

Exercises on symplectic geometry

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1 Symplectic linear algebra

In Exercises 1.1–1.4, we work out the details of the proof of Lemma 1.2 in the lecture notes. This will be used in the proof of the Darboux theorem in Section 6, and in particular shows that any symplectic vector space is even-dimensional.

Let (V, ω) be a symplectic vector space. Fix a nonzero $e_1 \in V$.

Exercise 1.1. Show that there is an $f_1 \in V$ such that

$$\omega(e_1, f_1) = 1.$$

Exercise 1.2. Show that the restriction of ω to

$$U = \mathbb{R}e_1 + \mathbb{R}f_1$$

is nondegenerate.

We now work out some details of the third paragraph on page 5 of the lecture notes.

Exercise 1.3. Let b be a nondegenerate bilinear form a finite-dimensional vector space V . Let $U \subset V$ be a subspace such that b is nondegenerate on U . Prove that

$$V = U \oplus U^b,$$

and that b is nondegenerate on U^b . To be more precise, show that

(a) $\dim U + \dim U^b = \dim V$;

Hint: Apply the rank-nullity theorem to the map

$$\psi : V \rightarrow U^*$$

given by

$$\psi(v) = b(v, -)|_U.$$

- (b) $U \cap U^b = \{0\}$;
- (c) b is nondegenerate on U^b .

Exercise 1.4. Use induction on the dimension of V to conclude from the preceding exercises that Lemma 1.2 is true.

2 Flows of vector fields

Exercise 2.1. (a) Let $v \in \mathfrak{X}(\mathbb{R}^2)$ be given by

$$v(x, y) = (1, 0).$$

What is the flow of v ?

(b) The same question for

$$v(x, y) = (y, 0).$$

Exercise 2.2. Let $v \in \mathfrak{X}(\mathbb{R}^2)$ be given by¹

$$v \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- (a) Draw a picture of the vector field v , and guess what the flow curves² of v are.
- (b) Let ϕ_t be the flow of v over time t . Prove that

$$\phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What does this mean for the shape of the flow curves?

(c) Let $f \in C^\infty(\mathbb{R}^2)$, and consider the two-form

$$\omega := f \, dy \wedge dx \in \Omega^2(\mathbb{R}^2).$$

Consider a point $m = \begin{pmatrix} x \\ 0 \end{pmatrix}$ on the x -axis, and two vectors

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in T_m \mathbb{R}^2 = \mathbb{R}^2.$$

¹In this exercise we denote points in \mathbb{R}^2 by column vectors, so we can apply matrices to them from the left.

²A flow curve of a vector field v on a manifold M is a curve γ of the form $\gamma(t) = \phi_t(m)$, for an $m \in M$.

Compute that

$$(\phi_t^* \omega)_m(v, w) = f \left(\begin{array}{c} x \cos(t) \\ x \sin(t) \end{array} \right) (v_2 w_1 - v_1 w_2).$$

The next exercise will later be an illustration for the exponential map of a Lie group.

Exercise 2.3. Let $M_n(\mathbb{R})$ be the vector space of real $n \times n$ matrices. For any $X \in M_n(\mathbb{R})$, consider the vector field \tilde{X} on $M_n(\mathbb{R})$ given by

$$\tilde{X}_g := g \cdot X \in M_n(\mathbb{R}) = T_g M_n(\mathbb{R}),$$

for $g \in M_n(\mathbb{R})$. Prove that the flow ϕ_t along \tilde{X} over time t is given by

$$\phi_t(g) = g e^{tX} := g \sum_{j=0}^{\infty} \frac{t^j X^j}{j!}, \quad (1)$$

for all $g \in M_n(\mathbb{R})$. (Here X^j is the j th matrix power of X .)

Hint: You may use the following result in analysis. If $(f_j)_{j=0}^{\infty}$ is a sequence of smooth maps

$$f_j : \mathbb{R} \rightarrow \mathbb{R}^n,$$

and if the series $\sum_{j=0}^{\infty} f_j$ converges pointwise, while the series $\sum_{j=0}^{\infty} f_j'$ converges uniformly, then the function defined by $\sum_{j=0}^{\infty} f_j$ is differentiable, and

$$\left. \frac{d}{dt} \right|_{t=s} \sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} f_j'(s). \quad (2)$$

In your solution, you may use the fact that the series (1) converges uniformly on compact subsets of \mathbb{R}^2 . Since (2) is a local statement, it is enough that $\sum_{j=0}^{\infty} f_j'$ converges uniformly on compact sets.

Exercise 2.4. Consider the vector field $v \in \mathfrak{X}(\mathbb{R}^2)$ given by

$$v(x, y) = ((x^2 + 1)(y^2 + 1), y),$$

for $(x, y) \in \mathbb{R}^2$. Let ϕ_t be the flow along v over time t .

(a) Show that

$$\phi_t(0, y) = (\tan((y^2 + 1)t), y),$$

for $t, y \in \mathbb{R}$ for which the right hand side is defined.

(b) Conclude that there is no single $t \neq 0$ such that ϕ_t is defined as a map defined on all of \mathbb{R}^2 .

3 Lie derivatives

Exercise 3.1. Let M be a smooth manifold, $f \in C^\infty(M)$ a smooth function, and v a vector field on M . Prove that

$$\mathcal{L}_v(f) = v(f).$$

The following fact is useful for computing the derivative of a function with respect to a real parameter that plays two different roles in the definition of the function. It is often called a chain rule, although the chain rule does not necessarily have to be applied to prove it.

Exercise 3.2. Let M be a smooth manifold,³ and let

$$F : \mathbb{R}^2 \rightarrow M$$

be a smooth map. Prove that

$$\left. \frac{d}{dt} \right|_{t=0} F(t, t) = \left. \frac{d}{dt} \right|_{t=0} F(t, 0) + \left. \frac{d}{dt} \right|_{t=0} F(0, t).$$

Exercise 3.3. Let M be a smooth manifold, $\phi : M \rightarrow M$ a diffeomorphism. Let $\alpha \in \Omega^p(M)$ be a p -form, and let $v \in \mathfrak{X}(M)$ be a vector field. Prove that

$$\phi^*(i_v \alpha) = i_{\phi^* v} \phi^* \alpha.$$

Exercise 3.4. Let M be a smooth manifold. For a vector field v on M , denote the Lie derivatives of differential forms by v by \mathcal{L}_v . Let i_v be contraction of differential forms by v . Prove that for all differential forms $\alpha \in \Omega(M)$ and all vector fields v, w on M ,

$$i_{[v, w]} \alpha = \mathcal{L}_v i_w \alpha - i_w \mathcal{L}_v \alpha. \quad (3)$$

Deduce from this equality that for all functions $f \in C^\infty(M)$,

$$[v, w](f) = v(w(f)) - w(v(f)).$$

Hint: See the paragraph below the proof of Theorem 2.1, on page 17 of the lecture notes. Explain all steps in the argument given there in detail.

³This exercise can be applied in the solutions of some other exercises, where the manifold M is often a vector bundle over another manifold N , such as $M = \wedge T^*N$.

4 Poisson brackets

Exercise 4.1. Let (M, ω) be a symplectic manifold, and let $\{-, -\}$ be the Poisson bracket defined by ω . Prove that for all functions $f, g \in C^\infty(M)$, one has

$$[v_f, v_g] = v_{\{f, g\}}.$$

Hint: This exercise is Theorem 4.2 in the lecture notes. A short proof is given just above that theorem, so it is only necessary to fill in some details, and briefly mention why all equalities are true.

A *Lie bracket* on a real vector space V is a map

$$V \times V \rightarrow V; \quad (v, w) \mapsto [v, w],$$

such that

1. (*bilinearity*) for all $u, v, w \in V$, $\lambda \in \mathbb{R}$:

$$[u + \lambda v, w] = [u, w] + \lambda[v, w];$$

2. (*anti-symmetry*) for all $v, w \in V$,

$$[v, w] = -[w, v];$$

3. (*Jacobi-identity*) for all $u, v, w \in V$,

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

Exercise 4.2. Let (M, ω) be a symplectic manifold. Prove that

- (a) the Poisson bracket $\{-, -\}$ defined by ω is Lie bracket on $C^\infty(M)$;
- (b) for $f, g, h \in C^\infty(M)$, one has

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

Remark 4.1. • In general, a Poisson bracket on a smooth manifold M is defined as a Lie bracket on $C^\infty(M)$ with the property in part (b). So Exercise 4.2 shows that the bracket defined by the symplectic form is in fact a Poisson bracket in this sense.

- Exercise 4.1 is exactly the statement that the map $f \mapsto v_f$ is a Lie algebra homomorphism from $C^\infty(M)$ to the Lie algebra of vector fields on M .

5 Examples of symplectic manifolds

5.1 Vector spaces

Exercise 5.1. Consider the situation of Exercise 1.4. Now view V as a smooth manifold, equipped with coordinates $x_j, \xi_j : V \rightarrow \mathbb{R}$, such that for all $v \in V$,

$$v = \sum_{j=1}^n x_j(v)e_j + \xi_j(v)f_j.$$

We view the symplectic form ω on the vector space V as a symplectic form on the manifold V by identifying all tangent spaces to V with V . Prove that

$$\omega = \sum_{j=1}^n d\xi_j \wedge dx_j.$$

So in particular, $\omega = d\theta$ is exact, hence closed, for

$$\theta = \sum_{j=1}^n \xi_j \wedge dx_j.$$

5.2 Cotangent bundles

Exercise 5.2. Prove the local equality

$$\theta = \sum_{j=1}^n \xi_j dx_j$$

above Definition 3.6 in the lecture notes. Conclude that $\omega = d\theta$ is indeed a symplectic form on $M = T^*N$.

5.3 Smooth projective manifolds

Exercise 5.3. Let h be a Hermitian form on a complex vector space V . Write $h = B + i\omega$, with B and ω real-valued. Prove that for all $v, w \in V$,

$$B(iv, w) = -\omega(v, w).$$

Exercise 5.4. In this exercise, we will make it plausible that the standard Hermitian form on \mathbb{C}^{n+1} indeed induces a symplectic form on the projective space $\mathbb{P}^n(\mathbb{C})$. (See Example 3.19 in the lecture notes.)

Let $h = B + i\omega$ be the standard Hermitian form on \mathbb{C}^{n+1} :

$$h(v, w) := \sum_{j=1}^{n+1} v_j \bar{w}_j,$$

for $v, w \in \mathbb{C}^{n+1}$. Consider the sphere S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$, and the restricted two-form $\omega|_{S^{2n+1}}$ on this sphere. Fix a point $p \in S^{2n+1}$.

(a) Consider the submanifold

$$U(1)p = \{e^{i\alpha}p; \alpha \in \mathbb{R}\} \subset S^{2n+1}.$$

Show that

$$T_p(U(1)p) = i\mathbb{R}p.$$

(b) Let $\ker \omega_p$ be the subspace of all vectors $v \in T_p S^{2n+1}$ such that for all $w \in T_p S^{2n+1}$,

$$\omega_p(v, w) = 0.$$

Prove that

$$\ker \omega_p = i\mathbb{R}p = T_p(U(1)p).$$

The conclusion is that, intuitively, the directions in which ω is degenerate are the tangent spaces to the sets $U(1)p$. Passing from S^{2n+1} to $\mathbb{P}^n(\mathbb{C})$, one therefore gets rid of all these degenerate directions, and obtains a nondegenerate form. This will be made more precise in the proof of the Marsden–Weinstein theorem that symplectic reduction is well-defined.

Hint: For part (b), use the fact that a tangent space to a sphere can be identified with the orthogonal complement of its base point on the sphere:

$$T_p S^{2n+1} = \{v \in \mathbb{C}^{n+1}; B_p(v, p) = 0\}.$$

Also, use Exercise 5.3.

6 Darboux theorem

In the lecture notes, Moser's proof of the Darboux theorem (Theorem 3.24) is given. In this set of exercises, we work out the details of this proof.

Let (M, ω) be a symplectic manifold, and fix a point $x \in M$. Let $V \subset M$ be a coordinate neighbourhood of x , with a chart

$$\kappa : V \rightarrow W \subset \mathbb{R}^{2n},$$

such that $\kappa(x) = (0, 0)$. Then

$$\omega_0 := (\kappa^{-1})^*(\omega|_V)$$

is a symplectic form on W . We write

$$\nu := (\omega_0)_{(0,0)}$$

for the symplectic form $(\omega_0)_{(0,0)}$ on $T_{(0,0)}W = \mathbb{R}^{2n}$. Let

$$\{e_1, \dots, e_n, f_1, \dots, f_n\}$$

be a basis of \mathbb{R}^{2n} as in the linear Darboux lemma (Lemma 1.2 in the lecture notes), with respect to this form ν . For $j = 1, \dots, n$, consider the linear coordinates

$$y_j, \eta_j : \mathbb{R}^{2n} \rightarrow \mathbb{R}$$

determined by

$$v = \sum_{j=1}^n y_j(v)e_j + \eta_j(v)f_j,$$

for all $v \in \mathbb{R}^{2n}$.

Let $\omega_1 \in \Omega^2(W)$ be the translation-invariant extension of ν to all of W , i.e. for all $p \in W$ and $v, w \in T_pW = \mathbb{R}^{2n}$,

$$(\omega_1)_p(v, w) = \nu(v, w).$$

Exercise 6.1. Prove that

$$\omega_1 = \sum_j d\eta_j \wedge dy_j.$$

The forms ω_0 and ω_1 on W are not the same in general; we only know they are the same at $(0, 0)$. In the remainder of the proof, we will deform ω_1 to ω_0 in a small enough neighbourhood of $(0, 0)$, in a way that allows us to prove the Darboux theorem.

For $r > 0$ we will write B_r for the open ball in \mathbb{R}^{2n} around $(0, 0)$ of radius r . Since W is open and contains $(0, 0)$, there is an $\varepsilon > 0$ such that $B_\varepsilon \subset W$.

Since B_ε is contractible, and

$$d(\omega_1 - \omega_0) = d\omega_1 - d\omega_0 = 0,$$

there is a one-form $\lambda \in \Omega^1(B_\varepsilon)$ such that on B_ε ,

$$\omega_1 - \omega_0 = d\lambda.$$

(This is the Poincaré lemma.)

Exercise 6.2. Show that one may choose λ such that $\lambda_{(0,0)} = 0$.

For $t \in [0, 1]$, set

$$\omega_t := (1 - t)\omega_0 + t\omega_1 \in \Omega^2(B_\varepsilon).$$

Exercise 6.3. Prove that there is a ζ with $0 < \zeta < \varepsilon$, such that for all $t \in [0, 1]$, the form ω_t is nondegenerate on B_ζ .

Hint: In any set of local coordinates $\{a_l\}_{l=1}^{2n}$ (such as $\{y_j, \eta_j\}$), the form ω_t can be written as

$$(\omega_t)_p = \sum_{l,m=1}^{2n} (M_{l,m})(p, t) da_l \wedge da_m.$$

This form is nondegenerate at p if and only if the matrix

$$M(p, t) := (M_{l,m}(p, t))_{l,m=1}^{2n}$$

has nonzero determinant. Now use the function

$$D : [0, 1] \times B_\varepsilon \rightarrow \mathbb{R}$$

given by

$$D(t, p) = \det M(p, t).$$

and the tube lemma in topology.

Exercise 6.4. Prove that for all $t \in [0, 1]$, there is a unique vector field v_t on B_ζ such that

$$i_{v_t}\omega_t + \lambda = 0. \tag{4}$$

Furthermore, the map

$$(t, p) \mapsto v_t(p) \tag{5}$$

from $[0, 1] \times B_\zeta$ to \mathbb{R}^{2n} , is continuous (smooth on $]0, 1[\times B_\zeta$).

Next, consider the family of smooth maps⁴

$$\phi : [0, 1] \times B_\zeta \rightarrow \mathbb{R}^{2n}$$

given by

$$\phi(t, p) = \phi_t(p),$$

such that for all $s \in [0, 1]$ and $p \in B_\zeta$,

$$\left. \frac{d}{dt} \right|_{t=s} \phi_t(p) = v_s(\phi_s(p)).$$

⁴Smooth on $]0, 1[\times B_\zeta$; continuous on $[0, 1] \times B_\zeta$. Existence of such a family of maps follows from a slight generalisation of the proof of existence of flow of vector fields.

Exercise 6.5. Prove that there is a $\delta > 0$ such that for all $t \in [0, 1]$,

$$\phi_t(B_\delta) \subset B_\zeta.$$

Hint: Analogously to Exercise 6.3, use the continuous map

$$\phi : [0, 1] \times B_\zeta \rightarrow \mathbb{R}^{2n}.$$

The following property of the maps ϕ_t is important to Moser's trick.

Exercise 6.6. In this exercise, we prove that for all differential forms $\alpha \in \Omega(B_\zeta)$, one has

$$\left. \frac{d}{dt} \right|_{t=s} \phi_t^* \alpha = \phi_s^* \mathcal{L}_{v_s} \alpha. \quad (6)$$

Note that for constant families of vector fields v_t , this relation is easier to prove.

In this exercise, you may use all properties of Lie derivatives.

- (a) Prove that (6) holds if $\alpha = f \in C^\infty(M)$ is a smooth function.
- (b) Prove that if (6) holds for a differential form $\alpha \in \Omega(M)$, then it also holds for $d\alpha$.
- (c) Prove that if (6) holds for two differential forms $\alpha, \beta \in \Omega(M)$, then it also holds for $\alpha \wedge \beta$.
- (d) Conclude from parts (a)–(c) that (6) holds for all $\alpha \in \Omega(B_\zeta)$.

We are now ready for the key step in Moser's proof of the Darboux theorem.

Exercise 6.7. Prove that for all $t \in [0, 1]$,

$$\phi_t^*(\omega_t|_{B_\zeta}) = \omega_0|_{B_\zeta}. \quad (7)$$

Hint: See the proof of Theorem 2.12 in the lecture notes.

Exercise 6.7 finally allows us to prove Darboux's theorem. Consider the coordinates

$$\begin{aligned} x_j &:= y_j \circ \phi_1 \circ \kappa; \\ \xi_j &:= \eta_j \circ \phi_1 \circ \kappa \end{aligned}$$

on $U := \kappa^{-1}(B_\delta)$.

Exercise 6.8 (Darboux theorem). Prove that

$$\omega|_U = \sum_j d\xi_j \wedge dx_j.$$

Exercise 6.9. Compute the Poisson bracket in Darboux coordinates.

7 Groups and actions

7.1 Proper maps

Exercise 7.1. Let X, Y, Z be topological spaces, and $\varphi : X \rightarrow Y \times Z$ a continuous map. Show that φ is proper if and only if for all compact subsets $C_Y \subset Y$ and $C_Z \subset Z$, the set $\varphi^{-1}(C_Y \times C_Z)$ is compact.

Exercise 7.2. Let G be a locally compact topological group, $H < G$ a subgroup. Consider the action by H on G by right multiplication, and let G/H be the orbit space of this action. Consider the quotient map

$$q : G \rightarrow G/H,$$

given by $q(g) = gH$. Prove that q is proper if and only if H is compact.

Hint: If H is compact, let $C \subset G/H$ be a compact subset. Show that there are finitely many relatively compact open sets $U_1, \dots, U_n \subset G$ such that

$$q^{-1}(C) \subset \bigcup_{j=1}^n H \cdot U_j.$$

7.2 Proper and free actions

Exercise 7.3. Prove that a proper action has compact stabilisers.

Exercise 7.4. Let G be a topological group acting on a topological space X . Prove the following statements.

- (a) If G is compact, the action is proper.
- (b) If X is compact, the action is proper if and only if G is compact.

Exercise 7.5. Let G be a topological group, $H < G$ a closed subgroup. Consider the action by H on G by right multiplication. Let G/H be the orbit space of this action. Consider the action by G on G/H defined by

$$g \cdot (g'H) = gg'H,$$

for $g, g' \in G$.

- (a) Show that the action by H on G by right multiplication is proper, so that G/H is a Hausdorff space.
- (b) If G is Lie group, show that G/H has a smooth manifold structure.

- (c) Determine the stabiliser of any point $gH \in G/H$.
- (d) Show that the action by G on G/H is free if and only if H is the trivial group.
- (e) Show that the action is proper if and only if H is compact.

Exercise 7.6. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-torus. For $\lambda \in \mathbb{R}$, define the subgroup

$$H := \{(a, \lambda a) + \mathbb{Z}^2; a \in \mathbb{R}\} < G.$$

Consider the action by H on G by multiplication (addition). For which values of λ is this action proper? And when is it free? What is the relation with Exercise 7.5 (a)?

Hint: Draw a picture of H for $\lambda = 1/2$, and try to draw one for $\lambda = 1/\sqrt{2}$. To draw these pictures, it is easiest to depict \mathbb{T}^2 as the square $[0, 1]^2$, with appropriate identifications on the boundaries.

8 Lie groups

8.1 General

Exercise 8.1. Let G be a Lie group, and consider the multiplication map

$$m : G \times G \rightarrow G.$$

Prove that its derivative at the identity element is give by addition:

$$T_e m(X, Y) = X + Y,$$

for all $X, Y \in \mathfrak{g}$.

Exercise 8.2. Consider the determinant map

$$\det : \mathrm{GL}(n) \rightarrow \mathbb{R}.$$

Prove that

$$T_I \det = \mathrm{tr}$$

is the trace map.

Exercise 8.3. Consider the Lie group

$$\mathrm{SL}(n) = \{g \in \mathrm{GL}(n); \det(g) = 1\}.$$

(a) Prove that the determinant map $\det : \mathrm{GL}(n) \rightarrow \mathbb{R}$ is a submersion at I .

(b) Prove that the Lie algebra $\mathfrak{sl}(n)$ of $\mathrm{SL}(n)$ equals

$$\mathfrak{sl}(n) = \{X \in M_n(\mathbb{R}); \mathrm{tr}(X) = 0\}.$$

Exercise 8.4. Consider the Lie group

$$\mathrm{O}(n) = \{g \in \mathrm{GL}(n); g^T g = I\}.$$

Here g^T is the transpose of g , and I is the identity matrix.

Let $S_n(\mathbb{R})$ be the vector space of all symmetric real $n \times n$ matrices,⁵ and let $\mathrm{SGL}(n)$ be the open subset

$$\mathrm{SGL}(n) := S_n(\mathbb{R}) \cap \mathrm{GL}(n).$$

(a) Prove that the map $f : \mathrm{GL}(n) \rightarrow \mathrm{SGL}(n)$ given by

$$f(g) = g^T g$$

has tangent map

$$T_I f : M_n(\mathbb{R}) \rightarrow S_n(\mathbb{R})$$

given by

$$T_I f(X) = X^T + X.$$

(b) Prove that f is a submersion at I .

(c) Prove that the Lie algebra $\mathfrak{o}(n)$ of $\mathrm{O}(n)$ equals

$$\mathfrak{o}(n) = \{X \in M_n(\mathbb{R}); X + X^T = 0\}.$$

Exercise 8.5. Consider the Lie group

$$\mathrm{SO}(n) = \mathrm{SL}(n) \cap \mathrm{O}(n) = \{g \in \mathrm{GL}(n); g^T g = I \text{ and } \det(g) = 1\}.$$

Prove that the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ equals

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}); X + X^T = 0\} = \mathfrak{o}(n).$$

(In fact, orthogonal matrices have determinant ± 1 . The group $\mathrm{O}(n)$ has two connected components: $\mathrm{SO}(n)$, and the subset of matrices with determinant -1 .)

⁵The symbols $S_n(\mathbb{R})$ and $\mathrm{SGL}(n)$ are not standard.

8.2 The exponential map

Exercise 8.6. Consider the Lie group $G = \mathbb{R}$, with addition as the group operation. Fix an element $X \in \mathfrak{g} = \mathbb{R}$.

- (a) What is the left invariant vector field v_X associated to X ?
- (b) What is the differential equation defining the flow curve α_X along v_X starting at $e = 0$?
- (c) What is the solution of this equation?
- (d) What is $\exp(X)$?

Exercise 8.7. Consider the Lie group $G = \mathbb{R}_+$ of positive real numbers, with multiplication as the group operation. Fix an element $X \in \mathfrak{g} = \mathbb{R}$.

- (a) What is the left invariant vector field v_X associated to X ?
- (b) What is the differential equation defining the flow curve α_X along v_X starting at $e = 1$?
- (c) What is the solution of this equation?
- (d) What is $\exp(X)$?

Exercise 8.8. Let $G < \text{GL}(n)$ be a linear group, and let $X \in \mathfrak{g} \subset M_n(\mathbb{R})$. Use Exercise 2.3 to show that

$$\exp(X) = e^X := \sum_{j=1}^{\infty} \frac{X^j}{j!}.$$

Exercise 8.9. Consider the Lie group $\text{SO}(2)$.

- (a) Show that the Lie algebra of $\text{SO}(2)$ equals

$$\mathfrak{so}(2) = \left\{ t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; t \in \mathbb{R} \right\}.$$

- (b) Prove that for all $j \in \mathbb{N}$:

$$\begin{aligned} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2j} &= (-1)^j I; \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2j+1} &= (-1)^j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

(c) Prove that

$$\exp\left(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Exercise 8.10. Let G and H be Lie groups, with Lie algebras \mathfrak{g} and \mathfrak{h} , unit elements e_G and e_H , and exponential maps \exp_G and \exp_H , respectively. Let

$$\varphi : G \rightarrow H$$

be a Lie group homomorphism.

Fix an element $X \in \mathfrak{g}$. Consider the left invariant vector field $v_{T_{e_G}\varphi(X)}$ on H associated to $T_{e_G}\varphi(X) \in \mathfrak{h}$.

(a) Prove that for all $s, t \in \mathbb{R}$,

$$\varphi(\exp_G((s+t)X)) = l_{\varphi(\exp_G(sX))}(\varphi(\exp_G(tX))).$$

(b) Let $\alpha_{T_{e_G}\varphi(X)}^H : \mathbb{R} \rightarrow H$ be the flow curve along the vector field $v_{T_{e_G}\varphi(X)}$, starting at the unit element e_H . Prove that for all t ,

$$\alpha_{T_{e_G}\varphi(X)}^H(t) = \varphi(\exp_G(tX)). \quad (8)$$

(c) Conclude that

$$\exp_H(T_e\varphi(X)) = \varphi(\exp_G(X)),$$

i.e. Lemma 3.9 in the lecture notes on Lie groups is true.

8.3 The Lie bracket

Exercise 8.11. Let G be a Lie group, and $\mathfrak{g} = T_eG$ its Lie algebra. For $X \in \mathfrak{g}$, we denote the associated left invariant vector field by v_X . In this exercise, we will show that for any two $X, Y \in \mathfrak{g}$, one has

$$v_{[X,Y]} = [v_X, v_Y], \quad (9)$$

the Lie bracket of the vector fields v_X and v_Y . The conclusion is that the space of left invariant vector fields with the usual Lie bracket of vector fields, is isomorphic, as a Lie algebra, to \mathfrak{g} with the Lie bracket defined via the adjoint action.

Fix $X, Y \in \mathfrak{g}$ and $g \in G$. For any Lie algebra element $Z \in \mathfrak{g}$, we denote the flow along v_Z over time t by ϕ_t^Z .

(a) Prove that

$$[v_X, v_Y]_g = \left. \frac{d}{dt} \right|_{t=0} T_e(\phi_{-t}^X \circ l_{\phi_t^X(g)})(Y).$$

(b) Prove that, for all $s, t \in \mathbb{R}$

$$(\phi_{-t}^X \circ l_{\phi_t^X(g)})(\exp(sY)) = g\alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$

Hint: At the start of the proof of Lemma 3.2 in the notes on Lie groups, it is shown that for all $Z \in \mathfrak{g}$, $h \in G$ and $u \in \mathbb{R}$,

$$\phi_u^Z(h) = h\alpha_Z(u). \quad (10)$$

(c) Conclude from parts (a) and (b) that

$$[v_X, v_Y]_g = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} g\alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$

(d) Prove that for all $t \in \mathbb{R}$,

$$\text{Ad}(\exp(tX))Y = \left. \frac{d}{ds} \right|_{s=0} \alpha_X(t)\alpha_Y(s)\alpha_X(-t).$$

(e) Prove that

$$(v_{[X,Y]})_g = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} g\alpha_X(t)\alpha_Y(s)\alpha_X(-t),$$

completing the proof of (9).

9 Infinitesimal actions

Exercise 9.1. Consider the natural action by $\text{SO}(3)$ on the two-sphere $S^2 \subset \mathbb{R}^3$ by rotations.

(a) Prove that the matrices

$$R_x := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_y := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$R_z := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{so}(3) = \text{Lie}(\text{SO}(3))$.

Hint: Use Exercise 8.5.

- (b) Generalise Exercise 8.9 to show that for all $t \in \mathbb{R}$,

$$\exp(tR_x) \in \text{SO}(3)$$

is rotation over angle t around the x -axis, and similarly for the exponentials of tR_y and tR_z . (Give explicit expressions for these exponentials.)

- (c) Let

$$m = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2$$

be given. Let $(R_x)_{S^2}$, $(R_y)_{S^2}$ and $(R_z)_{S^2}$ be the vector fields on S^2 induced by R_x , R_y and R_z via the infinitesimal action. Compute $(R_x)_{S^2}(m)$, $(R_y)_{S^2}(m)$ and $(R_z)_{S^2}(m)$.

- (d) For $j = 1, 2, 3$, let e_j be the j th standard basis vector of \mathbb{R}^3 . Conclude that in particular,

$$\begin{aligned} (R_x)_{S^2}(e_1) &= 0; \\ (R_y)_{S^2}(e_1) &= e_3; \\ (R_z)_{S^2}(e_1) &= e_2. \end{aligned}$$

In other words, the vector fields induced by R_x , R_y and R_z indeed point “in the direction of the action”.

Exercise 9.2. To generalise Exercise 9.1, let $G < \text{GL}(n)$ be a linear group, and let $M \subset \mathbb{R}^2$ be a G -invariant submanifold. Consider the natural action by G on M . Prove that, for all $X \in \mathfrak{g}$ and $m \in M$,

$$X_M(m) = X \cdot m.$$

Here X_M is the vector field on M induced by the infinitesimal action, and the dot in $X \cdot m$ denotes the product of the matrix X and the vector m .

10 The slice lemma

The next set of exercises is about a proof of the *slice lemma* (Lemma 13.7 in the notes on Lie groups). Since we focused on left actions, we will state this result for left actions rather than right actions.⁶

Let G be a Lie group acting smoothly, properly and freely on a smooth manifold M . We will prove the following result:

Lemma 10.1 (Slice lemma). *For every $m \in M$, there is a submanifold S of M containing m , such that the map*

$$\varphi : G \times S \rightarrow M$$

given by $\varphi(g, s) = g \cdot s$ for all $g \in G$ and $s \in S$, is an equivariant diffeomorphism onto an open, G -invariant neighbourhood U of M .

The slice lemma is the key step in the proof that proper, free, smooth actions have smooth quotients. Indeed, it implies that locally, M/G looks like the smooth manifold S . More on this in Exercise 10.7.

Fix a point $m \in M$. Consider the map

$$\alpha_m : G \rightarrow M$$

given by $\alpha_m(g) = g \cdot m$, for $g \in G$.

Exercise 10.1. Prove that α_m is injective.

Let \mathfrak{g} be the Lie algebra of G , and consider the tangent map

$$T_e \alpha_m : \mathfrak{g} \rightarrow T_m M.$$

Exercise 10.2. Let $X \in \mathfrak{g}$ be in the kernel of $T_e \alpha_m$:

$$T_e \alpha_m(X) = 0.$$

We will show that $X = 0$, so that $T_e \alpha_m$ is injective.

Consider the curve $c : \mathbb{R} \rightarrow M$ given by

$$c(t) = \exp(tX) \cdot m$$

for $t \in \mathbb{R}$.

⁶But recall that any left action α corresponds to a right action β (and vice versa) via $\beta(g) = \alpha(g^{-1})$.

- (a) Prove that $c'(0) = 0$.
- (b) Prove that $c'(t) = 0$ for all $t \in \mathbb{R}$.
- (c) Prove that $X = 0$.

Choose a linear subspace $\mathfrak{s} \subset T_m M$ such that

$$T_m M = \mathfrak{s} \oplus T_e \alpha_m(\mathfrak{g}).$$

Since $T_e \alpha_m$ is injective, \mathfrak{s} has dimension $\dim M - \dim G$. Choose a submanifold $S' \subset M$ containing m , such that

$$T_m S' = \mathfrak{s}.$$

Define the map $\varphi : G \times S' \rightarrow M$ by $\varphi(g, s) = g \cdot s$, for $g \in G$ and $s \in S'$.

Exercise 10.3. (a) Prove that for all $X \in \mathfrak{g}$ and $v \in \mathfrak{s}$,

$$T_{(e,m)}\varphi(X, v) = v + X_M(m),$$

where X_M is the vector field on M induced by X via the infinitesimal action.

- (b) Prove that $T_{(e,m)}\varphi$ is bijective.
- (c) Prove that $T_{(e,s)}\varphi$ is bijective for all s in an open neighbourhood S'' of m in S' .
- (d) Prove that $T_{(g,s)}\varphi$ is bijective for all $g \in G$ and $s \in S''$.

We conclude that, by the inverse function theorem, $\varphi : G \times S'' \rightarrow M$ is a local diffeomorphism onto its image. So there are open neighbourhoods \mathcal{O} of e in G , and S''' of m in S'' , such that

$$\varphi : \mathcal{O} \times S''' \rightarrow M$$

is a diffeomorphism onto its image. (We use the same notation for φ and its restrictions to various subsets.) Furthermore, if the closure \overline{S}''' of S''' is not compact, we can always replace S''' by a smaller neighbourhood of m , such that \overline{S}''' is compact. Assume this has been done.

Set

$$C := \{g \in G; g\overline{S}''' \cap \overline{S}''' \neq \emptyset\}.$$

This set is compact by properness of the action, and Lemma 13.3 in the notes on Lie groups. Set

$$C_0 := C \setminus \mathcal{O}.$$

Exercise 10.4. Prove that there is an open subset $S \subset S'''$ such that for all $g \in C_0$,

$$gS \cap S = \emptyset.$$

Hint: Use Lemma 13.6 in the notes on Lie groups.

Exercise 10.5. Prove that $\varphi : G \times S \rightarrow M$ is injective.

Set

$$U := \varphi(G \times S).$$

Exercise 10.6. Finish the proof of Lemma 10.1, by showing that U is a G -invariant open neighbourhood of m , and

$$\varphi : G \times S \rightarrow U$$

is an equivariant diffeomorphism.

As mentioned above, the slice lemma can be used to prove that M/G has the structure of a smooth manifold. Indeed, by the slice lemma, M can be covered by open subsets U_j for which there are submanifolds S_j of M and equivariant diffeomorphisms

$$\varphi_j : G \times S_j \rightarrow U_j,$$

given by $\varphi_j(g, s) = g \cdot s$ for $g \in G$ and $s \in S_j$. Consider the induced homeomorphisms on orbit spaces

$$\chi_j := (\varphi_j)_G^{-1} : U_j/G \rightarrow (G \times S_j)/G \cong S_j.$$

By shrinking the S_j , one may assume that S_j is diffeomorphic to an open subset of \mathbb{R}^n . Then the maps χ_j form a smooth atlas on M/G , if the transition maps $\chi_k \circ \chi_j^{-1}$ are smooth where they are defined.

Exercise 10.7. Let j, k be such that $U_j \cap U_k \neq \emptyset$. Use Lemma 12.4 in the notes on Lie groups to show that the map

$$\chi_k \circ \chi_j^{-1} : \chi_j(U_j/G \cap U_k/G) \rightarrow \chi_k(U_j/G \cap U_k/G)$$

is smooth.

11 Hamiltonian group actions

Exercise 11.1. Consider the manifold $M = \mathbb{R}^2$, with coordinates (q, p) . Let $\omega = dp \wedge dq$ be the standard symplectic form on M . Consider the Lie group $G = \mathbb{R}$, acting on M by translation in the q -direction:

$$g \cdot (q, p) = (q + g, p),$$

for $g \in G$, $(q, p) \in M$.

Prove that this action is Hamiltonian, by showing that the map $\mu : M \rightarrow \mathbb{R}$, given by

$$\mu(q, p) = p$$

is a momentum map. Here we identify $\mathfrak{g}^* \cong \mathbb{R}$, using the fact that any linear function on $\mathfrak{g} = \mathbb{R}$ is given by multiplication by a real number.

Exercise 11.2. Let a Hamiltonian action by a Lie group G on a symplectic manifold (M, ω) be given. Let $\mu : M \rightarrow \mathfrak{g}^*$ be a momentum map. Let $H < G$ be a closed subgroup, and consider the map

$$p : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$$

given by restricting linear functions on \mathfrak{g} to \mathfrak{h} .

Prove that the action by H on (M, ω) is Hamiltonian, with momentum map $\mu^H = p \circ \mu$.

Exercise 11.3. Let a Hamiltonian action by a Lie group G on a symplectic manifold (M, ω) be given. Let $\mu : M \rightarrow \mathfrak{g}^*$ be a momentum map. Let $N \subset M$ be a G -invariant symplectic submanifold, and consider the inclusion map

$$\iota : N \hookrightarrow M.$$

Then, by assumption, $\iota^*\omega$ is a symplectic form on N .

Prove that the action by H on N is Hamiltonian, with momentum map $\mu^N = \iota^*\mu$.

Exercise 11.4. Let (M, ω) be a symplectic manifold, and let a Hamiltonian action by a Lie group G on (M, ω) be given. Let $\mu : M \rightarrow \mathfrak{g}^*$ be a momentum map. Let $X \in \mathfrak{g}$, and let v_{μ_X} be the Hamiltonian vector field of μ_X .

- (a) What is the Hamiltonian vector field of the function μ_X , for $X \in \mathfrak{g}$?
- (b) Prove that for all $X, Y \in \mathfrak{g}$,

$$\{\mu_X, \mu_Y\} = -\mu_{[X, Y]}.$$

Exercise 11.5. Let $\varphi : G \rightarrow H$ be a Lie group homomorphism between two Lie groups G and H . Let (M, ω) be a symplectic manifold, and let a Hamiltonian action by H on (M, ω) be given. Let $\mu^H : M \rightarrow \mathfrak{h}^*$ be a momentum map for this action.

Consider the action by G on M given by

$$g \cdot m := \varphi(g) \cdot m,$$

for $g \in G$ and $m \in M$. Prove that this action is Hamiltonian, with momentum map $\mu^G : M \rightarrow \mathfrak{g}^*$ given by

$$(\mu^G(m))(X) = (\mu^H(m))(T_e\varphi(X)).$$

Remark 11.1. This exercise generalises Exercise 11.2, where φ is the inclusion map of a closed subgroup. (The roles of G and H are interchanged between the two exercises.)

12 Symplectic reduction

In this set of exercises, we prove a slight simplification of Theorem 5.17 in the lecture notes.

Let (M, ω) be a symplectic manifold, and let G be a Lie group. Let a Hamiltonian action by G on (M, ω) be given, and let $\mu : M \rightarrow \mathfrak{g}^*$ be a momentum map. Let $\xi \in \mathfrak{g}^*$ be a regular value of μ , i.e. for all m in the nonempty subset $\mu^{-1}(\xi) \subset M$, the tangent map

$$T_m\mu : T_mM \rightarrow \mathfrak{g}^*$$

is surjective. Then $\mu^{-1}(\xi)$ is a smooth submanifold of M , by the submersion theorem.

$$\iota_\xi : \mu^{-1}(\xi) \hookrightarrow M$$

be the inclusion map.

Let G_ξ be the stabiliser of ξ with respect to the coadjoint action, and suppose G_ξ acts properly and freely on $\mu^{-1}(\xi)$. Then

$$M_\xi := \mu^{-1}(\xi)/G_\xi$$

is a smooth manifold, as we saw in Section 10. Let

$$\pi_\xi : \mu^{-1}(\xi) \rightarrow M_\xi$$

be the quotient map.

Theorem 12.1 (Marsden–Weinstein). *There is a unique symplectic form ω_ξ on M_ξ , such that*

$$\pi_\xi^* \omega_\xi = \iota_\xi^* \omega \quad \in \Omega^2(\mu^{-1}(\xi)). \quad (11)$$

Note that the quotient map π_ξ is a submersion.⁷ Hence the tangent map

$$T_m \pi_\xi : T_m \mu^{-1}(\xi) \rightarrow T_{Gm} M_\xi$$

is surjective for all $m \in \mu^{-1}(\xi)$. Hence (11) indeed determines ω_ξ uniquely. Indeed, any two tangent vectors in $T_{Gm} M_\xi$ are of the form

$$T_m \pi_\xi(v), T_m \pi_\xi(w) \in T_{Gm} M_\xi,$$

for $v, w \in T_m \mu^{-1}(\xi)$. (Though these may be equal even if v and w are different.)

Exercise 12.1. Show that for all for $v, w \in T_m \mu^{-1}(\xi)$,

$$(\omega_\xi)_{Gm}(T_m \pi_\xi(v), T_m \pi_\xi(w)) = \omega_m(v, w), \quad (12)$$

where we consider $T_m \mu^{-1}(\xi)$ as a subspace of $T_m M$ via the map $T_m \iota_\xi$.

It therefore remains to show that

1. ω_ξ is well-defined by (12);
2. ω_ξ is nondegenerate;
3. ω_ξ is closed.

The first two points will follow from the facts in the following exercise.

Exercise 12.2. Fix $m \in \mu^{-1}(\xi)$.

(a) Show that

$$T_m(G \cdot m) = \{X_M(m); X \in \mathfrak{g}\}. \quad (13)$$

(b) Show that

$$T_m \mu^{-1}(\xi) = (T_m(G \cdot m))^{\omega_m},$$

where the superscript ω_m denotes the annihilator (orthogonal complement) of a space with respect to ω_m . (See page 5 in the lecture notes.)

⁷See the proof of Theorem 12.5 in the notes on Lie groups.

(c) Show that

$$T_m(G_\xi \cdot m) = \ker(T_m\pi_\xi).$$

Hint: use the slice lemma to prove that these spaces have equal dimensions.

Exercise 12.3. Prove that ω_ξ is well-defined by (12). I.e. if $m \in \mu^{-1}(\xi)$, $v, w \in T_m\mu^{-1}(\xi)$, and $T_m\pi_\xi(v) = 0$, then $\omega_m(v, w) = 0$. (And similarly if $T_m\pi_\xi(w) = 0$.)

Exercise 12.4. Let W be a vector space, and b a bilinear form on W . Let $V \subset W$ be a linear subspace. Prove that

$$(V^b)^b = V.$$

Hint: For the inclusion $(V^b)^b \subset V$, consider a vector $w \in (V^b)^b$, and the annihilator of the space $U := V + \mathbb{R}w$.

Exercise 12.5. Prove that ω_ξ is nondegenerate.

Exercise 12.6. (a) Argue that ω_ξ is closed if and only if $\pi_\xi^*(d\omega_\xi) = 0$.

(b) Prove that ω_ξ is closed.

13 Cotangent bundles

Let N be a smooth manifold, and consider its cotangent bundle T^*N . Let $\theta \in \Omega(T^*N)$ be the tautological one-form, and let $\omega := d\theta$ be the standard symplectic form on T^*N .

Let G be a Lie group acting smoothly on N . The induced action on T^*N is given by

$$(g \cdot \eta)(v) = \eta(T_{g \cdot n}g^{-1}(v)), \quad (14)$$

for $g \in G$, $n \in N$, $\eta \in T_n^*N$ and $v \in T_{g \cdot n}N$.

13.1 Momentum map

Exercise 13.1. (a) Prove that θ is G -invariant.

(b) Prove that the action is symplectic.

(c) Prove that for all $X \in \mathfrak{g}$,

$$d(i_{X_{T^*N}}\theta) = -i_{X_{T^*N}}\omega.$$

Here X_{T^*N} is the vector field on T^*N induced by X via the infinitesimal action.

Consider the map $\mu : T^*N \rightarrow \mathfrak{g}^*$ given by

$$(\mu(\eta))(X) = \eta(X_N(n)),$$

for $n \in N$, $\eta \in T_n^*N$ and $X \in \mathfrak{g}$. Let $\pi : T^*N \rightarrow N$ be the cotangent bundle projection.

Exercise 13.2. (a) Prove that for all $X \in \mathfrak{g}$, $n \in N$ and $\eta \in T_n^*N$,

$$T_\eta\pi(X_{T^*N}(\eta)) = X_N(n) \in T_nN.$$

(b) Prove that for all $X \in \mathfrak{g}$,

$$\mu_X = i_{X_{T^*N}}\theta \in C^\infty(T^*N).$$

(c) Prove that for all $X \in \mathfrak{g}$,

$$d\mu_X = -i_{X_{T^*N}}\omega.$$

It remains to check equivariance of μ .

Exercise 13.3. (a) Prove that for all $X \in \mathfrak{g}$, $n \in N$ and $g \in G$,

$$T_n g(X_N(n)) = (\text{Ad}(g)X)_N(g \cdot n).$$

(b) Prove that μ is equivariant.

13.2 Symplectic reduction

We have seen that the action by G on T^*N is Hamiltonian, with momentum map μ . We now suppose that G acts properly and freely on N . Then N/G is a smooth manifold. Let $\theta_G \in \Omega^1(T^*(N/G))$ be the tautological one-form on $T^*(N/G)$, and let $\omega_G := d\theta_G$ be the standard symplectic form on $T^*(N/G)$. We are going to construct a symplectomorphism

$$((T^*N)_0, \omega_0) \cong (T^*(N/G), \omega_G).$$

Here $((T^*N)_0, \omega_0)$ is the symplectic reduction at zero of T^*N by the action by G .

Consider the quotient map $q : N \rightarrow N/G$. For $n \in N$, the tangent map $T_n q$ dualises to

$$\begin{aligned} (T_n q)^* : T_{G \cdot n}^*(N/G) &\rightarrow T_n^*N, \\ ((T_n q)^*(\zeta))(v) &= \zeta(T_n q(v)), \end{aligned}$$

for $n \in N$, $\zeta \in T_{G \cdot n}^*(N/G)$ and $v \in T_nN$.

Exercise 13.4. Prove that for all $n \in N$

$$(T_n q)^*(T_{G \cdot n}^*(N/G)) \subset \mu^{-1}(0).$$

Consider the quotient map $p : \mu^{-1}(0) \rightarrow M_0 = \mu^{-1}(0)/G$, and the inclusion map $\iota : \mu^{-1}(0) \hookrightarrow T^*N$. Then, for all $n \in N$, we have the diagram

$$\begin{array}{ccc} T_{G \cdot n}^*(N/G) & \xrightarrow{(T_n q)^*} & \mu^{-1}(0) \xrightarrow{\iota} T^*N. \\ & \searrow p \circ (T_n q)^* & \downarrow p \\ & & (T^*N)_0 \end{array}$$

Consider the map

$$\Psi : T^*(N/G) \rightarrow (T^*N)_0$$

given by

$$\Psi(\zeta) = (p \circ (T_n q)^*)(\zeta),$$

for all $\zeta \in T_{G \cdot n}^*(N/G)$.

Exercise 13.5. Show that the map Ψ is well-defined, in the sense that for all $n \in N$, $\zeta \in T_{G \cdot n}^*(N/G)$ and $g \in G$,

$$(p \circ (T_{g \cdot n} q)^*)(\zeta) = (p \circ (T_n q)^*)(\zeta).$$

We first show that Ψ is a diffeomorphism.

Exercise 13.6. (a) Let $f : X \rightarrow Y$ be a submersion between smooth manifolds X and Y . Prove that, for each $x \in X$, the map

$$(T_x f)^* : T_{f(x)}^* Y \rightarrow T_x^* X,$$

defined in the same way as $T_n q$ above, is injective.

(b) Prove that Ψ is injective.

Next, we use the fact that $(T^*N)_0$ is a vector bundle over N/G . The vector bundle projection map is induced by the equivariant cotangent bundle projection $\pi : \mu^{-1}(0) \rightarrow N$.

Exercise 13.7. Prove that, at all $G \cdot n \in N/G$, the map Ψ is a linear isomorphism

$$\Psi_{G \cdot n} : T_{G \cdot n}^*(N/G) \xrightarrow{\cong} ((T^*N)_0)_{G \cdot n}.$$

The above exercise implies that Ψ is a bijection. It follows from the theory of vector bundles that Ψ , and its inverse, are smooth. Hence Ψ is indeed a diffeomorphism.

It remains to show that the diffeomorphism Ψ is in fact a symplectomorphism. Fix a point $n_0 \in N$. We will prove the equality

$$\Psi^*\omega_0 = \omega_G.$$

locally, in a neighbourhood n_0 . Since n_0 is an arbitrary point in N , this proves the claim.

Since the action by G on N is proper and free, the slice lemma implies that there is a G -invariant neighbourhood U of $G \cdot n_0$ in N/G such that there is an equivariant diffeomorphism

$$\tau : q^{-1}(U) \xrightarrow{\cong} U \times G.$$

This allows us to choose a smooth map

$$\sigma : U \rightarrow q^{-1}(U)$$

such that $\sigma(G \cdot n) \in G \cdot n$ for all $G \cdot n \in U$. For example, one can set

$$\sigma(G \cdot n) := \tau^{-1}(G \cdot n, e).$$

Using σ , we define the map

$$T_\sigma^*q : T^*(N/G) \rightarrow T^*N$$

by

$$T_\sigma^*q(\eta) = (T_{\sigma G \cdot n}q)^*\eta,$$

for all $\eta \in T_{G \cdot n}^*(N/G)$. By definition of the map Ψ , we have

$$\Psi|_U = p \circ T_\sigma^*q$$

on U .

Now consider the diagram

$$\begin{array}{ccccc} T^*(N/G) \supset U & \xrightarrow{(T_\sigma^*q)} & \mu^{-1}(0) & \xrightarrow{\iota} & T^*N. \\ & \searrow \Psi & \downarrow p & & \\ & & (T^*N)_0 & & \end{array}$$

Exercise 13.8. (a) Prove that, on U ,

$$q \circ \pi \circ T_\sigma^* q = \pi_G|_U,$$

where $\pi : T^*N \rightarrow N$ is the cotangent bundle projection of N , and $\pi_G : T^*(N/G) \rightarrow N/G$ is the cotangent bundle projection of N/G .

(b) Prove that

$$(\iota \circ T_\sigma^* q)^* \theta = \theta_G|_U.$$

(c) Prove that

$$(\Psi^* \omega_0)|_U = \omega_G|_U.$$

14 Coadjoint orbits

Let G be a Lie group, with Lie algebra \mathfrak{g} . Let \mathfrak{g}^* be the dual vector space of \mathfrak{g} . Consider the *coadjoint action* (representation)

$$\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$$

defined by

$$(\text{Ad}^*(g)\xi)(X) = \xi(\text{Ad}(g^{-1})X),$$

for all $g \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. This induces the infinitesimal coadjoint action

$$\text{ad}^* : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*),$$

given by

$$\text{ad}^* = T_e \text{Ad}^*.$$

That is,

$$(\text{ad}(X)\xi)(Y) = -\xi([X, Y]),$$

for all $X, Y \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

Definition 14.1. A *coadjoint orbit* of G is an orbit of the coadjoint action Ad^* . That is, a subset \mathcal{O} of \mathfrak{g}^* of the form

$$\mathcal{O} = \text{Ad}^*(G)\xi = \{\text{Ad}^*(g)\xi; g \in G\},$$

for some $\xi \in \mathfrak{g}^*$.

Exercise 14.1. Let \mathcal{O} be a coadjoint orbit of G , and let $\xi \in \mathcal{O}$ be a point on \mathcal{O} . Show that⁸

$$T_\xi \mathcal{O} = \{X_\xi; X \in \mathfrak{g}\} = \{\text{ad}^*(X)\xi; X \in \mathfrak{g}\} \subset \mathfrak{g}^*.$$

Let \mathcal{O} be a coadjoint orbit.

Definition 14.2. The *Kostant–Kirillov–Souriau* two-form ω on \mathcal{O} is given by

$$\omega_\xi(X_\xi, Y_\xi) := -\xi([X, Y]),$$

for all $\xi \in \mathcal{O}$ and $X, Y \in \mathfrak{g}$.

We will show that the form ω is a well-defined symplectic form on \mathcal{O} .

Exercise 14.2. (a) Prove that the above expression for ω is well-defined. I.e. if $\xi \in \mathcal{O}$, $X, Y \in \mathfrak{g}$ and $X_\xi = 0$, then $-\xi([X, Y]) = 0$, and similarly for Y .

(b) Prove that the form ω is nondegenerate.

(c) Prove that the form ω is closed.

Hint: For part (c), show that for all $X \in \mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*)$,

$$i_X \omega = -dX.$$

Then show that

$$i_X d\omega = 0$$

for all such X .

Exercise 14.3. Show that the symplectic form ω is G -invariant.

The coadjoint action by G on \mathcal{O} is in fact Hamiltonian.

Exercise 14.4. Prove that the inclusion map

$$\mu : \mathcal{O} \hookrightarrow \mathfrak{g}^*$$

is a momentum map for the coadjoint action by G on \mathcal{O} .

⁸In this section, we write X_ξ instead of $X_{\mathcal{O}}(\xi)$ for the value at ξ of the vector field $X_{\mathcal{O}}$ on \mathcal{O} induced by the infinitesimal action.

15 Smooth projective manifolds

Consider the complex vector space \mathbb{C}^n , equipped with the standard hermitian form h :

$$h(z, z') = \sum_{j=1}^n z_j \bar{z}'_j,$$

for $z, z' \in \mathbb{C}^n$. Then

$$h = B + i\omega,$$

with B the standard inner product on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, and ω the standard symplectic form on $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

The natural action by $U(n)$ on \mathbb{C}^n preserves h (by definition), hence also B and ω . Hence the action is symplectic.

Exercise 15.1. Prove that the Lie algebra of $U(n)$ is

$$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}); X^* + X = 0\}.$$

Here X^* is the conjugate transpose of X .

Hint: see Exercise 8.4.

We will show that the map

$$\mu : \mathbb{C}^n \rightarrow \mathfrak{u}(n)^*,$$

given by

$$(\mu(z))(X) = \frac{i}{2} h(Xz, z),$$

for $z \in \mathbb{C}^n$ and $X \in \mathfrak{u}(n)$, is a momentum map for this action.

Exercise 15.2. Prove that the map μ is equivariant.

Exercise 15.3. Prove that for all $z \in \mathbb{C}^n$, $v \in T_z \mathbb{C}^n \cong \mathbb{C}^n$ and $X \in \mathfrak{g}$,

$$(d\mu_X)_z(v) = -\omega(X_{\mathbb{C}^n}(z), v).$$

Hint: Use Exercise 9.2. What is the simplest curve γ in \mathbb{C}^n with $\gamma(0) = z$ and $\gamma'(0) = v$?

Exercise 15.4. Let G be a Lie group, and $\rho : G \rightarrow U(n)$ a Lie group homomorphism. Prove that the induced action by G on \mathbb{C}^n is Hamiltonian, and give a momentum map.

Hint: Use Exercise 11.4. Your answer may be short.