Harish-Chandra's Plancherel formula for $\mathrm{SL}(2,\mathbb{R})$

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Abstract

We introduce Harish-Chandra's Plancherel formula and some ideas in its proof. We work out details for $\mathrm{SL}(2,\mathbb{R})$. This is based on Knapp's book on representation theory of semisimple groups.

Contents

1 The Plancherel formula

Let G be a linear, connected, real semisimple Lie group, and $\hat{G}$ its unitary dual. Fix a (left and right) Haar measure dg on G.

1.1 The Plancherel measure

For $f \in C_c^\infty(G)$ and $\pi \in \hat{G}$, define the operator $\pi(f)$ on the representation space of π by

$$\pi(f) := \int_G f(g)\pi(g)\,dg.$$

This operator is trace-class by Theorem 10.2 in [1], and hence Hilbert–Schmid. Its trace is by definition $\Theta_\pi(f)$, the global character $\Theta_\pi \in \mathcal{D}'(G)$ applied to f. The distribution Θ_π is given by integration against a locally integrable function, which we also denote by Θ_π. This is a deep result, see Theorem 10.36 in [1].

Let the Fourier transform $\hat{f}$ of f be the function on $\hat{G}$ defined by

$$\hat{f}(\pi) = \pi(f),$$

for $\pi \in \hat{G}$. Then $\hat{f}(\pi)$ is a Hilbert Schmid-operator on the representation space of π, for all $\pi \in \hat{G}$.

Theorem 1.1. *There is a measure μ on $\hat{G}$ such that for all $f \in C_c^\infty(G)$,*

$$\|f\|^2_{L^2(G)} = \int_{\hat{G}} \mathrm{tr}(\hat{f}(\pi)^*\hat{f}(\pi))\,d\mu(\pi). \tag{1.1}$$

This is generalises to a statement about arbitrary locally compact groups [2]. This theorem states that the Fourier transform defines a unitary isomorphism

$$L^2(G) \xrightarrow{\cong} \int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*\,d\mu(\pi),$$

where $\mathcal{H}_\pi$ is the representation space of $\pi \in \hat{G}$, and the right hand side is defined as the space of L^2-sections of the field of Hilbert spaces $\coprod_{\pi\in\hat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* \to \hat{G}$ with respect to μ. (The Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}^*$ equals the Hilbert space of Hilbert–Schmid operators on a Hilbert space $\mathcal{H}$.)

Definition 1.2. The measure μ from Theorem 1.1 is the *Plancherel measure* on $\hat{G}$.

Harish-Chandra's Plancherel formula is an explicit expression for μ.

The condition Eq. (1.1) can be rewritten into a more practical form.

Lemma 1.3. *Suppose that every* $h \in C_c^\infty(G)$ *satisfies the Fourier inversion formula*

$$h(e) = \int_{\hat{G}} \Theta_\pi(h)\, d\mu(\pi). \tag{1.2}$$

Then Eq. (1.1) *holds for all* $f \in C_c^\infty(G)$.

Proof. Let $f \in C_c^\infty(G)$, and set $h := f^* * f$. Then $h(e) = \|f\|^2_{L^2(G)}$, and the right hand side of Eq. (1.2) equals the right hand side of Eq. (1.1). □

1.2 Cartan subgroups and root systems

For a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let H be the corresponding Cartan subgroup, defined as the centraliser of $\mathfrak{h}$ in G, hence possible disconnected. Let $R = R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the root system for $\mathfrak{h}$. Let $W(G,H) = N_G(H)/Z_G(H)$ be the analytic Weyl group of (G,H).

We will tacitly choose a positive root system $R^+ \subset R$ for every Cartan subgroup. A root is real or imaginary if it takes real or imaginary values on $\mathfrak{h}$. We write $R_R^+ := R_R^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ and $R_I^+ := R_I^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ for the sets of real and imaginary positive roots, respectively.

Let $\rho = \rho^H$ be half the sum of the positive roots. We will use the function e^ρ on H. This function is well-defined if ρ is integral. We will assume this from now on. We also need to be careful about the definition of e^ρ for disconnected H, so that the exponential map is not surjective. We assume that e^ρ is well-defined on $\exp(\mathfrak{h}^{\mathbb{C}})$, and define e^ρ as its restriction to H.

The *Weyl denominator* for H is the function on H given by

$$D_H = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}).$$

The functions $e^{-\alpha}$ are also defined via extensions to $\exp(\mathfrak{h}^{\mathbb{C}})$, as for e^ρ. Note that α is always integral. If the assumptions on ρ are not satisfied, one needs to check that expressions involving D_H are well-defined even when D_H itself is not.

We will use the function

$$\varepsilon_R^H := \operatorname{sgn} \prod_{\alpha \in R_R^+} (1 - e^{-\alpha})$$

on H. (As on p. 349 of [1].)

Lemma 1.4. *We have*

$$|D_H|^2 = (-1)^{\# R_I^+} D_H^2.$$

See (10.25a) in [1].

1.3 Weyl's integration formula

Let $\mathfrak{h}_1, \dots, \mathfrak{h}_n$ be a set of θ-stable representatives from each conjugacy class of Cartan subalgebras of $\mathfrak{g}$. Let $H_1, \dots, H_n$ be the corresponding Cartan subgroups of G. Let dh_j for a Haar measure on H_j, and $d(gH_j)$ a G-invariant measure on G/H_j, such that for all $f \in C_c^\infty(G)$,

$$\int_G f(g)\, dg = \int_{G/H_j} \int_{H_j} f(gh_j)\, dh_j\, d(gH_j).$$

Proposition 1.5 (Weyl's integration formula)**.** *For all* $f \in C_c(G)$,

$$\int_G f(g)\, dg = \sum_{j=1}^n \frac{1}{\# W(G, H_j)} \int_{H_j} \int_{G/H_j} f(gh_j g^{-1}) |D_{H_j}(h_j)|^2\, d(gH_j)\, dh_j.$$

See Proposition 5.27 in [1].

1.4 Orbital integrals

Fix a θ-stable Cartan $H < G$. Let $f \in C_c^\infty(G)$. A crucial role in Harish-Chandra's proof of his Plancherel formula is played by the function F_f^H on H given by

$$F_f^H(h) = \varepsilon_R^H(h) D_H(h) \int_{G/H} f(ghg^{-1})\, d(gH).$$

Lemma 1.6. *For every* $\pi \in \hat{G}$,

$$\Theta_\pi(f) = \sum_{j=1}^n \frac{(-1)^{\# R_I^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_j^{\mathbb{C}})}}{\# W(G, H_j)} \int_{H_j} \varepsilon_R^{H_j}(h_j) F_f^{H_j}(h_j) (D_H \Theta_\pi)(h_j)\, dh_j.$$

Proof. This follows from conjugation invariance of Θ_π, Lemma 1.4 and Proposition 1.5. $\square$

In Lemma 1.6, the function $D_H\Theta_\pi$ has a natural expression. See Theorem 10.35 in [1].

For $\alpha \in R$, let $H_\alpha \in \mathfrak{h}^{\mathbb{C}}$ be the vector such that for all $X \in \mathfrak{h}$, $\langle \alpha, X \rangle = B(H_\alpha, X)$. ([1], p. 65.) Consider the element

$$\tilde{\varpi}^H := \prod_{\alpha \in R^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})} H_\alpha \quad \in U(\mathfrak{h}^{\mathbb{C}}).$$

Let $\partial(\tilde{\varpi}^H)$ be the corresponding differential operator on $C^\infty(H)$.

Theorem 1.7. *Suppose* H *is* θ*-stable and maximally compact. Then there is a* $c \neq 0$ *such that for all* $f \in C_c^\infty(G)$*,*

$$(\partial(\tilde{\varpi}^H)F_f^H)(e) = cf(e).$$

See Theorem 11.17 in [1].

By Lemmas 1.3 and 1.6 and Theorem 1.7, Theorem 1.1 is equivalent to the equality

$$c^{-1}(\partial(\tilde{\varpi}^H)F_f^H)(e) = \sum_{j=1}^{n} \frac{(-1)^{\#R_I^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}_j^{\mathbb{C}})}}{\#W(G, H_j)} \int_{\hat{G}} \int_{H_j} \varepsilon_R^{H_j}(h_j) F_f^{H_j}(h_j)(D_H\Theta_\pi)(h_j)\, dh_j\, d\mu(\pi).$$

for all $f \in C_c^\infty(G)$, where $H < G$ is a maximally compact, θ-stable Cartan. The idea of the proof is to apply integration by parts and Fourier inversion on each Cartan subgroup. A crucial role is played by discontinuities of the functions $F_f^{H_j}$ on Weyl chamber walls.

2 Example: $SL(2, \mathbb{R})$

This proof of the Plancherel formula for $SL(2, \mathbb{R})$ is based on Section XI.3 in [1].

2.1 Setup

Let $G = SL(2,\mathbb{R})$, $T = SO(2)$. For $\theta, t \in \mathbb{R}$, we write

$$k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Let

$$A := \{\pm a_t; t \in \mathbb{R}\} < G.$$

Then A and T are two θ-stable representatives of the two conjugacy classes of Cartan subgroups of $SL(2,\mathbb{R})$.

The Haar measure da on A is normalised by

$$\int_A f(a)\, da = \frac{1}{2}\int_{\mathbb{R}} f(a_t)\, dt + \frac{1}{2}\int_{\mathbb{R}} f(-a_t)\, dt. \tag{2.1}$$

for all $f \in C_c(A)$.

Write

$$H_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.2}$$

and

$$H_2 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We choose $R^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) = \{\alpha\}$, where

$$\langle \alpha, H_1 \rangle = 2i.$$

And $R^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}}) = \{\beta\}$, where

$$\langle \beta, H_2 \rangle = 2.$$

For $n \in \mathbb{N}$ at least 2, let Θ_n be the global character of the direct sum of the discrete series representations D_n^+ and D_n^- with infinitesimal character $(n-1)\rho^T$.

For $\sigma \in \widehat{\mathbb{Z}/2}$ and $\nu \in \mathbb{R}$, let $\Theta_{\sigma,i\nu}$ be the global character of the principal series representation induced from the representation $\sigma \otimes e^{i\nu\rho^A} \otimes 1$ of the minimal parabolic of upper triangular matrices $P = MA_0N$, with $M = \{\pm I_2\}$, A_0 the identity component of A, and N the nilpotent group of 2×2 matrices with ones on the diagonal and zero in the lower left entry. Let σ_+ be the trivial representation of M, and σ_- the nontrivial irreducible one.

Our goal in this section is to prove the following result.

Theorem 2.1 (Plancherel formula for $SL(2,\mathbb{R})$)**.** *For all* $f \in C_c^\infty(G)$,

$$2\pi f(e) = \sum_{n=2}^{\infty}(n-1)\Theta_n(f)+\frac{1}{4}\int_{\mathbb{R}}\Theta_{\sigma_+,i\nu}(f)\nu\tanh(\pi\nu/2)\,d\nu+\frac{1}{4}\int_{\mathbb{R}}\Theta_{\sigma_-,i\nu}(f)\nu\coth(\pi\nu/2)\,d\nu.$$

2.2 Preliminaries

We now have $e^{\rho^A}(a_t) = e^t$, and, by the way in which e^ρ was defined for disconnected Cartans,

$$e^{\rho^A}(-a_t) = e^{\rho^A}(\exp((t+i\pi)H_2)) = e^{t+i\pi} = -e^t.$$

This complex-linear extension is also used to define functions e^α, for roots α. So, for example,

$$e^{-\beta}(-a_t) = e^{-\beta}(\exp((t+i\pi)H_2)) = e^{-2t-2\pi i} = e^{-2t} = e^{-\beta}(a_t).$$

So

$$\begin{aligned} D_T(k_\theta) &= D_T(\exp(\theta H_1)) = e^{i\theta} - e^{-i\theta}; \\ D_A(\pm a_t) &= D_A(\pm\exp(tH_2)) = \pm(e^t - e^{-t}). \end{aligned} \tag{2.3}$$

Furthermore, in this setting,

$$\begin{aligned} \varepsilon_R^T(k_\theta) &= \operatorname{sgn}(1-e^{-2i\theta}) = 1 \quad \text{if } \theta \notin \pi\mathbb{Z}; \\ \varepsilon_R^A(\pm a_t) &= \operatorname{sgn}(1-e^{-2t}) = \operatorname{sgn}(e^t - e^{-t}) = \operatorname{sgn}(t), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} R_I^+(\mathfrak{g}^{\mathbb{C}};\mathfrak{t}^{\mathbb{C}}) &= R^+(\mathfrak{g}^{\mathbb{C}};\mathfrak{t}^{\mathbb{C}}) = \{\alpha\}; \\ R_I^+(\mathfrak{g}^{\mathbb{C}};\mathfrak{a}^{\mathbb{C}}) &= \emptyset. \end{aligned} \tag{2.5}$$

Now $N_G(T) = Z_G(T) = T$, and $N_G(A)$ is generated by A and H_1, whereas $Z_G(A) = A$. So

$$\begin{aligned} W(G,T) &= \{e\}; \\ W(G,A) &= \mathbb{Z}/2. \end{aligned} \tag{2.6}$$

(See page 132 in [1].)

Remark 2.2. The algebraic Weyl groups of $R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and $R(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$ are both isomorphic to $\mathbb{Z}/2$, but we will need the analytic Weyl groups since they are used in Proposition 1.5.

Lemma 2.3. *For all* $t \in \mathbb{R} \setminus \{0\}$,

$$F_f^A(\pm a_t) = \pm e^t \int_{K \times N} f(\pm k a_t n k^{-1})\, dk\, dn.$$

See Lemma 10.10 in [1].

2.3 Character formulas

The proof of Theorem 2.1 that we give below is based on the following character formulas.

Proposition 2.4. *For all* $n \in \mathbb{N}$, $\theta, t \in \mathbb{R}$,

$$\Theta_{n+1}(k_\theta) = -\frac{e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}};$$
$$\Theta_{n+1}(\pm a_t) = (\pm 1)^{n+1} \frac{e^{nt}(1 - \operatorname{sgn} t) + e^{-nt}(1 + \operatorname{sgn} t)}{|e^t - e^{-t}|}.$$

See Proposition 10.12 in [1].

Proposition 2.5. *For all* $\sigma \in \widehat{\mathbb{Z}/2}$, $\theta, t \in \mathbb{R}$, $\nu \in \mathbb{C}$,

$$\Theta_{\sigma,\nu}(k_\theta) = 0;$$
$$\Theta_{\sigma,\nu}(\pm a_t) = \sigma(\pm 1) \frac{e^{\nu t} + e^{-\nu t}}{|e^t - e^{-t}|}.$$

See Corollary 10.13 in [1].

Note that by Eq. (2.3) and Propositions 2.4 and 2.5, the functions $D_H \Theta_n$ and $D_H \Theta_{\sigma, i\nu}$ in Lemma 1.6 indeed have a simple form.

2.4 Calculus on $\mathbb{R}$

Lemma 2.6. *Let* $g(t) = e^{-n|t|} \operatorname{sgn}(t)$. *Then*

$$\hat{g}(\nu) = \frac{-2i\nu}{n^2 + \nu^2}.$$

Proof. This is a direct computation. □

Lemma 2.7. *For all* $z \in \mathbb{C}$,

$$\sum_{n \in \mathbb{Z}} \frac{z}{z^2 - n^2} = \pi \cot(\pi z).$$

Lemma 2.8. *For all* $\nu \in \mathbb{C}$,

$$\sum_{n \in \mathbb{Z} \text{ even}} \frac{\nu^2}{n^2 + \nu^2} = \frac{\pi \nu}{2} \coth(\pi \nu / 2);$$

$$\sum_{n \in \mathbb{Z} \text{ odd}} \frac{\nu^2}{n^2 + \nu^2} = \frac{\pi \nu}{2} \tanh(\pi \nu / 2).$$

Proof. By Lemma 2.7,

$$\sum_{n \in \mathbb{Z} \text{ even}} \frac{\nu^2}{n^2 + \nu^2} = \frac{i\nu}{2} \sum_{n \in \mathbb{Z}} \frac{i\nu/2}{(i\nu/2)^2 - n^2} = \frac{\pi \nu}{2} \coth(\pi \nu / 2).$$

By Lemma 2.7 and the above calculation,

$$\begin{aligned} \sum_{n \in \mathbb{Z} \text{ odd}} \frac{\nu^2}{n^2 + \nu^2} &= \sum_{n \in \mathbb{Z}} \frac{\nu^2}{n^2 + \nu^2} - \sum_{n \in \mathbb{Z} \text{ even}} \frac{\nu^2}{n^2 + \nu^2} \\ &= i\pi\nu \cot(i\pi\nu) - \frac{\pi\nu}{2} \coth(\pi\nu/2) \\ &= \frac{\pi\nu}{2} \tanh(\pi\nu/2). \end{aligned}$$

□

2.5 Calculus on $\mathbb{R}^2$

Lemma 2.9. *Let* F *be a smooth function with compact support in*

$$\{(u, v) \in \mathbb{R}^2; |uv| < 1\}.$$

For $s \in \mathbb{R}$*, set*

$$I(s) := \pi s \int_0^\infty F(se^{2r}, se^{-2r})(e^{2r} - e^{-2r})\, dr.$$

Then

(a)

$$\lim_{s\downarrow 0} I(s) = \frac{\pi}{2}\int_0^\infty F(u,0)\,du;$$
$$\lim_{s\uparrow 0} I(s) = -\frac{\pi}{2}\int_{-\infty}^0 F(u,0)\,du.$$

(b) There are $c_1, c_2 > 0$ *such that for all nonzero* s *in a neighbourhood of zero,*

$$|\frac{dI}{ds}(s) + \pi F(s,s)| \le |s|(c_1 + c_2\log(|s|^{-1})).$$

In particular,

$$\lim_{s\to 0}\frac{dI}{ds}(s) = -\pi F(0,0).$$

Proof. See Lemma 11.5 in [1]. By substitutions, we find that

$$I(s) = \frac{\pi}{2}\int_s^\infty F(y, s^2y^{-1})\,dy - \frac{\pi}{2}\int_0^s F(s^2y^{-1}, y)\,dy. \tag{2.7}$$

if $s > 0$, and

$$I(s) = -\frac{\pi}{2}\int_{-\infty}^s F(y, s^2y^{-1})\,dy + \frac{\pi}{2}\int_s^0 F(s^2y^{-1}, y)\,dy. \tag{2.8}$$

if $s < 0$. By taking the limits as $s \downarrow 0$ or $s \uparrow 0$, we obtain part (a).

Taking derivatives in Eqs. (2.7) and (2.8) with respect to s and using boundedness of F and its support, we get (b).

For example, the derivative with respect to s of the second term on the right hand side of Eq. (2.7) equals

$$-\frac{\pi}{2}F(s,s) - \pi s\int_0^s (\partial_1 F)(s^2y^{-1}, y)\frac{dy}{y},$$

where ∂_1 is the derivative in the first component. Let $a > 0$ be such that all points in the support of $\partial_1 F$ have first coordinate in $[-a, a]$. Then the integrand in the second term above is zero if $s^2y^{-1} > a$, i.e. $y < s^2/a$. So, if $0 < s < a$, then

$$\begin{aligned}-\pi s\int_0^s (\partial_1 F)(s^2y^{-1}, y)\frac{dy}{y} &= -\pi s\int_{s^2/a}^s (\partial_1 F)(s^2y^{-1}, y)\frac{dy}{y} \\ &\le -\pi s\|\partial_1 F\|_\infty(\log(s) - \log(s^2/a)).\end{aligned}$$

The cases for the first term on the right hand side of Eq. (2.7) and for $s < 0$ (where one differentiates Eq. (2.8)) are analogous. $\square$

We will actually apply this lemma to a more general class of functions, to which the statement extends.

Example 2.10. Let F be the indicator function of the square $[-a, a] \times [-a, a] \subset \mathbb{R}^2$, for a small enough. Then

$$I(s) = \begin{cases} \frac{\pi a}{2} - \pi s + \frac{\pi s^2}{2a} & \text{if } 0 < s \leq a; \\ 0 & \text{if } s = 0; \\ -\frac{\pi a}{2} - \pi s - \frac{\pi s^2}{2a} & \text{if } -a \leq s < 0. \end{cases}$$

This implies claim (a) in the lemma. And for nonzero $s \in [-a, a]$,

$$\frac{dI}{ds}(s) + \pi F(s, s) = \frac{\pi |s|}{a},$$

so claim (b) also holds, with $c_1 = \pi/a$ and $c_2 = 0$.

2.6 Orbital integrals over G/T

From now on, we fix $f \in C_c^\infty(G)$. We may assume that f is invariant under conjugation by K, since both sides of the Plancherel formula are so we can apply averaging with respect to conjugation by K.

Writing $A^+ := \{a_t; t \geq 0\}$, we have $G = KA^+K$. We normalise the Haar measure dg so that for all $h \in C_c(G)$,

$$\int_G h(g)\, dg = 2\pi \int_0^{2\pi} \int_0^\infty \int_0^{2\pi} h(k_{\theta_1} a_t k_{\theta_2})\ \sinh(2t) \frac{d\theta_1}{2\pi}\, dt\, \frac{d\theta_2}{2\pi}.$$

Note that $\frac{d\theta}{2\pi}$ is the Haar measure on K giving it unit volume.

Lemma 2.11.

$$\begin{aligned} \lim_{\theta \downarrow 0} F_f^T(k_\theta) - \lim_{\theta \uparrow 0} F_f^T(k_\theta) &= i\pi F_f^A(a_0) \\ \lim_{\theta \downarrow \pi} F_f^T(k_\theta) - \lim_{\theta \uparrow \pi} F_f^T(k_\theta) &= i\pi F_f^A(-a_0); \\ \lim_{\theta \to 0} \frac{dF_f^T(k_\theta)}{d\theta}(k_\theta) &= -2\pi i f(e). \end{aligned} \tag{2.9}$$

Proof. By the top equalities in Eqs. (2.3) and (2.4), we have for all $\theta \in \mathbb{R}\backslash\pi\mathbb{Z}$,

$$F_f^T(k_\theta) = (e^{i\theta} - e^{-i\theta}) \int_{G/T} f(gk_\theta g^{-1})\, d(gT). \tag{2.10}$$

Since f is invariant under conjugation by K, the function $gT \mapsto f(gk_\theta g^{-1})$ is left K-invariant. So the right hand side of Eq. (2.10) equals

$$4\pi i \sin(\theta) \int_0^\infty f(a_t k_\theta a_{-t}) \sinh(2t)\, dt. \tag{2.11}$$

Define the function $F_\pm$ on $\{(u, v) \in \mathbb{R}^2; |uv| < 1\}$ by

$$F_\pm(u, v) := f \begin{pmatrix} \pm\sqrt{1-uv} & u \\ -v & \pm\sqrt{1-uv} \end{pmatrix}.$$

Then Eq. (2.11) equals

$$2\pi i \sin(\theta) \int_0^\infty F_\pm(\sin(\theta)e^{2t}, \sin(\theta)e^{-2t})(e^{2t} - e^{-2t})\, dt,$$

where use F_+ if $-\pi/2 < \theta < \pi/2$ and F_- if $\pi/2 < \theta < 3\pi/2$ (these ranges are taken modulo $2\pi\mathbb{Z}$).

Let $I_\pm$ be as in Lemma 2.9, defined with $F = F_\pm$. Then

$$F_f^T(k_\theta) = 2iI_\pm(\sin(\theta)),$$

with the sign $\pm$ depending on where θ lies, as above. So, by part (a) of Lemma 2.9

$$\begin{aligned}
\lim_{\theta\downarrow 0} F_f^T(k_\theta) &= 2i \lim_{s\downarrow 0} I_+(s) = i\pi \int_0^\infty F_+(u, 0)\, du; \\
\lim_{\theta\uparrow 0} F_f^T(k_\theta) &= 2i \lim_{s\uparrow 0} I_+(s) = -i\pi \int_{-\infty}^0 F_+(u, 0)\, du; \\
\lim_{\theta\downarrow \pi} F_f^T(k_\theta) &= 2i \lim_{s\uparrow 0} I_-(s) = -i\pi \int_{-\infty}^0 F_-(u, 0)\, du; \\
\lim_{\theta\uparrow \pi} F_f^T(k_\theta) &= 2i \lim_{s\downarrow 0} I_-(s) = i\pi \int_0^\infty F_-(u, 0)\, du.
\end{aligned}$$

Hence

$$\lim_{\theta\downarrow 0} F_f^T(k_\theta) - \lim_{\theta\uparrow 0} F_f^T(k_\theta) = i\pi \int_{\mathbb{R}} F_+(u,0)\,du = i\pi \int_{\mathbb{R}} f\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} ds;$$
$$\lim_{\theta\downarrow \pi} F_f^T(k_\theta) - \lim_{\theta\uparrow \pi} F_f^T(k_\theta) - i\pi \int_{\mathbb{R}} F_-(u,0)\,du = -i\pi \int_{\mathbb{R}} f\left(-\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) ds.$$

Lemma 2.12 and the fact that f is invariant under conjugation by K imply that

$$F_f^A(\pm a_0) = \pm \int_{\mathbb{R}} f\left(\pm\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) ds.$$

So the first two equalities in Eq. (2.9) follow.

And by part (b) of Lemma 2.9,

$$\lim_{\theta\to 0} \frac{dF_f^T(k_\theta)}{d\theta}(k_\theta) = 2i \lim_{\theta\to 0} \frac{dI_+(\sin(\theta))}{d\theta}(\sin(\theta)) = 2i \lim_{\theta\to 0} \cos(\theta) \frac{dI_+(s)}{ds}(\sin(\theta))$$
$$= -2\pi i F_+(0,0) = -2\pi i f(e).$$

□

2.7 Orbital integrals over G/A

Lemma 2.12. *For all* $t \in \mathbb{R}$,

$$F_f^A(a_t) = \frac{1}{2\pi} \int_{\mathbb{R}} (\Theta_{\sigma_-, i\nu}(f) + \Theta_{\sigma_+, i\nu}(f)) e^{-i\nu t}\,d\nu;$$
$$F_f^A(-a_t) = \frac{1}{2\pi} \int_{\mathbb{R}} (\Theta_{\sigma_-, i\nu}(f) - \Theta_{\sigma_+, i\nu}(f)) e^{-i\nu t}\,d\nu.$$

Proof. The equalities Eqs. (2.3) to (2.6) imply that for all $t \in \mathbb{R}$,

$$\frac{(-1)^{\#R_I^+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})}}{\#W(G,A)} (\varepsilon_R^A D_A)(\pm a_t) = \pm \frac{1}{2} |e^t - e^{-t}|. \tag{2.12}$$

So by Lemma 1.6, Proposition 2.5, and Eq. (2.1), we have for all $f \in C_c^\infty(G)$ and all $\sigma \in \widehat{\mathbb{Z}/2}$ and $\nu \in \mathbb{R}$,

$$\Theta_{\sigma, i\nu}(f) = \frac{1}{4} \int_{\mathbb{R}} F_f^A(a_t)(e^{i\nu t} + e^{-i\nu t})\,dt - \sigma(-1) \frac{1}{4} \int_{\mathbb{R}} F_f^A(-a_t)(e^{i\nu t} + e^{-i\nu t})\,dt. \tag{2.13}$$

With H_1 as in Eq. (2.2),

$$\pm a_{-t} = H_1(\pm a_t)H_1^{-1}.$$

So by conjugation invariance of F_f^A,

$$F_f^A(\pm a_t) = F_f^A(\pm a_{-t}).$$

Because of this equality, Eq. (2.13) equals

$$\frac{1}{2}\int_{\mathbb{R}} F_f^A(a_t)e^{i\nu t}\,dt - \sigma(-1)\frac{1}{2}\int_{\mathbb{R}} F_f^A(-a_t)e^{i\nu t}\,dt = \frac{1}{2}\int_{\mathbb{R}} (F_f^A(a_t) - \sigma(-1)F_f^A(-a_t))e^{i\nu t}\,dt.$$

By Fourier inversion, we find that

$$F_f^A(a_t) - F_f^A(-a_t) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\Theta_{\sigma_+,i\nu}(f)e^{-i\nu t}\,d\nu;$$
$$F_f^A(a_t) + F_f^A(-a_t) = \frac{1}{2\pi}\int_{\mathbb{R}} 2\Theta_{\sigma_-,i\nu}(f)e^{-i\nu t}\,d\nu,$$

from which the claim follows. □

2.8 Discrete series characters

Lemma 2.13. *For all* $n \in \mathbb{N}$,

$$n\Theta_{n+1}(f) = -\frac{1}{2\pi i}\int_0^{2\pi} (e^{in\theta} + e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta$$
$$+\frac{1}{2}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(a_t)}{dt}(a_t)\,dt + (-1)^n\frac{1}{2}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(-a_t)}{dt}(-a_t)\,dt.$$

Proof. The equalities Eqs. (2.3) to (2.6) imply that for all $\theta \in \mathbb{R}$,

$$\frac{(-1)^{\#R_I^+(\mathfrak{g}^{\mathbb{C}},\mathfrak{t}^{\mathbb{C}})}}{\#W(G,T)}(\varepsilon_R^T D_T)(k_\theta) = -(e^{i\theta} - e^{-i\theta}).$$

By this equality and Eqs. (2.1) and (2.12), Lemma 1.6 and Proposition 2.4 imply that

$$n\Theta_{n+1}(f) = \frac{n}{2\pi}\int_0^{2\pi} (e^{in\theta} - e^{-in\theta})F_f^T(k_\theta)\,d\theta$$
$$+\frac{n}{4}\int_{\mathbb{R}} \big(e^{nt}(1 - \operatorname{sgn} t) + e^{-nt}(1 + \operatorname{sgn} t)\big)(F_f^A(a_t) - (-1)^{n+1}F_f^A(-a_t))\,dt. \tag{2.14}$$

And by integration by parts, taking into account the discontinuities of F_f^T,

$$
\begin{aligned}
&n\int_0^{\pi}(e^{in\theta}-e^{-in\theta})F_f^T(k_\theta)\,d\theta = \\
&\quad -i\,\big((e^{in\theta}+e^{-in\theta})F_f^T(k_\theta)\big)\Big|_0^{\pi}+i\int_0^{\pi}(e^{in\theta}+e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta = \\
&\quad 2i\lim_{\theta\downarrow 0}F_f^T(k_\theta)-2i(-1)^n\lim_{\theta\uparrow\pi}F_f^T(k_\theta)+i\int_0^{\pi}(e^{in\theta}+e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&n\int_{\pi}^{2\pi}(e^{in\theta}-e^{-in\theta})F_f^T(k_\theta)\,d\theta = \\
&-2i\lim_{\theta\uparrow 0}F_f^T(k_\theta)+2i(-1)^n\lim_{\theta\downarrow\pi}F_f^T(k_\theta)+i\int_{\pi}^{2\pi}(e^{in\theta}+e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta.
\end{aligned}
$$

So

$$
\begin{aligned}
&n\int_0^{2\pi}(e^{in\theta}-e^{-in\theta})F_f^T(k_\theta)\,d\theta = \\
&\quad 2i\big(\lim_{\theta\downarrow 0}F_f^T(k_\theta)-\lim_{\theta\uparrow 0}F_f^T(k_\theta)\big)+2i(-1)^n\big(\lim_{\theta\downarrow\pi}F_f^T(k_\theta)-\lim_{\theta\uparrow\pi}F_f^T(k_\theta)\big) \\
&\qquad +i\int_0^{2\pi}(e^{in\theta}+e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta. \qquad (2.15)
\end{aligned}
$$

To evaluate the second term on the right hand side of (2.14), we again use integration by parts to find

$$
\begin{aligned}
&n\int_0^{\infty}\big(e^{nt}(1-\operatorname{sgn}t)+e^{-nt}(1+\operatorname{sgn}t)\big)(F_f^A(a_t)+(-1)^nF_f^A(-a_t))\,dt \\
&\quad = \int_0^{\infty}2ne^{-nt}(F_f^A(a_t)+(-1)^nF_f^A(-a_t))\,dt \\
&= \big(-2e^{-nt}(F_f^A(a_t)+(-1)^nF_f^A(-a_t))\big)\Big|_0^{\infty}+2\int_0^{\infty}e^{-nt}\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt \\
&= 2(F_f^A(a_0)+(-1)^nF_f^A(-a_0))+2\int_0^{\infty}e^{-nt}\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt.
\end{aligned}
$$

Here we used continuity of F_f^A. Similarly,

$$
\begin{aligned}
n\int_{-\infty}^{0}&\big(e^{nt}(1-\operatorname{sgn} t)+e^{-nt}(1+\operatorname{sgn} t)\big)(F_f^A(a_t)+(-1)^n F_f^A(-a_t))\,dt\\
&=\int_{-\infty}^{0} 2ne^{nt}(F_f^A(a_t)+(-1)^n F_f^A(-a_t))\,dt\\
&=\big(2e^{nt}(F_f^A(a_t)+(-1)^n F_f^A(-a_t))\big)\big|_{-\infty}^{0}-2\int_{-\infty}^{0} e^{nt}\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt\\
&=2(F_f^A(a_0)+(-1)^n F_f^A(-a_0))-2\int_{-\infty}^{0} e^{nt}\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt.
\end{aligned}
$$

So

$$
\begin{aligned}
n\int_{\mathbb{R}}&\big(e^{nt}(1-\operatorname{sgn} t)+e^{-nt}(1+\operatorname{sgn} t)\big)(F_f^A(a_t)+(-1)^n F_f^A(-a_t))\,dt\\
&=4(F_f^A(a_0)+(-1)^n F_f^A(-a_0))+2\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn} t\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt.
\end{aligned}
\tag{2.16}
$$

Using Eqs. (2.15) and (2.16), we conclude that the right hand side of Eq. (2.14) equals

$$
\begin{aligned}
&\frac{i}{\pi}\big(\lim_{\theta\downarrow 0}F_f^T(k_\theta)-\lim_{\theta\uparrow 0}F_f^T(k_\theta)\big)+\frac{i}{\pi}(-1)^n\big(\lim_{\theta\downarrow \pi}F_f^T(k_\theta)-\lim_{\theta\uparrow \pi}F_f^T(k_\theta)\big)\\
&\qquad+\frac{i}{2\pi}\int_0^{2\pi}(e^{in\theta}+e^{-in\theta})\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta\\
&+(F_f^A(a_0)+(-1)^n F_f^A(-a_0))+\frac{1}{2}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn} t\Big(\frac{dF_f^A(a_t)}{dt}+(-1)^n\frac{dF_f^A(-a_t)}{dt}\Big)(a_t)\,dt.
\end{aligned}
$$

By Lemma 2.11,

$$
\frac{i}{\pi}\big(\lim_{\theta\downarrow 0}F_f^T(k_\theta)-\lim_{\theta\uparrow 0}F_f^T(k_\theta)\big)+\frac{i}{\pi}(-1)^n\big(\lim_{\theta\downarrow \pi}F_f^T(k_\theta)-\lim_{\theta\uparrow \pi}F_f^T(k_\theta)\big)+(F_f^A(a_0)+(-1)^n F_f^A(-a_0))=0.
$$

Hence the claim follows. $\square$

2.9 Proof of Theorem 2.1

Let us prove Theorem 2.1. We start with a first expression for $f(e)$, which we later rewrite into Theorem 2.1.

Lemma 2.14.

$$2\pi f(e) = \sum_{n=1}^{\infty} n\Theta_{n+1}(f) + \frac{1}{2}(F_f^A(a_0) + F_f^A(-a_0))$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(a_t)}{dt}(a_t)\,dt - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(-a_t)}{dt}(-a_t)\,dt.$$

Proof. By Lemma 2.13,

$$\sum_{n=1}^{\infty} n\Theta_{n+1}(f) = \frac{-1}{2\pi i}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} e^{ik\theta}\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta + \frac{1}{2\pi i}\int_0^{2\pi}\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\,d\theta$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(a_t)}{dt}(a_t)\,dt + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(-a_t)}{dt}(-a_t)\,dt. \tag{2.17}$$

Splitting up the integral into integrals over $(0,\pi)$ and $(\pi,2\pi)$, we find that the second term on the right hand side equals

$$\frac{-1}{2\pi i}\big(\lim_{\theta\uparrow\pi} F_f^T(k_\theta) - \lim_{\theta\downarrow 0} F_f^T(k_\theta)\big) + \frac{-1}{2\pi i}\big(\lim_{\theta\uparrow 0} F_f^T(k_\theta) - \lim_{\theta\downarrow\pi} F_f^T(k_\theta)\big),$$

which by Lemma 2.11 equals

$$-\frac{1}{2}(F_f^A(a_0) + F_f^A(-a_0)).$$

Part (b) of Lemma 2.9 implies that

$$\begin{aligned}\left|\frac{dF_f^T(k_\theta)}{d\theta}(k_\theta)\right| &= 2|\cos\theta|\left|\frac{dI_\pm(s)}{ds}(\sin\theta)\right| \\ &\le 2\pi|F_\pm(0,0)| + |\sin\theta|(c_1 + c_2|\log((\sin\theta)^{-1})|)\end{aligned}$$

for certain constants $c_1, c_2 > 0$ and for s near zero. So $\frac{dF_f^T(k_\theta)}{d\theta}$ is a Hölder function, ands Dini's test implies that it satisfies Fourier inversion. So by the third equality in Lemma 2.11, the first term on the right hand side of Eq. (2.17) equals

$$i\frac{dF_f^T(k_\theta)}{d\theta}(e) = 2\pi f(e).$$

□

Lemma 2.15.

$$\frac{1}{2}F_f^A(a_0) - \frac{1}{2}\sum_{n=1}^{\infty}\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(a_t)}{dt}(a_t)\,dt$$
$$+\frac{1}{2}F_f^A(-a_0) - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(-a_t)}{dt}(-a_t)\,dt =$$
$$\frac{1}{4}\int_{\mathbb{R}}\Theta_{\sigma_+,i\nu}(f)\nu\tanh(\pi\nu/2)\,d\nu + \frac{1}{4}\int_{\mathbb{R}}\Theta_{\sigma_-,i\nu}(f)\nu\coth(\pi\nu/2)\,d\nu. \quad (2.18)$$

Proof. Define the functions g and $h_\pm$ on $\mathbb{R}$ by

$$g(t) := e^{-n|t|}\operatorname{sgn}(t);$$
$$h_\pm(\nu) = \frac{-i\nu}{2\pi}(\Theta_{\sigma_-,i\nu}(f) \pm \Theta_{\sigma_+,i\nu}(f)).$$

Differentiating both sides of the equalities in Lemma 2.12, we find that

$$\hat{h}_\pm(t) = \frac{dF_f^A(\pm a_t)}{dt}(\pm a_t).$$

Using this equality, Lemma 2.6, and the equality

$$\int_{\mathbb{R}}(g\hat{h})(t)\,dt = \int_{\mathbb{R}}(\hat{g}h)(\nu)\,d\nu,$$

we find that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} e^{-n|t|}\operatorname{sgn}(t)\frac{dF_f^A(\pm a_t)}{dt}(\pm a_t)\,dt = \frac{-1}{\pi}\int_{\mathbb{R}}\frac{\nu^2}{n^2+\nu^2}(\Theta_{\sigma_-,i\nu}(f) \pm \Theta_{\sigma_+,i\nu}(f))\,d\nu.$$

And by Lemma 2.12,

$$F_f^A(a_0) + F_f^A(-a_0) = \frac{1}{\pi}\int_{\mathbb{R}}\Theta_{\sigma_-,i\nu}(f)\,d\nu.$$

So the left hand side of Eq. (2.18) equals

$$\frac{1}{2\pi}\int_{\mathbb{R}} \Theta_{\sigma_-,i\nu}(f)\,d\nu + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{\mathbb{R}} \frac{\nu^2}{n^2+\nu^2}\left(\Theta_{\sigma_-,i\nu}(f)+\Theta_{\sigma_+,i\nu}(f)\right)d\nu \\ + \frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\int_{\mathbb{R}} \frac{\nu^2}{n^2+\nu^2}\left(\Theta_{\sigma_-,i\nu}(f)-\Theta_{\sigma_+,i\nu}(f)\right)d\nu.$$

This equals

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}\text{ even}}\int_{\mathbb{R}} \frac{\nu^2}{n^2+\nu^2}\Theta_{\sigma_-,i\nu}(f)\,d\nu + \frac{1}{2\pi}\sum_{n\in\mathbb{Z}\text{ odd}}\int_{\mathbb{R}} \frac{\nu^2}{n^2+\nu^2}\Theta_{\sigma_+ i\nu}(f)\,d\nu.$$

The claim now follows from Lemma 2.8. □

Theorem 2.1 follows from Lemmas 2.14 and 2.15.

References

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