + \cdots + x_n^3$ is smooth. Conclude that a generic cubic hypersurface in $\mathbb{P}^n(\mathbb{C})$ is smooth.

Exercise 3.12. *Let $\pi : U \twoheadrightarrow N$ be a tubular neighborhood of a submanifold $i : N \hookrightarrow M$ with smooth deformation retract (obtained by multiplication with the scalar $t \in [0,1]$ in the fibers) $\rho_t : U \to U, \rho_t \circ i = i$ for all $0 \leq t \leq 1$ and $\rho_0 = i \circ \pi, \rho_1 = \mathrm{Id}$. Show that for $\alpha \in \Omega^p(U)$ with $\mathrm{d}\alpha = 0$ and $i^*\alpha = 0$ there exists $\beta \in \Omega^{p-1}(U)$ with $\alpha = \mathrm{d}\beta$ and $i^*\beta = 0$. Hint: Let u_t be the smooth vector field on U, whose value at $y = \rho_t(x)$ is the tangent vector to the curve $s \mapsto \rho_s(x)$ at $s = t$. Hence u_t satisfies $\mathrm{d}/\mathrm{dt}(\rho_t^*\alpha) = \rho_t^*(\mathcal{L}_{u_t}\alpha)$ for any $\alpha \in \Omega^p(U)$. Show that the integral*

$$\beta = \int_0^1 \rho_t^*(i_{u_t}\alpha)\mathrm{d}t$$

is well defined and satisfies $\alpha = \mathrm{d}\beta$ and $i^*\beta = 0$ [4].

Exercise 3.13. *Suppose (M, ω) is symplectic manifold and $L \hookrightarrow M$ is a compact Lagrangian submanifold. Show by a variation of the Moser trick that a small tubular neighborhood of L is symplectomorphic with a small tubular neighborhood of L as the zero section in $(T^*L, d\theta)$. This exercise is due to Alan Weinstein, and the result is called the Weinstein Lagrangian neighborhood theorem [57],[18],[4].*

4 Hamilton Formalism

4.1 Poisson Brackets

Suppose (M, ω) is a symplectic manifold. For $f \in \mathcal{F}(M)$ a smooth function on M there is a unique vector field $v_f \in \mathcal{X}(M)$

$$\mathrm{i}_{v_f}\,\omega = -\,\mathrm{d}\,f$$

and v_f is called the Hamiltonian vector field of the function f. The integral curves of v_f are called the solution curves for the Hamiltonian system defined by f, and the corresponding one parameter group $\phi_t : M \to M$ is called the Hamiltonian flow of f.

In local Darboux coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ the integral curves of v_f are given by

$$\dot{q}_j = \frac{\partial f}{\partial p_j}\ ,\ \dot{p}_j = -\frac{\partial f}{\partial q_j}$$

which is called Hamilton's equation for the Hamiltonian system defined by f in canonical coordinates.

Definition 4.1. *For $f, g \in \mathcal{F}(M)$ the Poisson bracket $\{f, g\}$ is the smooth function on M defined by*

$$\{f, g\} = \mathcal{L}_{v_f}(g) = \mathrm{i}_{v_f}(\mathrm{d}\,g) = -\,\mathrm{i}_{v_f}(\mathrm{i}_{v_g}\,\omega) = \omega(v_f, v_g)$$

by using the Cartan formula in the second identity and the antisymmetry of ω in the fourth identity.

The right hand side of the above definition shows that the Poisson bracket is antisymmetric in the sense that

$$\{g, f\} = -\{f, g\}$$

for all $f, g \in \mathcal{F}(M)$. In canonical local coordinates the Poisson brackets are given by

$$\{f, g\} = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\Big)$$

as discussed in Example 3.1. The function space $\mathcal{F}(M)$ can be thought of as the space of classical observables on the phase space (M, ω). For an observable $g \in \mathcal{F}(M)$ the equation

$$\frac{\mathrm{d}\,g}{\mathrm{d}\,t} = \{f, g\}$$

is the abstract form of Hamilton's equation for the Hamiltonian f. The infinitesimal change of a given observable g under the Hamiltonian flow of f is equal to $\{f,g\}$. If $\{f,g\}=0$ on all of M then the observable g is called a constant of motion for the Hamiltonian system defined by f. Since $\{f,f\}=0$ the observable f is a constant of motion for the Hamiltonian f itself. If f is the total energy of a classical mechanical system, then this is the law of conservation of total energy.

The exterior derivative of $\{f,g\}$ is given by

$$\mathrm{d}\{f,g\}=\mathrm{d}(\mathcal{L}_{v_f}g)=\mathcal{L}_{v_f}(\mathrm{d}\,g)=-\mathcal{L}_{v_f}(\mathrm{i}_{v_g}\,\omega)=-\,\mathrm{i}_{[v_f,v_g]}\,\omega$$

using in the last identity $\mathcal{L}_v(\mathrm{i}_w\,\alpha)=\mathrm{i}_{[v,w]}\,\alpha+\mathrm{i}_w(\mathcal{L}_v\alpha)$ for all $v,w\in\mathcal{X}(M)$ and $\alpha\in\Omega^p(M)$. Moreover $\mathcal{L}_v\omega=0$ if $\mathrm{i}_v\,\omega$ is closed, which is the case if $v=v_f$. In turn we have proven the following result.

Theorem 4.2. *We have* $[v_f,v_g]=v_{\{f,g\}}$ *for all* $f,g\in\mathcal{F}(M)$.

If we let both sides of this identity act on a third function $h\in\mathcal{F}(M)$ then we get

$$\{f,\{g,h\}\}-\{g,\{f,h\}\}=\{\{f,g\},h\}$$

and using the antisymmetry of the Poisson bracket we obtain

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0$$

for all $f,g,h\in\mathcal{F}(M)$. This is the Jacobi identity for Poisson brackets. In other words, the Poisson bracket $\{\cdot,\cdot\}$ defines a Lie algebra structure on the vector space $\mathcal{F}(M)$, just like the commutator bracket $[\cdot,\cdot]$ is a Lie algebra structure on the vector space $\mathcal{X}(M)$. The above theorem states that the map $f\mapsto v_f$ is a Lie algebra homomorphism of the Lie algebra $(\mathcal{F}(M),\{\cdot,\cdot\})$ to the Lie algebra $(\mathcal{X}(M),[\cdot,\cdot])$.

The pointwise multiplication on $\mathcal{F}(M)$ relates to the Poisson bracket by

$$\{f,gh\}=\{f,g\}h+g\{f,h\}$$

for all $f,g,h\in\mathcal{F}(M)$, turning $\mathcal{F}(M)$ into a so called Poisson algebra.

The next historical remark I owe to Duistermaat [10]. The Jacobi identity for the Poisson bracket goes back to Jacobi in a posthumously published article from 1862 [25]. Jacobi mentioned that the Jacobi identity implies a theorem of Poisson from 1809, which states that for f and g two constants of motion for a Hamiltonian system defined by h the Poisson bracket $\{f,g\}$ is a third constant of motion [59].

For someone interested in the history of mathematics no knowledge of the Latin language is required to appreciate the formulas in volume V of the collected works of Jacobi [25]. On page 45 he defines for any two functions f, φ on phase space

> $\cdots$ Designabo sequentibus per $[f, \varphi]$ expressionem sequentum
>
> $$\begin{aligned}[f, \varphi] = &\frac{\partial f}{\partial q_1}\frac{\partial \varphi}{\partial p_1} + \frac{\partial f}{\partial q_2}\frac{\partial \varphi}{\partial p_2} + \cdots + \frac{\partial f}{\partial q_n}\frac{\partial \varphi}{\partial p_n} \\ &- \frac{\partial f}{\partial p_1}\frac{\partial \varphi}{\partial q_1} - \frac{\partial f}{\partial p_2}\frac{\partial \varphi}{\partial q_2} - \cdots - \frac{\partial f}{\partial p_n}\frac{\partial \varphi}{\partial q_n}\end{aligned}$$
>
> unde erit
>
> $$[f, f] = 0, [f, \varphi] = -[\varphi, f] \ \cdots$$

and so $[f, \varphi] = -\{f, \varphi\}$ is just minus the Poisson bracket in our notation. The modern sign convention is chosen such that $[v_f, v_g] = v_{\{f,g\}}$ and so the map $f \mapsto v_f$ is a Lie algebra homomorphism from the Poisson algebra of smooth functions to the Lie algebra of smooth vector fields.

From a direct calculation he concludes on page 46 that for any three functions f, φ, ψ on phase space

> $\cdots$ in hanc abit:
>
> $$[[f, \psi], \varphi] - [[f, \varphi], \psi] = [[\varphi, \psi], f]$$
>
> quae concinnius sic exhibetur:
>
> $$[[f, \varphi], \psi] + [[\varphi, \psi], f] + [[\psi, f]\varphi] = 0 \ \cdots$$

which ever since has been called the "Jacobi identity". On the next page 47 its relevance is explained, since any two constants of motion φ, ψ for a Hamiltonian f produce a third constant of motion $[\varphi, \psi]$ for f.

4.2 Integrable Systems

Let (M, ω) be a symplectic manifold of dimension $2n$, which we might think of as the phase space of some physical system.

Definition 4.3. *A set of n smooth functions $(f_1, \cdots, f_n)$ on M is called an integrable system if*

$$\{f_j, f_k\} = 0$$

for all $1 \leq j,k \leq n$, and if the regular locus M^r, where the differentials

$$\mathrm{d}\, f_1, \cdots, \mathrm{d}\, f_n \in \Omega^1(M)$$

are linearly independent, is a dense open subset of M, and if the flows of the Hamiltonian vector fields v_j of f_j on M are complete for all $j = 1, \cdots, n$.

An integrable system on M gives rise to a smooth map

$$f = (f_1, \cdots, f_n) : M \to \mathbb{R}^n$$

called the action coordinates map. The locus $\mathcal{D} = f(M - M^r)$ of all singular values, also called the discriminant of the integrable system, is a null set by the Sard theorem. Its complement $\mathcal{R} = f(M) - \mathcal{D}$ is the set of regular values of the integrable system. Clearly the image of f

$$f(M) = \mathcal{R} \sqcup \mathcal{D}$$

is a disjoint union of the locus of regular values and the discriminant. The open dense subset $M^{rf} = f^{-1}(\mathcal{R})$ of $M^r \subset M$ is the regular fiber locus.

Because the Hamiltonian vector fields $v_1, \cdots, v_n$ have commuting flows $\phi_{1,t}, \cdots, \phi_{n,t}$ we get an action

$$\mathbb{R}^n \times M \to M \ , \ (t_1, \cdots, t_n)x = \phi_{1,t_1}(\cdots(\phi_{n,t_n}(x))\cdots)$$

for all x in M. The action of the additive group $(\mathbb{R}^n, +)$ on M preserves the level sets $M_c = f^{-1}(c)$ for all $c \in \mathbb{R}^n$.

If $c \in \mathcal{R}$ then the level set M_c is contained in M^r, and M_c is a smooth submanifold of M of dimension n invariant under the action of $\mathbb{R}^n$ on M. Since

$$\omega(v_j, v_k) = \{f_j, f_k\} = 0$$

for all $1 \leq j,k \leq n$ and the vector fields v_i are linearly independent on M_c we conclude that M_c is a Lagrangian submanifold of M.

In addition let us assume that all regular level sets M_c are connected. The action of $\mathbb{R}^n$ on M_c is locally free, in the sense that the stabilizer subgroup $\mathbb{R}^n_x$ of $x \in M_c$ is a discrete subgroup of the additive group $\mathbb{R}^n$. Therefore each orbit of $\mathbb{R}^n$ in M_c is open, but then also closed as the complement of the remaining ones. Since M_c is connected it is just a single orbit of $\mathbb{R}^n$. Any discrete subgroup of the additive group $\mathbb{R}^n$ is of the form $L = \mathbb{Z}B$, the integral span of a linearly independent set B in $\mathbb{R}^n$. Hence $M_c \cong \mathbb{R}^n/L_x$ as homogeneous spaces for $\mathbb{R}^n$ with $x \in M_c$. Because $\mathbb{R}^n$ is Abelian we have $L_x = L_y$ if $f(x) = f(y) = c$ and we write $L_x = L_y = L_c$. The conclusion is that $M_c \cong \mathbb{R}^n/L_c$ as homogeneous spaces for $\mathbb{R}^n$. But this identification $M_c \cong \mathbb{R}^n/L_c$ is only possible after picking an origin $x \in M_c$.

Finally, if in addition $f : M \to \mathbb{R}^n$ is a proper map, then $M_c \cong \mathbb{R}^n/L_c$ is compact for all $c \in \mathcal{R}$, which in turn implies that $L_c = \mathbb{Z}B$ with B a basis of $\mathbb{R}^n$, which by definition means that L_c is a lattice in $thbbR^n$. Hence we have the following result, which is called the Arnold–Liouville theorem.

Theorem 4.4. *If $f : M \to \mathbb{R}^n$ is an integrable system with compact connected fibers then the flows of the Hamiltonian vector fields v_i induce a diffeomorphism $M_c \cong \mathbb{R}^n/L_c$ for all $c \in \mathcal{R}$. The lattice L_c in $\mathbb{R}^n$ is called the period lattice and $M_c \cong \mathbb{R}^n/L_c$ is called the Liouville torus.*

A smooth map $s : U \to M$ defined on a sufficiently small open ball $U \subset \mathcal{R}$ around a fixed base point $b \in \mathcal{R}$ with $f(s(c)) = c$ for all $c \in U$ is called a local Lagrangian section for $f : M \to \mathbb{R}^n$ around b if $s(U)$ is a Lagrangian submanifold of (M, ω). By the Darboux theorem applied around the point $x = s(b)$ local Lagrangian sections always exist around regular values $b \in \mathcal{R}$. Choose a local Lagrangian section $s : U \to M$ around $b \in \mathcal{R}$. Choose a basis $l_i(c)$ of the lattice $L_c/2\pi \subset \mathbb{R}^n$ for $i = 1, \cdots, n$ depending smoothly on $c \in U$.

Define an action of the torus $(\mathbb{R}/2\pi\mathbb{Z})^n$ on $f^{-1}(U)$ by

$$(\varphi, x) = ((\varphi_1, \cdots, \varphi_n), x) \mapsto \varphi * x = (\sum \varphi_i l_i(f(x))s(f(x))$$

and observe that this action commutes with the original Hamiltonian action $(t, x) \mapsto tx$ of $\mathbb{R}^n$ on $f^{-1}(U)$. Define angle coordinates

$$\alpha : f^{-1}(U) \to (\mathbb{R}/2\pi\mathbb{Z})^n, \ \alpha(x) = \varphi \ \Leftrightarrow \ x = \varphi * s(f(x))$$

and observe that the level sets $\alpha^{-1}(\varphi)$ are Lagrangian submanifolds. Indeed, the zero level set $\alpha^{-1}(0) = s(U)$ is Lagrangian by assumption, and the Hamiltonian action $\mathbb{R}^n \times f^{-1}(U) \to f^{-1}(U)$ permutes these level sets and preserves Lagrangian submanifolds. Therefore the angle coordinates α_i satisfy $\{\alpha_i, \alpha_j\} = 0$ for $i, j = 1, \cdots, n$.

In addition there exist action coordinates $a : f^{-1}(U) \to \mathbb{R}^n$ of the form $a(x) = I(f(x))$ with $c \mapsto I(c)$ a suitable coordinate transformation around $b \in U \subset \mathbb{R}^n$, such that the Hamilton flow of the action coordinate a_i on M_c is equal to the angle action of $\varphi_i \in \mathbb{R}/2\pi\mathbb{Z}$. The action coordinates are unique if we require $I(b) = 0$. The action together with the angle coordinates (a, α) on $f^{-1}(U)$ are called action-angle coordinates. It is clear that

$$\{a_i, a_j\} = \{\alpha_i, \alpha_j\} = 0 \ , \ \{a_i, \alpha_j\} = \delta_{ij}$$

for all $i, j = 1, \cdots, n$ [2].

Theorem 4.5. *Suppose $f : M \to \mathbb{R}^n$ is an integrable system on a symplectic manifold (M, ω) of dimension $2n$. Suppose $f : M \to \mathbb{R}^n$ is a proper map, and the fibers M_c are connected for all $c \in \mathcal{R}$. Then the smooth map*

$$f : M^{rf} \to \mathcal{R}$$

is a principal Lagrangian fibration with structure group a torus $(\mathbb{R}/2\pi\mathbb{Z})^n$ of dimension n and base space $\mathcal{R}$ the locus of regular values, and there exist action-angle coordinates $(a, \alpha) : f^{-1}(U) \to \mathbb{R}^n \times (\mathbb{R}/2\pi\mathbb{Z})^n$ with U a small ball in $\mathcal{R}$ around b as above for which $\omega = \sum \mathrm{d}a_i \wedge \mathrm{d}\alpha_i$.

Corollary 4.6. *Deforming the period lattice L_c in $\mathbb{R}^n$ with c moving along curves $\gamma : [0, 1] \to \mathcal{R}$ with a fixed base point $\gamma(0) = \gamma(1) = b \in \mathcal{R}$ gives a representation*

$$\rho : \Pi_1(\mathcal{R}, b) \to \mathrm{GL}(L_b) \cong \mathrm{GL}_n(\mathbb{Z})$$

which is called the monodromy representation of the principal Lagrangian fibration.

Proof. For $\gamma : [0, 1] \to \mathcal{R}$ a curve in $\mathcal{R}$ we obtain by following vectors in the period lattice $L_{\gamma(t)}$ along the curve a homomorphism $\rho(\gamma) : L_{\gamma(0)} \to L_{\gamma(1)}$. If $\delta : [0, 1] \to \mathcal{R}$ is another curve in $\mathcal{R}$ with $\delta(1) = \gamma(0)$ then the composition $\gamma\delta$ is defined by tracing out δ and subsequently γ, and $\rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$ is clear by definition. If γ_s is a homotopy of curves with fixed base point $b \in \mathcal{R}$ then $\rho(\gamma_s) \in \mathrm{End}(L_b)$ is an integral matrix, which varies continuously with $s \in [0, 1]$, and therefore is constant in $s \in [0, 1]$. Hence we obtain an induced homomorphism $\rho : \Pi_1(\mathcal{R}, b) \to \mathrm{GL}(L_b)$ on the level of the fundamental group. □

The monodromy representation being not trivial is an obstruction for having global action angle coordinates [9].

Definition 4.7. *A Hamiltonian system (M, ω, H) is a pair of a symplectic manifold (M, ω) and a smooth function $H \in \mathcal{F}(M)$ called the Hamiltonian.*

Thinking of (M, ω) as the phase space of states of a physical system the Hamiltonian H describes the dynamics of states in time via the flow $\phi_t : M \to M$ of its Hamiltonian vector field v_H.

Definition 4.8. *A Hamiltonian system (M, ω, H) is called a completely integrable system if there exists an integrable system*

$$f = (f_1, \cdots, f_n) : M \to \mathbb{R}^n$$

with $\{H, f_j\} = 0$ for $j = 1, \cdots, n$.

Under the additional assumptions of the Arnold–Liouville theorem the Hamiltonian flow of the vector field v_H becomes linearized on each Liouville torus $M_c \cong \mathbb{R}^n/L_c$, and so its integral curves describe periodic or quasi-periodic motion. Indeed on $M^{rf} = f^{-1}(\mathcal{R})$ the Hamiltonian H is constant on Liouville tori, and so we have $H(x) = F(f(x))$ for some smooth function $F : \mathcal{R} \to \mathbb{R}$. Hence $\mathrm{d}\, H = \sum_j (\partial F/\partial f_j)\, \mathrm{d}\, f_j$ which in turn implies that

$$v_H = \sum_j \frac{\partial F}{\partial f_j}(c) v_j$$

is a constant vector field on the Liouville torus $M_c \cong \mathbb{R}^n/L_c$ for all $c \in \mathcal{R}$. It is ample time to discuss these abstract ideas in a concrete example.

4.3 Spherical Pendulum

In problems of classical mechanics in a Euclidean space $\mathbb{R}^n$ it is a standard convention in handwritten text to denote vectors with an arrow or also an underline. In print the typesetting for vectors is boldface, so $\mathbf{q} \in \mathbb{R}^n$ while q denotes the length of the vector $\mathbf{q}$. The scalar product and vector product of vectors $\mathbf{q}, \mathbf{p}$ in $\mathbb{R}^3$ are denoted $\mathbf{q} \cdot \mathbf{p}$ and $\mathbf{q} \times \mathbf{p}$ respectively. We shall adopt this convention in this chapter. For example

$$\mathbf{q} \cdot \mathbf{p} = qp \cos\theta \ , \ |\mathbf{q} \times \mathbf{p}| = qp \sin\theta$$

with θ the angle between two nonzero vectors $\mathbf{q}, \mathbf{p} \in \mathbb{R}^3$

The configuration space $\mathbb{S}^2 = \{\mathbf{q} \in \mathbb{R}^3; q = 1\}$ of the spherical pendulum has phase space $T^*\mathbb{S}^2 = \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3; q = 1, \mathbf{q} \cdot \mathbf{p} = 0\}$. The Hamiltonian

$$H = p^2/2 + q_3$$

describes the motion of a point particle $\mathbf{q}$ of unit mass constraint to the unit sphere $\mathbb{S}^2$ under influence of a constant gravitational field of unit length in the vertical downward direction (with identification $T^*\mathbb{S}^2 \cong T\mathbb{S}^2$).

The circle group $\mathrm{SO}_2(\mathbb{R})$ of rotations around the third axis leaves the Hamiltonian invariant. The infinitesimal vector field of this action is the Hamiltonian vector field of the function

$$J = L_3 = q_1 p_2 - q_2 p_1$$

which is the third component of the angular momentum vector $\mathbf{L} = \mathbf{q} \times \mathbf{p}$. The fact that H is invariant for this action is equivalent to $\{H, J\} = 0$. We claim that $(H, J) : T^*\mathbb{S}^2 \to \mathbb{R}^2$ is an integrable system, and so the spherical pendulum is a completely integrable system. More explicitly we have the following result of Duistermaat [9].

Theorem 4.9. *The image under the energy-momentum map*

$$(H, J) : T^*\mathbb{S}^2 \to \mathbb{R}^2$$

is equal to $\{(x, y) \in \mathbb{R}^2; x \geq r(y)\}$ *with* $r : \mathbb{R} \to \mathbb{R}$ *the function*

$$r(z^{-1} - z^3) = \tfrac{1}{2}z^{-2} - \tfrac{3}{2}z^2$$

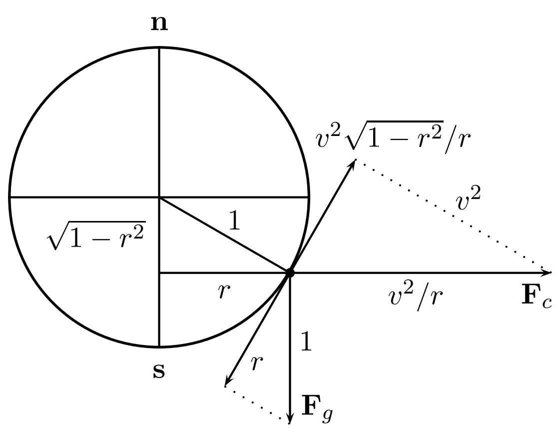
for $0 < |z| \leq 1$. *The locus of regular values of the energy-momentum map is equal to*

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2; x > r(y)\} - \{(1, 0)\}$$

with the additional singular value $(1, 0)$ *deleted as the image of the unstable immobile north pole.*

Proof. The Hamiltonian vector field v_J of J on $M = T^*\mathbb{S}^2$ has just two zeros at the north pole $\mathbf{n}$ and the south pole $\mathbf{s}$, viewed as points of M with zero cotangent vector. On the complement $M - \{\mathbf{n}, \mathbf{s}\}$ the Hamiltonian flow of J has no fixed points and is periodic with period 2π. The Hamiltonian vector field v_H also vanishes at these two poles, which implies that the energy-momentum map has rank 0 on M precisely at $\{\mathbf{n}, \mathbf{s}\}$ with singular values $\{(H, J) = (\pm 1, 0)\}$.

The energy-momentum map has rank 1 at those points of $M - \{\mathbf{n}, \mathbf{s}\}$ where the vector field v_H is a multiple λ of v_J. This scalar function λ in the equation $v_H = \lambda v_J$ is constant along the flow lines of v_J, which implies that the Hamiltonian flow of H is a horizontal circular motion in the southern hemisphere with such speed that the centrifugal force and the gravitational force cancel out. These point are called the stable relative equilibria of the spherical pendulum.



A planar circular motion $\mathbf{r}(t) = (r \cos \omega t, r \sin \omega t)$ around the origin with angular velocity $\omega > 0$ over a circle with radius $r > 0$ has velocity $\mathbf{v}(t) = (-r\omega \sin \omega t, r\omega \cos \omega t)$ and is traversed with constant speed $v = r\omega$ and has period $T = 2\pi r/v = 2\pi/\omega$. By Huygens and Newton the centripetal force $\mathbf{F} = \ddot{\mathbf{r}} = -\omega^2 \mathbf{r}$ has length $F = v^2/r$.

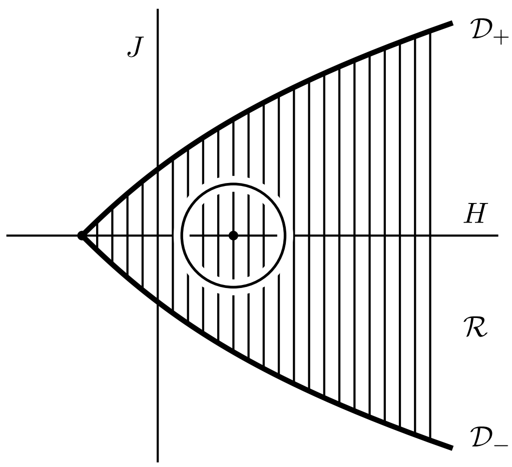
We use this to analyze the stable relative equilibria of the spherical pendulum, with $\mathbf{q}$ moving uniformly over a circle at constant negative height. A horizontal circular motion of the spherical pendulum at height $-1 < q_3 < 0$ with square radius $r^2 = 1 - q_3^2$ and speed v gives a horizontal centrifugal force of length $F_c = v^2/r$. Its tangential component has length $v^2\sqrt{1-r^2}/r$ and should be equal to the length r of the tangential component of the vertical gravitational force of length $F_g = 1$. In other words, stable relative equilibria do occur if

$$v^2\sqrt{1-r^2}/r = r \iff v^2 = r^2/\sqrt{1-r^2}\ .$$

If we denote $z = \pm\sqrt{-q_3} = \pm\sqrt[4]{1-r^2}$ for $-1 \le q_3 < 0$ and $0 \le r < 1$ then $r^2 = 1 - z^4$ and stable relative equilibria do occur for $v^2 = z^{-2} - z^2$. We find (with $z \in [-1, 0) \cup (0, 1] \cup \{\pm \mathrm{i}\}$)

$$H = \tfrac{1}{2}v^2 + q_3 = \tfrac{1}{2}z^{-2} - \tfrac{3}{2}z^2\ ,\ J = \pm rv = z^{-1} - z^3$$

as a (rational) parametrization of the discriminant. The theorem follows using the Lagrange multiplyer theorem as indicated in the remark below. $\square$



The above picture describes the image under the energy-momentum map. The fat drawn curve together with the point $(1, 0)$ is the discriminant $\mathcal{D}$ and the shaded locus $\mathcal{R}$ of regular values is bounded by $\mathcal{D}$.

Remark 4.10. *So the discriminant locus $\mathcal{D}$ of the spherical pendulum becomes*

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2; x = r(y)\} \sqcup \{(1, 0)\}$$

with $r : \mathbb{R} \to \mathbb{R}$ given by $r(z^{-1} - z^3) = \frac{1}{2}z^{-2} - \frac{3}{2}z^2$ for $|z| \in (0, 1]$. Note that for $z = \pm \mathrm{i}$ we get the singular value $(1, 0)$. The discriminant has two singular points $(\pm 1, 0)$ as images of the unstable equilibrium $\mathbf{n}$ and the stable equilibrium $\mathbf{s}$. The regular part of $\mathcal{D}$ has two connected components

$$\mathcal{D}_\pm = \{(x, y) \in \mathbb{R}^2; x = r(y), \pm y > 0\}$$

corresponding to the stable relative equilibria as described in the above proof. These are the images of the energy-momentum map where v_J is nonzero and $v_H = \lambda v_J$ for some scalar function λ. We claim that $\lambda > 0$ for $\mathcal{D}_-$ and $\lambda < 0$ for $\mathcal{D}_+$. Clearly $v_H = \lambda v_J$ is equivalent to $\mathrm{d}\, H = \lambda \, \mathrm{d}\, J$. On $\mathcal{D}_+$ the function H has a minimum and J has a maximum, while on $\mathcal{D}_-$ both H and J have a minimum. Hence $\lambda < 0$ on $\mathcal{D}_+$ and $\lambda > 0$ on $\mathcal{D}_-$. Note that by the Lagrange multiplyer theorem the set

$$\mathcal{D}_+ \sqcup \mathcal{D}_- \sqcup \{(-1, 0)\}$$

is just the image under the energy-momentum map of those points $(\mathbf{q}, \mathbf{p})$ where H is minimal under the constraint that J is constant.

Remark 4.11. *It is easy to check that the two assumptions of the Arnold–Liouville theorem (the map $f : M \to \mathbb{R}^n$ is proper and the regular fibers M_c are connected) are satisfied in our example of the spherical pendulum. Indeed the energy-momentum map $f = (H, J) : T^*\mathbb{S}^2 \to \mathbb{R}^2$ is proper because its first coordinate H is already a proper function on $M = T^*\mathbb{S}^2$. The orbits of v_H go through the north pole $\mathbf{n}$ or the south pole $\mathbf{s}$ at some speed if and only if these orbits lie in some plane through $\mathbf{n}$ and $\mathbf{s}$, which in turn is equivalent to $J = 0$. From this it is clear that the regular fiber M_c for $c = (h, 0)$ and $h > 1$ is a connected torus. Since $\mathcal{R}$ is connected this proves that the general regular fiber M_c is connected.*

For $\epsilon > 0$ small the curves

$$\theta_\pm(t) = (1 + \epsilon \cos \pi t, \pm \epsilon \sin \pi t)$$

for $t \in [0, 1]$ are half circles in $\mathcal{R}$ with begin point $b = c_+ = (1+\epsilon, 0)$ and end point $c_- = (1 - \epsilon, 0)$. The composition $\theta = \theta_-^{-1} \circ \theta_+$ is the natural generator of the fundamental group $\Pi_1(\mathcal{R}, b)$ compatible with the natural orientation of $\mathbb{R}^2$. If we denote the induced lattice transformations by

$$T_\pm : L_{c_+} \to L_{c_-} \ , \ T = T_-^{-1} \circ T_+ : L_b \to L_b$$

then T is the monodromy transformation which we like to compute.

For any $c \in \mathcal{R}$ the lattice L_c contains the element $\delta = (0, 2\pi)$. Indeed the Hamiltonian flow of the function J on $M^r = M - \{\mathbf{n}, \mathbf{s}\}$ with $M = T^*\mathbb{S}^2$ is periodic with period 2π. Clearly $T_\pm(\delta) = \delta$ and therefore $T(\delta) = \delta$. A second generator for L_c with $c = (h, 0)$ with either $h > 1$ or $-1 < h < 1$ can be found as follows.

For $h > 1$ the flow of the Hamiltonian field v_H on the fiber M_c over $c = (h, 0)$ has a periodic orbit in a vertical plane through north and south pole. At the north pole $\mathbf{n}$ the speed is equal to $p = \sqrt{2(h-1)}$ and at the south pole $\mathbf{s}$ its speed equals $p = \sqrt{2(h+1)}$. If $\tau(h) > 0$ is the period of this orbit for $h > 1$ then $\gamma = (\tau(h), 0)$ lies in L_c. In fact the lattice L_c is equal to $\mathbb{Z}\gamma + \mathbb{Z}\delta$. Note that $\tau(h) \uparrow \infty$ if $h \downarrow 1$.

For $-1 < h < 1$ the flow of the Hamiltonian field v_H on the fiber M_c over $c = (h, 0)$ has again a periodic orbit of v_H in a vertical plane through north and south pole, making a swing forth and back. At the south pole $\mathbf{s}$ its velocity equals $p = \sqrt{2(h+1)}$. However it does not quite reach the north pole, but comes only up to height h with zero speed $p = 0$ to swing back. If $2\tau(h) > 0$ is the period of this orbit of v_H then $(2\tau(h), 0)$ lies in L_c. However the lattice $\mathbb{Z}(2\tau(h), 0) + \mathbb{Z}(0, 2\pi)$ has index two in L_c since $(\tau(h), (2k+1)\pi) \in L_c$ for $k \in \mathbb{Z}$. Indeed, just make a half swing for v_H (in a plane through $\mathbf{n}$ and $\mathbf{s}$) and at the same time a half integral turn for v_J around the third axis. Note that $\tau(h) \uparrow \infty$ if $h \uparrow 1$.

We shall take $\gamma = (\tau(1+\epsilon), 0)$ and $\delta = (0, 2\pi)$ as a basis of the lattice L_b over the base point $b = c_+ = (1+\epsilon, 0)$. We claim that

$$T_\pm(\tau(1+\epsilon), 0) = (\tau(1-\epsilon), \mp\pi) \ .$$

In words, the continuation along θ_+ transforms the periodic orbit of v_H on M_{c_+} into a loop on M_{c_-} by making just one swing forth of v_H and at the same time a half turn of $-v_J$. The reason is that during the continuation in

$$\mathcal{R}_+ = \{(H, J) \in \mathcal{R}; J > 0\}$$

the image under the natural projection map

$$\pi : T^*\mathbb{S}^2 \cap (H, J)^{-1}(\mathcal{R}_+) \to \mathbb{S}^2 - \{\mathbf{n}, \mathbf{s}\}$$

of the transported loop of γ is homotopic inside $\mathbb{S}^2 - \{\mathbf{n}, \mathbf{s}\}$ to the equator traversed in negative direction relative to the positive third axis. This follows from Remark 4.10 since we found for the stable relative equilibria above $\mathcal{D}_+$ that $v_H = \lambda v_J$ with $\lambda < 0$. The integral curve of v_H over the point $(h, 0)$ with $h > 1$ is a periodic orbit passing through north and south pole with positive speed. The integral curve of v_H over the point (h, δ) (with $h > 1$ and $\delta > 0$ small) passes the north and south pole on the left side. Correcting with a small rotation around the third axis the deformed loop of γ remains a closed curve, and after the projection π remains homotopic to the equator traversed in negative direction. If we deform (h, c) in $\mathcal{R}_+$ so that $-1 < h < 1$ and $c = \delta > 0$ dropping down to 0 then γ deforms in a half swing in a plane through north and south pole plus a rotation around the third axis over an angle $-\pi$. In other words, we get $T_+(\tau(1+\epsilon), 0) = (\tau(1-\epsilon), -\pi)$ as claimed.

Likewise continuation along θ_- transforms the periodic orbit of v_H on M_{c_+} into a periodic orbit on M_{c_-} by making just one swing forth of v_H and at the same time a half turn of v_J. This proves the above claim.

Using this claim we arrive at

$$T(\tau(1+\epsilon), 0) = T_-^{-1} \circ T_+(\tau(1+\epsilon), 0) = T_-^{-1}(\tau(1-\epsilon), -\pi) = (\tau(1+\epsilon), -2\pi)$$

and with our notation $\gamma = (\tau(1+\epsilon), 0)$ and $\delta = (0, 2\pi)$ as basis of L_b the conclusion is

$$T(\gamma) = \gamma - \delta \ , \ T(\delta) = \delta \ .$$

This is just the classical Picard-Lefschetz formula [32],[5]. Indeed the integral symplectic form $\langle \cdot, \cdot \rangle$ on $L_c = \mathbb{Z}\gamma + \mathbb{Z}\delta$, defined by

$$\langle \gamma, \delta \rangle = 1 \ , \ \langle \gamma, \gamma \rangle = \langle \delta, \delta \rangle = 0$$

is just the intersection form on $L_c = H_1(M_c, \mathbb{Z})$ coming from the natural orientation of $M_c = \mathbb{R}^2/L_c$. Hence we obtain the following result, which was originally proved by Duistermaat by an analytic argument using symmetry reduction [9], by topological reasoning.

Theorem 4.12. *For $\lambda \in L_b$ the monodromy T of a small positive loop θ in $\mathcal{R}$ around $(1, 0)$ is given by the Picard-Lefschetz formula*

$$T(\lambda) = \lambda - \langle \lambda, \delta \rangle \delta$$

with $\langle \cdot, \cdot \rangle$ the intersection form on the lattice L_b and $\delta \in L_b$ the so called vanishing cycle above $(1, 0) \in \mathcal{D}$.

The fiber $M_{(1,0)}$ over $(1,0)$ has a singular point at the north pole $\mathbf{n}$ viewed as point of $M = T^*\mathbb{S}^2$. The complement $M_{(1,0)} - \{\mathbf{n}\}$ has the structure of the homogeneous space $\mathbb{R}^2/(0, 2\pi\mathbb{Z})$. This is a cylinder $\mathbb{R} \times \mathrm{U}_1(\mathbb{C})$ with the flow of v_H acting by translations in the first factor and the flow of v_J acting by rotations in the second factor. The argument is just the same as for the Arnold–Liouville theorem. The singular fiber $M_{(1,0)}$ is obtained from the nearby regular fiber $M_{(1+\epsilon,0)}$ by pinching a cycle representing δ. This explains the terminology vanishing cycle for δ.

Remark 4.13. *The spherical pendulum was first studied by Huygens in 1673 (so 14 years before the appearance of the Principia Mathematica of Newton in 1687) [24]. Our description of the stable relative equilibria is due to Huygens. It was for this purpose that he found his well known formula $F_c = mv^2/r$ for the centrifugal force of a uniform circular motion. The monodromy for the spherical pendulum was shown to be nontrivial by Cushman and subsequently computed by Duistermaat [9]. During his lecture in 1980 he expressed his surprise that for such a classical problem this computation had not been done long before.*

4.4 Kepler Problem

The Kepler problem is concerned with planetary motion around the sun. By a center of mass reduction one is led to the Newtonian differential equation

$$m\ddot{\mathbf{q}} = -k\mathbf{q}/q^3$$

with $m = m_1 m_2/(m_1 + m_2) > 0$ the reduced mass, $k = G m_1 m_2 > 0$ the coupling constant and $\mathbf{q} = (\mathbf{q}_1 - \mathbf{q}_2) \in \mathbb{R}^3$ the radius vector from the sun to the planet. The method of solution is the search for conserved quantities.

The first conserved quantity follows from the fact that the Newtonian force field $\mathbf{F} = -k\mathbf{q}/q^3$ is central. In this case the angular momentum

$$\mathbf{L} = \mathbf{q} \times \mathbf{p}$$

is conserved, with $\mathbf{p} = m\dot{\mathbf{q}}$ the linear momentum. Since $\dot{\mathbf{L}} = \mathbf{0}$ we conclude that the motion takes place in the plane perpendicular to the constant vector $\mathbf{L}$. If $A = A(t)$ is the area of the region traced out by the radius vector $\mathbf{q}$ from a fixed initial time on then the time derivative $\mathrm{d}\,A/\,\mathrm{d}\,t$ satisfies

$$\dot{A} = |\mathbf{q} \times \dot{\mathbf{q}}|/2 = \frac{L}{2m}$$

by elementary calculus. Therefore the area of the region traced out by the radius vector in a unit time interval is constant. Equal areas in equal times is the area law of Kepler.

The second conserved quantity is total energy H, valid in a spherically symmetric central force field $\mathbf{F}(\mathbf{q}) = f(q)\mathbf{q}/q$ for some smooth function f on $\mathbb{R}$. Indeed, it is easy to check by elementary calculus that

$$H = \frac{p^2}{2m} + V(q) \ , \ V(q) = -\int f(q)dq$$

satisfies $\dot{H} = 0$, which is the law of conservation of total energy.

Theorem 4.14. *The Hamiltonian system*

$$(\mathbb{R}^3 \times \mathbb{R}^3, \sum \mathrm{d}\, p_j \wedge \mathrm{d}\, q_j, H = \frac{p^2}{2m} + V(q))$$

of a spherically symmetric Hamiltonian is completely integrable.

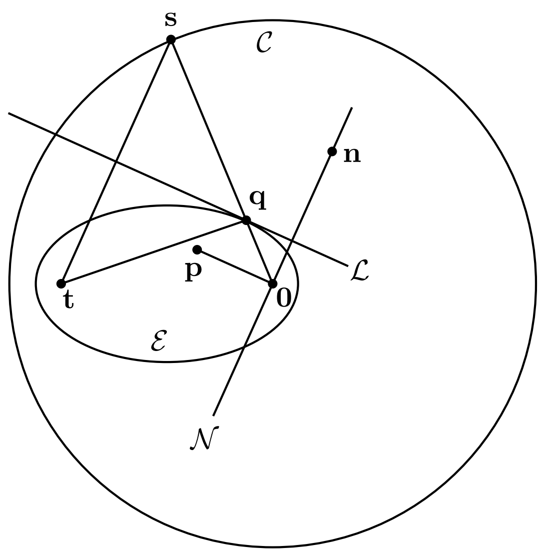
Proof. By a straightforward calculation using $\{p_j, q_k\} = \delta_{jk}$ one checks that the components of the angular momentum vector $\mathbf{L}$ satisfy the Poisson brackets relations

$$\{L_i, L_j\} = -\epsilon_{ijk} L_k$$

with ϵ_{ijk} the Levi-Civita symbol. These are the commutation relations for the Lie algebra $\mathfrak{so}_3(\mathbb{R})$. From this it is easy to check that the length L of the angular momentum vector Poisson commutes with all L_j for $j = 1, 2, 3$. In turn this implies that $(H, L, L_3) : \mathbb{R}^6 \to \mathbb{R}^3$ is an integrable system, at least under suitable completeness conditions for the flow of H. $\square$

We shall now discuss a solution of the Kepler problem with Hamiltonian $H = p^2/2m - k/q$ in terms of Euclidean geometry. Let us suppose that $H < 0$ is fixed. Since $k/q = p^2/2m - H \geq -H > 0$ we get $q < -k/H$, and so the motion is bounded inside a sphere of radius $-k/H > 0$. Under the assumption that $L > 0$ the motion takes place in the plane through the origin $\mathbf{0}$ perpendicular to $\mathbf{L}$.

The circle $\mathcal{C}$ with center $\mathbf{0}$ and radius $-k/H$ is the boundary of a disc in which motion with fixed energy $H < 0$ can take place. Points that fall from $\mathcal{C}$ onto the origin have the same energy H, and for this reason $\mathcal{C}$ is called the fall circle. Let $\mathbf{s} = -k\mathbf{q}/qH$ be the projection of $\mathbf{q}$ from the origin $\mathbf{0}$ on this circle $\mathcal{C}$. The line $\mathcal{L}$ through $\mathbf{q}$ with direction vector $\mathbf{p}$ is the tangent line to the orbit $\mathcal{E}$ at the point $\mathbf{q}$. Let $\mathbf{t}$ be the orthogonal projection of $\mathbf{s}$ in the line $\mathcal{L}$.



Theorem 4.15. *The point* $\mathbf{t}$ *is equal to* $\mathbf{K}/mH$ *with*

$$\mathbf{K} = \mathbf{p} \times \mathbf{L} - km\mathbf{q}/q$$

the Lenz vector. In addition $\dot{\mathbf{K}} = \mathbf{0}$ *and so* $\mathbf{K}$ *is conserved.*

Proof. The line $\mathcal{N}$ spanned by $\mathbf{n} = \mathbf{p} \times \mathbf{L}$ is perpendicular to $\mathcal{L}$. The point $\mathbf{t}$ is obtained from $\mathbf{s}$ by subtracting twice the ortogonal projection of $\mathbf{s} - \mathbf{q}$ on the line $\mathcal{N}$, and therefore

$$\mathbf{t} = \mathbf{s} - 2((\mathbf{s} - \mathbf{q}) \cdot \mathbf{n})\mathbf{n}/n^2.$$

Since $\mathbf{s} = -k\mathbf{q}/qH$ as projection of $\mathbf{q}$ on $\mathcal{C}$ we find

$$(\mathbf{s} - \mathbf{q}) \cdot \mathbf{n} = -(H + k/q)\mathbf{q} \cdot (\mathbf{p} \times \mathbf{L})/H = -(H + k/q)L^2/H$$

$$n^2 = p^2 L^2 = 2m(H + k/q)L^2 \ ,$$

and therefore

$$\mathbf{t} = -k\mathbf{q}/qH + \mathbf{n}/mH = \mathbf{K}/mH$$

with $\mathbf{K}$ the Lenz vector as given in the theorem. The fact that $\dot{\mathbf{K}} = \mathbf{0}$ follows by a straightforward calculation. □

Corollary 4.16. *The orbit* $\mathcal{E}$ *is an ellipse with foci* $\mathbf{0}$ *and* $\mathbf{t}$*, and long axis equal to* $2a = -k/H$.

Proof. Since $\dot{\mathbf{K}} = \mathbf{0}$ we get $\dot{\mathbf{t}} = \mathbf{0}$, and so

$$|\mathbf{t} - \mathbf{q}| + |\mathbf{q} - \mathbf{0}| = |\mathbf{s} - \mathbf{q}| + |\mathbf{q} - \mathbf{0}| = |\mathbf{s} - \mathbf{0}| = -k/H.$$

Hence $\mathcal{E}$ is an ellipse with foci $\mathbf{0}$ and $\mathbf{t}$, and long axis $2a = -k/H$. □

The Lenz vector $\mathbf{K} = (K_1, K_2, K_3)$ exhibits a remarkable symmetry of the Kepler problem, first observed by Pauli [44]. Both results are obtained by (sometimes unpleasant but) straightforward algebraic computations.

Theorem 4.17. *The angular momentum* $\mathbf{L}$ *and Lenz vector* $\mathbf{K}$ *satisfy the Poisson bracket relations*

$$\{L_i, L_j\} = -\epsilon_{ijk} L_k \ , \ \{L_i, K_j\} = -\epsilon_{ijk} K_k \ , \ \{K_i, K_j\} = (2mH)\epsilon_{ijk} L_k \ .$$

Theorem 4.18. *We have* $(-2mH)L^2 + K^2 = k^2 m^2$.

Corollary 4.19. *On the region* $H < 0$ *let us write* $\mathbf{M} = \mathbf{K}/\sqrt{-2mH}$. *In turn this gives the Poisson bracket relations*

$$\{L_i, L_j\} = -\epsilon_{ijk} L_k \ , \ \{L_i, M_j\} = -\epsilon_{ijk} M_k \ , \ \{M_i, M_j\} = -\epsilon_{ijk} L_k$$

which are the commutation relations for the standard basis of $\mathfrak{so}_4(\mathbb{R})$. *All these six functions Poisson commute with the Hamiltonian* $H = p^2/2m - k/q$ *of the Kepler problem. Five of them are functionally independent. Because we have more than three functionally independent conserved functions the Kepler problem is sometimes called superintegrable.*

Although the rescaled Lenz vector $\mathbf{t} = \mathbf{K}/mH$ has a clear geometric meaning as the second focus of the elliptical orbit the above Poisson bracket relations for $\mathbf{L}$ and the other rescaling $\mathbf{M} = \mathbf{K}/\sqrt{-2mH}$ only follow from algebraic calculations. A geometric explanation of the commutation relations for the components of $\mathbf{L}$ and $\mathbf{M}$ (using Moser's regularization of the Kepler problem [38]) was found by Heckman and de Laat [22].

The momentum-Lenz map

$$(\mathbf{L}, \mathbf{M}) : \{(\mathbf{q}, \mathbf{p}); H(\mathbf{q}, \mathbf{p}) < 0\} \to \mathbb{R}^6$$

has image contained in the quadric hypersurface $\{(\mathbf{L}, \mathbf{M}); \mathbf{L} \cdot \mathbf{M} = 0\}$ of dimension 5. The one dimensional fibers are the Kepler ellipses for the Hamiltonian H given by

$$-2H = k^2 m/(L^2 + M^2)$$

which is the explanation of the remarkable fact that for $H < 0$ all orbits are closed. Note that for each $H < 0$ there are the collision orbits, where the flow of v_H is no longer complete. The collision orbits for $H < 0$ are exactly those orbits that lie on the hypersurface $L^2 = 0$.

Remark 4.20. *The Kepler problem was solved in 1687 by Isaac Newton with a beautiful proof [41]. For a modern exposition of this proof, and a discussion of various other proofs we refer to an article by van Haandel and Heckman [19]. The Lenz vector* $\mathbf{K}$ *became popular after its use by Pauli in 1926 for the quantum mechanics of the Kepler problem [44]. Lenz was a teacher of Pauli and had rediscovered this vector, like several other people (Runge, Hamilton, Laplace) before him. I learned from Alain Albouy that the Lenz vector can be traced back to Lagrange in 1781, see pages 131 and 132 of [31].*

4.5 Three Body Problem

Since the appearance of Newton's masterpiece [41] in 1687 many examples of integrable systems were found during the following two centuries. It was (and in my opinion still is) a piece of mathematical craftsmanship to find the Poisson commuting integrals of motion for the given Hamiltonian, and thereby essentially solving the given system.

However Bruns in 1887 and Poincaré in 1890 found that the three body problem is not an integrable system. By a center of mass reduction

$$\sum_1^3 m_i \mathbf{q}_i = 0$$

the configuration space N has dimension 6, and so the phase space $M = T^*N$ is of dimension 12. It turns out that the only algebraic (by Bruns) or even analytic (by Poincaré) integrals of motion for the Hamiltonian

$$H = \sum_i p_i^2/2m_i - \sum_{i \neq j} \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}$$

are the familiar ones, namely total angular momentum

$$\mathbf{L} = \sum_i \mathbf{q}_i \times \mathbf{p}_i$$

and the total energy H itself. The generic integral curves of the Hamiltonian vector field v_H fill out densily a submanifold of dimension $12 - 4 = 8$ rather than of dimension $6 = 12/2$ as would be the case for an integrable system. From this moment on people have realized that the integrable systems are the rare exceptions, while generically nonintegrability is omnipresent [49].

4.6 Exercises

Exercise 4.1. *A Poisson algebra $\mathcal{P}$ is both an associative algebra, denoted $\mathcal{P} \ni f, g \mapsto fg \in \mathcal{P}$, and a Lie algebra, denoted $\mathcal{P} \ni f, g \mapsto \{f, g\} \in \mathcal{P}$, with the Leibniz rule*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

as compatibility condition. For a vector space V the symmetric algebra SV and the algebra PV^* of polynomial functions on the dual vector space V^* are naturally identified. Show that for a Lie algebra $\mathfrak{g}$ the commutative algebra $\mathcal{P} = S\mathfrak{g} = P\mathfrak{g}^*$ has a natural Lie bracket turning $\mathcal{P}$ into a Poisson algebra. Hint: Pick a basis (e_j) of $\mathfrak{g}$ with dual basis (f_j) of $\mathfrak{g}^*$, and so $\partial e_j/\partial f_k = \delta_{jk}$. Define the Poisson bracket by

$$\{f, g\} = \sum_{j,k} \frac{\partial f}{\partial f_j}\frac{\partial g}{\partial f_k}[e_j, e_k]$$

and check that this definition satisfies the Leibniz rule, is independent of the choice of the basis and is a Lie bracket. Here $\partial/\partial f_j$ denotes the directional derivative in the direction f_j.

Exercise 4.2. *Let $L = \mathbb{Z}B$ be a lattice in $\mathbb{R}^n$ with B a basis of $\mathbb{R}^n$, and let $\mathbb{R}^n/L$ be the associated torus. Show that the map*

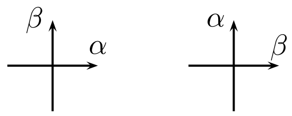
$$L \to H_1(\mathbb{R}^n/L)\ ,\ \lambda \mapsto [\sigma_\lambda]\ ,\ \sigma_\lambda(1-t, t) = t\lambda + L\ ,\ 0 \le t \le 1$$

realizes L as sublattice of the first homology space $H_1(\mathbb{R}^n/L)$.

Exercise 4.3. *Let $L = \mathbb{Z}\gamma + \mathbb{Z}\delta$ be the lattice in $\mathbb{R}^2$ with $\gamma = (\tau, 0)$ for some $\tau > 0$ and $\delta = (0, 2\pi)$. The torus $\mathbb{R}^2/L$ inherits a natural orientation from the standard orientation of $\mathbb{R}^2$. The de Rham pairing between $H_1(\mathbb{R}^2/L)$ and $H^1_{\mathrm{dR}}(\mathbb{R}^2/L)$ and the Poincaré form $H^1_{\mathrm{dR}}(\mathbb{R}^2/L) \times H^1_{\mathrm{dR}}(\mathbb{R}^2/L) \to \mathbb{R}$ give a nondegenerate symplectic pairing*

$$\langle \cdot, \cdot \rangle : H_1(\mathbb{R}^2/L) \times H_1(\mathbb{R}^2/L) \to \mathbb{R}$$

which is called the intersection form on the middle homology of $\mathbb{R}^2/L$. Suppose $\alpha, \beta : [0, 1] \to \mathbb{R}^2/L$ are smooth closed curves intersecting transversally. Check that the intersection number $\langle [\alpha], [\beta] \rangle$ of the corresponding classes counts the number of intersection points of the two representing curves, with a plus or a minus sign depending on whether



respectively. This explains the terminology intersection form.

Exercise 4.4. *Show that for $h \geq -1$ the solution of the spherical pendulum with angular momentum $J = 0$ and energy $H = h$ correspond to motion in a plane through north pole $\mathbf{n}$ and south pole $\mathbf{s}$, with speed $\sqrt{h+1}$ at $\mathbf{s}$ and (for $h \geq 1$) with speed $\sqrt{h-1}$ at $\mathbf{n}$. Conclude that the natural projection $\pi : T^*\mathbb{S}^2 \to \mathbb{S}^2$ maps the subset $\{J > 0\}$ of $T^*\mathbb{S}^2$ inside $\mathbb{S}^2 - \{\mathbf{n}, \mathbf{s}\}$.*

Exercise 4.5. *Check that the scalar and vector product on $\mathbb{R}^3$ have the compatibility relations*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \ , \ \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Show that the map $\mathbb{R}^3 \to \mathfrak{so}(\mathbb{R}^3)$ sending $\mathbf{u}$ to the antisymmetric linear operator $\mathbf{v} \mapsto \mathbf{u} \times \mathbf{v}$ is an isomorphism of Lie algebras from $(\mathbb{R}^3, \times)$ to $(\mathfrak{so}(\mathbb{R}^3), [\cdot, \cdot])$. If L_j denotes the image under the standard basis vector $\mathbf{e}_j \in \mathbb{R}^3$ then Lie brackets in $\mathfrak{so}_3(\mathbb{R})$ become $[L_i, L_j] = \epsilon_{ijk} L_k$.

Exercise 4.6. *Show that $\{L_i, L_j\} = -\epsilon_{ijk} L_k$ implies $\{L_i, L^2\} = 0$, and conclude that $\{L_i, L\} = 0$.*

Exercise 4.7. *Let $H = p^2/2m + V(q)$ be a spherically symmetric Hamiltonian on $\mathbb{R}^3 \times \mathbb{R}^3$. Show that the gradient of $-V(q)$ is equal to $\mathbf{F} = f(q)\mathbf{q}/q$ with $f = -\,\mathrm{d}V/\,\mathrm{d}q$. In other words, a spherically symmetric central force field is conservative.*

Exercise 4.8. *Prove the formulas of Theorem 4.17 and Theorem 4.18.*

Exercise 4.9. *Show that the matrices*

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ L_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \ M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \ M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

satisfy the commutation relations

$$[L_i, L_j] = \epsilon_{ijk} L_k \ , \ [L_i, M_j] = \epsilon_{ijk} M_k \ , \ [M_i, M_j] = \epsilon_{ijk} L_k$$

and form a basis of $\mathfrak{so}_4(\mathbb{R})$.

Exercise 4.10. *In the notation of the previous exercise show that the three components of the two vectors*

$$\mathbf{I} = (\mathbf{L} + \mathbf{M})/2 \ , \ \mathbf{J} = (\mathbf{L} - \mathbf{M})/2$$

define a new basis of $\mathfrak{so}_4(\mathbb{R})$ *with commutation relations*

$$[I_i, I_j] = \epsilon_{ijk} I_k \ , \ [J_i, J_j] = \epsilon_{ijk} J_k \ , \ [I_i, J_j] = 0 \ .$$

Conclude that $\mathfrak{so}_4(\mathbb{R}) \cong \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$ *as a direct sum of Lie algebras. Show that the quadric hypersurface* $\mathbf{L} \cdot \mathbf{M} = 0$ *in the new coordinates takes the form* $I^2 = J^2$. *Hence the image of the momentum-Lenz map is just the cone over the direct product* $\mathbb{S}^2 \times \mathbb{S}^2$ *of two unit spheres of dimension* 2. *This exercise goes back to Pauli [44]. One can show that the Lie algebra* $\mathfrak{so}_n(\mathbb{R})$ *for* $n \geq 2$ *is a simple Lie algebra (the only Lie ideals are the two trivial ideals) with the sole exception of the "odd" number* $n = 4$.

Exercise 4.11. *Consider the Kepler Hamiltonian* $H = p^2/2m - k/q$ *in the region of phase space with* $H < 0$. *Let* $2a$ *and* $2b$ *be the major and minor axis of the elliptical orbit, and define* $c > 0$ *by* $a^2 = b^2 + c^2$. *Check that* $2a = -k/H$, $2c = K/mH$ *and* $2b = 2L/\sqrt{-2mH}$. *Prove that the period* T *and semimajor axis* a *are related by Kepler's harmonic law*

$$T^2/a^3 = 4\pi^2 m/k = 4\pi^2/G(m_1 + m_2)$$

with m_1 *the mass of the planet and* m_2 *the mass of the sun. Since* $m_2 \gg m_1$ *we conlude that the ratio* T^2/a^3 *is (almost) the same for all planets. This observation was made by Kepler in 1619. Hint: The area of the ellipse equals* $\pi ab = LT/2m$.

5 Moment Map

5.1 Lie Groups

A Lie group G is at the same time a manifold and a group, and these two structures are compatible, in the sense that multiplication and inversion

$$G \times G \to G, (a, b) \mapsto ab \qquad G \to G, a \mapsto a^{-1}$$

are smooth maps for all $a, b \in G$. A Lie group is a differential geometric object concerning symmetry. A Lie algebra $\mathfrak{g}$ is a vector space with a binary operation (called the Lie bracket)

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (X, Y) \mapsto [X, Y]$$

satisfying antisymmetry and Jacobi identity

$$[X, Y] + [Y, X] = 0 \;, \;\; [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. A Lie algebra is just an algebraic object. The Norwegian mathematician Sophus Lie (1842-1899) discovered the intimate relation between these two notions.

Let us denote by $\lambda_a : G \to G$ the smooth map of left multiplication by $a \in G$. Clearly λ_a is a diffeomorphism since $\lambda_{a^{-1}}$ is its inverse. A vector field X on G is called left invariant if (upper star is pull back)

$$\lambda_a^* X = X$$

(or equivalently $\mathcal{L}_X$ and λ_a^* commute on $\mathcal{F}(G)$ by Exercise 2.5) for all $a \in G$. It is clear that the Lie bracket of two left invariant vector fields is again left invariant. Any left invariant vector field X on G is determined by its value X_e at the identity element e of G, and conversely any tangent vector X_e at the identity extends to a left invariant vector field X on G. The conclusion is that the set $\mathfrak{g}$ of all left invariant vector fields is a vector space of dimension equal to the dimension of G. Moreover $\mathfrak{g}$ inherits from $\mathcal{X}(G)$ a Lie bracket turning $\mathfrak{g}$ into a finite dimensional Lie algebra, called the Lie algebra of the Lie group G.

Let $(G, \mathfrak{g})$ be a Lie group with its Lie algebra. For $X \in \mathfrak{g}$ the integral curve through the identity e is denoted $t \mapsto \exp(tX)$. In turn we get for all smooth functions f on G

$$\mathcal{L}_X f(x) = \frac{\mathrm{d}}{\mathrm{d}\, t} \Big\{ f(x \exp(tX)) \Big\}_{t=0}$$

for all x in G. Since left invariant vector fields are complete it follows that $t \mapsto \exp(tX)$ is defined for all $t \in \mathbb{R}$. The homomorphism $t \mapsto \exp(tX)$ is called the one-parameter subgroup of G with infinitesimal generator $X \in \mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is smooth, and its tangent map at the identity e is equal to $\mathrm{Id} : \mathfrak{g} \to \mathfrak{g}$. By the inverse function theorem $\exp : \mathfrak{g} \to G$ is a diffeomorphism of an open neighborhood of 0 in $\mathfrak{g}$ onto an open neighborhood of e in G.

The beautiful discovery of Lie is that the Lie algebra $\mathfrak{g}$ captures a great deal of the structure of its Lie group G, and so questions about symmetry in differential geometry can often be dealt with by algebraic computations. An important example of this principle is the following theorem of Lie. Suppose $(G, \mathfrak{g})$ and $(H, \mathfrak{h})$ are both Lie groups with corresponding Lie algebras. A map $\phi : G \to H$ is called a Lie group homomorphism if it is both smooth and a homomorphism. A map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if it is both linear and preserves the Lie brackets. The next fundamental result is due to Lie, and we refer to the text books by Duistermaat–Kolk or Warner for a proof [12],[56].

Theorem 5.1. *A Lie group homomorphism $\phi : G \to H$ induces a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ (by abuse of notation) such that the diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\phi} & H \end{array}$$

is commutative. The Lie algebra homomorphism is obtained by differentiation at the identity of the Lie group homomorphism, and using the linear isomorphisms $T_eG \cong \mathfrak{g}$ and $T_eH \cong \mathfrak{h}$. If G is connected then the Lie group homomorphism $\phi : G \to H$ is completely determined by the Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$. Conversely, if G is connected and simply connected then any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ yields by integration a unique Lie group homomorphism $\phi : G \to H$ for which the above diagram is commutative.

In the particular case that $(H, \mathfrak{h})$ is equal to the general linear group and algebra $(\mathrm{GL}(V), \mathfrak{gl}(V))$ on a finite dimensional vector space V then we get analoguous relations between a Lie group representation $\rho : G \to \mathrm{GL}(V)$ and its corresponding Lie algebra representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

An action of a Lie group G on a smooth manifold M is a smooth map $G \times M \to M, (a, x) \mapsto ax$ with $(ab)x = a(bx)$ and $ex = x$ for all a, b in G

and e the unit element of G. In turn we get a Lie group representation of G on $\mathcal{F}(M)$ by

$$\rho : G \to \mathrm{Aut}(\mathcal{F}(M)) \ , \ (\rho(a)f)(x) = f(a^{-1}x)$$

with $\mathrm{Aut}(\mathcal{F}(M))$ the group of invertible linear operators of $\mathcal{F}(M)$ preserving the structure of multiplication of smooth functions. The corresponding Lie algebra representation becomes

$$\rho : \mathfrak{g} \to \mathrm{Der}(\mathcal{F}(M)) \ , \ (\rho(X)f)(x) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{f(\exp(-tX)x)\Big\}_{t=0} \ .$$

The inverse and minus signs in the above formulas are there to ensure that the homomorphism property

$$\rho(ab) = \rho(a)\rho(b) \ , \ \rho([X,Y]) = [\rho(X), \rho(Y)]$$

holds for all $a, b \in G$ and $X, Y \in \mathfrak{g}$. Here we have written

$$\mathrm{Der}(\mathcal{F}(M)) = \{D : \mathcal{F}(M) \to \mathcal{F}(M); D(fg) = D(f)g + fD(g) \ \forall \ f, g\}$$

for the Lie subalgebra of $\mathrm{End}(\mathcal{F}(M))$ of all derivations of $\mathcal{F}(M)$. Clearly $\rho(X)$ is a derivation as Lie derivative of a vector field acting on functions. For $x \in M$ the stabilizer group $G_x = \{a \in G; ax = x\}$ of x in G is a closed Lie subgroup, with Lie subalgebra $\mathfrak{g}_x = \{X \in \mathfrak{g}; (\rho(X)f)(x) = 0 \ \forall f \in \mathcal{F}(M)\}$.

The conjugation action of G on itself is denoted by

$$\mathrm{C}_a(x) = axa^{-1}$$

for $a, x \in G$. Since $\mathrm{C}_a(e) = e$ is a fixed point for any conjugation the tangent map $\mathrm{Ad}(a) = T_e \, \mathrm{C}_a : \mathfrak{g} \to \mathfrak{g}$ defines an important representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) \ , \ \mathrm{Ad}(a)X = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\{a \exp(tX)a^{-1}\Big\}_{t=0}$$

which is called the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The associated Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ fits in a commuative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

and is given by $\mathrm{ad}(X)Y = [X, Y]$ for all $X, Y \in \mathfrak{g}$. For linear Lie groups the adjoint representation is just the conjugation representation of G on $\mathfrak{g}$, and the latter formula follows by a power series expansion. It is not the adjoint representation, but its dual representation that we are interested in. This so-called coadjoint representation

$$\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)\ ,\ \mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$$

is defined by

$$\mathrm{Ad}^*(a) = (\mathrm{Ad}\, a^{-1})^*\ ,\ \mathrm{ad}^*(X) = (-\,\mathrm{ad}\, X)^*$$

for $a \in G$ and $X \in \mathfrak{g}$.

For $\xi \in \mathfrak{g}^*$ the antisymmetric bilinear form on $\mathfrak{g}$

$$\omega_\xi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\ ,\ \omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle = \langle -\,\mathrm{ad}^*(X)\xi, Y \rangle$$

has kernel equal to the stabilizer Lie subalgebra

$$\mathfrak{g}_\xi = \{X \in \mathfrak{g}; \mathrm{ad}^*(X)(\xi) = 0\}$$

in $\mathfrak{g}$ of the point $\xi \in \mathfrak{g}^*$. Because of the natural isomorphism $T_e(G/G_\xi) \cong \mathfrak{g}/\mathfrak{g}_\xi$ this gives a nondegenerate 2-form on any coadjoint orbit $G\xi \cong G/G_\xi$. This 2-form on a coadjoint orbit is closed by the theorem and remark below, and so any coadjoint orbit carries a natural symplectic form.

Theorem 5.2. *Let ω be a 2-form on a manifold M that is nondegenerate at each point. For $f \in \mathcal{F}(M)$ let $v = v_f \in \mathcal{X}(M)$ be defined by $\mathrm{i}_v\, \omega = -\,\mathrm{d}\, f$ like the definition of Hamilton vector field in case that ω is a symplectic form on M. Likewise define on $\mathcal{F}(M)$ the antisymmetric bracket $\{f, g\} = \omega(v_f, v_g)$ as for the Poisson bracket. Then ω is a closed (and hence a symplectic) form on M if and only if the Jacobi identity*

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

holds for all $f, g, h \in \mathcal{F}(M)$.

Proof. By definition $\{f, g\} = \omega(v_f, v_g) = \mathrm{i}_{v_f}(\mathrm{d}\, g) = \mathcal{L}_{v_f}(g)$ for all functions $f, g \in \mathcal{F}(M)$. In turn this implies

$$\mathrm{d}\{f, g\} = \mathrm{d}(\mathcal{L}_{v_f} g) = \mathcal{L}_{v_f}(\mathrm{d}\, g) = -\mathcal{L}_{v_f}(\mathrm{i}_{v_g}\, \omega) = -\,\mathrm{i}_{[v_f, v_g]}\, \omega - \mathrm{i}_{v_g}(\mathcal{L}_{v_f}\omega)\ ,$$

and so $[v_f, v_g] = v_{\{f,g\}}$ for all $f, g \in \mathcal{F}(M)$ if and only if

$$\mathcal{L}_{v_f}\omega = 0$$

for all $f \in \mathcal{F}(M)$. This is equivalent to $\mathrm{d}\, \omega = 0$ since $\mathcal{L}_{v_f}\omega = \mathrm{i}_{v_f}\, \mathrm{d}\, \omega$. The theorem follows because $[v_f, v_g] = v_{\{f,g\}}$ for all $f, g \in \mathcal{F}(M)$ is equivalent to the Jacobi identity for the Poisson bracket $\{\cdot, \cdot\}$. ☐

The fact that the nondegenerate 2-form on a coadjoint orbit as defined above is closed follows from Exercise 4.1 and the following remark.

Remark 5.3. *Let* (e_j) *and* (f_j) *be dual bases of a Lie algebra* $\mathfrak{g}$ *and its dual* $\mathfrak{g}^*$ *respectively. For* f, g *smooth functions on* $\mathfrak{g}^*$ *the formula for the Poisson bracket (as derived in Exercise 4.1)*

$$\{f, g\} = \sum_{j,k} \frac{\partial f}{\partial f_j} \frac{\partial g}{\partial f_k} [e_j, e_k]$$

coincides, after restriction to a coadjoint orbit, with the definition of the Poisson bracket for functions on that coadjoint orbit as in the previous theorem. Indeed let M *be the coadjoint orbit* $\mathrm{Ad}^*(G)\xi$ *through* $\xi \in \mathfrak{g}^*$*. It is sufficient to check this for* $f = X \in \mathfrak{g}$ *and* $g = Y \in \mathfrak{g}$ *viewed as linear functions on* $\mathfrak{g}^*$*. In that case*

$$\{f, g\}(\xi) = \sum_{j,k} f_j(X) f_k(Y) \xi([e_j, e_k]) = \xi([X, Y])$$

coincides with $\omega_\xi(X, Y)$*.*

Remark 5.4. *Suppose* G *is a connected Lie group with Lie algebra* $\mathfrak{g}$*. The center of the Poisson algebra* $S\mathfrak{g} = P\mathfrak{g}^*$ *of polynomial functions on* $\mathfrak{g}^*$ *is equal to the subalgebra* $(S\mathfrak{g})^G$ *of polynomial functions on* $\mathfrak{g}^*$ *invariant under* G *for the coadjoint action. An invariant polynomial on* $\mathfrak{g}^*$ *is also called a Casimir function, after the Dutch physicist Hendrik Casimir. Casimir was the first to clearly see the relevance of these invariant functions (in his thesis written with Niels Bohr and Paul Ehrenfest), both for Lie group theory itself and for questions of symmetry in quantum mechanics [6].*

Definition 5.5. *The natural symplectic form on a coadjoint orbit of a Lie algebra will be denoted* ω_{KK}*, and is called the Kirillov–Kostant symplectic form (going back to the thesis of Alexander Kirillov from 1962 on the orbit method for nilpotent Lie groups [27], and to lecture notes by Bertram Kostant from 1970 on geometric quantization [29]).*

All in all, we have seen three natural classes of symplectic manifolds: cotangent bundles with the exterior derivative $\omega = \mathrm{d}\,\theta$ of the tautological 1-form θ as the canonical symplectic form, Kähler manifolds and complex projective manifolds with respect to the restriction of the Fubini–Study symplectic form ω_{FS} as particular examples, and coadjoint orbits with the Kirillov–Kostant symplectic form ω_{KK}. These three classes might overlap in examples.

5.2 Moment Map

Suppose (M, ω) is a connected symplectic manifold, and

$$G \times M \to M \ , \ (a, x) \mapsto ax$$

is an action of a connected Lie group G on M. For $X \in \mathfrak{g}$ we denote by X_M the vector field on M whose flow is given by left multiplication on M with the one parameter group $t \mapsto \exp(tX)$. This vector field X_M on M is called the infinitesimal vector field of $X \in \mathfrak{g}$ for the action of G on M. We shall denote by $X_x \in T_x M$ the value of the vector field X_M at the point $x \in M$.

One should pay attention to the fact that the linear map

$$\mathfrak{g} \to \mathcal{X}(M) \ , \ X \mapsto X_M$$

is an antihomomorphism of Lie algebras, in the sense that

$$[X, Y]_M = -[X_M, Y_M]$$

for all $X, Y \in \mathfrak{g}$. Indeed, in the notation of the previous section

$$\mathcal{L}_{X_M} = \rho(-X)$$

as linear operators on $\mathcal{F}(M)$, and the map $\rho : \mathfrak{g} \to \mathrm{Der}(\mathcal{F}(M))$ is a Lie algebra homomorphism by a formal application of Theorem 5.1.

The symplectic form ω is invariant under the action of the Lie group G if and only if ω is infinitesimally invariant under the Lie algebra $\mathfrak{g}$, in the sense that

$$\mathcal{L}_{X_M} \omega = 0$$

for all $X \in \mathfrak{g}$. Therefore ω is invariant under this action if and only if

$$\mathcal{L}_{X_M} \omega = (\mathrm{d}\, \mathrm{i}_{X_M} + \mathrm{i}_{X_M}\, \mathrm{d})\omega = \mathrm{d}(\mathrm{i}_{X_M}\, \omega) = 0$$

for all $X \in \mathfrak{g}$, so if the 1-forms $\mathrm{i}_{X_M}\, \omega$ are closed for all $X \in \mathfrak{g}$. This will certainly be the case if the 1-forms $\mathrm{i}_{X_M}\, \omega$ are exact for all $X \in \mathfrak{g}$. This brings us to the following definition.

Definition 5.6. *The action $G \times M \to M$ of the Lie group G on the symplectic manifold (M, ω) is called Hamiltonian, if there exists a map $\mu : M \to \mathfrak{g}^*$, whose coordinate functions $\mu_X : M \to \mathbb{R}$, which are given by $\mu_X(x) = \langle \mu(x), X \rangle$ for $X \in \mathfrak{g}$, satisfy*

$$\mathrm{i}_{X_M}\, \omega = -\,\mathrm{d}\, \mu_X \ , \ \{\mu_X, \mu_Y\} = -\mu_{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$. The map $\mu : M \to \mathfrak{g}^*$ is called the moment map (or momentum map) of the Hamitonian action of G on M.

The first condition says that the infinitesimal vector field X_M for $X \in \mathfrak{g}$ is equal to the Hamiltonian vector field of some function $\mu_X : M \to \mathbb{R}$. Because M is connected this determines the function μ_X up to a constant depending on $X \in \mathfrak{g}$. In addition we require that

$$\mathfrak{g} \to \mathcal{F}(M) \ , \ X \mapsto \mu_X$$

is a linear map, so there exists a function $\mu : M \to \mathfrak{g}^*$ with coordinate functions μ_X for all $X \in \mathfrak{g}$.

Under these assumptions the second condition $\{\mu_X, \mu_Y\} = -\mu_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$ implies that the moment map $\mu : M \to \mathfrak{g}^*$ is equivariant for the two actions. Indeed, the infinitesimal vector field for the coadjoint action is the linear vector field

$$X_{\mathfrak{g}^*} = (-\operatorname{ad} X)^* : \mathfrak{g}^* \to \mathfrak{g}^* \ , \ X_\xi = X_{\mathfrak{g}^*}\xi$$

or, more explicitly

$$\langle X_\xi, Y \rangle = \langle \xi, -[X, Y] \rangle$$

for $\xi \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$. Taking $\xi = \mu(x)$ yields by Definition 5.6

$$(\mathcal{L}_{X_M}\mu_Y)(x) = \{\mu_X, \mu_Y\}(x) = -\mu_{[X,Y]}(x) = -\langle \xi, [X, Y] \rangle \ ,$$

for all $x \in M$ and $X, Y \in \mathfrak{g}$, or equivalently

$$(\mathcal{L}_{X_M}\mu)(x) = X_{\mu(x)}$$

for all $x \in M$ and $X \in \mathfrak{g}$. Hence $(T_x\mu)X_x = X_{\mu(x)}$ which implies that μ intertwines the Hamiltonian action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. In fact the second condition $\{\mu_X, \mu_Y\} = -\mu_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$ is easily seen to be equivalent with the equivariance of the moment map $\mu : M \to \mathfrak{g}^*$.

Example 5.7. *Let $T^*\mathbb{R}^3 = \{(\mathbf{q}, \mathbf{p}); \mathbf{q}, \mathbf{p} \in \mathbb{R}^3\}$ with standard symplectic form $\omega = \sum \mathrm{d}\, p_j \wedge \mathrm{d}\, q_j$. The action $\mathbb{R}^3 \times T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ given by*

$$\mathbf{x}(\mathbf{q}, \mathbf{p}) = (\mathbf{x} + \mathbf{q}, \mathbf{p})$$

is the natural action on phase space induced by the action on configuration space of the translation group $\mathbb{R}^3$. We claim that a (rather than the) moment map $\mu : T^*\mathbb{R}^3 \to \mathbb{R}^3$ is given by the linear momentum $\mu(\mathbf{q}, \mathbf{p}) = \mathbf{p}$. Indeed the function $\mu_j(\mathbf{q}, \mathbf{p}) = p_j$ has Hamiltonian vector field $\partial/\partial q_j$ as shown in Example 3.1, and these vector fields are the infinitesimal generators for the above action of the additive group $\mathbb{R}^3$ acting on the configuration space by translations.

Example 5.8. *Let* $T^*\mathbb{R}^3 = \{(\mathbf{q},\mathbf{p}); \mathbf{q},\mathbf{p} \in \mathbb{R}^3\}$ *with standard symplectic form* $\omega = \sum \mathrm{d}\, p_j \wedge \mathrm{d}\, q_j$*. The action* $\mathrm{SO}_3(\mathbb{R}) \times T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ *given by*

$$a(\mathbf{q},\mathbf{p}) = (a\mathbf{q}, a\mathbf{p})$$

is the natural action on phase space induced by the action on configuration space of the rotation group $\mathrm{SO}_3(\mathbb{R})$*. We claim that the moment map is given by* $(\mathbf{q},\mathbf{p}) \mapsto \mathbf{L} = (\mathbf{q} \times \mathbf{p})$ *which is just the angular momentum vector. For example* $L_3 = q_1p_2 - q_2p_1$ *has Hamilton vector field*

$$v_{L_3} = -q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} - p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}$$

which is the infinitesimal generator of the circle group action by rotations in the planes with coordinates (q_1, q_2) *and* (p_1, p_2)*. Using the formula*

$$a(\mathbf{q} \times \mathbf{p}) = \det(a)(a\mathbf{q} \times a\mathbf{p})$$

for all $a \in \mathrm{O}_3(\mathbb{R})$ *and* $\mathbf{q},\mathbf{p} \in \mathbb{R}^3$ *we also have equivariance for* $\mathrm{SO}_3(\mathbb{R})$*.*

Hence the simplest examples of moment maps are linear and angular momentum. This explains the terminology momentum map for $\mu : M \to \mathfrak{g}^*$ as above. Linear momentum is also called momentum and angular momentum is also called moment of momentum. The word moment reminds one of the rotation group $\mathrm{SO}_3(\mathbb{R})$ of the configuration space $\mathbb{R}^3$, whereas momentum relates to the translation group $\mathbb{R}^3$ of the configuration space $\mathbb{R}^3$. Since the applications of Hamiltonian actions in case of compact Lie groups are the most interesting I prefer the word moment map rather than momentum map. Another (admittedly not so strong) argument is that moment map is just shorter than momentum map.

The moment map is functorial with respect to symmetry breaking. The easy proof is left to the reader as an exercise.

Theorem 5.9. *Suppose a Lie group* G *acts on a symplectic manifold* (M,ω) *in a Hamiltonian way with moment map* $\mu : M \to \mathfrak{g}^*$*. Let* H *be a closed Lie subgroup of* G *with Lie algebra* $\mathfrak{h}$ *as Lie subalgebra of* $\mathfrak{g}$*. Then the action of* H *on* (M,ω) *is Hamiltonian with moment map the composition*

$$\pi \circ \mu : M \to \mathfrak{h}^*$$

of the original moment map $\mu : M \to \mathfrak{g}^*$ *with the natural projection map* $\pi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ *as the dual of the natural inclusion map* $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$*.*

Theorem 5.10. *Let N be a connected manifold and $G \times N \to N$ a smooth action of a connected Lie group G on N. Then the corresponding action $G \times T^*N \to T^*N$ on the cotangent bundle $(T^*N, \omega = \mathrm{d}\,\theta)$ is a Hamiltonian action. For $q \in N$ and $p \in T_q^*N$ the moment map $\mu : T^*N \to \mathfrak{g}^*$ is given by*

$$\mu_X(q,p) = \langle p, X_q \rangle$$

with $X_q \in T_qN$ the infinitesimal vector field of $X \in \mathfrak{g}$ at the point $q \in N$.

Proof. If the action of $a \in G$ on N is denoted by $\lambda_a : N \to N$ then the corresponding action of G on $M = T^*N$, also denoted by $\lambda_a : M \to M$, is given by

$$\lambda_a(q,p) = (\lambda_a q, p \circ T_{\lambda_a q}\lambda_{a^{-1}})$$

for $q \in N$ and $p \in T_q^*N$.

First observe that the tautological 1-form θ, defined by $\theta_\xi = \xi \circ T_\xi \pi$ for all $\xi \in M$, is invariant under the action of G on M. Indeed, for all $a \in G$, $p \in T_q^*N$ and $\xi = (q,p) \in M$ we have

$$(\lambda_a^*\theta)_\xi = \theta_{\lambda_a\xi} \circ T_\xi\lambda_a = ((\lambda_a\xi) \circ T_{\lambda_a\xi}\pi) \circ T_\xi\lambda_a =$$
$$(p \circ T_{\lambda_a q}\lambda_{a^{-1}}) \circ T_{\lambda_a\xi}\pi \circ T_\xi\lambda_a = p \circ T_\xi(\lambda_{a^{-1}} \circ \pi \circ \lambda_a) = \xi \circ T_\xi\pi = \theta_\xi$$

because the projection map $\pi : M \to N$ is equivariant for the actions of G on M and N.

Hence $\mathcal{L}_{X_M}\theta = 0$ for all $X \in \mathfrak{g}$, which by the Cartan formula implies

$$\mathrm{d}(\iota_{X_M}\theta) + \iota_{X_M}(\mathrm{d}\,\theta) = 0\ .$$

Because $\omega = \mathrm{d}\,\theta$ we find that the infinitesimal vector field X_M is equal to the Hamiltonian vector field of the function $\mu_X = \iota_{X_M}\theta$. The equivariance of this map μ follows from

$$\mu_{\mathrm{Ad}(a)X}(\lambda_a\xi) = \langle p \circ T_{\lambda_a q}\lambda_{a^{-1}}, (\mathrm{Ad}(a)X)_{\lambda_a q} \rangle = \langle p, X_q \rangle = \mu_X(\xi)$$

for all a in G. $\square$

The above theorem is the natural generalization of the examples of linear and angular momentum as moment maps for the actions of the translation and rotation groups on a Euclidean vector space.

Theorem 5.11. *Let the symplectic manifold (M, ω) be a finite dimensional complex vector space (V, J) with symplectic form ω the imaginary part of a Hermitian form $h : V \times V \to \mathbb{C}$. Let K be a connected compact Lie group* *and $\rho : K \to \mathrm{U}(V, J, h)$ a unitary representation of K on (V, J, h). Then this action of K on (M, ω) is Hamiltonian with moment map*

$$\mu : M \to \mathfrak{k}^* \ , \ \mu_X(v) = \mathrm{i}h(\rho(X)v, v)/2$$

for $X \in \mathfrak{k}$ and $v \in V$.

Proof. The equivariance of this map $\mu : M \to \mathfrak{k}^*$ follows from

$$\mu_{\mathrm{Ad}(a)X}(\rho(a)v) = \mu_X(v)$$

for $a \in K$, $X \in \mathfrak{k}$ and $v \in V$, which is clear since ρ is a unitary representation. An alternative proof of the equivariance of μ follows from

$$\begin{aligned}\{\mu_X, \mu_Y\}(v) &= \omega(X_V, Y_V)(v) = \Im(h(\rho(X)v, \rho(Y)v)) \\ &= (h(\rho(X)v, \rho(Y)v) - h(\rho(Y)v, \rho(X)v))/2\mathrm{i} \\ &= h(\rho([X, Y])v, v)/2\mathrm{i} = -\mu_{[X,Y]}(v)\end{aligned}$$

for all $X, Y \in \mathfrak{k}$.

In addition this component function μ_X satisfies

$$\langle (\mathrm{d}\,\mu_X)_v, w\rangle = \mathrm{i}(h(\rho(X)w, v) + h(\rho(X)v, w))/2 = -\omega(\rho(X)v, w)$$

since $h(\rho(X)w, v) = -h(w, \rho(X)v) = -\overline{h(\rho(X)v, w)}$ for all $v, w \in V$. Since the infinitesimal vector field X_M of $X \in \mathfrak{k}$ at the point $v \in V$ is equal to $X_v = \rho(X)v$ we get

$$\mathrm{i}_{X_M}\,\omega = -\,\mathrm{d}\,\mu_X$$

for all $X \in \mathfrak{k}$. Hence the Hamiltonian vector field of the function μ_X is equal to X_M. □

Corollary 5.12. *Let $\rho : K \to \mathrm{U}(V, J, h)$ be a unitary representation of a connected compact Lie group K on a finite dimensional Hilbert space (V, J, h). Consider a holomorphic submanifold (M, ω) of the projective space $\mathbb{P}(V)$ as a Kähler manifold for the restriction of the Fubini–Study form ω_{FS}. If M is invariant under K then the action of K on (M, ω) is Hamiltonian with moment map*

$$\mu : M \to \mathfrak{k}^* \ , \ \mu_X([v]) = \frac{h(\rho(X)v, v)}{2\pi\mathrm{i}h(v, v)}$$

for all $X \in \mathfrak{k}$ and $[v] \in M$ with nonzero representative $v \in V$.

Proof. Consider V as symplectic manifold with respect to the normalized symplectic form $-\Im(h)/\pi$. The standard action $\mathrm{U}_1(\mathbb{C}) \times V \to V$ of the circle group $\mathrm{U}_1(\mathbb{C})$ on V is given by $\exp(\mathrm{i}\theta)v = \cos(\theta)v + \sin(\theta)Jv$.

The unit sphere $\iota : \mathbb{S}(V) = \{v \in V; h(v,v) = 1\} \hookrightarrow V$ is invariant under this action of $\mathrm{U}_1(\mathbb{C})$ with quotient $\pi : \mathbb{S}(V) \twoheadrightarrow \mathbb{P}(V) = \mathbb{S}(V)/\,\mathrm{U}_1(\mathbb{C})$. The Fubini–Study form ω_{FS} was defined by $\pi^*\omega_{\mathrm{FS}} = -\iota^*\Im(h)/\pi$. According to Exercise 3.8 the Fubini–Study form ω_{FS} gives a projective line in $\mathbb{P}(V)$ unit area.

All in all, the moment map $\mu : M \to \mathfrak{k}^*$ for the Hamiltonian action of K on (M, ω) with ω the restriction of the Fubini-Study form to M is given by the previous theorem as

$$\mu_X(v) = \frac{h(\rho(X)v, v)}{2\pi\mathrm{i}h(v,v)}$$

under the restriction $h(v,v) = 1$. Indeed moving the i from numerator to denominator gives a minus sign, and together with the factor π in the denominator matches with the normalization of the Fubini–Study form ω_{FS}. This proves the formula. □

It really took a long time to come to the insight that the correct notion for symmetry of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω) is that of a Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$. The oldest examples of momentum (for the translation group on $\mathbb{R}^3$) and angular momentum (for the rotation group on $\mathbb{R}^3$) can be traced back to Galilei and Newton. A fundamental paper by Emmy Noether from 1918 was the first to discuss arbitrary Lie group symmetries in connection with conservation laws [42]. Noether worked in the Lagrangian formalism with variational calculus. A recent book by Yvette Kosmann–Schwarzbach discusses the relevance of this work by Noether throughout the past century [28]. In the Hamiltonian formalism the Noether theorem takes the following form.

Theorem 5.13. *Suppose that the connected Lie group G with Lie algebra $\mathfrak{g}$ acts on a symplectic manifold (M, ω) in a Hamiltonian way with moment map $\mu : M \to \mathfrak{g}^*$. Then a Hamiltonian $H : M \to \mathbb{R}$ is invariant under G if and only if all components μ_X of the moment map Poisson commute with H.*

In particular, invariance of a Hamiltonian H under a Hamiltonian action of a Lie group G leads to as many independent conserved quantities as the dimension of the Lie group G. In short, symmetry is the primary cause for conservation laws.

The breakthrough of the present concept of the moment map μ as a map from the symplectic manifold M to the dual $\mathfrak{g}^*$ of a Lie algebra came in 1970 after its advocation in the text book by Jean-Marie Souriau [52] and the lecture notes by Bertram Kostant [29].

5.3 Symplectic Reduction

Suppose (M, ω) is a connected symplectic manifold, and $G \times M \to M$ is a Hamiltonian action of a connected Lie group G with Lie algebra $\mathfrak{g}$, with moment map $\mu : M \to \mathfrak{g}^*$. By Definition 5.6 this means that

$$\omega_x(X_x, v) = -(\mathrm{d}\,\mu_X)_x(v) = -\langle (T_x\mu)v, X\rangle$$

for all $v \in T_xM$ and $X \in \mathfrak{g}$. This formula is crucial for understanding the rest of this section.

For $x \in M$ the stabilizer algebra $\mathfrak{g}_x = \{X \in \mathfrak{g}; X_x = 0\}$ of x in $\mathfrak{g}$ is the Lie algebra of the stabilizer group $G_x = \{a \in G; ax = x\}$ of x in G. Now the tangent map $T_x\mu : T_xM \to \mathfrak{g}^*$ has kernel and image equal to

$$\mathrm{Ker}(T_x\mu) = (T_x(Gx))^{\omega_x} \ , \ \mathrm{Im}(T_x\mu) = \mathfrak{g}_x^{\perp}$$

with $T_x(Gx) = \{X_x; X \in \mathfrak{g}\}$ the tangent space at $x \in M$ to the orbit of Gx and the superscript ω_x denotes the orthogonal complement with respect to ω_x on T_xM, and the superscript $\perp$ denotes the annihilator of $\mathfrak{g}_x$ in the dual space $\mathfrak{g}^*$. Therefore the following lemma is clear.

Lemma 5.14. *The Hamiltonian action of G on (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$ is locally free at $x \in M$ (by definition $\mathfrak{g}_x = 0$) if and only if x is a regular point of μ (by definition $T_x\mu$ is a surjection).*

Let M^r be the set of all regular points of the moment map μ in M. By equivariance M^r is invariant under G, and the action of G on M^r is locally free. For $x \in M^r$ and $\xi = \mu(x)$ the locus

$$M^r \cap \mu^{-1}(\xi)$$

is a smooth submanifold of M^r by the implicit function theorem, and the restriction of ω_x to the tangent space

$$T_x(\mu^{-1}(\xi)) = \mathrm{Ker}(T_x\mu) = (T_x(Gx))^{\omega_x}$$

has kernel

$$T_x(Gx) \cap (T_x(Gx))^{\omega_x} = T_x(Gx) \cap \mathrm{Ker}(T_x\mu) = \{X_x; X \in \mathfrak{g}_\xi\} = T_x(G_\xi x)$$

with $\mathfrak{g}_\xi$ the stabilizer algebra and G_ξ the stabilizer group of $\xi \in \mathfrak{g}^*$ for the coadjoint action on $\mathfrak{g}^*$. Hence the next lemma is again clear.

Lemma 5.15. *If $\iota_\xi : M^r \cap \mu^{-1}(\xi) \hookrightarrow M$ is the natural inclusion map, then the pull-back $\iota_\xi^*\omega$ is a closed 2-form on $M^r \cap \mu^{-1}(\xi)$, and the leaves of the null foliation of $\iota_\xi^*\omega$ are just the orbits of G_ξ on $M^r \cap \mu^{-1}(\xi)$.*

We need the following result on Lie group actions.

Theorem 5.16. *If a compact Lie group G acts freely on a manifold M then the quotient space M/G has a natural manifold stucture, and the quotient map $\pi : M \twoheadrightarrow M/G$ is a principal fibration.*

If the action of the compact Lie group G on the manifold M is no longer free, then the quotient space M/G might get singular points. However it is a Hausdorff topological space. In differential geometry these orbit spaces are called orbifolds, a terminolgy that goes back to Thurston. Even though they might have singular points a good deal of the manifold properties goes through. This result has the following consequence in the situation at hand.

Theorem 5.17. *Let us suppose that the action of G on M is effective, which we tacitly will assume by replacing G by the factor group G/N with $N = \cap_x G_x$ a normal subgroup of G. Let M^r be the set of regular points for μ in M, where the action of G is locally free. Let M^{sr} be the subset of M^r of strongly regular points for μ in M, where the action of G is free. Assume that $M^{sr} \hookrightarrow M^r \hookrightarrow M$ are open dense subsets.*

Suppose that the value $\xi \in \mu(M^{sr})$ has a compact stabilizer group G_ξ. By the above theorem the quotient space $M_\xi^{sr} = \{M^{sr} \cap \mu^{-1}(\xi)\}/G_\xi$ is a manifold, and the quotient map

$$\pi_\xi : \{M^{sr} \cap \mu^{-1}(\xi)\} \twoheadrightarrow M_\xi^{sr}$$

is a principal fibration. Moreover the manifold M_ξ^{sr} inherits a natural symplectic form ω_ξ characterized by

$$\pi_\xi^*\omega_\xi = \iota_\xi^*\omega$$

with

$$\iota_\xi : \{M^{sr} \cap \mu^{-1}(\xi)\} \hookrightarrow M$$

the natural embedding. The symplectic manifold (M_ξ^{sr}, ω_ξ) is called (the strongly regular part of) the reduced phase space or the symplectic quotient at $\xi \in \mathfrak{g}^*$. Likewise denote

$$M_\xi^r = \{M^r \cap \mu^{-1}(\xi)\}/G_\xi \ , \ M_\xi = \{\mu^{-1}(\xi)\}/G_\xi$$

as topological Hausdorff spaces.

Finally if the fiber $\mu^{-1}(\xi)$ is compact then

$$M_\xi^{sr} \hookrightarrow M_\xi^r \hookrightarrow M_\xi$$

gives a compactification of the reduced symplectic manifold (M_ξ^{sr}, ω_ξ) by a topological Hausdorff space. The partial compactification $M_\xi^{sr} \hookrightarrow M_\xi^r$ adds so called finite quotient singular points, whereas $M_\xi^r \hookrightarrow M_\xi$ adds worse singular points.

This theorem describes the symplectic reduction method. It is quite a mouthful, but the proof is really short given our discussion before the theorem. The only thing that might not be obvious is whether ω_ξ is a closed form. Because π_ξ is a submersion the pullback π_ξ^* is an injection on differential forms. So $\mathrm{d}\,\omega_\xi = 0$ if and only if $\pi_\xi^*(\mathrm{d}\,\omega_\xi) = 0$, which follows from

$$\pi_\xi^*(\mathrm{d}\,\omega_\xi) = \mathrm{d}(\pi_\xi^*\omega_\xi) = \mathrm{d}(\iota_\xi^*\omega) = \iota_\xi^*(\mathrm{d}\,\omega)$$

and the fact that ω is closed.

For general Hamiltonian Lie group actions on symplectic manifolds the symplectic reduction theorem goes back (independently of each other) to Marsden–Weinstein [34] and to Meyer [36]. However in particular examples the reduction procedure had been carried out long before. For example, for a Hermitian vector space (V, J, h) the action of the circle group $\mathrm{U}_1(\mathbb{C})$ on V by $\exp(\mathrm{i}\theta)v = \cos(\theta)v + \sin(\theta)Jv$ with infinitesimal generator $\mathrm{d}\,/\,\mathrm{d}\,\theta = J$ is Hamiltonian with moment map $\mu_J(v) = -h(v,v)/2$ by Theorem 5.11. The construction of the Fubini–Study form ω_{FS} on $\mathbb{P}(V)$ in Example 3.19 is just the symplectic reduction method on the inverse image $\mu_J^{-1}(-1/2\pi)$.

Lemma 5.18. *Suppose $G \times M \to M$ is a Hamiltonian action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. If $\eta = \mathrm{Ad}^*(a)\xi$ then the action*

$$\lambda_a : \{M^{sr} \cap \mu^{-1}(\xi)\} \to \{M^{sr} \cap \mu^{-1}(\eta)\}$$

by $a \in G$ induces a natural symplectomorphism

$$(M_\xi^{sr}, \omega_\xi) \to (M_\eta^{sr}, \omega_\eta)$$

of reduced symplectic manifolds.

This lemma is obvious since the moment map is equivariant for the action of G on M and the coadjoint action of G on $\mathfrak{g}^*$. The next theorem is a further elaboration of the Noether theorem using the concept of symplectic reduction.

Theorem 5.19. *Suppose $G \times M \to M$ is a Hamiltonian action of a Lie group G with Lie algebra $\mathfrak{g}$ on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. Suppose the Hamiltonian function $H \in \mathcal{F}(M)$ is invariant under G. Then the Hamiltonian flow of v_H leaves $\mu^{-1}(\xi)$ invariant, and commutes with the action of G_ξ on $\mu^{-1}(\xi)$, and so it induces a canonical flow on the reduced symplectic manifold (M_ξ^{sr}, ω_ξ). This flow on M_ξ^{sr} is Hamiltonian with reduced Hamiltonian H_ξ characterized by*

$$H_\xi \circ \pi_\xi = H \circ \iota_\xi$$

with $\iota_\xi : \{M^{sr} \cap \mu^{-1}(\xi)\} \hookrightarrow M$ and $\pi_\xi : \{M^{sr} \cap \mu^{-1}(\xi)\} \twoheadrightarrow M_\xi^{sr}$ the natural inclusion and projection maps of the previous theorem.

Proof. Replace M by M^{sr} and hence M_ξ by M_ξ^{sr}. The symplectic form ω_ξ on the reduced phase space M_ξ was defined by the sequence

$$M \xleftarrow{\iota_\xi} \mu^{-1}(\xi) \xrightarrow{\pi_\xi} M_\xi = \mu^{-1}(\xi)/G_\xi$$

and the relation $\pi_\xi^* \omega_\xi = \iota_\xi^* \omega$. Let ϕ_t be the flow on M of the Hamilton vector field v_H of the function H. Since H is invariant under G all component functions μ_X of the moment map Poisson commute with H. Hence the flow ϕ_t preserves the fiber $\mu^{-1}(\xi)$ and commutes with the action of G_ξ on $\mu^{-1}(\xi)$. The induced flow $\phi_{\xi,t}$ on M_ξ is defined by $\phi_{\xi,t}\pi_\xi = \pi_\xi \iota_\xi^{-1} \phi_t \iota_\xi$. Hence

$$\pi_\xi^* \phi_{\xi,t}^* \omega_\xi = \iota_\xi^* \phi_t^* \iota_\xi^{*-1} \pi_\xi^* \omega_\xi = \iota_\xi^* \phi_t^* \omega = \iota_\xi^* \omega = \pi_\xi^* \omega_\xi$$

and because π_ξ is a submersion we conclude that $\phi_{\xi,t}^* \omega_\xi = \omega_\xi$. Therefore the flow $\phi_{\xi,t}$ on M_ξ preserves the reduced symplectic form ω_ξ as should.

It remains to check that the infinitesimal generator of the flow $\phi_{\xi,t}$ is equal to the Hamilton field of H_ξ, where the function H_ξ on M_ξ is characterized by $H_\xi \pi_\xi = H \iota_\xi$. If we denote for a point $x \in \mu^{-1}(\xi)$ and a tangent vector $v_x \in T_x \mu^{-1}(\xi)$ by $v_{\pi_\xi x} \doteq (T_x \pi_\xi) v_x \in T_{\pi_\xi x} M_\xi$ the image of v_x under $T_x \pi_\xi$ then

$$\begin{gathered}(\mathrm{d}\, H_\xi)_{\pi_\xi x} v_{\pi_\xi x} = (\mathrm{d}(H_\xi \pi_\xi))_x v_x = (\mathrm{d}\, H \iota_\xi)_x v_x = (\iota_\xi^* \,\mathrm{d}\, H)_x v_x = \\ -(\iota_\xi^* \omega)_x((v_H)_x, v_x) = -(\pi_\xi^* \omega_\xi)_x((v_H)_x, v_x) = -(\omega_\xi)_{\pi_\xi x}((v_H)_{\pi_\xi x}, v_{\pi_\xi x})\end{gathered}$$

which in turn implies that the Hamilton field of the function H_ξ on (M_ξ, ω_ξ) is equal to the image under $T\pi_\xi$ of the vector field v_H restricted to $\mu^{-1}(\xi)$. In other words, the Hamiltonian flow ϕ_t of H on $\mu^{-1}(\xi)$ projects via π_ξ to the Hamiltonian flow $\phi_{\xi,t}$ of H_ξ on M_ξ. $\square$

Definition 5.20. *Equilibrium points of the reduced Hamiltonian system $(M_\xi^{sr}, \omega_\xi, H_\xi)$ are called relative equilibria of the original Hamiltonian system (M, ω, H) with symmetry group G.*

The notion of relative equilibria was already encountered (by Christiaan Huygens) in the example of the spherical pendulum.

5.4 Symplectic Reduction for Cotangent Bundles

Let N be a connected manifold and let $G \times N \to N$ be a smooth action of a connected Lie group G on N. The induced action $G \times T^*N \to T^*N$ on the cotangent bundle $(M = T^*N, \omega = \mathrm{d}\,\theta)$ is Hamiltonian by Theorem 5.10 with moment map $\mu : T^*N \to \mathfrak{g}^*$ given by (the equality of functions on N)

$$\mu_X \circ \alpha = \mathrm{i}_{X_N}\, \alpha$$

with $\alpha \in \Omega^1(N)$ also viewed as a section in the cotangent bundle $M \to N$ and $X_N \in \mathcal{X}(N)$ the infinitesimal vector field of $X \in \mathfrak{g}$ on N. Under a suitable condition on the regular value $\xi \in \mu(M)$, namely the existence of a smooth 1-form α_ξ on N as below, the reduced symplectic manifold (M_ξ, ω_ξ) is given by the following theorem of Abraham and Marsden [1].

Theorem 5.21. *Suppose $G \times N \to N$ is a smooth action with induced moment map $\mu : T^*N \to \mathfrak{g}^*$ given by*

$$\mu_X \circ \alpha = \mathrm{i}_{X_N}\, \alpha$$

with $\alpha \in \Omega^1(N)$ and $X_N \in \mathcal{X}(N)$ the infinitesimal vector field of $X \in \mathfrak{g}$. Let $\xi \in \mu(M)$ be a regular value of the moment map, and suppose that there exists a G_ξ-invariant 1-form $\alpha_\xi \in \Omega^1(N)$ with $\mu \circ \alpha_\xi$ equal to the constant function ξ on N. If G_ξ is compact and acts freely on $\mu^{-1}(\xi)$ then there exists a smooth symplectic embedding

$$\phi_\xi : M_\xi \hookrightarrow T^*(N/G_\xi)$$

from the reduced symplectic manifold (M_ξ, ω_ξ) at the point $\xi \in \mathfrak{g}^*$ into the cotangent bundle $T^*(N/G_\xi)$ equipped with the twisted symplectic form

$$\mathrm{d}\,\theta + \pi^*\,\mathrm{d}\,\alpha_\xi = \mathrm{d}(\theta + \pi^*\alpha_\xi)\ .$$

Here $\mathrm{d}\,\theta$ is the canonical symplectic form on $T^*(N/G_\xi)$ and

$$\pi : T^*(N/G_\xi) \to N/G_\xi$$

is the natural projection map.

Proof. For $0 \in \mathfrak{g}^*$ a (strongly) regular value of the moment map μ the space

$$\mu^{-1}(0) = \{p \in T_q^* N; p(X_q) = 0 \; \forall X \in \mathfrak{g}\} = \{p \in T_q^* N; p(T_q(Gq)) = 0\}$$

is a smooth submanifold of M, and the natural map

$$\phi_0 : M_0 = \mu^{-1}(0)/G \to T^*(N/G)$$

is a diffeomorphism, and even a symplectomorphism from the symplectic reduction (M_0, ω_0) to the cotangent bundle $(T^*(N/G), \mathrm{d}\,\theta)$. Likewise for $\xi \in \mathfrak{g}^*$ we have an embedding

$$\phi_{0,\xi} : \mu^{-1}(0)/G_\xi \to T^*(N/G_\xi)$$

with the property that $\pi_{0,\xi}^* \phi_{0,\xi}^*(\mathrm{d}\,\theta) = \iota_0^* \omega$ on $\mu^{-1}(0)$. By abuse of notation $\mathrm{d}\,\theta$ is now the canonical symplectic form on $T^*(N/G_\xi)$, and the maps

$$\iota_0 : \mu^{-1}(0) \hookrightarrow M \quad , \quad \pi_{0,\xi} : \mu^{-1}(0) \to \mu^{-1}(0)/G_\xi$$

are the natural embedding and the natural submersion respectively.

Under the hypothetical existence of a differential $\alpha_\xi \in \Omega^1(N)$, which is invariant under G_ξ and satisfies $\mu \circ \alpha_\xi = \xi$, the translation map

$$\Omega^1(N) \to \Omega^1(N) \; , \; \alpha \mapsto (\alpha - \alpha_\xi)$$

induces under the usual assumption that both 0 and ξ are (strongly) regular values of the moment map μ a diffeomorphism

$$\psi_\xi : \mu^{-1}(\xi) \to \mu^{-1}(0)$$

of manifolds. Since ψ_ξ is equivariant for the locally free (free) action of G_ξ there is a natural embedding

$$\phi_\xi : M_\xi \hookrightarrow T^*(N/G_\xi)$$

such that the composition $\phi_{0,\xi}\pi_{0,\xi}\psi_\xi = \phi_\xi\pi_\xi$ holds. Here $\pi_\xi : \mu^{-1}(\xi) \to M_\xi$ is the quotient map for the action of G_ξ in the usual notation. The discussion so far can be summarized in a commutative diagram

$$\begin{array}{ccc} \mu^{-1}(\xi) & \xrightarrow{\psi_\xi} & \mu^{-1}(0) \\ {\scriptstyle \pi_\xi}\downarrow & & \downarrow{\scriptstyle \phi_{0,\xi}\pi_{0,\xi}} \\ M_\xi & \xrightarrow{\phi_\xi} & T^*(N/G_\xi) \end{array}$$

The twisted cotangent bundle was introduced in Exercise 3.6. Using Exercise 5.9 the diffeomorphism $\psi_\xi : \mu^{-1}(\xi) \to \mu^{-1}(0)$ induced by the map

$$\Omega^1(N) \to \Omega^1(N)\ ,\ \alpha \mapsto (\alpha - \alpha_\xi)$$

relates the restriction to $\mu^{-1}(\xi)$ of the canonical symplectic form ω on T^*N to the restriction to $\mu^{-1}(0)$ of the twisted symplectic form ω_{α_ξ} on T^*N. This induces a smooth symplectic embedding

$$\phi_\xi : M_\xi \hookrightarrow T^*(N/G_\xi)$$

relative to reduced symplectic form ω_ξ on the reduced space M_ξ and the twisted symplectic form $\mathrm{d}\,\theta+\pi^*\,\mathrm{d}\,\alpha_\xi$ on the cotangent bundle $T^*(N/G_\xi)$. □

Remark 5.22. *The following remark concerning the symplectic reduction of the cotangent bundle is due to Ortega and Ratiu [43]. Under the above assumptions there is a commutative diagram*

$$\begin{array}{ccccc}
\mu^{-1}(\xi) & \longrightarrow & M_\xi = \mu^{-1}(\xi)/G_\xi & \xrightarrow{\phi_\xi} & T^*(N/G_\xi) \\
\Big\downarrow \iota & & \cong \Big\downarrow \delta & & \Big\downarrow \pi \\
\mu^{-1}(G\xi) & \longrightarrow & M_\xi \cong \mu^{-1}(G\xi)/G & \xrightarrow{\phi_{G\xi}} & T^*(N/G)
\end{array}$$

with the top horizontal line as discussed in the above theorem. The left vertical arrow in an immersion, the middle vertical arrow a diffeomorphism and the right vertical arrow a submersion. The top horizontal map ϕ_ξ is an immersion with codimension of the image equal to the dimension of G/G_ξ. The bottom horizontal map $\phi_{G\xi}$ is a submersion with dimension of the fiber equal to the dimension of G/G_ξ. In turn this realizes the reduced manifold M_ξ as a fiber bundle with base $T^*(N/G)$ and fiber the coadjoint orbit $G\xi$. The map ϕ_ξ is a diffeomorphism of the reduced manifold M_ξ onto the cotangent bundle $T^*(N/G)$ if and only if $G_\xi = G$.

Under suitable conditions there is a natural choice for the smooth 1-form α_ξ on N with values in $\mu^{-1}(\xi)$ as required in the above theorem. Suppose that the stabilizer group G_ξ of $\xi \in \mathfrak{g}^*$ is equal to all of G. In addition let G be compact and act freely on N, so that the quotient space N/G is a manifold. Under these assumptions the map $\phi_\xi : M_\xi \hookrightarrow T^*(N/G)$ is a diffeomorphism. The element ξ defines a bi invariant 1-form ξ on G and likewise an invariant 1-form ξ on any orbit of G on N. Let g be a Riemannian metric on N, which is invariant under the action of G. Using the Riemannian structure there exists a unique 1-form α_ξ on N, which vanishes in the direction normal to the orbits and restricts to ξ along the orbits of G on N. By Theorem 5.10 we get $\mu \circ \alpha_\xi = \xi$ as desired.

Theorem 5.23. *Let (N, g) be a Riemannian manifold with a compact Lie group G acting freely on N by isometries. Let $\xi \in \mathfrak{g}^*$ be a regular value of the moment map $\mu : T^*N \to \mathfrak{g}^*$ with $G_\xi = G$. Then the smooth map $\phi_\xi : M_\xi \to T^*(N/G)$ is a symplectomorphism from the reduced symplectic manifold (M_ξ, ω_ξ) onto the cotangent bundle $T^*(N/G)$ with its canonical symplectic form.*

Proof. By our discussion above we only have to show that $\mathrm{d}\,\alpha_\xi = 0$, which in turn implies that the twisted symplectic form and the canonical symplectic form on $T^*(N/G)$ coincide.

Any left and right invariant 1-form α on a Lie group is closed. Indeed, this follows from the Cartan formula $\mathcal{L}_v\alpha = \mathrm{d}\,\mathrm{i}_v\,\alpha + \mathrm{i}_v\,\mathrm{d}\,\alpha$. If v is a left invariant vector field X on G and α a left invariant 1-form on G, then $\mathrm{i}_X\,\alpha$ is a left invariant and hence constant function on G. Therefore $\mathrm{d}\,\mathrm{i}_X\,\alpha = 0$. On the other hand, the flow of X is given by right multiplication with $\exp(tX)$. Since α is also right invariant we get $\mathcal{L}_X\alpha = 0$. Hence $\mathrm{i}_X\,\mathrm{d}\,\alpha = 0$ for all $X \in \mathfrak{g}$ and so $\mathrm{d}\,\alpha = 0$.

Applied to our setting this means that for a regular value $\xi \in \mathfrak{g}^*$ with $\mathfrak{g}_\xi = \mathfrak{g}$ the 1-form ξ on G is closed. In turn this implies that the 1-form $\alpha_\xi \in \Omega^1(N)$ is also closed. Indeed, in tubular neighborhood coordinates around an orbit Gx in N the form α_ξ is just the pullback of ξ under the normal bundle projection map $N(Gx) \to Gx$. Since pullback commutes with exterior derivative we conclude that $\mathrm{d}\,\alpha_\xi = 0$, which in turn implies that the twisted symplectic structure on the cotangent bundle $T^*(N/G)$ is just the canonical symplectic structure. □

Conclusion 5.24. *Let $G \times N \to N$ be an action of a compact connected Lie group on a Riemannian manifold (N, g) preserving the Riemannian metric. Let θ be the tautological 1-form on T^*N and $\omega = \mathrm{d}\,\theta$ the canonical symplectic form. The action of G on T^*N is Hamiltonian with moment map*

$$\mu_X \circ \alpha = \mathrm{i}_{X_N}\,\alpha$$

for all $X \in \mathfrak{g}$ and $\alpha \in \Omega^1(N)$, also viewed as section of the natural cotangent bundle projection $\pi : T^*N \to N$. The Riemannian metric induces a vector bundle isomorphism $g : TN \to T^*N$ mapping a vector field $v \in \mathcal{X}(N)$ to the corresponding 1-form $\alpha = g(v) \in \Omega^1(N)$. Let $K(\alpha) = g(v, v)/2$ be *the kinetic energy viewed as function on T^*N, which was also encountered before in Theorem 3.12. Let $V \in \mathcal{F}(N)$ be a potential energy function, which is invariant under the action of G. If the Hamiltonian H is defined as the sum of kinetic and potential energy*

$$H = (K + V \circ \pi) : T^*N \to \mathbb{R}$$

then the Hamiltonian system

$$(T^*N, \mathrm{d}\,\theta, H)$$

is called the Newtonian system on the Riemannian manifold (N, g) with potential $V \in \mathcal{F}(N)$.

Let the action $G \times N \to N$ be free, and let $\xi \in \mathfrak{g}^*$ be a regular value of the moment map μ with stabilizer group G_ξ equal to all of G. Then the symplectic reduction at the point ξ of the Newtonian system $(T^*N, \mathrm{d}\,\theta, H)$ is again a Newtonian system. The reduced Newtonian system lives on the cotangent bundle $T^*(N/G)$ of N/G with its canonical symplectic structure. The reduced Riemannian metric on N/G is the natural one induced from the Riemannian metric on N. The reduced Hamiltonian H_ξ has reduced kinetic energy corresponding to this reduced Riemannian metric. However the reduced (or effective or amended) potential energy is given by

$$V_\xi = V + K(\alpha_\xi)$$

with V the natural function on N/G obtained from the original invariant potential V on N. Here we use that $g(v, w) = 0$ for $g(v) = \alpha_\xi$ and $g(w)$ the pull back under the quotient map $N \twoheadrightarrow N/G$ of an element of $\Omega^1(N/G)$. The stationary points of the effective potential V_ξ are equilibrium points for the reduced Newtonian system and are called relative equilibria for the original Newtonian system.

The above result in the setting of a general Lie group symmetry goes back to the work of Smale [50] with refinements due to Satzer [48] and Kummer [30]. But in particular examples, notably with symmetry group the circle group $\mathrm{U}_1(\mathbb{C})$ or the rotation group $\mathrm{SO}_3(\mathbb{R})$, this is the truely centuries old idea that polar or spherical coordinates in problems of circular or rotational symmetry are helpful to reduce the number of variables.

Example 5.25. *The spherical coordinates on $\mathbb{S}^2$ minus north and south pole are given by*

$$(\mathbb{R}/2\pi\mathbb{Z}) \times (0, \pi) \ni (\phi, \theta) \mapsto \mathbf{r} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \in \mathbb{S}^2$$

and in these coordinates the Riemannian metric becomes

$$(\mathrm{d}\, s)^2 = (\sin\theta)^2(\mathrm{d}\,\phi)^2 + (\mathrm{d}\,\theta)^2$$

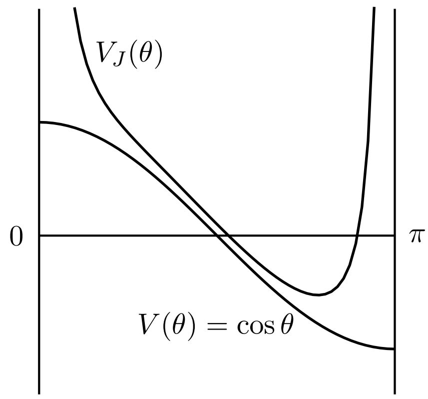
by the chain rule. The Lie group $G = \mathbb{R}/2\pi\mathbb{Z}$ *acts on* $\mathbb{S}^2$ *by rotations around the third axis. The Lie algebra* $\mathfrak{g} = \mathbb{R}$ *has standard generator the vector field* $\mathrm{d}\,/\,\mathrm{d}\,\phi$*, and the dual vector space* $\mathfrak{g}^* = \mathbb{R}$ *has standard generator the* 1*-form* $\mathrm{d}\,\phi$*. In these coordinates the total energy* H *and the angular momentum* $J = L_3$ *of the spherical pendulum become*

$$H = (\dot{\phi}\sin\theta)^2/2 + (\dot{\theta})^2/2 + \cos\theta\ ,\ J = \dot{\phi}(\sin\theta)^2$$

and so the effective potential $V_J(\theta)$ *is given by*

$$H_J = (\dot{\theta})^2/2 + V_J(\theta)\ ,\ V_J(\theta) = \cos\theta + \frac{J^2}{2(\sin\theta)^2}$$

relative to the canonical symplectic form $\mathrm{d}\,\dot{\theta} \wedge \mathrm{d}\,\theta$*.*



The stationary points of the effective potentials V_J *for* $J \neq 0$ *correspond to the stable relative equilibria of the spherical pendulum as found by Huygens. Indeed, the stability of these relative equilibria is a consequence of the fact that these stationary points are nondegenerate minima.*

5.5 Geometric Invariant Theory

Let K be a closed connected subgroup of the unitary group $\mathrm{U}(V,h)$ of a complex vector space V, equipped with a Hermitian inner product h. In turn K is a compact connected Lie group, and any such K occurs in the above way for some pair (V, h). The complexification G of K is a holomorphic (and even algebraic) subgroup of the general linear group $\mathrm{GL}(V)$. By definition any such G is called a complex reductive algebraic group. The remarkable interplay between a connected compact linear Lie group K and its complex connected reductive algebraic complexification G goes back to Hermann Weyl, who called it the "unitary trick". The unitary trick gives a bridge between the smooth topological world of connected compact Lie groups and the algebraic geometric world of complex connected reductive algebraic groups.

Theorem 5.26. *Let $\rho : G \to \mathrm{GL}(W)$ be a representation of a reductive complex Lie group G on a finite dimensional complex vector space W. Then any invariant linear subspace U of W has an invariant complement $U^\perp$, and so $W = U \oplus U^\perp$ is a direct sum of two subrepresentations.*

Proof. Given a compact real form K of G we can average an arbitrary Hermitian inner product on W over K with respect to the normalized Haar measure, and obtain a Hermitian inner product that is invariant under K. Let $U^\perp$ be the orthogonal complement of U with respect to this Hermitian form. Then the direct sum $W = U \oplus U^\perp$ is invariant under K, or equivalenty the orthogonal projection $P : W \to U$ (with kernel $U^\perp$) commutes with K. Let $\sigma : G \to \mathrm{GL}(\mathrm{End}(W))$ be the natural representation defined by $\sigma(a)A = \rho(a)A\rho(a^{-1})$ for $a \in G$ and $A \in \mathrm{End}(W)$. Hence $P \in \mathrm{End}\, W$ is a fixed vector for K, and hence also for G. In turn this implies that G preserves the decomposition $W = U \oplus U^\perp$. $\square$

The following theorem was obtained by Weyl as an application of the unitary trick.

Theorem 5.27. *Let $\rho : G \to \mathrm{GL}(W)$ be a representation of a reductive complex Lie group G on a finite dimensional complex vector space W. Let $PW = \oplus P^d W$ be the commutative algebra of polynomial functions on W, and let $P\rho = \oplus P^d \rho : G \to \mathrm{GL}(PW)$ be the natural representation of G on PW, defined by $P\rho(a)f(w) = f(\rho(a^{-1})w)$. Let $(PW)^G$ be the algebra of invariant polynomials on W. Then there exists a linear (Reynolds) operator $R : PW \to PW$, such that $R(P^d W) \subset (P^d W)^G$ for all d (so R preserves the degree), $R^2 = R$ (so R is a projection operator) and $R(fg) = fR(g)$ for $f \in (PW)^G$ and $g \in PW$.*

Proof. Just take for the Reynolds operator

$$Rf(w) = \int_K f(\rho(a^{-1})w)\mathrm{d}\mu(a)$$

with μ the normalized Haar measure on K, and use that $(PW)^G = (PW)^K$. $\square$

Using the Hilbert basis theorem we obtain the finite generation for the algebra of invariant polynomials of a reductive group.

Theorem 5.28. *Let $\rho : G \to \mathrm{GL}(W)$ be a representation of a reductive complex Lie group G on a finite dimensional vector space W. Then the commutative algebra $(PW)^G$ is finitely generated.*

Proof. Let I^+ be the ideal of PW generated by the homogeneous invariant polynomials of positive degree. By the Hilbert basis theorem any ideal of $P(W)$ is finitely generated, so

$$I^+ = f_1 PW + \cdots + f_k PW$$

for some homogeneous invariant polynomials $f_1, \cdots, f_k$ of positive degree. We claim that $f_1, \cdots, f_k$ generate the commutative algebra $(PW)^G$. Indeed, let $f \in (PW)^G$ be a homogeneous invariant polynomial of degree $d > 0$. Then $f \in I^+$ and therefore

$$f = f_1 g_1 + \cdots + f_k g_k$$

for some homogeneous polynomials $g_1, \cdots, g_k$ in PW of degree strictly less than d. By application of the Reynolds operator R we can assume that $g_1, \cdots, g_k$ are invariant polynomials of degree strictly less than d. Hence the result follows by induction on the degree d of the invariant polynomial f. $\square$

The projective space $\mathbb{P}(V)$ has the polynomial algebra PV as the graded coordinate algebra. Suppose M is a compact holomorphic (hence complex algebraic by the GAGA principle of Serre) submanifold of $\mathbb{P}(V)$ with coordinate algebra $PV/I(M)$ and $I(M)$ the graded ideal of polynomials vanishing on M. If G leaves the space M invariant, then we like to describe the "points" of the quotient space $M /\!\!/ G$ with coordinate algebra $(PV)^G/I(M)$, which is finiteley generated by Weyl's theorem.

Mumford's answer to this question was given in his book on Geometric Invariant Theory from 1965 (when Mumford was just 28 years old), and goes as follows. A vector $[v] \in M$ is called unstable if the orbit Gv contains the origin $0 \in V$ in its closure. If Gv does not contain 0 in its closure then $[v] \in M$ is called semistable. In the latter case $[v] \in M$ is called stable if both the orbit Gv is closed in V and the stabilizer group G_v is finite. If

$[v] \in M$ is semistable but not stable then it is called polystable (or strictly semistable). Therefore we have disjoint unions

$$M = M^{ss} \sqcup M^{us} \ , \ M^{ss} = M^{s} \sqcup M^{ps}$$

of the semistable locus M^{ss} and the unstable locus M^{us} (also called the nilcone) for M, and of the stable locus M^s and the polystable locus M^{ps} for M^{ss}.

The minimal semistable locus M^{mss} is the set of all semistable points $[v] \in M$ with Gv closed in V. Clearly

$$M^{mss} = M^{s} \sqcup M^{mps}$$

with M^{mps} the (minimal polystable) complement of M^s in M^{mss}. In general the algebra $(PV)^G$ of invariant polynomials need not separate the orbits of G on M. Indeed, if $G[v]$ lies in the closure of $G[w]$ for $[v], [w] \in M$ then the graded algebra $(PV)^G = \oplus (P^d V)^G$ takes the same value on $G[v]$ and $G[w]$. Mumford showed that essentially this is the only exception for separation of orbits by invariants. In other words, the algebra of invariants is the coordinate algebra of

$$M /\!\!/ G = M^{mss}/G$$

whose "points" are the orbits of G in M^{mss}. The results below together with Theorem 5.17 imply that this "GIT-quotient" is smooth at those orbits $G[v]$ with Gv closed in V and G_v trivial. It is possibly mildly singular (with finite quotient singularities) at orbits in M^s, and most singular at orbits of M^{mps}. The following result is due to Kempf and Ness [26], and we refer to lecture notes by Woodward for an exposition of the proof [60]. Another recent reference is Georgoulas, Robbin and Salamon [14].

Theorem 5.29. *For $K < \mathrm{U}(V,h)$ a compact connected linear Lie group and $G < \mathrm{GL}(V)$ its reductive complexification the restriction of the norm function $v \mapsto h(v,v)/2$ on V to an orbit Gv has as critical points only minima, and such critical points exist if and only if Gv is a closed subset of V. If this minimum is attained at v then Kv consists of all minima of the norm function on Gv and the transverse Hessian of the norm function along Kv is nondegenerate.*

The Kempf–Ness theorem implies that $[v] \in M$ is minimal semistable if and only if Gv is closed in V. The following result of Guillemin and Sternberg [17] is an important consequence. It can be considered as an elaboration of the unitary trick for GIT.

Theorem 5.30. *Under suitable regularity assumptions we have an isomorphism $M /\!\!/ G \cong M_0$ between the GIT-quotient $M /\!\!/ G$ and the symplectic quotient $M_0 = \mu^{-1}(0)/K$. Here $\mu : M \to \mathfrak{k}^*$ is the moment map for the Hamiltonian action of K on the symplectic manifold M with repect to the Fubini–Study form ω_{FS} on M. Finally the symplectic quotient M_0 inherits a natural structure of Kähler manifold with Kähler form ω_0 obtained by symplectic reduction at $0 \in \mathfrak{k}^*$ of the Fubini–Study form ω_{FS} on M.*

Proof. The regularity assumption is that 0 is a regular value of the moment map $\mu : M \to \mathfrak{k}^*$ and therefore $\mu^{-1}(0)$ is a smooth submanifold of M and the action of K on $\mu^{-1}(0)$ is locally free. By Corollary 5.12 the moment map is given by

$$\mu : M \to \mathfrak{k}^* \ , \ \mu_X([v]) = \frac{h(Xv, v)}{2\pi\mathrm{i}h(v, v)}$$

for all $X \in \mathfrak{k}$ and $[v] \in M$.

In turn we get $\mu_X([v]) = 0$ for $[v] \in M$ and all $X \in \mathfrak{k}$ if and only if (say $h(v, v) = 1$) we have $h(Xv, v) = 0$ for all $X \in \mathfrak{k}$, or equivalently if and only if $h(Zv, v) = 0$ for all $Z \in \mathfrak{g}$. But this means that the norm function on Gv has a critical point at v. By the theorem of Kempf and Ness this implies that $[v] \in M^{mss}$ and we conclude that $\mu([v]) = 0$ if and only if the norm function of the orbit Gv has a minimum along Kv. Hence $\mu^{-1}(0)/K = M_0 \cong M /\!\!/ G$. $\square$

The Guillemin–Sternberg theorem is expressed as the principle that "quantization commutes with reduction", or in short $[Q, R] = 0$. The quantization of the symplectic manifold $(M, \omega_{\mathrm{FS}})$ is the coordinate algebra $PV/I(M)$ and its quantum reduction, which is the algebra $(PV)^G/I(M)^G$ of invariants in $PV/I(M)$, equals the coordinate algebra (or quantization) of the classical symplectic reduction (M_0, ω_0). However, the two natural Hilbert space structures on $PV/I(M)$ and $(PV)^G/I(M)^G$ coming from the Liouville volume forms on $(M, \omega_{\mathrm{FS}})$ and (M_0, ω_0) respectively do not match, and presumably a correction term coming from the Fubini theorem needs to be added [20].

5.6 Exercises

Exercise 5.1. *Show that a left invariant vector field on a Lie group is always complete.*

Exercise 5.2. *Show that the connected component of the identity G° of a Lie group G is the subgroup of G generated by* $\exp(\mathfrak{g})$. *Under the condition* *that G is a connected Lie group conclude that in Theorem 5.1 the Lie group homomorphism $\phi : G \to H$ is completely determined by the Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$.*

Exercise 5.3. *Show that the formula $(\rho(a)f)(x) = f(a^{-1}x)$ for $f \in \mathcal{F}(M)$ and $a, x \in G$ satisfies $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in G$ with the product on the right hand side being composition of linear operators on $\mathcal{F}(M)$.*

Exercise 5.4. *Show that $\mathrm{Ad}(ab) = \mathrm{Ad}(a)\,\mathrm{Ad}(b)$ for all elements a, b in a Lie group G.*

Exercise 5.5. *Check the relation $\mathrm{ad}(X)Y = [X, Y]$ for all X, Y in the Lie algebra $\mathfrak{g}$ of a linear Lie group G.*

Exercise 5.6. *Prove Theorem 5.9.*

Exercise 5.7. *Verify that the moment map in Theorem 5.10 given by the formula $\mu_X = \iota_{X_M}\theta$ is indeed equivariant.*

Exercise 5.8. *Show that the quotient space M/G for a continuous action of a compact topological group G on a topological Hausdorff space M is again Hausdorff.*

Exercise 5.9. *Let $\pi : M = T^*N \to N$ be a cotangent bundle with its canonical symplectic form $\omega = \mathrm{d}\,\theta$. Let $\alpha \in \Omega^1(N)$ be a fixed 1-form on N and consider $\alpha : N \to T^*N$ also as a section in the cotangent bundle $\pi : T^*N \to N$. Let*

$$t_\alpha : M \to M \ , \ t_\alpha(\xi) = \xi + \alpha_{\pi(\xi)}$$

be the translation over α in the fibers of the cotangent bundle. Show that t_α is a diffeomorphism of M. Show that

$$t_\alpha^*\theta = \theta + \pi^*\alpha$$

and conclude that

$$t_\alpha^*\omega = \omega_\alpha$$

with $\omega_\alpha = \omega + \pi^*(\mathrm{d}\,\alpha) = \mathrm{d}(\theta + \pi^*\alpha)$ the twisted symplectic form on M. Hint: Rewrite $(t_\alpha^*\theta)_\xi = \theta_{t_\alpha(\xi)} \circ T_\xi t_\alpha = \cdots = \theta_\xi + (\pi^*\alpha)_\xi$ for $\xi \in T_xN$.

Exercise 5.10. *Show that the moment map for a coadjoint orbit $(M, \omega_{\mathrm{KK}})$ in $\mathfrak{g}^*$ for the coadjoint action of G is equal to minus the identity.*

Exercise 5.11. *Show that for a transitive Hamiltonian action $G \times M \to M$ of a connected Lie group G on a connected symplectic manifold (M, ω) the moment map is a local diffeomorphism onto a coadjoint orbit.*

Exercise 5.12. *Show that in the coordinates on the two sphere $\mathbb{S}^2$ minus north and south pole*

$$(\mathbb{R}/2\pi\mathbb{Z}) \times (0, \pi) \ni (\phi, \theta) \mapsto \mathbf{r} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \in \mathbb{S}^2$$

the Riemannian metric on $\mathbb{S}^2$ induced by the embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is given by

$$(\mathrm{d}\, s)^2 = (\sin\theta)^2 (\mathrm{d}\, \phi)^2 + (\mathrm{d}\, \theta)^2 \ .$$

Moreover the angular momentum $\mathbf{L}$ on $T^*\mathbb{S}^2 \cong T\mathbb{S}^2$ is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times (\mathbf{r}_\phi \dot{\phi} + \mathbf{r}_\theta \dot{\theta})$$

with in turn implies that $J = L_3 = (\sin\theta)^2 \dot{\phi}$. Conclude that the effective potential for the spherical pendulum is equal to

$$V_J(\theta) = \cos\theta + \frac{J^2}{2(\sin\theta)^2}$$

as given in Example 5.25. Check that for all $J \neq 0$ the effective potential V_J has a nondegenerate minimum corresponding to the stable relative equilibria as found by Huygens.

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