Dirac operators

Peter Hochs

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1 Introduction

Dirac operators were introduced by Dirac [6, 7] in 1928 to treat the electron quantum mechanically. The idea was that to make this treatment consistent with Lorentz transformations in special relativity, the Laplace operator that occurs in the Schrödinger equation should be written as the square of some first-order differential operator D.

Dirac considered this problem on four-dimensional space-time with the Minkowski metric. Let us now consider the case of n-dimensional Euclidean space. Then the Laplacian is (up to a sign convention)

$$\Delta = \sum_{j=1}^{n} -\frac{\partial^2}{(\partial x^j)^2}.$$

For a first-order operator with constant coefficients of the form

$$D = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j},$$

a short computation shows that on smooth functions, we have the desired relation $D^2 = \Delta$ if and only if for all j and k,

$$a^j a^k + a^k a^j = -2\delta_{jk},$$

where δ_{jk} is the Kronecker δ. This is clearly impossible if D and Δ act on scalar functions (so the coefficients a^j are numbers), so one needs to consider vector-valued functions (so the coefficients a^j are matrices).

Apart from their motivation from physics, Dirac operators have turned out to be very relevant to several areas of mathematics. These include representation theory [1, 15], existence of Riemannian metrics of positive scalar curvature [14], and geometry and topology more broadly. Many applications involve the Atiyah–Singer index theorem [3, 2], which relates the space of solutions of the equation $Ds = 0$ to the geometry and topology of the space under consideration.

The goal of this course is to introduce Dirac operators and their index theory. We discuss the important special case of Spin-Dirac operators. We state the Atiyah–Singer index theorem for such operators, and deduce a consequence to existence of Riemannian metrics of positive scalar curvature.

Prerequisites are basic theory of (Riemannian) manifolds and vector bundles, and bounded and compact operators on Hilbert spaces.

Standard references on Dirac operators are [4, 8, 9, 13]. We will cite these in various places.

Notation

If X is a set, then we write Id_X for the identity map on X.

We write $M_r(\mathbb{C})$ for the space of complex $r \times r$ matrices, and $\mathrm{End}(V)$ for the space of linear endomorphisms of a finite-dimensional vector space V.

If M is a smooth manifold, then $C^\infty(M)$ denotes the space of smooth functions on M, and $C^\infty_c(M)$ denotes the space of compactly supported smooth functions on M. More generally, if V is a finite-dimensional real vector space, then $C^\infty(M, V)$ denotes the vector space of smooth functions from M to V.

If $E \to M$ is a smooth vector bundle, then $\Gamma^\infty(E)$ denotes the space of smooth sections of E, and $\Gamma^\infty_c(E)$ denotes the space of compactly supported smooth sections of E. We write $\Omega^k(M; E) := \Gamma^\infty(\bigwedge^k T^*M \otimes E)$ for the space of differential forms of degree k with values in E. The endomorphism bundle of E is denoted by $\mathrm{End}(E) = E \otimes E^* \to M$.

2 Dirac operators

Throughout these notes, M is a smooth manifold of dimension n, with a Riemannian metric g. (Some constructions and results extend to pseudo-Riemannian manifolds.) Furthermore, we consider a complex vector bundle $S \to M$ of rank r.

Definition 2.1. A *first order, linear differential operator* on S is a linear map $D\colon \Gamma^\infty(S) \to \Gamma^\infty(S)$ such that every point in M has an open neighbourhood U that admits local coordinates $(x^1, \ldots, x^n)$ and a trivialisation of S, such that there are smooth functions $a^1, \ldots, a^n;, b\colon U \to M_r(\mathbb{C})$, so that for all $s \in \Gamma^\infty(S)$, supported in U,

$$Ds = \sum_{j=1}^{n} a^j \frac{\partial s}{\partial x^j} + bs, \tag{2.1}$$

if s is viewed as a smooth function from an open set in $\mathbb{R}^n$ to $\mathbb{C}^r$ via the local coordinates and trivialisation on U.

Lemma 2.2. *A linear operator* $A\colon \Gamma^\infty(S) \to \Gamma^\infty(S)$ *that commutes with pointwise multiplication by smooth functions is given by a vector bundle endomorphism of* S.

Lemma 2.3. *Let* D *be a first order, linear differential operator on* S*.*

(a) For all $f \in C^\infty(M)$*, viewed as an operator on* $\Gamma^\infty(S)$ *by pointwise multiplication, the commutator* $[D, f]$ *is given by a vector bundle endomorphism of* S*.*

(a) If $m \in M$*, and* $f_1, f_2 \in C^\infty(M)$ *satisfy* $d_m f_1 = d_m f_2$*, then the vector bundle endomorphisms* $[D, f_1]$ *and* $[D, f_2]$ *of* S *are equal at* m*.*

See Exercise 2.1.

Definition 2.4. Let D be a first order, linear differential operator on S. The *principal symbol* of D is the vector bundle homomorphism $\sigma_D \colon T^*M \to \mathrm{End}(S)$ such that for all $f \in C^\infty(M)$ and $m \in M$,

$$\sigma_D(d_m f) = [D, f]_m,$$

where the right hand side is the value of the endomorphism $[D, f]$ of S at m.

Lemma 2.5. *In local coordinates and a trivialisation, where* D *is given by* (2.1), *we have*

$$\sigma_D(\xi) = \sum_{j=1}^{n} a^j \xi_j$$

for all $m \in U$ *and* $\xi = \sum_{j=1}^n \xi_j d_m x^j \in T^*_m M$.

See Exercise 2.2.

Definition 2.6. A first order, linear differential operator D on S is a *Dirac operator* if for all $m \in M$ and $\xi \in T^*_m M$,

$$\sigma_D(\xi)^2 = -g_m(\xi, \xi)\,\mathrm{Id}_{S_m}.$$

Here g_m is the inner product on $T^*_m M$ induced the the inner product g_m on $T_m M$.

From now on, we will assume that a smooth Hermitian metric $(-,-)_S$ on S is given. We also assume that M is oriented.

We write vol_g for the Riemannian volume form associated to g. We consider the inner product $(-,-)_{L^2(S)}$ on $\Gamma^\infty_c(S)$ given by

$$(s_1, s_2)_{L^2(S)} := \int_M (s_1, s_2)_S \,\mathrm{vol}_g, \tag{2.2}$$

for all $s_1, s_2 \in \Gamma_c^\infty(S)$. We denote the completion of $\Gamma_c^\infty(S)$ in this inner product by $L^2(S)$. A first-order differential operator D^* is a *formal adjoint* of a first-order, linear differential operator D if for all $s_1, s_2 \in \Gamma_c^\infty(S)$,

$$(Ds_1, s_2)_{L^2(S)} = (s_1, D^* s_2)_{L^2(S)}.$$

(Differential operators between different vector bundles and their formal adjoints can be defined analogously; this is used in Proposition 3.8 and Theorem 8.10.)

Lemma 2.7. *Let* $S = \bigwedge T^*M \otimes \mathbb{C} \to M$, *and* $D = d$, *the exterior derivative. Then*

(a) d *has a formal adjoint* d^*, *and*

(b) $d + d^*$ *is a Dirac operator.*

Proof. For part (a), see Definition 4.1 and (4.4) in [20]. Part (b) is Exercise 2.6. □

The operator $d+d^*$ is the (complexification of the) *Hodge–Dirac operator*.

If $f \in C^\infty(M)$, then the endomorphism $\sigma_D(df)$ of S defines an operator on $\Gamma_c^\infty(S)$. If the operator $\sigma_D(df)$ is bounded with respect to the inner product (2.2), then we denote its operator norm by $\|\sigma_D(df)\|$. We denote the Riemannian distance on M by d.

Proposition 2.8. *If* D *is a Dirac operator on* M, *then for all* $m, m' \in M$, *and any Hermitian metric on* S,

$$d(m, m') = \sup\{|f(m) - f(m')|; f \in C^\infty(M), \|\sigma_D(df)\| \leq 1\}.$$

Proof. See Proposition 9.12 in [10] or Formula 1 on page 544 of [5]. See also Exercise 2.7 for the inequality in one direction, for the other inequality one can use smooth approximations of the function $f(m') = d(m, m')$, for a given $m \in M$. □

Exercises

Exercise 2.1. Prove Lemma 2.3. Hint: use the local expression for D.

Exercise 2.2. Prove Lemma 2.5.

Exercise 2.3. Let $M = \mathbb{R}^n$ and $S = M \times \mathbb{C}^r$. Let $a_1, \ldots, a_n \in M_r(\mathbb{C})$. Define $D\colon \Gamma^\infty(S) \to \Gamma^\infty(S)$ by (2.1), with $b = 0$.

(a) Prove that D is a Dirac operator for the Euclidean metric on M if and only if for all j, k,

$$a^j a^k + a^k a^j = -2\delta_{jk} I_r,$$

where δ_{jk} is the Kronecker δ, and I_r is the $r \times r$ identity matrix.

(b) Prove that if D is a Dirac operator, then

$$D^2 = -\sum_{j=1}^{n} \frac{\partial^2}{(\partial x^j)^2}.$$

(c) In the case $n = r = 1$, conclude that $i\frac{d}{dx}$ is a Dirac operator on $\mathbb{R}$.

(d) In the case $n = r = 2$, let

$$a_1 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad a_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Use these matrices to prove that the operator $D\colon C^\infty(\mathbb{R}^2, \mathbb{C}^2) \to C^\infty(\mathbb{R}^2, \mathbb{C}^2)$ given by

$$D\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 2i \begin{pmatrix} \frac{\partial s_2}{\partial z} \\ \frac{\partial s_1}{\partial \bar{z}} \end{pmatrix},$$

for $s_1, s_2 \in C^\infty(\mathbb{R}^2, \mathbb{C})$, is a Dirac operator on $\mathbb{R}^2 \cong \mathbb{C}$. Here

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);$$
$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Exercise 2.4. Let D by any first-order differential operator on S, and D^* a formal adjoint of D. Prove that for all $\xi \in T^*M$,

$$\sigma_{D^*}(\xi) = -\sigma_D(\xi)^*,$$

where the star on the right is the fibre-wise adjoint of vector bundle endomorphisms on S.

Exercise 2.5. Let V be a finite-dimensional vector space with an inner product $(-,-)_V$. For all $k \in \mathbb{Z}_{\geq 0}$, consider the inner product on $\bigwedge^k V^*$ such that

$$\{e^{j_1} \wedge \cdots \wedge e^{j_k}; j_1 < \cdots < j_k\}$$

is an orthonormal basis of $\bigwedge^k V^*$, for an orthonormal basis $\{e^1, \ldots, e^n\}$ of V^*. Let $v \in V$, and let $\xi := (v, -)_V \in V^*$. Let

$$\begin{aligned} \xi \wedge -\colon \bigwedge^k V^* &\to \bigwedge^{k+1} V^*; \\ \iota_v\colon \bigwedge^{k+1} V^* &\to \bigwedge^k V^* \end{aligned}$$

be give by exterior multiplication and contraction, respectively. Prove that these two maps are each other's adjoints.

Exercise 2.6. Let $S = \bigwedge T^*M \otimes \mathbb{C} \to M$, and $D = d$, the exterior derivative.

(a) Prove that d is a first order, linear differential operator.

(b) Prove that the principal symbol of d is given by

$$\sigma_d(\xi)\omega = \xi \wedge \omega,$$

for all $m \in M$, $\xi \in T^*_m M$ and $\omega \in \bigwedge T^*_m M$.

(d) Prove part (b) of Lemma 2.7. (Hint: use earlier exercises.)

Exercise 2.7. Let D be a Dirac operator on M.

(a) Prove that for all $f \in C^\infty(M)$,

$$\|\sigma_D(df)\| = \sup_{m \in M} \|d_m f\|,$$

where $\|d_m f\|$ is the operator norm of $d_m f$ as a linear map from $T_m M$ to $\mathbb{R}$.

(b) Prove that for all $m, m' \in M$, and all $f \in C^\infty(M)$ with $\|\sigma_D(df)\| \leq 1$,

$$|f(m) - f(m')| \leq d(m, m').$$

3 Clifford actions

Definition 3.1. A *Clifford action* is a vector bundle homomorphism $c\colon T^*M \to \operatorname{End}(S)$ such that for all $m \in M$ and $\xi \in T^*_m M$,

$$c(\xi)^2 = -g(\xi,\xi)\,\mathrm{Id}_{S_m}.$$

If c is a Clifford action, then we denote the composition of the isomorphism $TM \cong T^*M$ defined by g with c by c as well.

A connection on S is *Hermitian* if for all $s_1, s_2 \in \Gamma^\infty(S)$ and smooth vector fields v on M,

$$v((s_1,s_2)_S) = (\nabla_v s_1, s_2)_S + (s_1, \nabla_v s_2)_S$$

We denote the Levi–Civita connection on TM for g by ∇^g.

Definition 3.2. Let c be a Clifford action. A *Clifford connection* on S is a Hermitian connection ∇ such that for all smooth vector fields v and w on M, and all $s \in \Gamma^\infty(S)$,

$$\nabla_v c(w)s = c(w)\nabla_v s + c(\nabla^g_v w)s.$$

If c is a Clifford action on S, then we also write c for the map from $T^*M \otimes S$ to S given by

$$c(\xi \otimes x) = c(\xi)x,$$

for $m \in M$, $\xi \in T^*_m M$ and $x \in S_m$.

Definition 3.3. Given a Clifford action c and a Clifford connection ∇ on S, the associated Dirac operator is the composition

$$D\colon \Gamma^\infty(S) \xrightarrow{\nabla} \Gamma^\infty(T^*M \otimes S) \xrightarrow{c} \Gamma^\infty(S).$$

Lemma 3.4. *The Dirac operator associated to a Clifford action and a Clifford connection is indeed a Dirac operator.*

See Exercise 3.2.

Lemma 3.5. *Let $U \subset M$ be an open set admitting a local frame $\{e_1, \ldots, e_n\}$ for TM. Let $\{e^1, \ldots, e^n\}$ be the dual frame for T^*M. Then, on U, the Dirac operator associated to a Clifford action c and a Clifford connection ∇ is given by*

$$D|_{\Gamma^\infty(S|_U)} = \sum_{j=1}^n c(e^j)\nabla_{e_j}.$$

See Exercise 3.3.

Example 3.6. The Dirac operator $D = d + d^*$ in Lemma 2.7 is associated to a Clifford action and a Clifford connection; see (4.16) in [20].

Lemma 3.7. *Let* D *be the Dirac operator associated to a Clifford action and a Clifford connection. Then for all* $s_1, s_2 \in \Gamma^\infty_c(S)$*,*

$$(Ds_1, s_2)_{L^2(S)} = (s_1, Ds_2)_{L^2(S)}.$$

Proof. See the proposition on page 69 of [8], or Proposition 3.44 in [4]. □

Proposition 3.8. *Let* c *be a Clifford action on* S*, and* ∇ *a Clifford connection. Let* D *be the Dirac operator associated to these data. Let* R^S *be the curvature tensor of* ∇*. Then in terms of any local orthonormal frame* $\{e_1, \ldots, e_n\}$ *of* TM*, we locally have*

$$D^2 = \nabla^*\nabla + \frac{1}{2}\sum_{j,k} c(e_j)c(e_k)R^S(e_j, e_k), \tag{3.1}$$

for a formal adjoint $\nabla^* \colon \Omega^1(M; S) \to \Gamma^\infty(S)$ *of* ∇*.*

Proof. See page 73 of [8]. □

Exercises

Exercise 3.1. Let D be a Dirac operator on S. Use D to define a Clifford action on S.

Exercise 3.2. Prove Lemma 3.4.

Exercise 3.3. Prove Lemma 3.5.

4 Essential self-adjointness and resolvents

Definition 4.1. Let H be a Hilbert space, and $W \subset H$ a dense linear subspace. Let $T \colon W \to H$ be a linear map.

(a) The operator T is *closable* if the closure of its graph in $H \times H$ is the graph of a linear map $\bar{T}$. Then $\bar{T}$ is the *closure* of T.

(b) The operator T is *symmetric* if for all $v, w \in W$, we have $(Tv, w)_H = (v, Tw)_H$.

(c) The operator T is *self-adjoint* if it is symmetric, and all vectors $v \in H$ such that the linear functional $w \mapsto (v, Tw)$ on W is bounded lie in W.

(d) The operator T is *essentially self-adjoint* if it is closable, and its closure is self-adjoint.

Proposition 4.2. *Let* $T\colon W \to H$ *be a self-adjoint operator. Then the operators* $T \pm i\colon W \to H$ *are invertible, with bounded inverses.*

See Theorem VIII.3 in [16].

Let D be the Dirac operator associated to a Clifford action and a Clifford connection. By Lemma 3.7, the operator

$$D\colon \Gamma^\infty_c(S) \to L^2(S) \tag{4.1}$$

is symmetric. In fact, something stronger is true if M is complete.

Theorem 4.3 (Wolf, 1973)**.** *If* M *is complete, then the operator* (4.1) *is essentially self-adjoint.*

Proof. The original result is in [19]. See also Proposition 10.2.10 in [11]. □

For $k = 0, 1, \ldots$, let $W^k_D(S)$ be the completion of $\Gamma^\infty_c(S)$ in the inner product

$$(s_1, s_2)_{W^k_D(S)} := \sum_{j=0}^{k} (D^j s_1, D^j s_2)_{L^2(S)}.$$

Lemma 4.4. *The closure of the operator* (4.1) *is the continuous extension of* (4.1) *to* $W^1_D(S)$.

See Exercise 4.1.

Definition 4.5. Suppose that M is complete. Then the closure of (4.1) plus i is invertible by Proposition 4.2 and Theorem 4.3. The *resolvent* of the operator (4.1) is the bounded operator

$$(\bar{D} + i)^{-1}\colon L^2(S) \to L^2(S).$$

Exercises

Exercise 4.1. Let D be the Dirac operator associated to a Clifford action and a Clifford connection, viewed as an operator from $\Gamma_c^\infty(S)$ to $L^2(S)$.

(a) Let $p\colon \operatorname{graph}(D) \to \Gamma_c^\infty(S)$ be projection onto the first factor. Prove that p extends to a unitary isomorphism from $\overline{\operatorname{graph}(D)}$ to $W_D^1(S)$.

(b) Prove that the domain of $\bar{D}$ is $W_D^1(S)$.

(c) Prove Lemma 4.4.

Exercise 4.2. We prove Theorem 4.3 in the example where $M = S^1$, $S = S^1 \times \mathbb{C}$ and $D = i\frac{d}{dx}$.

(a) Prove directly that the operator (4.1) is symmetric in this example.

(b) Use the Fourier transform and Lemma 4.4 to prove that D is essentially self-adjoint.

5 The index of a Dirac operator

Let D be a Dirac operator associated to a Clifford action and a Clifford connection.

Theorem 5.1 (Rellich lemma). *Suppose that M is compact. For all k, the inclusion map $W_D^{k+1}(S) \to W_D^k(S)$ is a compact operator.*

Proof. See 10.4.3 and 10.4.4 in [11] for the case $k = 0$, or Lemmas 1.3.4(a) and 1.3.5 in [9] in general. □

Corollary 5.2. *If M is compact, then the resolvent of D is a compact operator on $W_D^k(S)$ for all k.*

See Exercise 5.3.

Theorem 5.3 (Atkinson's lemma). *Suppose that H_1 and H_2 are Hilbert spaces, and that $T\colon H_1 \to H_2$ is a bounded operator. Then the following are equivalent:*

1. *there is a bounded operator $Q\colon H_2 \to H_1$ such that the operators $QT - \mathrm{Id}_{H_1}$ on H_1 and $TQ - \mathrm{Id}_{H_2}$ on H_2 are compact;*

2. $\mathrm{im}(T)$ *is closed and* $\ker(T)$ *and* $H_2/\mathrm{im}(T)$ *are finite-dimensional.*

Proof. See Remark 2.1.3 and Theorem 2.1.4 in [11]. $\square$

Corollary 5.4. *Suppose that* M *is compact. Then the operator*

$$\bar{D}\colon W^1_D(S) \to L^2(S) \tag{5.1}$$

is Fredholm.

See Exercise 5.4.

Theorem 5.5 (Elliptic regularity)**.** *The kernel of the operator* (5.1) *consists of smooth sections.*

This is a special case of elliptic regularity; for the general version see Lemma 1.3.5 in [9]. In the setting of Theorem 5.5, if $s \in W^1_D(S)$ satisfies $Ds = 0$, then it is immediate that $s \in \bigcap_{j=0}^{\infty} W^k_D(S)$. It then remains to show that the latter space consists of smooth sections, using the Gårding inequality (see Lemma 1.3.1(c) in [9] or 10.4.4 in [11]) and the Sobolev embedding theorem (see Lemma 1.3.4(b) in [9]).

From now on, we suppose that the vector bundle S is $\mathbb{Z}/2\mathbb{Z}$*-graded*; i.e. that it decomposes as an orthogonal direct sum of sub-bundles $S = S^+ \oplus S^-$. Suppose that the Clifford connection used to define D interchanges S^+ and S^-, whereas the Clifford action preserves the grading. Then D maps sections of S^+ to sections of S^- and vice versa. If M is compact, then by Corollary 5.4 and Theorem 5.5, the kernel of D in $\Gamma^\infty(S)$ is finite-dimensional.

Definition 5.6. If M is compact, then the *index* of D is

$$\mathrm{index}(D) = \dim(\ker(D) \cap \Gamma^\infty(S^+)) - \dim(\ker(D) \cap \Gamma^\infty(S^-)).$$

Example 5.7. Suppose that M is compact. Let $S = \bigwedge T^*M$ and $D = d + d^*$ as in Lemma 2.7. Consider the grading on S by parity of degrees: S^+ is the direct sum of the even-degree exterior powers of T^*M, and S^- is the direct sum of it odd-degree exterior powers. By the Hodge theorem (see Theorem 6.11 in [18]; here the Rellich lemma and elliptic regularity are used),

$$\ker(D) \cap \Gamma^\infty(S^+) \cong \bigoplus_{k \text{ even}} H^k_{dR}(M);$$
$$\ker(D) \cap \Gamma^\infty(S^-) \cong \bigoplus_{k \text{ odd}} H^k_{dR}(M).$$

So $\mathrm{index}(D) = \sum_k (-1)^k \dim H^k_{dR}(M)$ is the *Euler characteristic* of M.

Exercises

Exercise 5.1. We prove the Rellich lemma in an example. Let $M = S^1$, $S = S^1 \times \mathbb{C}$ and $D = i\frac{d}{d\theta}$. Let $\hat{W}^1_D(S)$ be the space of $f \in l^2(\mathbb{Z})$ such that

$$n \mapsto (1+n^2)^{1/2} f(n)$$

lies in $l^2(\mathbb{Z})$. Consider the inner product on this space given by

$$(f_1, f_2)_{\hat{W}^1_D(S)} := \sum_{n\in\mathbb{Z}} f_1(n)\bar{f}_2(n)(1+n^2).$$

(a) Prove that $\hat{W}^1_D(S)$ is a Hilbert space with this inner product, and that the Rellich lemma for $k = 0$ in this case is equivalent to compactness of the inclusion map $j \colon \hat{W}^1_D(S) \hookrightarrow l^2(\mathbb{Z})$.

(b) For $n \in \mathbb{N}$, let $p_n \colon \hat{W}^1_D(S) \to l^2(\mathbb{Z})$ be given by

$$(p_n(f))(k) = \begin{cases} f(k) & \text{if } |k| \le n; \\ 0 & \text{if } |k| > n. \end{cases}$$

Prove that, in the operator norm of bounded operators from $\hat{W}^1_D(S)$ to $l^2(\mathbb{Z})$,

$$\|j - p_n\|_{\mathcal{B}(\hat{W}^1_D(S), l^2(\mathbb{Z}))} \le \frac{1}{(1+(n+1)^2)^{1/2}}.$$

(c) Prove the Rellich lemma in this case. (You may use that a bounded operator is compact if and only if it can be approximated in operator norm by operators with finite-dimensional images.)

Exercise 5.2. We show with an example that compactness is important in Theorem 5.1. Let $M = \mathbb{R}$ and $D = i\frac{d}{dx}$, on $S = \mathbb{R} \times \mathbb{C}$. Let $s \in C^\infty_c(\mathbb{R})$. For $j \in \mathbb{N}$, define $s_j \in C^\infty_c(\mathbb{R})$ by $s_j(x) = s(x - j)$.

(a) Prove that the sequence $(s_j)_{j=1}^\infty$ is bounded in $W^1_D(S)$.

(b) Prove that the sequence $(s_j)_{j=1}^\infty$ does not have a convergent subsequence in $L^2(S)$.

(c) Prove that the inclusion $W^1_D(S) \to L^2(S)$ is *not* a compact operator.

Exercise 5.3. Prove Corollary 5.2.

Exercise 5.4. Prove Corollary 5.4.

6 Spin-groups

For $n \geq 3$, the group $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$. Because $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$, $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$. It can be constructed in terms of Clifford algebras.

Definition 6.1. Let V be a finite-dimensional real vector space, with a quadratic form Q. The *Clifford algebra* $\mathrm{Cl}(V, Q)$ of V with respect to Q is the quotient of the tensor algebra

$$T(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j}$$

(where $V^{\otimes j}$ is the tensor product of j copies of V if $j \geq 1$, and $V^{\otimes 0} := \mathbb{R}$), with the tensor product as multiplication, by the two-sided ideal generated by the set

$$\{v \otimes v - Q(v); v \in V\}.$$

If $V = \mathbb{R}^n$ and Q is minus the Euclidean norm-squared function, then we write $\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n, Q)$.

The Clifford algebra $\mathrm{Cl}(V, Q)$ is finite-dimensional, of dimension $2^{\dim(V)}$; see the second proposition on page 7 of [8]. The inclusion map $V = V^{\otimes 1} \hookrightarrow T(V)$ induces an injective linear map $V \hookrightarrow \mathrm{Cl}(V, Q)$; see the corollary on page 5 pf [8]. We will use this map to identify V with a linear subspace of $\mathrm{Cl}(V, Q)$.

Definition 6.2. The group $\mathrm{Spin}(n)$ consists of products in Cl_n of even numbers of unit vectors in $\mathbb{R}^n$.

Lemma 6.3. *The set* $\mathrm{Spin}(n)$ *is a group with respect to the multiplication in* Cl_n.

See Exercise 6.1.

Proposition 6.4. *Let* V *be a finite-dimensional real vector space, with a quadratic form* Q. *There is a unique linear map* $\gamma\colon \mathrm{Cl}(V, Q) \to \mathrm{Cl}(V, Q)$ *such that* $\gamma^2 = \mathrm{Id}_{\mathrm{Cl}(V,Q)}$, $\gamma|_V = \mathrm{Id}_V$. *and for all* $x, y \in \mathrm{Cl}(V, Q)$,

$$\gamma(x \cdot y) = \gamma(y) \cdot \gamma(x).$$

Proof. See the proposition on page 6 of [8]. $\square$

Lemma 6.5. *For all $v \in \mathbb{R}^n$ and $x \in \mathrm{Spin}(n)$,*

$$x \cdot v \cdot \gamma(x) \in \mathbb{R}^n.$$

Proof. See the lemma on page 15 of [8]. □

Proposition 6.6. *For all $x \in \mathrm{Spin}(n)$, the map $\lambda(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$\lambda(x)v = x \cdot v \cdot \gamma(x),$$

for $v \in \mathbb{R}^n$, lies in $\mathrm{SO}(n)$. The map $\lambda\colon\ \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a surjective group homomorphism, and $\ker(\lambda) = \{-1, 1\}$. The group $\mathrm{Spin}(n)$ is connected if $n \geq 2$, and simply connected if $n \geq 3$.

Proof. See the proposition on page 16 of [8]. □

A certain standard representation of $\mathrm{Spin}(n)$ will play an important role. We discuss the most relevant case, where n is even. Consider the matrices

$$\begin{aligned} I_2 &:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \\ A_{-1} &:= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \\ A_1 &:= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \\ B &:= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Proposition 6.7. *Suppose that n is even. There is a unique isomorphism of complex algebras*

$$\mathrm{Cl}_n \otimes \mathbb{C} \to M_2(\mathbb{C})^{\otimes n/2} = \mathrm{End}(\mathbb{C}^{2^{n/2}})$$

mapping the jth standard basis vector of $\mathbb{R}^n$ to

$$I_2 \otimes \cdots \otimes I_2 \otimes A_{(-1)^j} \otimes B \otimes \cdots \otimes B, \tag{6.1}$$

where the number of factors B is $\lfloor (j-1)/2 \rfloor$.

Proof. See the proposition on page 13 of [8]. □

Definition 6.8. Suppose that n is even. The vector space $\mathbb{C}^{2^{n/2}}$, equipped with the representation of $\mathrm{Cl}_n \otimes \mathbb{C}$ in Proposition 6.7, is denoted by Δ_n.

Lemma 6.9. *Suppose that n is even. Let $\{e_1, \ldots, e_n\}$ be an oriented orthonormal basis of $\mathbb{R}^n$. The element*

$$i^k e_1 \cdots e_n \in \mathrm{Cl}_n \otimes \mathbb{C} \tag{6.2}$$

squares to 1 and commutes with all elements of $\mathrm{Spin}(n)$.

See Exercise 6.6. For even n, let $\alpha \in \mathrm{End}(\Delta_n)$ be the image of (6.2) under the representation from Proposition 6.7. By Lemma 6.9, the only possible eigenvalues of α are ± 1, and α commutes with the representation of $\mathrm{Cl}_n \otimes \mathbb{C}$. So α defines a $\mathrm{Cl}_n \otimes \mathbb{C}$-invariant $\mathbb{Z}/2\mathbb{Z}$-grading

$$\Delta_n = \Delta_n^+ \oplus \Delta_n^-, \tag{6.3}$$

where $\Delta_n^\pm$ is the ± 1 eigenspace of α, and the subspaces $\Delta_n^\pm \subset \Delta_n$ are invariant under the representation of $\mathrm{Spin}(n)$.

Exercises

Exercise 6.1. Prove that the subset $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$ is a group.

Exercise 6.2. Verify explicitly that $\mathrm{Spin}(2)$ is the circle, and that the map $\lambda\colon \mathrm{Spin}(2) \to \mathrm{SO}(2)$ in Proposition 6.6 maps an element of the circle to its square.

Exercise 6.3. Let V be a finite-dimensional real vector space with an inner product $(-,-)_V$. Let $Q(v) = -(v,v)_V$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Prove that in $\mathrm{Cl}(V, Q)$,

$$v_j v_k + v_k v_j = -2\delta_{jk},$$

for all j, k.

Exercise 6.4. Let E_j be the tensor product of matrices (6.1). Prove that $E_j E_k + E_k E_j = -2\delta_{jk},$. Explain why this relation is necessary for Proposition 6.7 to be true.

Exercise 6.5. Write out the representation of $\mathrm{Spin}(2)$ in $\Delta_2 = \mathbb{C}^2$ explicitly.

Exercise 6.6. Prove Lemma 6.9.

7 Spin-manifolds

We still suppose that M is oriented. Let $\mathrm{SOF}(TM) \to M$ be the oriented, orthonormal frame bundle of M. Its fibre at $m \in M$ is the set of oriented, orthogonal linear isomorphisms $\mathbb{R}^n \to T_mM$. This is a principal $\mathrm{SO}(n)$-bundle. The fibred product $\mathrm{SOF}(TM) \times_{\mathrm{SO}(n)} \mathbb{R}^n$ is the quotient of the Cartesian product $\mathrm{SOF}(TM) \times \mathbb{R}^n$ by the action by $\mathrm{SO}(n)$ given by

$$x \cdot (f, v) := (f \circ x^{-1}, xv),$$

for $x \in \mathrm{SO}(n)$, $f \in \mathrm{SOF}(TM)$ and $v \in \mathbb{R}^n$. This is a vector bundle over M. The map $(f, v) \mapsto f(v)$ descends to a vector bundle isomorphism $\mathrm{SOF}(TM) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong TM$. Under this isomorphism, the Riemannian metric on M corresponds to the Euclidean inner product on $\mathbb{R}^n$.

A Spin-structure on M is a variation on this construction, where $\mathrm{SO}(n)$ is replaced by $\mathrm{Spin}(n)$. If this exists, then it allows us to define an important type of Dirac operator: the Spin-Dirac operator.

If G is a Lie group, $P \to M$ a principal G-bundle, and V a finite-dimensional representation space of G, then we write $P \times_G V$ for the corresponding associated vector bundle over M. This is the quotient of $P \times V$ by the diagonal action by G. If $p \in P$ and $v \in V$, then we denote the class of (p, v) in $P \times_G V$ by $[p, v]$. We consider $\mathbb{R}^n$ as a representation space of $\mathrm{Spin}(n)$ via the covering homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$.

Definition 7.1. A Spin-*structure* on a smooth manifold M is a pair (P, ψ), where $P \to M$ is a principal $\mathrm{Spin}(n)$-bundle, and $\psi\colon P \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to TM$ a vector bundle isomorphism. A Spin-*manifold* is a manifold together with a Spin-structure.

The orientation on M induced by a Spin-structure (P, ψ) is the one corresponding to the standard orientation on $\mathbb{R}^n$ via ψ. The Riemannian metric on M induced by the Spin-structure (P, ψ) is the one corresponding to the Euclidean metric on $\mathbb{R}^n$ via ψ. If an orientation and a Riemannian metric on M are given, then a Spin-structure on M is *compatible* with these data if the orientation and Riemannian metric induced by the Spin-structure agree with the given ones.

Let M be an oriented, Riemannian manifold as before. The *second Stiefel-Whitney class* of M is an invariant $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$, see Definition II.1.6 in [13].

Theorem 7.2. *There is a* Spin*-structure on* M *compatible with the given orientation and Riemannian metric if and only if* $w_2(M) = 0$.

Proof. See Theorem II.2.1 in [13] or Lemma 3.3.1(a) in [9]. □

Example 7.3. Every manifold M with trivialisable tangent bundle has the Spin-structure $(M \times \mathrm{Spin}(n), \psi)$, where ψ is the vector bundle isomorphism

$$\psi\colon (M \times \mathrm{Spin}(n)) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \cong M \times \mathbb{R}^n \cong TM.$$

This includes all Lie groups.

Example 7.4. The sphere $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ has the Spin-structure $(\mathrm{Spin}(n+1), \psi)$, where the double covering map $\mathrm{Spin}(n+1) \to \mathrm{SO}(n+1)$ induces

$$\psi\colon \mathrm{Spin}(n+1) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to \mathrm{SO}(n+1) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong TS^n.$$

Example 7.5. A complex manifold M has a Spin-structure of and only if the image of $c_1(TM) \in H^2(M;\mathbb{Z})$ in $H^2(M;\mathbb{Z}/2\mathbb{Z})$ is zero. Indeed, this image is $w_2(M)$; see Remark II.1.8 in [13].

Example 7.6. The complex projective space $\mathbb{CP}^k$ admits a Spin-structure if and only if k is odd; see the proposition on page 42 of [8], or Lemma 3.3.2(c) in [9].

8 Spin-Dirac operators

Definition 8.1. Suppose that n is even. Suppose that (P, ψ) is a Spin-structure on M. The *spinor bundle* associated to this structure is $S_P := P \times_{\mathrm{Spin}(n)} \Delta_n \to M$. We consider the Hermitian metric on S_P corresponding to the standard Hermitian metric on $\Delta_n = \mathbb{C}^{2^{n/2}}$.

The Clifford action $c\colon TM \to \mathrm{End}(S_P)$ is defined by

$$c(\psi([f, v]))[f, a] := [f, c(v)a], \tag{8.1}$$

for $f \in P$, $v \in \mathbb{R}^n$ and $a \in \Delta_n$. On the right hand side, $c(v)$ is the action by $v \in \mathbb{R}^n \hookrightarrow \mathrm{Cl}_n \hookrightarrow \mathrm{Cl}_n \otimes \mathbb{C}$ on Δ_n from Proposition 6.7. Also, we have identified $T^*M \cong TM$ via the Riemannian metric.

We consider the $\mathbb{Z}/2\mathbb{Z}$-grading on S_P induced by (6.3).

To construct a Dirac operator on a spinor bundle, we will use a Clifford connection canonically induced by the Levi–Civita connection.

Definition 8.2. Let G be a Lie group, and $P \to M$ a principal G-bundle. A *connection one-form* on P is an $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ such that

1. for all $g \in G$, we have $(g^* \otimes \mathrm{Ad}(g))\omega = \omega$; and
2. for all $X \in \mathfrak{g}$,
$$\langle \omega, X^P \rangle = X.$$
Here X^P is the vector field on P induced by $X \in \mathfrak{g}$; at $p \in P$ it equals
$$X^P_p := \left.\frac{d}{dt}\right|_{t=0} \exp(tX)p.$$

If $\{e_1, \dots, e_n\}$ is a local orthonormal, oriented frame for TM, on an open set U, then we write $\omega_{j,k}$ for the one-forms on U such that for all j,
$$\nabla^g e_j = \sum_{k=1}^{n} \omega_{j,k} \otimes e_k.$$
The frame $\{e_1, \dots, e_n\}$ defines a section of $\mathrm{SOF}(TU)$, which we denote by e.

Let $\{e_1^{\mathbb{R}^n}, \dots, e_n^{\mathbb{R}^n}\}$ be the standard basis of $\mathbb{R}^n$. Let $E_{j,k} \in \mathfrak{so}(n)$ be the basis element given by
$$E_{j,k}(v) = v_j e_k^{\mathbb{R}^n} - v_k e_j^{\mathbb{R}^n}$$
for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$.

Proposition 8.3. *There is a unique connection one-form ω on $\mathrm{SOF}(TM)$ such that for all local orthonormal, oriented frames $\{e_1, \dots, e_n\}$ for TM,*
$$e^*(\omega|_{\mathrm{SOF}(TU)}) = \sum_{j<k} \omega_{j,k} \otimes E_{j,k} \quad \in \Omega^1(U) \otimes \mathfrak{so}(n).$$

Proof. See Proposition II.4.4 in [13]. □

Fix a Spin-structure (P, ψ) on M for the rest of this section, assuming it exists. If $p \in P_m$, then we obtain an oriented, orthogonal linear isomorphism $q(p)\colon \mathbb{R}^n \to T_mM$, given by
$$q(p)v = \psi([p, v]),$$

for $v \in \mathbb{R}^n$. This defines a double covering map $q\colon P \to \mathrm{SOF}(TM)$. Let $\omega \in \Omega^1(\mathrm{SOF}(TM)) \otimes \mathfrak{so}(n)$ be any connection one-form on $\mathrm{SOF}(TM)$. Because $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$, the Lie algebra $\mathfrak{spin}(n)$ of $\mathrm{Spin}(n)$ equals $\mathfrak{so}(n)$. Hence we obtain

$$q^*\omega \in \Omega^1(P) \otimes \mathfrak{spin}(n).$$

Lemma 8.4. *This element* $q^*\omega$ *is a connection one-form on* P.

See Exercise 8.2.

Let G be a Lie group, $P \to M$ a principal G-bundle, and V a finite-dimensional representation space of G. Let ω be a connection one-form on P. Via the derivative of π, also denoted by π, this induces

$$\pi \circ \omega \in (\Omega^1(P) \otimes \mathrm{End}(V))^G.$$

Define

$$d + \pi \circ \omega\colon (C^\infty(P) \otimes V)^G \to (\Omega^1(P) \otimes V)^G$$

by

$$((d + \pi \circ \omega)s)(p) = d_p s + (\pi \circ \omega)_p(s(p)) \quad \in T_p^*P \otimes V,$$

for all $s \in (C^\infty(P) \otimes V)^G$ and $p \in P$. Let $E = P \times_G V \to M$ be the vector bundle associated to P and π. We write $(\Omega^1(P) \otimes V)^G_{\mathrm{hor}}$ for the space of $\omega \in (\Omega^1(P) \otimes V)^G$ such that for all $X \in \mathfrak{g}$,

$$\langle \omega, X^P \rangle = 0.$$

Proposition 8.5. *The image of* $d + \pi \circ \omega$ *lies in* $(\Omega^1(P) \otimes V)^G_{\mathrm{hor}}$. *Via the isomorphisms*

$$\Gamma^\infty(E) \cong (C^\infty(P) \otimes V)^G;$$
$$\Gamma^\infty(T^*M \otimes E) \cong (\Omega^1(P) \otimes V)^G_{\mathrm{hor}},$$

the operator $d + \pi \circ \omega$ *defines a connection on* E.

Definition 8.6. In the setting of Proposition 8.5, the connection on E defined by $d + \pi \circ \omega$ is denoted by ∇^ω.

Now let ω be as in Proposition 8.3. Let $q^*\omega$ be as in Lemma 8.4. Applying Proposition 8.5 with $G = \mathrm{Spin}(n)$ and $V = \Delta_n$, we obtain a connection $\nabla^{q^*\omega}$ on S_P.

Definition 8.7. The connection $\nabla^{q^*\omega}$ is the *connection on* S_P *induced by the Levi–Civita connection*.

Proposition 8.8. *The connection* $\nabla^{q^*\omega}$ *on* S_P *is a Clifford connection.*

Proof. See Proposition II.4.11 in [13]. □

Definition 8.9. The Dirac operator on S_P associated to c and $\nabla^{q^*\omega}$ as in Definition 3.3 is the Spin-*Dirac operator* on S_P.

Let κ be the scalar curvature associated to g via ∇^g. In terms of the Riemann tensor R, we have for any $m \in M$ and any local orthonormal frame $\{e_1, \ldots, e_n\}$ for TM near m,

$$\kappa(m) = \sum_{j,k=1}^{n} g(R(e_j, e_k)e_k, e_j).$$

Alternatively, if $B_r^M(m)$ denotes the geodesic ball in M with radius r and centre m, and $B_r^{\mathbb{R}^n}(0)$ is the Euclidean ball of radius r around the origin, then for all $m \in M$, $\kappa(m)$ is determined by

$$\frac{\operatorname{vol} B_r^M(m)}{\operatorname{vol} B_r^{\mathbb{R}^n}} = 1 - \frac{\kappa(m)}{6(n+2)} r^2 + \mathcal{O}(r^4)$$

as $r \downarrow 0$. This can be proved via an asymptotic expansion of the Riemannian density in suitable coordinates; see Lemma 5.3.4 in [17].

Theorem 8.10 (Lichnerowicz)**.** *If* D *is the* Spin-*Dirac operator, then*

$$D^2 = \nabla^*\nabla + \kappa/4,$$

for a formal adjoint ∇^* *of* ∇.

Proof. See Theorem II.8.8 in [13], the proposition on page 74 of [8], or [14]. The idea is to prove that the curvature term in (3.1) equals $\kappa/4$. □

Corollary 8.11. *If* M *is compact and* κ *is positive everywhere, then* $\ker(D) = \{0\}$.

See Exercise 8.4.

Exercises

Exercise 8.1. Prove that (8.1) is a well-defined Clifford action on S_P.

Exercise 8.2. Prove Lemma 8.4.

Exercise 8.3. Suppose that n is even. Consider the trivial Spin-structure on $\mathbb{R}^n$ from Example 7.3.

(a) Prove that the Clifford connection on $S = \mathbb{R}^n \times \Delta_n \to \mathbb{R}^n$ is the trivial connection $d \otimes 1_{\Delta_n}$.

(b) For $j = 1, \ldots, n$, let $\gamma^j \in M_{2^{n/2}}(\mathbb{C})$ be the image of (6.1). Prove that the Spin-Dirac operator on $\mathbb{R}^n$ is

$$D = \sum_{j=1}^{n} \gamma^j \frac{\partial}{\partial x^j} \colon C^\infty(\mathbb{R}^n, \Delta_n) \to C^\infty(\mathbb{R}^n, \Delta_n).$$

Exercise 8.4. Prove Corollary 8.11.

9 The Atiyah–Singer index theorem and positive scalar curvature

There is a well-defined fibre-wise trace map

$$\mathrm{tr} \colon \Gamma^\infty(\mathrm{End}(TM)) \to C^\infty(M). \tag{9.1}$$

Indeed, in terms of a local frame for TM, a section of $\mathrm{End}(TM)$ is a matrix-valued function. Its trace does not depend on the local frame by conjugation-invariance of the matrix trace, and hence is well-defined globally.

For every k, (9.1) extends to a unique map

$$\mathrm{tr} \colon \Omega^k(M; \mathrm{End}(TM)) \to \Omega^k(M) \tag{9.2}$$

such that for all $\alpha \in \Omega^k(M)$ and $s \in \Gamma^\infty(\mathrm{End}(M))$,

$$\mathrm{tr}(\alpha \otimes s) = \mathrm{tr}(s)\alpha.$$

There is a unique bilinear product

$$\Omega^k(M; \mathrm{End}(TM)) \times \Omega^l(M; \mathrm{End}(TM)) \to \Omega^{k+l}(M; \mathrm{End}(TM))$$

such that for all $\alpha_1, \alpha_2 \in \Omega^k(M)$ and $s_1, s_2 \in \Gamma^\infty(\mathrm{End}(M))$,

$$(\alpha_1 \otimes s_1)(\alpha_2 \otimes s_2) = \alpha_1 \wedge \alpha_2 \otimes (s_1 \circ s_2).$$

Let $f(x) = \sum_{j=0}^\infty a_j x^j$ be any formal power series. Then for any $\omega \in \Omega^k(M; \mathrm{End}(TM))$, the terms in the sum

$$f(\omega) = \sum_{j=0}^\infty a_j \omega^j \in \bigoplus_j \Omega^{kj}(M; \mathrm{End}(TM)),$$

for which $kj > n$ are zero. So this sum is well-defined, without convergence issues.

Let $\hat{a}$ be the Taylor series of the function

$$x \mapsto \frac{1}{2} \log \frac{x/2}{\sinh(x/2)}.$$

Then we obtain a map

$$\hat{A}\colon \Omega^2(M; \mathrm{End}(TM)) \to \bigoplus_{j=0}^\infty \Omega^{4j}(M), \tag{9.3}$$

given by

$$\hat{A}(\omega) = \exp(\mathrm{tr}(\hat{a}(\omega))).$$

The exponential function on the right is defined via the Taylor series of the exponential map. The degrees of forms in the image of (9.3) are divisible by 4 because the power series $\hat{a}$ only contains even powers of x.

Suppose that M is compact and even-dimensional. Let $R \in \Omega^2(M; \mathrm{End}(TM))$ be the Riemann curvature tensor associated to g via ∇^g. Let

$$\int_M \hat{A}(R) \tag{9.4}$$

be the integral over M of the top-degree part of $\hat{A}(R)$; this is zero if $\dim(M)$ is not divisible by 4.

Proposition 9.1. *The number* (9.4) *is independent of* g.

Proof. See Theorem 1.11 in [20]. The idea is that if R' is the curvature for a different Riemannian metric, then $\hat{A}(R) - \hat{A}(R')$ is exact. The claim then follows from Stokes' theorem. □

Definition 9.2. The number

$$\hat{A}(M) = \frac{1}{(2\pi i)^{n/2}} \int_M \hat{A}(R)$$

is the $\hat{A}$-*genus* of M.

Remark 9.3. The construction of invariants like the $\hat{A}$-genus is the subject of *Chern–Weil theory* [20].

Theorem 9.4 (Atiyah–Singer, 1963)**.** *Suppose that* M *is a compact, even-dimensional* Spin*-manifold. Let* D *be the* Spin*-Dirac operator on* M*. Then*

$$\operatorname{index}(D) = \hat{A}(M).$$

Proof. See Theorem 5.3 in [2], or page 151 of [4]. □

Corollary 9.5. *The* $\hat{A}$*-genus of a compact* Spin*-manifold is an integer.*

Example 9.6. If k is even, then the complex projective space $\mathbb{CP}^k$ is not Spin (see Example 7.6), and its $\hat{A}$-genus is not an integer. For example, $\hat{A}(\mathbb{CP}^2) = -1/8$ (see the example on page 111 of [8]).

Kazdan and Warner showed that any smooth function on a compact manifold of dimension at least 3 that is negative somewhere occurs as the scalar of some Riemannian metric, see Theorem 1.1. in [12]. It still an open question what compact manifolds admit Riemannian metrics whose scalar curvature is positive everywhere. The following result by Lichnerowicz [14] initiated the use of index theory of Dirac operators to study this problem.

Corollary 9.7 (Lichnerowicz, 1963)**.** *If a compact* Spin*-manifold* M *has nonzero* $\hat{A}$*-genus, then it does not admit any Riemannian metric with positive scalar curvature.*

Proof. This follows from Corollary 8.11 and Theorem 9.4. □

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