Approximation of Higher Degree Spectra Results for Twisted Laplace Operators

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Abstract

This thesis examines the eigenvalues of the connection Laplacian acting on differential forms with values in a Hermitian vector bundle with connection over a closed Riemannian manifold. Specifically, building upon previous work by Whitney, Dodziuk, Patodi and Zahariev, a combinatorial analogue of the connection Laplacian is defined via triangulations of the manifold whereby differential forms are associated to cochains. Using the min-max principle as a key ingredient, this reduces the infinite dimensional analytic eigenvalue problem to a finite dimensional combinatorial one. In theory, this allows the eigenvalues to be calculated with numerical methods and sufficient computational power.

In this thesis, I prove that the eigenvalues of the analytic Laplacian are bounded below by the eigenvalues of the combinatorial Laplacian for differential forms and cochains of arbitrary degree with values in a trivial complex line bundle provided an assumption is met. This is achieved via an explicit calculation of the growth rate of the Whitney map under standard subdivisions.

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

I give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

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Matthias Fresacher

Matthias Fresacher

Date: 21st June, 2021

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Part I

Preliminaries

CHAPTER 1

1749 – 1827 PIERRE-SIMON LAPLACE *(French)*

INTRODUCTION

Given a closed Riemannian manifold and differential forms with values in a given vector bundle over the manifold, various Laplacian operators can be defined on these forms. A basic quantity of interest of these Laplacians are their eigenvalues.

In this thesis, I aim to approximate the eigenvalues of the Laplacian in an analytic setting by reformulating the analytic eigenvalue problem to an associated eigenvalue problem in a combinational setting. In essence, every object in the analytic setting is associated with a corresponding object in the combinatorial setting. The manifold is associated with a triangulation or simplicial complex, differential forms with cochains, the exterior derivative with the coboundary operator, and the analytic Laplacian is associated with a combinatorial Laplacian. This transportation from the infinite dimensional analytic setting to the finite dimensional combinatorial setting reduces the complexity of calculating eigenvalues but in the process introduces technicalities that must be managed as the combinatorial setting only *approximates* the analytic one. Making this approximation precise is the main challenge of this method and the subject of this thesis. The min-max principle is used in this method to provide a way to express the eigenvalues of the Laplacian in terms of inner products and hence norms of differential forms or cochains and the Laplacian acting on these differential forms or cochains.

1.1 History

In 1931, de Rham provided the foundation of combinatorial approximations to analytic differential geometric objects by defining the de Rham map as part of his isomorphism theorem of cohomologies [7]. This map takes analytic objects and maps them to combinational ones. However, the missing ingredient, a map from combinational objects to analytic ones, was not formalised until 1957 when Whitney [28, Chapter VII.11 (16)] presented the Whitney map. These two maps provide the foundation of approximating analytic problems with combinatorial ones by mapping between the two settings 'almost' interchangeably.

The de Rham map composed with the Whitney map is the identity map [28, Chapter VII.11 (5)], whilst the Whitney map composed with the de Rham map is not the identity map. Therefore, the two settings are only 'almost' interchangeable. Promisingly though, the Whitney map composed with the de Rham map approximates the identity map to arbitrary precision for a sufficiently refined triangulation of the manifold [9, Corollary 3.27].

This approximation of the identity map is precisely what allows the combinatorial setting to be used to calculate values of interest in the analytic setting, whilst at the same time being the main obstacle for such calculations. The difficulty lies in keeping track of the size of the error terms in the approximation. In principle, using the combinatorial setting to obtain analytic results should be readily achievable, in practice however, it requires careful analysis and, in some cases, has proven to be unpredictable and difficult.

For instance, little is known about the convergence of the adjoint of the coboundary operator to the adjoint of the exterior derivative. In 1991, Smits proved the two operators converge but only for degree one on a surface with a specific subdivision scheme [24, 5 Theorem]. Two decades later, in 2014, Arnold, Falk, Guzman and Tsogtgerel showed that this convergence for degree one can be generalised to manifolds of arbitrary dimension, but also provided counter examples when convergence failed in the setting of other subdivision schemes and separately, in the case of higher degree differential forms and cochains [3, Page 5489]. This leaves open the question of what, if anything, can be stated about the relationship of the adjoint of the coboundary operator to the adjoint of the exterior derivative for higher degrees. Considering these adjoint operators appear frequently in this area including in the definition of the Laplacian, many suspected approximation problems including ones tackled by Wilson in 2005 remain open. In his 2005 paper, Wilson [29] examined a combinatorial Hodge star operator and found that in some compositions with the coboundary operator, the combinatorial operators approximate their respective analytic operators, whilst results for other compositions are still unknown and 'seemingly harder to come by' [29, Page 19]. Wilson also speculated based on example calculations on a circle, whether the square of the combinatorial Hodge star approximates the identity operator up to a sign as is the case for the analytic Hodge star operator. In 2015, Tanabe proved this result in the affirmative [26, Theorem 3.1].

Returning to the problem considered in this thesis, the first use of solving associated combinatorial problems to analytic eigenvalue problems is found in the 1973 thesis by Dodziuk [9]. It was Dodziuk who provided the exact error term in the composition of the Whitney map with the de Rham map compared to the identity map [9, Corollary 3.27].

Additionally, Dodziuk established the convergence of eigenvalues of the Laplacian acting on functions [9, Theorem 5.7] as well as a technical result. This technical result, roughly speaking, says that given a Hodge decomposition in both the analytic and combinatorial settings, then for a given differential form and the Hodge decomposition of that form, the corresponding components of the decomposition in the combinatorial setting converge to their respective analytic components [9, Theorem 4.9].

The following year, 1974, Patodi proved a subtle generalisation of Dodziuk's technical result. Again roughly speaking, Patodi showed that given a Hodge decomposition in both the analytic and combinatorial settings, then for a given cochain (instead of a differential form) and its Hodge decomposition, the corresponding components of the decomposition converge [17, Theorem 2]. He then used his and Dodziuk's Hodge decomposition results to generalise Dodziuk's convergence of eigenvalues for functions to differential forms of arbitrary degree [17, Theorem 3].

The next advancement in this area was published in 1975 when Dodziuk and Patodi jointly combined their previous work to prove that the eigenvalues of the Laplace operator acting on differential forms with values in a flat real vector bundle converge [10, Theorem 3.7]. In their proof of the convergence of eigenvalues, they proved the generalisation of their earlier technical Hodge decomposition results for differential forms and cochains with values in a flat real vector bundle.

Following these three papers, there is a lull in research in this area[†] until 2007 when Zahariev,

[†]Possibly due to Patodi's early death in 1976.

supervised by Dodziuk, published the results of his thesis[‡]. Zahariev proved the convergence of eigenvalues of the Laplace operator acting on degree zero differential forms and cochains with values in a complex vector bundle with arbitrary connection [30, Theorem 5.4].

Whilst Dodziuk and Patodi have been the main contributors for the results so far in this area, it is important to note that the aforementioned series of results from the 1970s may not have eventuated without one person. Isadore Singer provided Dodziuk with support during the development of his thesis [9, Page 104], he brought the Dodziuk thesis to Patodi's attention [17, Page 43] and most importantly, brought Dodziuk and Patodi together for their substantive joint paper [10, Page 3].

Patodi's connection to Singer may well explain Patodi's motivation for working in this area. Patodi wanted to prove the Ray-Singer conjecture [19], posed in 1971. The conjecture states that the Reidemeister-Franz torsion, a function based on a triangulation of the manifold, can be expressed in terms of analytic quantities. Ray and Singer proposed the so called analytic torsion, which is defined in terms of the eigenvalues of the Laplace operator, for this analytic quantity. Dodziuk and Patodi in their joint paper [10] attempted to prove this conjecture but ultimately they were not successful and published their work on combinatorial approaches to the analytic eigenvalue problem instead. They believed their technical results for the Hodge decomposition would be the key to unlocking the conjecture. The conjecture was proved a few years after the Dodziuk and Patodi paper in 1977 by Cheeger [6] who did not use the work of Dodziuk and Patodi. However the following year, 1978, Müller [14], independently to Cheeger, used the work of Dodziuk and Patodi to prove the conjecture.

Analysing the four papers discussed that deal with the various versions of the eigenvalue problem, it is noteworthy that they can be split into two categories. Proofs that use the technical Hodge decomposition results as the basis for the convergence of eigenvalues as is the case in Patodi [17] and later Dodziuk and Patodi [10], and proofs that do not use the technical Hodge decomposition results as is the case in Dodziuk [9] and Zahariev [30]. Further, the proofs that use a Hodge decomposition prove convergence for arbitrary degree, whilst the ones that do not use a Hodge decomposition only prove convergence for degree zero.

The results of these four papers are formally stated in Chapter 3 following the definition of the notation used in Chapter 2.

1.2 Aim

The aim of this thesis is to investigate to what extent it is possible to generalise the work of Zahariev [30] on the convergence of eigenvalues of the combinatorial Laplace operator acting on degree zero cochains with values in an arbitrary complex vector bundle to the eigenvalues of the corresponding analytic Laplace operator to the setting of higher degrees.

Therefore, the procedure employed is the same as that used by Zahariev [30] in his work for degree zero. Specifically, a three step approach is used that initially, (1), considers the case of the Laplace operator acting on differential forms and cochains with values in a trivial complex line bundle. This is generalised to the case of, (2), differential forms and cochains with values in a trivial complex vector bundle. This leads to the final generalisation, (3), is for differential forms and cochains with values in an arbitrary complex vector bundle. This last step makes use of an embedding of the arbitrary vector bundle into a suitable higher rank *trivial* vector bundle, thus allowing the previous generalisation, (2), to be applied.

[‡]Zahariev later published a paper [31] covering the results of his thesis.

This thesis examines the possibility of generalising the first step of Zahariev's proof to higher degrees. That is, the convergence of eigenvalues for Laplace operators acting on higher degree differential forms and cochains with values in a trivial complex line bundle. This thesis shows that it is possible to partially generalise Zahariev's proof as long as an assumption is met. The main result of this thesis along with a discussion of the assumption is found in Chapter 4.

The two key obstacles for generalising Zahariev's proof to higher degrees are as follows:

1. the image of the Whitney map for zero degree cochains lies in the subset of L^2 forms whose derivatives are L^2 functions whilst for higher degrees this is not the case; and

2. the Laplace operator contains additional terms for positive degrees compared to the case of degree zero where the adjoint of the exterior derivative is identically equal to zero.

The key technical result of this thesis is developed to overcome the first obstacle encountered. This result is an explicit computation of the growth rate of the Whitney map under the standard subdivision scheme, covered in Chapters 5 and 6.

Chapter 7 provides technical estimates which are used to prove the lower bound in Chapter 8. The difficulties encountered in proving the upper bound are detailed in Chapter 9.

The notation required to study and hence generalise the second and third steps of Zahariev's proof to higher degree is defined in Chapter 10. This allows for a discussion about future work that can be performed using this thesis as a basis in Chapter 11.

1.3 Original Contribution

The key result of this thesis is Theorem 4.4.1. For a detailed explanation of this result and my original contributions, see Chapter 4.

1.4 Inclusion

The author believes in the active inclusion of everybody in the mathematics community regardless of any personal attribute. As the author was unable to follow up with every referenced author about their preferred pronoun, the author uses the gender natural pronoun they/them/their.

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Chapter 2 — 1912 – 1954 — Alan Mathison Turing (English)

Problem Setup

This chapter explains the basic construction of the objects considered throughout this thesis. In doing so, it covers the notation that will be used throughout as well as relevant basic results from the literature.

This thesis uses two kinds of definitions. Ones entitled 'Definition' are definitions that apply for the entire thesis whilst ones entitled 'Chapter Definition' are definitions that only apply to the chapter they appear in. The reason for this is that the number of symbols used in this thesis exceeds the Latin and Greek alphabets and hence some letters correspond to different objects depending on which chapter they are used in.

A summary of the notation used in this thesis can be found in the Symbols Appendix (Appendix C), or on a digital version by using the blue hyperlinked symbols. Terms written in red can be looked up via the Index (Appendix D).

2.1 Analytic Setting

Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N. Denote by $\Omega^q(M, \mathbb{C})$ the vector space of q-th degree smooth complex valued differential forms on M.

Let $d^q : \Omega^q(M, \mathbb{C}) \to \Omega^{q+1}(M, \mathbb{C})$ denote the exterior derivative. Where the degree is clear from context, d is used. Let $\wedge : \Omega^{q_1}(M, \mathbb{C}) \times \Omega^{q_2}(M, \mathbb{C}) \to \Omega^{q_1+q_2}(M, \mathbb{C})$ denote the exterior product.

Definition 2.1.1. *Define the inner product $\langle \cdot, \cdot \rangle_0$ on $\Omega^q(M, \mathbb{C})$ by*

$$\langle \omega_1, \omega_2 \rangle_0 = \int_M \overline{\omega}_1 \wedge \star\omega_2 \,, \tag{2.1.1}$$

where $\star : \Omega^q(M, \mathbb{C}) \to \Omega^{N-q}(M, \mathbb{C})$ is the Hodge star operator and $\overline{\omega}_1$ is complex conjugation.

With respect to the inner product of Definition 2.1.1, let $(d^q)^* : \Omega^{q+1}(M, \mathbb{C}) \to \Omega^q(M, \mathbb{C})$ be the (formal) adjoint of the exterior derivative d^q.

Theorem 2.1.2 (Nakahara [15, Theorem 7.4])**.** *For all $\omega \in \Omega^q(M, \mathbb{C})$ it holds that*

$$\star\star\omega = (-1)^{q(N-q)}\,\omega \,. \tag{2.1.2}$$

Lemma 2.1.3 (Nakahara [15, Equation (7.179)]). *For all $\omega_1, \omega_2 \in \Omega^q(M, \mathbb{C})$ it holds that*

$$\omega_1 \wedge \star\omega_2 = \omega_2 \wedge \star\omega_1 \,. \tag{2.1.3}$$

Consider a coordinate chart $\left(U, x^1, x^2, \ldots, x^N\right)$, a point $p \in U$ and a differential form $\omega \in \Omega^q(M, \mathbb{C})$. Let $\Upsilon^1, \Upsilon^2, \ldots, \Upsilon^N$ be a local orthonormal frame of $T^*M|_U$ with respect to the metric g. Then locally

$$\omega|_U = \sum_{1 \le \nu_1 < \nu_2 < \cdots < \nu_q \le N} f_{\nu_1, \nu_2, \ldots, \nu_q} \Upsilon^{\nu_1} \wedge \Upsilon^{\nu_2} \wedge \cdots \wedge \Upsilon^{\nu_q} \,, \tag{2.1.4}$$

for some collection of smooth functions $f_{\nu_1, \nu_2, \ldots, \nu_q} : U \to \mathbb{C}$.

Definition 2.1.4. *Define the pointwise norm $|\cdot|_p$ of a differential form $\omega \in \Omega^q(M, \mathbb{C})$ at a point p to be*

$$|\omega|_p = \left(\sum_{1 \le \nu_1 < \nu_2 < \cdots < \nu_q \le N} \left| f_{\nu_1, \nu_2, \ldots, \nu_q}(p) \right|^2 \right)^{\frac{1}{2}} , \tag{2.1.5}$$

where $|\cdot|$ is the absolute value of complex numbers.

It should be noted that the above definition is independent of the chosen local orthonormal frame.

From the Riemannian metric g, let ∇^g be the Levi-Civita connection on TM. This connection canonically induces a connection on T^*M and hence also induces a canonical connection on $\Lambda^q T^*M$. Let all of these connections be denoted by $\nabla^g : \Lambda^q T^*M \to \Lambda^1 T^*M \otimes \Lambda^q T^*M$ and hence let

$$\left(\nabla^g\right)^j : \Lambda^q T^*M \to \left(\bigotimes_{k=1}^{j} \left(\Lambda^1 T^*M\right) \right) \otimes \Lambda^q T^*M \,. \tag{2.1.6}$$

Similarly, let $|\cdot|_g$ be the pointwise norm constructed on $\left(\bigotimes_{k=1}^{j} \left(\Lambda^1 T^*M\right)\right) \otimes \Lambda^q T^*M$ from the Riemannian metric g. It is clear by definition that when $j = 0$ then $|\cdot|_g = |\cdot|_p$. Using these constructions, define the following norm.

Definition 2.1.5. *Define $\|\cdot\|_r$ to be the Sobolev norm on $\Omega^q(M, \mathbb{C})$ with natural number*[†] *parameter $r \in \mathbb{N}$ to be*

$$\|\omega\|_r = \left(\sum_{j=0}^{r} \int_M \left(\left| \left(\nabla^g\right)^j \omega \right|_g \right)^2 d\,\mathrm{Vol}_g \right)^{\frac{1}{2}} , \tag{2.1.7}$$

where $d\,\mathrm{Vol}_g = \star 1$ is the volume form associated to the Riemannian metric g.

By a short computation, for $\Omega^q(M, \mathbb{C})$, the norm associated to $\langle \cdot, \cdot \rangle_0$ is $\|\cdot\|_0$.

Definition 2.1.6. *Define the Sobolev space $\mathcal{H}^q_r(M, \mathbb{C})$ to be the completion of $\Omega^q(M, \mathbb{C})$ with respect to the norm $\|\cdot\|_r$.*

Hence $\mathcal{H}^q_0(M, \mathbb{C})$ is the L^2 completion of $\Omega^q(M, \mathbb{C})$ with respect to the inner product $\langle \cdot, \cdot \rangle_0$. That is, the space of square integrable differential q-forms on M.

[†]The set of natural numbers includes the number zero.

Theorem 2.1.7 (Pythagorean Theorem [8, Theorem 1.9])**.** *For all $j = 1, 2, \ldots$, and for all $\omega_1, \omega_2, \ldots, \omega_j \in \Omega^q(M, \mathbb{C})$, if $\omega_1, \omega_2, \ldots, \omega_j$ are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_0$, then it holds that*

$$\left(\left\| \sum_{k=1}^{j} \omega_j \right\|_0 \right)^2 = \sum_{k=1}^{j} \left(\|\omega_j\|_0 \right)^2 . \tag{2.1.8}$$

Lemma 2.1.8. *For all $\omega \in \Omega^q(M, \mathbb{C})$ and for all integers $0 \le r \le s$ it holds that*

$$\|\omega\|_r \le \|\omega\|_s . \tag{2.1.9}$$

Proof. This result follows immediately from Definition 2.1.5 as the integrand of the right hand side contains more nonnegative terms than the integrand of the left hand side. □

Fix a real one-form $A \in \Omega^1(M, \mathbb{R})$.

Definition 2.1.9. *Define the twisted exterior derivative $d_A^q : \Omega^q(M, \mathbb{C}) \to \Omega^{q+1}(M, \mathbb{C})$ to be*

$$d_A^q = d^q + iA \wedge , \tag{2.1.10}$$

where i is the imaginary unit.

With respect to the inner product $\langle \cdot, \cdot \rangle_0$, let the (formal) adjoint of the twisted exterior derivative be $(d_A)^*$.

Theorem 2.1.10 (Nakahara [15, Theorem 7.5])**.** *The adjoint of d_A^q is*

$$(d_A^q)^* = (-1)^{qN+1} \star d_A \star . \tag{2.1.11}$$

Definition 2.1.11. *Define the twisted Laplacian to be the map $\Delta_A^q : \Omega^q(M, \mathbb{C}) \to \Omega^q(M, \mathbb{C})$ where*

$$\Delta_A^q = (d_A^q)^* \, d_A^q + d_A^{q-1} (d_A^{q-1})^* . \tag{2.1.12}$$

Where clear from context, the twisted Laplace operator (twisted Laplacian) is also denoted by Δ_A.

Theorem 2.1.12 (Shubin [23, Theorem 8.3])**.** *The Laplacian operator Δ_A^q has purely discrete spectrum and all eigenvalues are real, bounded from below and there are infinitely many of them.*

The eigenvalues of Δ_A^q are denoted by

$$\lambda_1^q \le \lambda_2^q \le \cdots \le \lambda_j^q \le \cdots , \tag{2.1.13}$$

where eigenvalues are repeated according to their multiplicity. Where clear from context, the notation λ_j is used.

Theorem 2.1.13 (Min-Max Principle [21, Theorem XIII.2])**.** *Let u be a bounded from below self-adjoint linear operator with purely discrete spectrum acting on a Hilbert space H with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$, repeated according to their multiplicities. Then the j-th eigenvalue is given by*

$$\lambda_j = \sup_{\substack{f_k \in H \\ k=1,2,\ldots,j-1}} \; \inf_{\substack{f \in (H \setminus \{0\}) \cap Q(u) \\ \langle f, f_k \rangle = 0 \\ k=1,2,\ldots,j-1}} \frac{\langle uf, f \rangle}{\langle f, f \rangle} , \tag{2.1.14}$$

where $Q(u)$ is the domain of the quadratic form[‡] associated to u as defined in Section VIII.6 of Reed and Simon's book [20]. It should be noted that the f_k-s may be linearly dependent.

[‡] The quadratic form associated to u is $\langle u \cdot, \cdot \rangle : H \times H \to \mathbb{C}$.

2.2 Combinatorial Setting

2.2.1 Triangulation of a Manifold

Let K be a smooth oriented triangulation of M, a smooth compact manifold as per Section 2.1. Loosely speaking, a triangulation is a union of sets K^q for $q = 0, 1, \ldots, N$, where each K^q is a set of q-dimensional simplices σ. Intuitively for K to be a triangulation, all simplices of all dimensions must fit together appropriately to approximate the manifold in some sense. More formally:

Definition 2.2.1. *K is a* smooth triangulation *of M if K is an N-dimensional simplicial complex in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ such that there exists a homeomorphism $F : K \to M$ with the property that for all simplices $\sigma \in K$, there exists a coordinate chart (U_σ, φ) which contains $F(\sigma)$ and $\varphi \circ F$ restricted to σ is an affine map.*

Here an affine map is a map such that for all $q = 0, 1, \ldots, N$, a q-simplex is mapped to a q-simplex.

Theorem 2.2.2 (Whitehead [27, Theorem 7])**.** *For all smooth manifolds there exists a smooth triangulation.*

Lemma 2.2.3. *For all compact M and for all triangulations K of M, it holds that K is finite, that is, K only contains a finite number of simplices.*

What it means for K to be oriented requires more notation to be set up before it can be defined in Section 2.2.5.

Given a q-simplex σ, it is completely characterised by its ordered $q + 1$ vertices, hence the following notation is used to denote the simplex $\sigma = \left[p_0, p_1, \ldots, p_q\right]$. Again, more definitions are required to specify how this ordering is defined and chosen. See Section 2.2.5 for more details.

2.2.2 Cochains

Let $C^q(K, \mathbb{C})$ be the vector space of complex valued q-th degree cochains on K. Cochains can be thought of in a number of ways which will be interchanged as required. A q-cochain c can be defined as a function $c : K^q \to \mathbb{C}$ that assigns to each q-simplex a complex number. Alternatively, it can be defined as a formal sum of q-simplices

$$c = \sum_{\sigma \in K^q} c_\sigma \sigma \,, \tag{2.2.1}$$

for some complex coefficients c_σ. In fact, the translation between the two approaches is given by $c(\sigma) = c_\sigma$ for all $\sigma \in K^q$.

Definition 2.2.4. *For all $q = 0, 1, \ldots, N$, define the corresponding* unity cochain*, $\mathbb{1}_{K^q}$, to be the cochain where all coefficients are 1,*

$$\mathbb{1}_{K^q} = \sum_{\sigma \in K^q} \sigma \,. \tag{2.2.2}$$

Definition 2.2.5. *Define the* inner product *$\langle \cdot, \cdot \rangle_C$ on $C^q(K, \mathbb{C})$ by*

$$\langle c_1, c_2 \rangle_C = \sum_{\sigma \in K^q} \overline{c_1(\sigma)} c_2(\sigma) \,. \tag{2.2.3}$$

This gives an associated norm, $\|\cdot\|_{C,2}$, on cochains.

Definition 2.2.6. *Define the supremum norm, $\|\cdot\|_{C,\infty}$ on cochains to be*

$$\|c\|_{C,\infty} = \max_{\sigma\in K^q} \left|c(\sigma)\right| . \tag{2.2.4}$$

2.2.3 Barycentric Coordinates

The following definition of simplicial complex barycentric coordinates as used in this thesis is based on one given by Whitney [28, Appendix II.2][§]. Let $p_1, p_2, \ldots, p_{\left|K^0\right|}$ be the vertices of K and therefore every q-simplex σ has the form $\sigma = \left[p_{\nu_0}, p_{\nu_1}, \ldots, p_{\nu_q}\right]$ for some $\left\{\nu_0, \nu_1, \ldots, \nu_q\right\} \subseteq \left\{1, 2, \ldots, \left|K^0\right|\right\}$. Then all points $p \in K$ can be uniquely written in the form [28, Appendix II.2]

$$p = \sum_{j=1}^{\left|K^0\right|} \mu_j(p)\, p_j \,, \tag{2.2.5}$$

where $\mu_j : K \to [0,1]$ for all j, with the property that $\sum_{j=1}^{\left|K^0\right|} \mu_j(p) = 1$ and that the following property holds.

Property 2.2.7. *For all $q = 0, 1, \ldots, N$, if $p \in \left[p_{\nu_0}, p_{\nu_1}, \ldots, p_{\nu_q}\right]$ then the barycentric coordinate $\mu_j(p) = 0$ for $j \in \left\{1, 2, \ldots, \left|K^0\right|\right\} \setminus \left\{\nu_0, \nu_1, \ldots, \nu_q\right\}$.*

Definition 2.2.8. *The collection of functions μ_j for all $j = 1, 2, \ldots, \left|K^0\right|$ are the barycentric coordinates of a point p in the simplicial complex K and for a particular value of j, the function μ_j is the* barycentric coordinate *associated to the point p_j or the vertex v_j.*

Alternative definitions of simplicial complex barycentric coordinates include ones by authors such as Adams [1, Section 2], Albeverio and Zegarlinski [2, Section 1.1] and Müller [14, Section 2]. Some authors such as Arnold, Falk and Winther [4, Section 2.1] refer to simplicial complex barycentric coordinates as piecewise linear finite element basis functions.

Figure 2.1 provides a pictorial illustration of barycentric coordinates for a two dimensional surface.

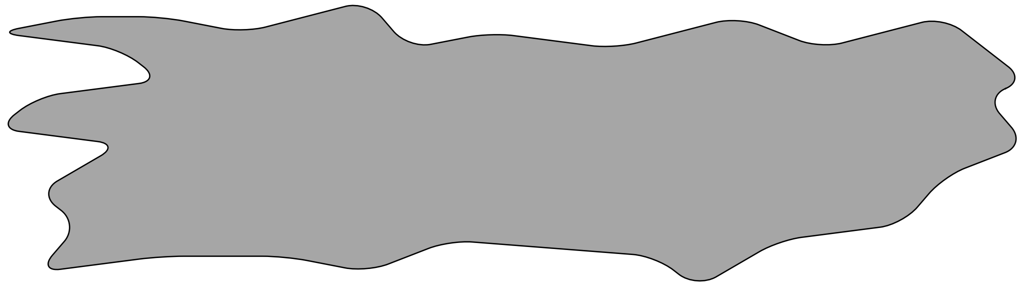
2.2.4 The Whitney Map

Consider a q-simplex $\sigma = \left[p_0, p_1, \ldots, p_q\right]$ and let μ_j be the barycentric coordinate function corresponding to the vertex at p_j for $j = 0, 1, \ldots, q$. As here the vertices of σ require an ordering, which is equivalent to an orientation for σ and this is yet to be defined, temporarily order the vertices

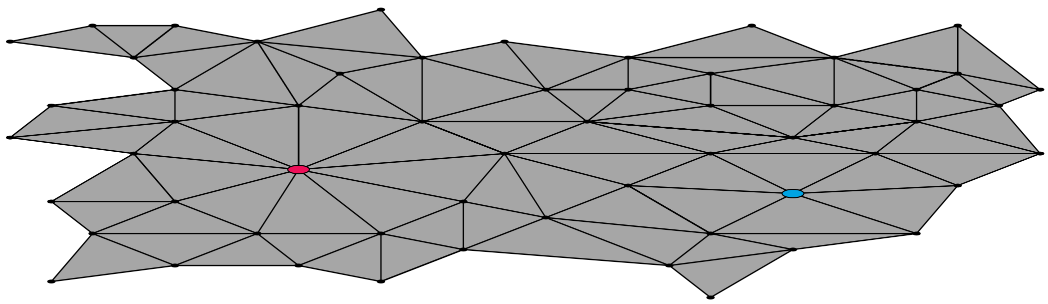
$$p_0 < p_1 < \cdots < p_q \,, \tag{2.2.6}$$

and then set a final ordering that will be used from then on when orientation is defined using the Whitney map. Through the use of this temporary ordering to define the final ordering, there is no circular argument in the definition of the Whitney map and orientation.

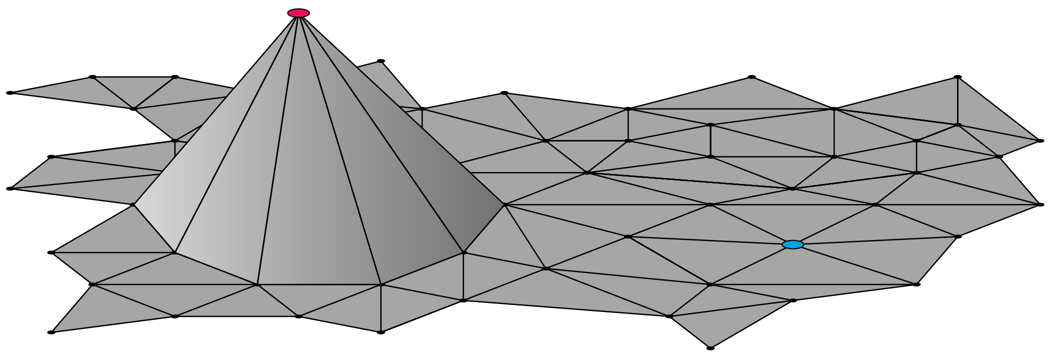
[§]Whitney defines both the simplex and simplicial complex barycentric coordinates in various places in his book [28]. Caution should be used to ascertain which version of barycentric coordinates are used.



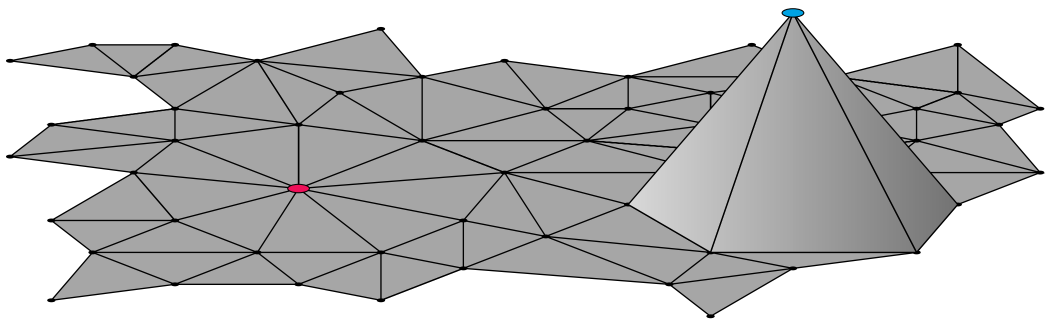
(a) Two dimensional surface.



(b) Triangulation of surface.



(c) Barycentric coordinate for the red vertex.



(d) Barycentric coordinate for the blue vertex.

Figure 2.1: Pictorial representation of the simplicial complex barycentric coordinate for a triangulation of a two dimensional surface at selected points or vertices. The height axis represents the value of the barycentric coordinate.

Definition 2.2.9. *The Whitney map [28, Chapter VII.11 (16)] $W_n^q : C^q(K_n, \mathbb{C}) \to \mathcal{H}_0^q(M, \mathbb{C})$ is defined to be*

$$W_n^q \sigma = q! \sum_{j=0}^{q} (-1)^j \mu_j \, d\mu_0 \wedge d\mu_1 \wedge \cdots \wedge d\mu_{j-1} \wedge d\mu_{j+1} \wedge \cdots \wedge d\mu_q \,, \tag{2.2.7}$$

for $q > 0$. When $q = 0$, define

$$W_n^0 \sigma = \mu_0 \,. \tag{2.2.8}$$

This definition extends linearly to any cochain by viewing the cochain as a formal sum.

It is clear from this definition that the Whitney map is invariant under even permutations on the ordering given to the vertices of σ and an odd permutation reverses the sign of the Whitney map.

Context permitting, the Whitney map may be denoted by W_n and W. In the definition the Whitney map, n denotes the n-th subdivision of K as defined in Section 2.2.7.

2.2.5 Orientation

This thesis defines the orientation of a simplex as follows. First, the case of an N-dimensional simplex is defined followed by the case of a q-dimensional simplex where $q < N$.

Consider a simplex $\sigma \in K^N$ where $\sigma = \left[p_{\nu_0}, p_{\nu_1}, \ldots, p_{\nu_N}\right]$ with associated barycentric coordinates $\left\{\mu_{\nu_0}, \mu_{\nu_1}, \ldots, \mu_{\nu_N}\right\}$ and let $m = \left|K^0\right|$. Then by the definition of the Whitney map

$$\begin{aligned}
W\sigma &= N! \sum_{j=0}^{N} (-1)^j \mu_{\nu_j} \, d\mu_{\nu_0} \wedge d\mu_{\nu_1} \wedge \cdots \wedge d\mu_{\nu_{j-1}} \wedge d\mu_{\nu_{j+1}} \wedge \cdots \wedge d\mu_{\nu_N} \\
&= N! \, \mu_{\nu_0} \, d\mu_{\nu_1} \wedge d\mu_{\nu_2} \wedge \cdots \wedge d\mu_{\nu_N} \\
&\quad + N! \sum_{j=1}^{N} (-1)^j \mu_{\nu_j} \, d\mu_{\nu_0} \wedge d\mu_{\nu_1} \wedge \cdots \wedge d\mu_{\nu_{j-1}} \wedge d\mu_{\nu_{j+1}} \wedge \cdots \wedge d\mu_{\nu_N} \\
&= N! \, \mu_{\nu_0} \, d\mu_{\nu_1} \wedge d\mu_{\nu_2} \wedge \cdots \wedge d\mu_{\nu_N} \\
&\quad + N! \sum_{j=1}^{N} (-1)^j \mu_{\nu_j} \, d\left(1 - \sum_{\substack{k=1 \\ k \neq \nu_0}}^{m} \mu_k\right) \wedge d\mu_{\nu_1} \wedge \cdots \wedge d\mu_{\nu_{j-1}} \wedge d\mu_{\nu_{j+1}} \wedge \cdots \wedge d\mu_{\nu_N} \,,
\end{aligned} \tag{2.2.9}$$

where the last equality comes from the fact that the barycentric coordinates sum to one.

Consider a barycentric coordinate on its region of support. On this region, it is a non-constant piecewise linear function and hence it is clear that the support of the barycentric coordinate and its derivative are identical. Therefore if $p \in \operatorname{int}(\sigma)$, where $\operatorname{int}(\cdot)$ is the interior of a subset, then by Property 2.2.7 for all $j \in \{1, 2, \ldots, m\} \setminus \{\nu_0, \nu_1, \ldots, \nu_N\}$ it holds that

$$\mu_j(p) = 0 \,. \tag{2.2.10}$$

Hence

$$\sum_{j=0}^{N} \mu_{\nu_j} = 1 \,, \tag{2.2.11}$$

and (2.2.9) restricted to $\operatorname{int}(\sigma)$ becomes

$$\begin{aligned} W\sigma|_{\operatorname{int}(\sigma)} &= N!\,\mu_{\nu_0}\,d\mu_{\nu_1}\wedge d\mu_{\nu_2}\wedge\cdots\wedge d\mu_{\nu_N}\big|_{\operatorname{int}(\sigma)} \\ &\qquad - N!\sum_{j=1}^{N}(-1)^j\,\mu_{\nu_j}\,d\mu_{\nu_j}\wedge d\mu_{\nu_1}\wedge\cdots\wedge d\mu_{\nu_{j-1}}\wedge d\mu_{\nu_{j+1}}\wedge\cdots\wedge d\mu_{\nu_N}\big|_{\operatorname{int}(\sigma)} \\ &= N!\,d\mu_{\nu_1}\wedge d\mu_{\nu_2}\wedge\cdots\wedge d\mu_{\nu_N}\big|_{\operatorname{int}(\sigma)} \quad (\text{by } (2.2.11))\,. \end{aligned} \tag{2.2.12}$$

Therefore it is clear that $W\sigma|_{\operatorname{int}(\sigma)}$ is simply a multiple of the volume form $d\operatorname{Vol}_g|_{\operatorname{int}(\sigma)}$.

Definition 2.2.10. *Define an ordering, up to even permutations, on the set of vertices of* $\sigma = \left[p_{\nu_0}, p_{\nu_1}, \dots, p_{\nu_N}\right]$ *such that* $W\sigma|_{\operatorname{int}(\sigma)}$ *is a positive multiple of the volume form* $d\operatorname{Vol}_g|_{\operatorname{int}(\sigma)}$.

The case of $q < N$ is treated in the same way as the top dimensional case. Specifically, let $\left(U_\sigma, x^1, x^2, \dots, x^N\right)$ be a local coordinate neighbourhood of σ in M. View $\operatorname{int}(\sigma)$ as a closed submanifold of an open subset of M. Then there exists local coordinates $\left(\operatorname{int}(\sigma), y^1, y^2, \dots, y^q\right)$ for σ that are compatible with the chosen orientation on $\operatorname{int}(\sigma)$, viewed as a manifold, such that $x^j = y^j$ for all $j = 1, 2, \dots, q$ and $x^j = 0$ for all $j = q+1, q+2, \dots, N$. Let $\iota_{\operatorname{int}(\sigma)} : \sigma \to M$ denote this inclusion map. Then the barycentric coordinates and Whitney forms pulled back to $\operatorname{int}(\sigma)$ via $\iota_{\operatorname{int}(\sigma)}$, gives an analogous equation of (2.2.12) for dimension q where the right hand side is a nowhere zero q-form.

Definition 2.2.11. *For all* $q = 0, 1, \dots, N-1$*, define an ordering, up to even permutations, on the set of vertices of* $\sigma = \left[p_0, p_1, \dots, p_q\right]$ *such that* $\iota^*_{\operatorname{int}(\sigma)}\left(W\sigma|_{\operatorname{int}(\sigma)}\right)$ *is a positive multiple of* $dy^1 \wedge dy^2 \wedge \cdots \wedge dy^q$.

It should be noted that this final ordering of vertices of σ may be different to the temporary ordering of

$$p_0 < p_1 < \cdots < p_q\,, \tag{2.2.13}$$

that was placed on σ (in Section 2.2.4) in order to apply the Whitney map and hence define the final ordering. In the final ordering it may well be the case that $p_j > p_k$ when $j < k$.

2.2.6 Ordering for a Subdivision

In Section 2.2.5 each simplex is given an ordering, up to even permutations, on its vertices and hence an orientation. This orientation and ordering is used for all constructions used in this thesis except for one, when using the standard subdivision which is about to be defined in Section 2.2.7.

To apply the standard subdivision, an ordering must be placed on the set of vertices of the whole simplicial complex, not just individual simplices as was the case in Section 2.2.5. This is because otherwise the standard subdivision may not produce a simplicial complex.

There is no canonical ordering on the simplicial complex that can be induced from the ordering of the individual simplices. In fact it is not clear if there even is a consistent way of obtaining an ordering on the simplicial complex from the ordering of vertices on individual simplices. Hence it is assumed that the initial simplicial complex has a fixed given ordering just for the purpose of applying the standard subdivision, independent and separate to all of the orderings of Section 2.2.5.

For subsequent subdivisions, the following definition (using notation about to be defined in Section 2.2.7) provides an ordering of the vertices of the simplicial complex that is again independent and separate to all of the orderings of Section 2.2.5.

Definition 2.2.12. *The ordering given to the vertex set of the standard subdivision*[††]*, $\mathfrak{S}K$, of a triangulation K, where each vertex is of the form*[‡‡] *$p_{j,k}$, is the lexicographical ordering of the pair (j, k).*

2.2.7 Standard Subdivision

In order to improve the approximation of the manifold, the triangulation is iteratively refined or subdivided. There are numerous subdivision schemes to choose from including the well-known barycentric subdivision [13, Page 119] from Algebraic Topology. However, as will be explained subsequently (see Assumption 2.2.30 and Lemmas 2.2.31 and 2.2.32), the barycentric subdivision does not have properties required for this thesis. As the standard subdivision does have these properties, it is used instead.

Let K_0 be a given initial triangulation. This thesis uses the standard subdivision of a simplicial complex as defined by Whitney [28, Appendix II.4]. The standard subdivision of a simplicial complex K is denoted by $\mathfrak{S}K$ and is constructed as follows. This description is based on the one given by Dodziuk [9, Section 2].

2.2.7.1 Standard Subdivision of a Simplex

Definition 2.2.13. *Given a q-simplex $\sigma = [p_0, p_1, \ldots, p_q]$ (with ordering as per Section 2.2.6), for integers $0 \le j \le k \le q$, define*

$$p_{j,k} = \frac{1}{2}p_j + \frac{1}{2}p_k \,, \tag{2.2.14}$$

which means $p_{j,j} = p_j$.

Let $\{p_{j,k} | 0 \le j \le k \le q\}$ be the set of vertices of the standard subdivision, $\mathfrak{S}\sigma$, with the following partial ordering

$$p_{j,k} \le p_{r,s} \iff j \ge r \text{ and } k \le s \,. \tag{2.2.15}$$

The m-dimensional simplices of $\mathfrak{S}\sigma$ for $m = 0, 1, \ldots, q$ are strictly increasing sequences of vertices of $\mathfrak{S}\sigma$ with respect to the partial ordering comprising of $m + 1$ elements.

The partial ordering can be visualised in a Hasse diagram as seen in Figure 2.2 (or Appendix A for a diagram with more terms). The simplices of the standard subdivision are then the upward travelling paths of all possible lengths, which includes paths that skip nodes in the Hasse diagram.

It is important to note that the ordering of vertices of the initial simplex σ can have a significant effect on the resultant standard subdivision as an example in Figure 2.3 shows for a triangle. The standard subdivision of a tetrahedron is shown in Figure 2.4.

2.2.7.2 Standard Subdivision of a Simplicial Complex

The following lemma is an original contribution that gives the triangulation some properties that will be used later in the thesis.

Lemma 2.2.14. *If $\widehat{K}$ is a smooth triangulation of M contained in $\mathbb{R}^{\widehat{k}}$ for some $\widehat{k} \in \mathbb{N}$, then it can be modified into a smooth triangulation K of M contained in $\mathbb{R}^k$ where $k = \widehat{k} + \left|\widehat{K}^0\right| - 1$ such that $\left\{p_2, p_3, \ldots p_{\left|K^0\right|}\right\}$ is a linearly independent set when each vertex is viewed as a vector.*

[††] See Section 2.2.7.1 for the details of the notation used.

[‡‡] See Definition 2.2.13.

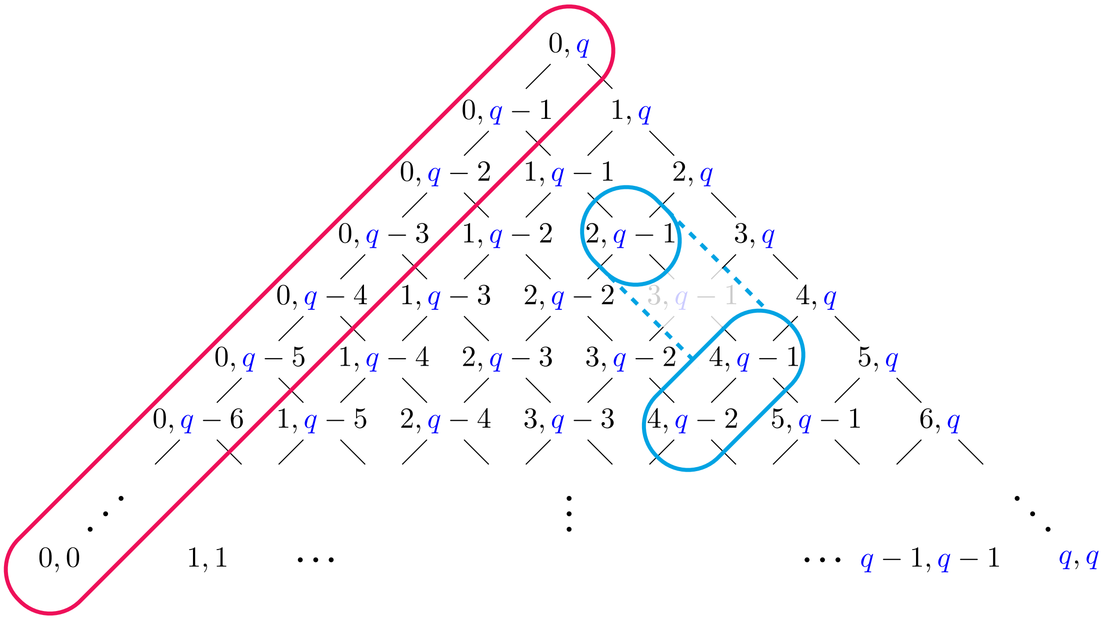


Figure 2.2: Hasse diagram for the partial ordering of vertices of $\mathfrak{S}\sigma$. For example, the red encircled points are the vertices of a q-dimensional simplex, whilst the three points encircled in blue are the vertices of a triangle and are an example of an upward travelling path with a skipped node.

In Lemma 2.2.14, K is said to have the 'linear independence' property.

Proof. Consider the set of vertices $\widehat{K}^0 = \left\{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_{\left|\widehat{K}^0\right|}\right\}$ inside the affine space $\mathbb{R}^{\widehat{k}}$. For each $j = 1, 2, \ldots, \left|\widehat{K}^0\right|$, let

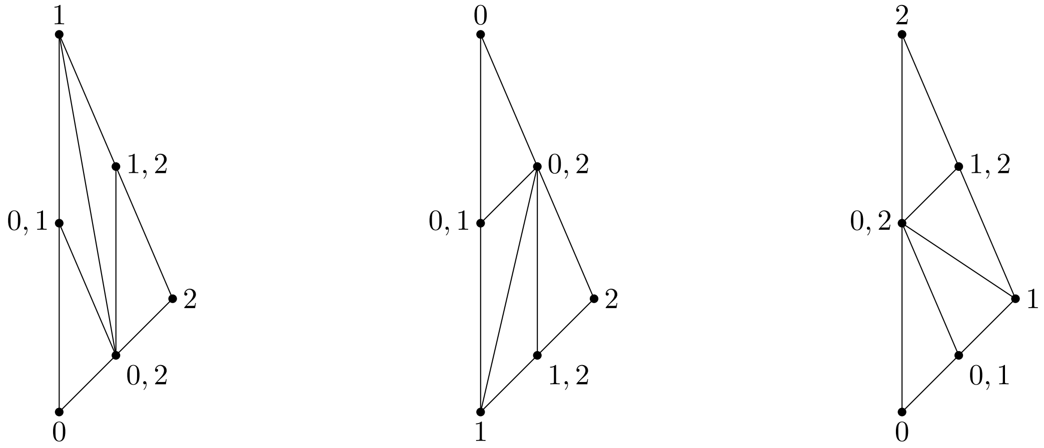
$$\widehat{p}_j = \left(\widehat{x}_j^1, \widehat{x}_j^2, \ldots, \widehat{x}_j^{\widehat{k}}\right) . \tag{2.2.16}$$

Define a new triangulation K of M contained in $\mathbb{R}^k$ where $k = \widehat{k} + \left|\widehat{K}^0\right| - 1$ as follows. For each $j = 2, 3, \ldots, \left|\widehat{K}^0\right|$, define

$$\begin{aligned} p_j &= \left(x_j^1, x_j^2, \ldots, x_j^k\right) \\ &= \left(\widehat{x}_j^1, \widehat{x}_j^2, \ldots, \widehat{x}_j^{\widehat{k}}, \underbrace{0, \ldots, 0}_{j-2}, 1, \underbrace{0, \ldots, 0}_{\left|\widehat{K}^0\right|-j}\right) , \end{aligned} \tag{2.2.17}$$

and define

$$\begin{aligned} p_1 &= \left(x_1^1, x_1^2, \ldots, x_1^k\right) \\ &= \left(\widehat{x}_1^1, \widehat{x}_1^2, \ldots, \widehat{x}_1^{\widehat{k}}, 0, \ldots, 0\right) . \end{aligned} \tag{2.2.18}$$



(a) Point $0, 2$ on shortest edge. (b) Point $0, 2$ on middle length edge. (c) Point $0, 2$ on longest edge.

Figure 2.3: Different standard subdivisions of a triangle due to the choice of ordering on its vertices.

Define $\widehat{F} : \widehat{K} \to K$ as follows. Let $\widehat{\sigma} \in \widehat{K}^q$ where $\widehat{\sigma} = \left[\widehat{p}_{\alpha_0}, \widehat{p}_{\alpha_1}, \ldots, \widehat{p}_{\alpha_q}\right]$. Then define $\sigma \in K^q$ such that $\widehat{F}(\widehat{\sigma}) = \sigma$ where for all $j = 0, 1, \ldots, q$, $\widehat{F}\left(\widehat{p}_{\alpha_j}\right) = p_{\alpha_j}$ and σ is the convex hull of $\left\{p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_q}\right\}$.

By the property that the composition of affine homeomorphisms is affine homeomorphic, it follows that K is a smooth triangulation of M with the required 'linear independence' property. $\square$

To subdivide a simplicial complex K, an ordering on the set of vertices is required as defined in Section 2.2.6. This naturally gives an ordering of the vertices of any simplex $\sigma \in K$. Subdivide each simplex separately and by construction, the subdivided simplices will still fit together to form a simplicial complex, $\widehat{K}$. This simplicial complex $\widehat{K}$ is then modified via Lemma 2.2.14 to possess the 'linear independence' property and this is the standard subdivision $\mathfrak{S}K$ of K.

2.2.7.3 Iterative Standard Subdivision

This thesis considers a given initial triangulation K_0 which is iteratively subdivided using the standard subdivision.

Definition 2.2.15. *For $n = 1, 2, \ldots,$ define K_n to be the n-*th subdivision *of K_0 where*

$$K_n = \mathfrak{S}^n K_0\,, \tag{2.2.19}$$

and $\mathfrak{S}^n$ is $\mathfrak{S}$ applied n times.

Lemma 2.2.16 (Dodziuk [9, Page 86])**.** *For all $q = 0, 1, \ldots, N$ and for all q-simplices σ, there are 2^q many q-simplices in $\mathfrak{S}\sigma$.*

Corollary 2.2.17. *For all $q = 0, 1, \ldots, N$ and for all $n = 0, 1, \ldots,$ it holds that*

$$\left|K_n^q\right| = 2^{nq}\left|K_0^q\right|. \tag{2.2.20}$$

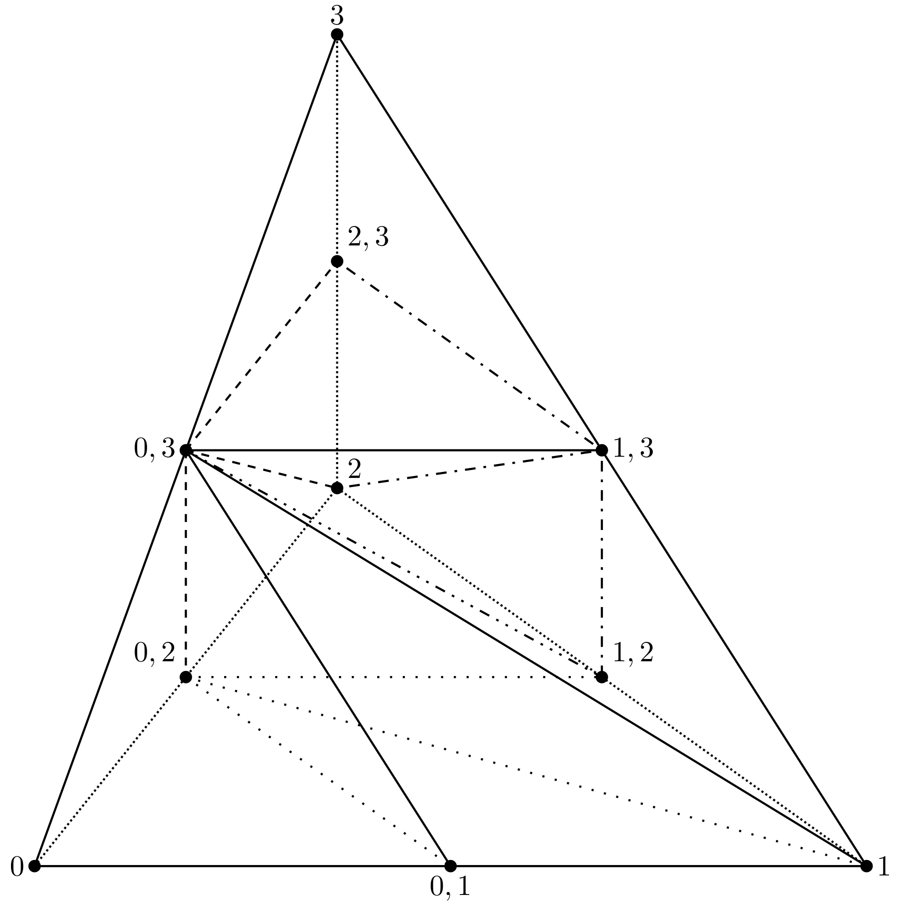


Figure 2.4: Standard subdivision of a tetrahedron.

Lemma 2.2.18 (Dodziuk [9, Lemma 3.4])**.** *There exists a finite set $\mathcal{U}$ of coordinate charts of M such that for all $n = 0, 1, \ldots,$ and for all $\tau \in K_n^N$, there exists a coordinate chart $\left(U, \varphi = (x^1, x^2, \ldots, x^N)\right) \in \mathcal{U}$ and there exists a $\sigma \in K_0^N$ for which it holds that $\tau \subseteq \mathrm{cl}\,(\sigma) \subseteq U$ where $\mathrm{cl}\,(\cdot)$ is the closure of a subset.*

Definition 2.2.19. *Define and fix $\mathcal{U}$ to be the set of finite coordinate charts as defined in Lemma 2.2.18.*

2.2.8 Triangulation Parameters

Definition 2.2.20. *For all $q = 1, 2, \ldots, N$ and for all $j = 0, 1, \ldots, q$, define $\partial_j^q : K^q \to K^{q-1}$ to be the face maps of a simplex (using the orientation as per Section 2.2.5) as follows*

$$\partial_j^q \left([p_0, p_1, \ldots, p_q] \right) = [p_0, p_1, \ldots, p_{j-1}, p_{j+1} \ldots, p_q] \ . \tag{2.2.21}$$

Hence for all $q = 2, 3 \ldots, N$ and for all integers $0 \leq j < k \leq q$, the face maps satisfy

$$\partial_j^{q-1} \circ \partial_k^q = \partial_{k-1}^{q-1} \circ \partial_j^q \, . \tag{2.2.22}$$

Where clear from context, ∂_j will be used.

Definition 2.2.21. *Define the simplicial coboundary operator $\delta^q : C^q(K,\mathbb{C}) \to C^{q+1}(K,\mathbb{C})$ to be*

$$\left(\delta^q(c)\right)(\sigma) = \sum_{j=0}^{q} (-1)^j\, c\left(\partial_j^{q+1}\sigma\right), \tag{2.2.23}$$

where $c \in C^q(K,\mathbb{C})$ and $\sigma \in K^{q+1}$.

Where clear from context δ is used.

Consider $\sigma \in K^N$ and a corresponding coordinate chart (U_σ, φ) that contains σ. Let g_σ be the pullback of the Euclidean metric to U_σ via φ.

Definition 2.2.22. *Define $d_{g_\sigma}(\cdot,\cdot)$ to be the local distance on U_σ induced by the metric g_σ.*

Definition 2.2.23. *Define the mesh of the n-th subdivision, h_n, to be*

$$h_n = \sup_{e\in K_n^1} d_{g_\sigma}(\partial_0 e, \partial_1 e). \tag{2.2.24}$$

Intuitively the mesh is a measure of the fineness of the triangulation.

Lemma 2.2.24 (Dodziuk [9, Lemma 3.6])**.** *It holds that the mesh of the triangulation, h_n, satisfies*

$$\lim_{n\to\infty} h_n = 0. \tag{2.2.25}$$

Of interest for this thesis is the reduction in mesh size for each subdivision. Let $\widehat{\beta}_n \le 1$ be the reduction in mesh size from h_{n-1} to h_n, that is

$$h_n = \widehat{\beta}_n h_{n-1}, \tag{2.2.26}$$

and let $\beta_n \le 1$ be the reduction in mesh size from h_0 to h_n, specifically

$$h_n = \beta_n h_0. \tag{2.2.27}$$

The following properties of the mesh reduction parameter are of interest as they provide information on the behaviour of the error terms in the later derived estimates.

Lemma 2.2.25. *For the sequence of $\widehat{\beta}_n$ as defined in (2.2.26), for all $n = 1, 2, \ldots$, a lower bound is given by*

$$\widehat{\beta}_n \ge \frac{1}{2}. \tag{2.2.28}$$

Proof. By construction of the standard subdivision, each edge is cut in half. □

Corollary 2.2.26. *For the sequence of β_n as defined in (2.2.27), for all $n = 1, 2, \ldots$, a lower bound is given by*

$$\beta_n \ge \left(\frac{1}{2}\right)^n. \tag{2.2.29}$$

Proof. The result is immediate by the iterative construction of β_n from $\widehat{\beta}_n$. □

Lemma 2.2.27. *For the sequence of β_n as defined in (2.2.27), it holds that*

$$\lim_{n\to\infty} \beta_n = 0. \tag{2.2.30}$$

Proof. The result is immediate from Lemma 2.2.24 as $h_0 \neq 0$. □

Definition 2.2.28. *The volume, $\mathrm{vol}(\cdot)$, of an N-simplex is defined to be*

$$\mathrm{vol}(\sigma) = \int_\sigma d\,\mathrm{Vol}_g\,. \tag{2.2.31}$$

Definition 2.2.29. *The fullness, θ_n, of K_n is defined to be*

$$\theta_n(K_n) = \inf_{\sigma \in K_n^N} \frac{\mathrm{vol}(\sigma)}{(h_n)^N}\,. \tag{2.2.32}$$

Assumption 2.2.30. *There exists a positive constant κ such that for all $n = 0, 1, \ldots$, it holds that*

$$\theta_n(K_n) \geq \kappa\,. \tag{2.2.33}$$

In other words, the initial triangulation and subdivision scheme used has a bounded from below fullness. This assumption is assumed from now on.

Lemma 2.2.31 (Whitney [28, Appendix II.4 Lemma 4b])**.** *The standard subdivision satisfies Assumption 2.2.30.*

Lemma 2.2.32 (Whitney [28, Appendix II.4])**.** *The barycentric subdivision does not satisfy Assumption 2.2.30.*

Proposition 2.2.33 (Dodziuk & Patodi [10, Proposition 2.2b])**.** *If Assumption 2.2.30 holds, then for all $\sigma \in K_n^N$ it holds that*

$$\mathrm{vol}(\sigma) \leq (h_n)^N\,. \tag{2.2.34}$$

In light of Lemma 2.2.32 and the importance of Proposition 2.2.33 in the method of this thesis, it is clear that the more well-known barycentric subdivision scheme is not a suitable choice for this thesis and hence the standard subdivision is used.

2.2.9 Combinatorial Norms

Using an analogous construction as was used for Definition 2.1.5 whereby g is replaced with g_σ, define the following seminorm.

Definition 2.2.34. *Given $\sigma \in K^N$, define the local seminorm $\|\cdot\|_r^\sigma$ on differential forms with parameter $r \in \mathbb{N}$ dependent on a given triangulation K to be*

$$\|\omega\|_r^\sigma = \left(\sum_{j=0}^{r} \int_\sigma \left(\left| (\nabla^{g_\sigma})^j \, \omega \right|_{g_\sigma} \right)^2 d\,\mathrm{Vol}_g \right)^{\frac{1}{2}}\,. \tag{2.2.35}$$

Definition 2.2.35. *Define the global norm $\|\cdot\|_r^K$ on differential forms with parameter $r \in \mathbb{N}$ dependent on a given triangulation K to be*

$$\|\omega\|_r^K = \left(\sum_{\sigma \in K^N} \left(\|\omega\|_r^\sigma \right)^2 \right)^{\frac{1}{2}}\,. \tag{2.2.36}$$

2.2.10 The de Rham Map

Definition 2.2.36. *The* de Rham map*,* $R_n^q : \Omega^q(M,\mathbb{C}) \to C^q(K_n,\mathbb{C})$*, is defined by de Rham [7] as*

$$\left(R_n^q(\omega)\right)(\sigma) = \int_\sigma \omega\,, \tag{2.2.37}$$

where $\omega \in \Omega^q(M,\mathbb{C})$*,* $\sigma \in K^q$ *and* $q > 0$*. When* $q = 0$*, the de Rham map is defined as the evaluation of the differential form at the vertices.*

Where clear from context, R_n and R will be used. An important property of the de Rham map is as follows.

Lemma 2.2.37 (Wilson [29, Page 19])**.** *It holds that*

$$Rd = \delta R\,. \tag{2.2.38}$$

2.2.11 Simplicial Cup Product

The following definition is by Adams [1, Section 2] and Wilson [29, Definition 5.1].

Definition 2.2.38. *Define the* cup product $\cup : C^{q_1}(K,\mathbb{C}) \times C^{q_2}(K,\mathbb{C}) \to C^{q_1+q_2}(K,\mathbb{C})$ *on two cochains to be*

$$c_1 \cup c_2 = R\left(Wc_1 \wedge Wc_2\right)\,. \tag{2.2.39}$$

An alternative but equivalent definition [30, Chapter 3] is given by authors such as Birmingham and Rakowski [5, Section 2] and Dupont [11, Equation (2.17)].

2.2.12 Combinatorial Laplace Operator

Definition 2.2.39. *Define the* Whitney inner product $\langle\cdot,\cdot\rangle_W$ *on* $C^q(K,\mathbb{C})$ *by*

$$\langle c_1, c_2\rangle_W = \langle Wc_1, Wc_2\rangle_0\,. \tag{2.2.40}$$

This inner product is nondegenerate as W is injective by Corollary 2.3.4. The associated norm is denoted by $\|\cdot\|_W$.

Recall that A is a fix real one-form.

Definition 2.2.40. *Define* $a \in C^1(K,\mathbb{C})$ *such that*

$$a = RA\,. \tag{2.2.41}$$

Definition 2.2.41. *Define the* twisted combinatorial exterior derivative*,* δ_a^q*, to be the map* $\delta_a^q : C^q(K,\mathbb{C}) \to C^{q+1}(K,\mathbb{C})$ *given by*

$$\delta_a^q = \delta^q + ia\cup\,. \tag{2.2.42}$$

Where context permits, the superscript is omitted.

With respect to the inner product $\langle\cdot,\cdot\rangle_W$, let the adjoint of the twisted combinatorial exterior derivative be $(\delta_a)^*$.

Definition 2.2.42. *Define the* twisted combinatorial Laplacian $\blacktriangle_a^q : C^q(K,\mathbb{C}) \to C^q(K,\mathbb{C})$ *to be*

$$\blacktriangle_a^q = (\delta_a^q)^*\,\delta_a^q + \delta_a^{q-1}(\delta_a^{q-1})^*\,. \tag{2.2.43}$$

Where clear from context, the twisted combinatorial Laplace operator is also denoted by $\blacktriangle_a$. The eigenvalues of $\blacktriangle_a^q$ are denoted by

$$\gamma_1^q \le \gamma_2^q \le \cdots \le \gamma_j^q \le \cdots \le \gamma^q_{\dim\left(C^q(K,\mathbb{C})\right)}, \tag{2.2.44}$$

where eigenvalues are repeated according to their multiplicity.

It should be noted that generally speaking, the combinatorial operators considered in this thesis are acting on the cochains of the n-th triangulation. Therefore the operators and eigenvalues have a dependence on n. This dependence is not explicit in the notation but clear from context.

2.3 Properties of the de Rham and Whitney Maps

Proposition 2.3.1 (Dodziuk [9, Equation (1.5)]). *For all $c \in C^q(K, \mathbb{C})$, Wc is a smooth differential form when restricted to the interior of any $\sigma \in K^N$, which extends to a smooth differential form on $\mathrm{cl}(\sigma)$, denoted by $Wc|_{\mathrm{cl}(\sigma)}$. Furthermore, if $\sigma, \sigma' \in K^N$ have $\tau \in K^q$ as a common face*[§§]*, then*

$$\iota^*\left(Wc|_{\mathrm{cl}(\sigma)}\right) = (\iota')^*\left(Wc|_{\mathrm{cl}(\sigma')}\right), \tag{2.3.1}$$

where $\iota : \mathrm{cl}(\tau) \to \mathrm{cl}(\sigma)$ and $\iota' : \mathrm{cl}(\tau') \to \mathrm{cl}(\sigma')$ are the respective inclusion maps.

It should be noted that Proposition 2.3.1 does not imply $Wc|_{\mathrm{cl}(\sigma)}$ and $Wc|_{\mathrm{cl}(\sigma')}$ are equal on τ, which is not the case.

In light of Proposition 2.3.1, it makes sense to compose the de Rham map with the Whitney map, as well as compose the exterior derivative on the complement of the $(N-1)$-dimensional skeleton of K with the Whitney map.

Proposition 2.3.2 (Dodziuk & Patodi [10, Proposition 1.10(a)]). *It holds that*

$$RW = \mathrm{Id}, \tag{2.3.2}$$

where Id is the identity operator on cochains.

Corollary 2.3.3. *The de Rham map is surjective.*

Corollary 2.3.4. *The Whitney map is injective.*

In light of Corollary 2.3.3, fix a linear right inverse of the de Rham map, notated as follows.

Notation 2.3.5. *Let $\left(R_n^q\right)^{-1} : C^q(K_n, \mathbb{C}) \to \Omega^q(M, \mathbb{C})$ be a linear right inverse of the de Rham map.*

Where clear from context, $(R_n)^{-1}$ will be used.

It should be noted that the difference between W and R^{-1} is the fact that R^{-1} maps to smooth differential forms whilst this is not the case for W.

Definition 2.3.6. *Define the map $P_n^q : \Omega^{q+1}(M, \mathbb{C}) \to \Omega^q(M, \mathbb{C})$ such that*

$$P_n^q\omega = (R_n^q)^{-1}(\delta_a^q)^* R_n^{q+1}\omega - (R_n^q)^{-1} R_n^q (d_A^q)^*\omega. \tag{2.3.3}$$

Where clear from context, P_n will be used.

[§§]For clarity, a face of a q-simplex σ is any j-dimensional simplex that is contained in σ for all $j = 0, 1, \ldots, q$.

Proposition 2.3.7 (Dodziuk [9, Equation (1.4)]). *It holds that*

$$W\delta = dW\,, \tag{2.3.4}$$

where d is applied on the complement of the $(N-1)$-dimensional skeleton of K.

Lemma 2.3.8 (Zahariev [30, Equation (6.4)]). *There exist positive constants κ_1 and κ_2 independent of n such that for all $c \in C^q(K_n, \mathbb{C})$ it holds that*

$$\kappa_1 (h_n)^{N-2q} \left(\|c\|_{C,2}\right)^2 \leq \left(\|W_n c\|_0\right)^2 = \left(\|c\|_W\right)^2 \leq \kappa_2 (h_n)^{N-2q} \left(\|c\|_{C,2}\right)^2 . \tag{2.3.5}$$

Proposition 2.3.2 is in contrast to Dodziuk's work which showed that WR can only approximate the identity map to arbitrary degree for a sufficiently fine triangulation. This approximation result lies at the heart of using combinatorial problems to approximate analytic ones.

Theorem 2.3.9 (Dodziuk [9, Theorem 3.7]). *There exists a positive constant κ independent of n such that for all $\omega \in \Omega^q(M, \mathbb{C})$ it holds that*

$$|W_n R_n \omega - \omega|_p \leq \kappa h_n \max_{j=1,2,\ldots,N} \left|\frac{\partial \omega}{\partial x^j}\right|_p , \tag{2.3.6}$$

where $x^1, x^2, \ldots, x^N$ are any local coordinates defined in a neighbourhood of p from the finite set $\mathcal{U}$ of coordinate charts as defined in Definition 2.2.19 and $\left|\frac{\partial \omega}{\partial x^j}\right|_p$ is defined via Definition 2.1.4.

The proof is based on considering a σ inside of $\mathbb{R}^m$ for some $m \in \mathbb{N}$ such that the side lengths of the simplex depend on h. An application of the mean value theorem and some explicit calculations then yield the desired bound.

Dodziuk builds on the pointwise estimate by integrating over the manifold to obtain a global estimate. Zahariev later provides a more explicit bound by applying a Sobolev embedding theorem of the following form.

Theorem 2.3.10 (Zahariev [30, Theorem 2.4]). *There exists a positive constant κ such that for all integers $r > \frac{N}{2} + 1$, for all $\sigma \in K^N$, for all $q = 0, 1, \ldots, N$ and for all $c \in C^q(K, \mathbb{C})$, it holds that*

$$\sup_{p \in \sigma} |Wc|_p \leq \kappa \|Wc\|_r^\sigma , \tag{2.3.7}$$

and

$$\max_{j=1,2,\ldots,N} \sup_{p \in \sigma} \left|\frac{\partial Wc}{\partial x^j}\right|_p \leq \kappa \|Wc\|_r^\sigma , \tag{2.3.8}$$

where $x^1, x^2, \ldots, x^N$ are any local coordinates defined in a neighbourhood of σ from $\mathcal{U}$.

This results in the following fundamental estimate of WR, presented in the form used by Zahariev and this thesis [30, Equation (2.2)].

Corollary 2.3.11 (Dodziuk [9, Corollary 3.27]). *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C})$ it holds that*

$$\|W_n R_n \omega - \omega\|_0 \leq \kappa h_n \|\omega\|_r . \tag{2.3.9}$$

Definition 2.3.12. *For all* $r = 0, 1, \ldots,$ *define the positive constant* $\kappa_{A,r} \geq 0$ *depending on* A *by*

$$\kappa_{A,r} = \max \left\{ 1, \sup_{p\in M} |A|_p \, , \max_{j=1,2,\ldots,N} \sup_{p\in M} \left| \frac{\partial A}{\partial x^j} \right|_p , \|iA \wedge\|_{\mathrm{op}} \right\} , \tag{2.3.10}$$

where $x^1, x^2, \ldots, x^N$ *are any local coordinates defined in a neighbourhood of* p *from* $\mathcal{U}$ *and* $\|\cdot\|_{\mathrm{op}}$ *is the operator norm with respect to the* $\|\cdot\|_0$ *norm on the domain and the* $\|\cdot\|_r$ *norm on the range*[†††].

Lemma 2.3.13. *For all integers* $0 \leq j \leq k$ *it holds that*

$$\kappa_{A,j} \leq \kappa_{A,k} \, . \tag{2.3.11}$$

Proof. This result follows immediately by Lemma 2.1.8. □

Definition 2.3.14. *Define the positive constant* $\kappa_A \geq 0$ *depending on* A *by*

$$\kappa_A = \kappa_{A,j} \, , \tag{2.3.12}$$

where j *is equal to* $\frac{N}{2} + 2$*, rounded up if* N *is odd.*

Proposition 2.3.15 (Zahariev [30, Proposition 4.3a]). *There exists a positive constant* κ *independent of* n *and* A *such that for all* $\omega \in \Omega^q(M, \mathbb{C})$*, for all* $\sigma \in K_n^N$ *and for all* $p \in \sigma$*, it holds that*

$$|W_n \delta_a R_n \omega - d_A W_n R_n \omega|_p \leq \kappa \kappa_{A,0} h_n \left(\sup_{p'\in\sigma} |\omega|_{p'} + \max_{j=1,2,\ldots,N} \sup_{p'\in\sigma} \left| \frac{\partial \omega}{\partial x^j} \right|_{p'} \right) , \tag{2.3.13}$$

where $x^1, x^2, \ldots, x^N$ *are any local coordinates defined in a neighbourhood of* σ *from* $\mathcal{U}$*.*

Proposition 2.3.16 (Zahariev [30, Proposition 4.3b]). *There exists a positive constant* κ *independent of* n *and* A *such that for all integers* $r > \frac{N}{2} + 1$ *and for all* $\omega \in \Omega^q(M, \mathbb{C})$ *it holds that*

$$\|W_n \delta_a R_n \omega - d_A \omega\|_0 \leq \kappa \kappa_{A,0} h_n \left(\|\omega\|_r + \|d\omega\|_r \right) . \tag{2.3.14}$$

2.4 Additional Definitions

Definition 2.4.1. *Define the map* $\Sigma_n : C^q(K_n, \mathbb{C}) \to \Omega^{q+1}(M, \mathbb{C})$ *to be*

$$\Sigma_n(c) = W_n(ia \cup c) - iA \wedge W_n c \, . \tag{2.4.1}$$

Context permitting, this map is denoted by Σ.

Definition 2.4.2. *Define the map* $\Theta_n : \Omega^q(M, \mathbb{C}) \to \mathcal{H}_0^q(M, \mathbb{C})$ *to be*

$$\Theta_n(\omega) = W_n R_n \omega - \omega \, . \tag{2.4.2}$$

Context permitting, this map is denoted by Θ.

[†††]This constant is finite as by Hölder's Inequality $\|iA \wedge\|_{\mathrm{op}}$ is bounded by the supremum norm of A.

Definition 2.4.3. *For all* $r = 0, 1, \ldots$ *and for all* $k = 1, 2, 3$*, define the following functions* $\widetilde{\vartheta}_n^k : C^q(K_n, \mathbb{C}) \setminus \{0\} \to \mathbb{R}$ *dependent on* n *as follows*

$$\widetilde{\vartheta}_n^1(c) = \frac{\|d_A W_n c\|_0}{\|W_n c\|_0}, \tag{2.4.3}$$

$$\widetilde{\vartheta}_n^2(c) = \frac{\|(d_A)^* W_n c\|_0 \left(\left\|(R_n)^{-1} c\right\|_r + \left(\left\|(R_n)^{-1} c\right\|_r \left\|P_n (R_n)^{-1} c\right\|_r\right)^{\frac{1}{2}}\right)}{\left(\|W_n c\|_0\right)^2}, \tag{2.4.4}$$

and

$$\widetilde{\vartheta}_n^3(c) = \left(\frac{\left\|(R_n)^{-1} c\right\|_r + \left(\left\|(R_n)^{-1} c\right\|_r \left\|P_n (R_n)^{-1} c\right\|_r\right)^{\frac{1}{2}}}{\|W_n c\|_0}\right)^2, \tag{2.4.5}$$

where d_A *and* $(d_A)^*$ *are applied to the smooth regions of* $W_n c$*.*

It should be noted that the dependence on r for $\widetilde{\vartheta}_n^k$ is not notated. The value of r used is clear from context.

Definition 2.4.4. *For all* $k = 1, 2, 3$ *and for all* $m = 1, 2, \ldots, \dim\left(C^q(K_n, \mathbb{C})\right)$*, define the following constant dependent on* n

$$\widehat{\vartheta}_n^k = \sup_{\substack{c_j \in C^q(K_n, \mathbb{C}) \\ j = 1, 2, \ldots, m-1}} \inf_{\substack{c \in C^q(K_n, \mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j = 1, 2, \ldots, m-1}} \widetilde{\vartheta}_n^k(c). \tag{2.4.6}$$

Lemma 2.4.5. *The constants* $\widehat{\vartheta}_n^1, \widehat{\vartheta}_n^2$ *and* $\widehat{\vartheta}_n^3$ *are well defined and positive.*

Proof. These constants are all trivially positive and bounded above as

$$\sup_{\substack{c_j \in C^q(K_n, \mathbb{C}) \\ j = 1, 2, \ldots, m-1}} \inf_{\substack{c \in C^q(K_n, \mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j = 1, 2, \ldots, m-1}} \leq \sup_{c \in C^q(K_n, \mathbb{C}) \setminus \{0\}}, \tag{2.4.7}$$

and hence the expressions for each $\widetilde{\vartheta}_n^k$ is bounded above by the respective operator norm or a sum/product of operator norms. As the operators are linear operators acting on a finite dimensional space $(C^q(K_n, \mathbb{C}))$ the operators are trivially bounded. □

Definition 2.4.6. *For all* $m = 1, 2, \ldots, \dim\left(C^q(K_n, \mathbb{C})\right)$*, define the following constant dependent on* n

$$\vartheta_n^{1,2} = \max\left\{\widehat{\vartheta}_n^1, \widehat{\vartheta}_n^2\right\}, \tag{2.4.8}$$

and

$$\vartheta_n^3 = \min\left\{1, \widehat{\vartheta}_n^3\right\}. \tag{2.4.9}$$

It should be noted that the dependence on m for $\widehat{\vartheta}_n$ and ϑ_n is not notated. The value of m used is clear from context.

CHAPTER 3

1862 – 1943 DAVID HILBERT (German)

Motivation

This chapter provides an overview of the previously established results regarding the eigenvalues of Laplace operators and their combinatorial approximations. Specifically, the sequence of generalisations that has been obtained is presented. In doing so, the aim of this thesis to extend these results to the next generalisation is motivated.

3.1 Differential Form Laplacian

The combinatorial approximation to the analytic eigenvalue problem was first used in 1973 by Dodziuk in his thesis [9]. Here the real non-twisted setting ($A = 0$) was considered and hence the Laplacians considered by Dodziuk were $\Delta_A = d^*d + dd^*$ and $\blacktriangle_a = \delta^*\delta + \delta\delta^*$ acting on real differential forms and cochains of degree zero.

Dodziuk showed that the eigenvalues of the Laplace operator acting on functions in the combinatorial setting converge to the eigenvalues of the Laplace operator in the analytic setting.

Theorem 3.1.1 (Dodziuk [9, Theorem 5.7])**.** *Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N and let K_0 a smooth initial triangulation of M. Then for sufficiently many subdivisions of the initial triangulation, n, there exists a positive constant κ independent of n such that for all $j = 1, 2, \ldots, \dim\left(C^0\left(K_n, \mathbb{R}\right)\right)$, the eigenvalues λ_j^0 of the operator $\Delta_A^0 = (d^0)^* d^0 : \Omega^0\left(M, \mathbb{R}\right) \to \Omega^0\left(M, \mathbb{R}\right)$, and the eigenvalues γ_j^0 of the operator $\blacktriangle_a^0 = (\delta^0)^*\delta^0 : C^0\left(K_n, \mathbb{R}\right) \to C^0\left(K_n, \mathbb{R}\right)$, satisfy the following inequalities*

$$\gamma_j^0 - \kappa h_n \leq \lambda_j^0 \leq \gamma_j^0 . \tag{3.1.1}$$

This work was generalised from degree zero to arbitrary degree by Patodi [17] in 1974.

Theorem 3.1.2 (Patodi [17, Theorem 3])**.** *Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N and let K_0 a smooth initial triangulation of M. Then for sufficiently many subdivisions of the initial triangulation, n, there exists a positive constant κ independent of n and a positive integer r such that for all $q = 0, 1, \ldots, N$ and for all $j = 1, 2, \ldots, \dim\left(C^q\left(K_n, \mathbb{R}\right)\right)$, the eigenvalues λ_j^q of the operator*

$$\Delta_A = d^*d + dd^* : \Omega^q\left(M, \mathbb{R}\right) \to \Omega^q\left(M, \mathbb{R}\right), \tag{3.1.2}$$

and the eigenvalues γ_j^q of the operator

$$\blacktriangle_a = \delta^*\delta + \delta\delta^* : C^q\left(K_n, \mathbb{R}\right) \to C^q\left(K_n, \mathbb{R}\right), \tag{3.1.3}$$

satisfy the following inequalities

$$\gamma_j^q\left(1-\kappa h_n\left(\lambda_j^q\right)^r\right)\leq\lambda_j^q\leq\frac{\gamma_j^q}{1-\kappa\gamma_j^q\left(h_n\log\left(h_n\right)\right)^2}\,. \tag{3.1.4}$$

3.2 Flat Vector Bundle Laplacian

In 1975, Dodziuk and Patodi joined their research efforts and published a joint paper [10] that established the convergence of eigenvalues for the Laplacian acting on differential forms with values in a real flat vector bundle.

Their motivation for this problem was their desire to prove the Ray-Singer conjecture [19], posed in 1971.

The following summary of the notation suffices to understand the convergence results about to be presented. For the full technical details of Dodziuk and Patodi's problem setup, see Chapter 10.

Let $\mathcal{O}$ be a representation of the fundamental group $\pi_1(M)$ by orthogonal $\varepsilon\times\varepsilon$ matrices and let $E(\mathcal{O})$ be the associated real flat vector bundle over M. This is constructed as follows.

Let $\widetilde{M}$ be the universal cover of M, then the fundamental group $\pi_1(M)$ has a natural action on $\widetilde{M}$ and via the representation $\mathcal{O}$, it also has a natural action on $\mathbb{C}^\varepsilon$. This action then defines an equivalence relation $\sim$ on $\widetilde{M}\times\mathbb{C}^\varepsilon$ where for all $\widetilde{p}\in\widetilde{M}$, for all $v\in\mathbb{C}^\varepsilon$ and for all $g\in\pi_1(M)$

$$(\widetilde{p}g,v)\sim(\widetilde{p},gv)\;. \tag{3.2.1}$$

It then follows that

$$E(\mathcal{O})=\left(\widetilde{M}\times\mathbb{C}^\varepsilon\right)\big/\sim\,, \tag{3.2.2}$$

is a real flat vector bundle of M. See Guichard [12, Section 4.3] for more details.

The connection, ∇, on the vector bundle is d and the Laplacians considered are $\Delta=d^*d+dd^*$ and $\blacktriangle=\delta^*\delta+\delta\delta^*$, where d and δ have been trivially extended to $E(\mathcal{O})$.

Dodziuk and Patodi proved the convergence of eigenvalues for arbitrary degree.

Theorem 3.2.1 (Dodziuk & Patodi [10, Theorem 3.7])**.** *Let (M,g) be a connected smooth closed oriented Riemannian manifold of dimension N, let K_0 be a smooth initial triangulation of M and let $E(\mathcal{O})$ be a flat real vector bundle over M. Then for sufficiently many subdivisions of the initial triangulation, n, there exists a positive constant κ independent of n and a positive integer r such that for all $q=0,1,\ldots,N$ and for all $j=1,2,\ldots,\dim\left(C^q\left(K_n,E(\mathcal{O})\right)\right)$, the eigenvalues λ_j^q of the operator $\Delta:\Omega^q\left(M,E(\mathcal{O})\right)\to\Omega^q\left(M,E(\mathcal{O})\right)$, and the eigenvalues γ_j^q of the operator $\blacktriangle:C^q\left(K_n,E(\mathcal{O})\right)\to C^q\left(K_n,E(\mathcal{O})\right)$, satisfy the following inequalities*

$$\frac{\gamma_j^q}{1+\kappa h_n\left(1+\lambda_j^q\right)^r}\leq\lambda_j^q\leq\gamma_j^q\left(1-\kappa h_n\left|\log\left(h_n\right)\right|\left(1+\sqrt{\gamma_j^q}\right)\right)\,. \tag{3.2.3}$$

3.3 Trivial Line Bundle Laplacian

In 2007, Zahariev in his thesis [30] established the convergence of eigenvalues for the Laplacian acting on differential forms with values in a trivial complex line bundle with arbitrary connection for degree zero.

Zahariev considered the setting where, given a real one-form A, $\Delta_A^q = (d_A^q)^* d_A^q + d_A^{q-1}(d_A^{q-1})^*$ and $\blacktriangle_a^q = (\delta_a^q)^* \delta_a^q + \delta_a^{q-1}(\delta_a^{q-1})^*$.

In this setting, Zahariev proves the convergence of eigenvalues for degree zero.

Theorem 3.3.1 (Zahariev [30, Theorem 4.6]). *Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N and let K_0 a smooth initial triangulation of M. Then for sufficiently many subdivisions of the initial triangulation, n, there exists a positive constant κ independent of n and A such that for all $j = 1, 2, \ldots, \dim\left(C^0\left(K_n, \mathbb{C}\right)\right)$, the eigenvalues λ_j^0 of the operator $\Delta_A^0 : \Omega^0\left(M, \mathbb{C}\right) \to \Omega^0\left(M, \mathbb{C}\right)$, and the eigenvalues γ_j^0 of the operator $\blacktriangle_a^0 : C^0\left(K_n, \mathbb{C}\right) \to C^0\left(K_n, \mathbb{C}\right)$, satisfy the following inequalities*

$$\gamma_j^0 - \kappa\left(\kappa_{A,0}\right)^2 h_n \leq \lambda_j^0 \leq \left(\frac{\sqrt{\gamma_j^0} + \kappa\kappa_{A,0}h_n}{1 - \kappa\kappa_{A,0}h_n}\right)^2 . \tag{3.3.1}$$

Zahariev reasoned that this result can be extended to a trivial vector bundle of arbitrary rank through an appropriate generalisation of his construction and proof.

3.4 Arbitrary Vector Bundle Laplacian

Zahariev, also in his thesis [30], then used his convergence of eigenvalues in the trivial line bundle case to show convergence in the setting of an arbitrary vector bundle.

The following summary of the notation suffices to understand the convergence results about to be presented. For the full technical details of Zahariev's problem setup, see Chapter 10.

Zahariev defined a new combinatorial Laplacian, $\blacktriangle$, in order for him to generalise his convergence result for trivial line bundles to arbitrary vector bundles.

It must be noted however, that Zahariev's combinatorial Laplacian does not reduce to the combinatorial Laplacian used by Dodziuk and Patodi in the case of a flat vector bundle. The details of this discrepancy are discussed in Section 10.2.4.

On the analytic side, Zahariev considered $\Delta^q = (\nabla^q)^* \nabla^q + \nabla^{q-1}(\nabla^{q-1})^*$, where ∇ is the connection of the vector bundle.

Zahariev proved the convergence of eigenvalues for arbitrary connection for degree zero.

Theorem 3.4.1 (Zahariev [30, Theorem 5.4]). *Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N, let K_0 be a smooth initial triangulation of M and let E be a Hermitian complex vector bundle over M with a connection. Then for sufficiently many subdivisions of the initial triangulation, n, there exists a positive constant κ independent of n such that for all $j = 1, 2, \ldots, \dim\left(C^0\left(K_n, E\right)\right)$, the eigenvalues λ_j^0 of the operator*

$$\Delta^0 : \Omega^0\left(M, E\right) \to \Omega^0\left(M, E\right), \tag{3.4.1}$$

and the eigenvalues γ_j^0 of the operator

$$\blacktriangle^0 : C^0\left(K_n, E\right) \to C^0\left(K_n, E\right), \tag{3.4.2}$$

satisfy the following inequalities

$$\gamma_j^0 - \kappa h_n \leq \lambda_j^0 \leq \left(\frac{\sqrt{\gamma_j^0} + \kappa h_n}{1 - \kappa h_n}\right)^2 . \tag{3.4.3}$$

Part II

Core Analysis

CHAPTER 4

180 4 1851 — CARL GUSTAV JACOB JACOBI (German)

Key Aim and Result

4.1 Aim of Thesis

Given the sequence of results presented in Chapter 3, the natural next generalisation is to extend Zahariev's results for arbitrary non-flat vector bundles from degree zero to arbitrary degree. To what extent this is possible is the problem studied by this thesis.

The aim of this thesis is to see how far Zahariev's method of proving convergence on the trivial line bundle for degree zero can be generalised to the setting of any degree.

Specifically the aim is to prove an analogue of Theorem 3.3.1 for arbitrary degree.

4.2 Original Contribution

A key obstacle to generalising Theorem 3.3.1 to arbitrary degree was identified by Zahariev in his thesis [30]. Specifically, it is that Zahariev's Proposition 4.3c only holds for degree zero. Zahariev noted that 'there is no obvious analogue of the estimate [Proposition 4.3c] for cochains c of positive degree as in this case $[Wc]$ does not lie in $[\mathcal{H}_1^q(M,\mathbb{C})]$ anymore' [30, Remark 4.4].

This obstacle is overcome in this thesis by replacing the Sobolev norm $\|\cdot\|_r$ with a norm dependent on the triangulation, $\|\cdot\|_r^K$, whereby the norm of Wc is well defined for all degrees.

Additionally, the growth rate of the Whitney map under subdivisions is calculated in thesis. This allows resulting error terms that are dependent on K_n to be pulled back to be expressed in terms of the initial triangulation and hence become a constant independent of n.

These two original ideas make it possible to derive a new bound, Proposition 7.1.4, for Zahariev's Proposition 4.3c that holds for all degrees. With this new bound, the proofs of Zahariev's thesis for the lower bound of the trivial line bundle case are able to be adapted to the setting of all degrees.

4.3 Difficulties Encountered

The obstacle identified by Zahariev in his thesis [30] is not the only obstacle to generalising his result to higher degrees. As Zahariev considered the setting of degree zero, his Laplacian has the form $\Delta_A^0 = (d_A^0)^* d_A^0$ compared to the general form $\Delta_A^q = (d_A^q)^* d_A^q + d_A^{q-1}(d_A^{q-1})^*$. The additional $d_A^{q-1}(d_A^{q-1})^*$ term means that all of the key technical estimates used in Zahariev's method (Proposition 7.1.4 and Corollary 8.1.15) need to be replicated in an analogous way for the adjoint operators $(d_A)^*$ and $(\delta_a)^*$ (Proposition 7.4.10 and Corollary 8.2.7 respectively).

Deriving the required technical estimates in the adjoint setting is considerably more difficult compared to the nonadjoint setting. The missing link or estimate is Assumption 7.3.4, restated here for ease of reading.

Assumption 7.3.4. *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M, \mathbb{C})$ it holds that*

$$\left\| d\star (W_n R_n \omega - \omega) \right\|_0 \leq \kappa h_n \|\omega\|_r . \tag{4.3.1}$$

Whilst this obstacle was not able to be overcome, there are published results that provide a means to sidestepping this obstacle. For more details of this obstacle and possible solutions, see Section 4.5.

4.4 Main Result

The key result of this thesis is the following theorem. It provides a link between the combinatorial and analytic eigenvalue problem in the setting of a Laplacian operator acting on differential forms of any degree with values in a trivial complex line bundle.

Theorem 4.4.1. *Let (M, g) be a connected smooth closed oriented Riemannian manifold of dimension N. Then there exists a smooth initial triangulation K_0 of M contained in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ such that the first vertex p_1 of K_0 is the origin and $\left\{ p_2, p_3, \ldots p_{|K^0|} \right\}$ is a linearly independent set when each vertex is viewed as a vector.*

For such an initial triangulation, there exists a positive constant κ independent of n, the number of subdivisions of the triangulation, and A such that for all $q = 0, 1, \ldots, N$ and for all $j = 1, 2, \ldots, \dim\left(C^q(K_n, \mathbb{C})\right)$, if Assumption 7.3.4 holds then for sufficiently many subdivisions of the initial triangulation, n, the eigenvalues λ_j^q of the operator

$$\Delta_A : \Omega^q(M, \mathbb{C}) \to \Omega^q(M, \mathbb{C}) , \tag{4.4.1}$$

and the eigenvalues γ_j^q of the operator

$$\blacktriangle_a : C^q(K_n, \mathbb{C}) \to C^q(K_n, \mathbb{C}) , \tag{4.4.2}$$

satisfy the following inequality

$$\gamma_j^q - \kappa (\kappa_A)^2 \beta_n - 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \kappa_A - 2^{nq} (\beta_n)^q \kappa \leq \lambda_j^q , \tag{4.4.3}$$

where κ_A is a constant dependent on A as defined in Definition 2.3.14 and β_n is the mesh parameter of (2.2.27).

The details of the proof of Theorem 4.4.1 are the subject of the rest of this thesis. The existence of the required initial triangulation is proved by Lemmas 2.2.14 and 5.1.1. For the proof of the eigenvalue bound, see Section 8.3.

4.5 Discussion of Assumption 7.3.4

Initial investigations of whether Assumption 7.3.4 holds or not suggest that this problem has not been previously researched. The main obstacle being that d is not a bounded operator in the respective norms.

This assumption appears to not be the only way of generalising Proposition 7.1.4 and Corollary 8.1.15 to the adjoint setting. Smits established that the adjoint of the coboundary operator does approximate the adjoint of the exterior derivative as required but only for degree one on a surface ($N = 2$) and with the regular standard subdivision [24, Theorem 5], as opposed to the standard subdivision used in this thesis.

Arnold, Falk, Guzman and Tsogtgerel subsequently showed that this approximation between the analytic and combinatorial side for degree one can be generalised to manifolds of arbitrary dimension, but also provided counter examples when convergence failed in the setting of other subdivision schemes such as the standard subdivision and separately, in the case of higher degree differential forms and cochains [3, Page 5489].

The work by Wilson [29] and later Tanabe [26] on a combinatorial analogue of the Hodge star operator could be an additional path to overcoming Assumption 7.3.4. Wilson and separately Tanabe showed that the combinatorial Hodge star operator approximates the analytic Hodge star operator in a number of ways. Considering the definition of the adjoint of the exterior derivative involves the Hodge star, it is conceivable that using the combinatorial Hodge star for defining the adjoint of the coboundary operator allows for the required estimates, generalisations of Proposition 7.1.4 and Corollary 8.1.15 to the adjoint setting to be derived.

Therefore there is evidence to suggest that the missing link, the generalising of Proposition 7.1.4 and Corollary 8.1.15 to the adjoint setting, of this thesis can be overcome with further research efforts but that the settings in which this is possible may well be quite restrictive.

4.6 Discussion of Convergence

Given Corollary 2.2.26, at best $\beta_n = 2^{-n}$ and hence the error term

$$\kappa \left(\kappa_A\right)^2 \beta_n + 2^{\frac{nq}{2}} \left(\beta_n\right)^{\frac{q}{2}} \kappa\kappa_A + 2^{nq} \left(\beta_n\right)^q \kappa\,, \tag{4.6.1}$$

in Theorem 4.4.1 approaches the constant $\kappa\kappa_A + \kappa$ in this optimal case, and in all other cases the error term grows in size as $n \to \infty$.

4.7 Dependence on Choices

4.7.1 Ordering of Vertices

In the construction of the standard subdivision, the vertices set is given an ordering. As was discussed in Section 2.2.7.1, a different choice of ordering can result in quite a different standard subdivision. Additionally Definition 2.2.12 or other choices of how to order the vertices of the subdivision will likewise have a significant effect on the sequence of subdivisions performed.

All of these choices result in potential changes to the mesh size h_n and its related parameters β_n and $\widehat{\beta}_n$. Clearly these changes to β_n will have a direct impact on the bound of Theorem 4.4.1.

4.7.2 Linear Right Inverse of the de Rham Map

In Notation 2.3.5, there is a choice about which linear right inverse of the de Rham map is used. This choice then affects Definition 2.3.6. However this choice does not affect the bound in any significant way as by Proposition 8.2.6, a different choice would simply result in a different constant κ.

4.7.3 Basis for Eigenspace

In Chapter Definition 8.1.11, there is a choice of basis vectors for the Eigenspace of the analytic Laplacian. However, this choice does not affect Theorem 4.4.1 in any significant way as Propositions 8.1.14 and 8.2.6 show the only effect is a change in the constant κ of Theorem 4.4.1.

CHAPTER 5

1879 – 1955 Albert EINSTEIN (German)

Barycentric Coordinate Relationship

This chapter proves that the barycentric coordinates of a triangulation are a linear combination of the barycentric coordinates of its standard subdivision. In essence this result is a direct consequence of simple linear algebra techniques.

5.1 Simplicial Complex Construction

This section constructs a simplicial complex with certain properties which will be critical for later steps.

Lemma 5.1.1. *For all M, there exists a smooth triangulation K of M contained in $\mathbb{R}^k$ for some $k \in \mathbb{N}$ such that the first vertex p_1 of K is the origin.*

Proof. Let M be given. Then by Theorem 2.2.2 there exists a smooth triangulation $\widehat{K}$ of M contained in $\mathbb{R}^{\widehat{k}}$ for some $\widehat{k} \in \mathbb{N}$. Define a new smooth triangulation K of M contained in $\mathbb{R}^k$ where $k = \widehat{k}$ via the affine homeomorphism $F : \widehat{K} \to K$ where

$$\widehat{F}\left(\widehat{p}\right) = \widehat{p} - \widehat{p}_1 \, , \tag{5.1.1}$$

where $\widehat{p}_1$ is the first vertex of $\widehat{K}$. By the property that the composition of affine homeomorphisms is affine homeomorphic, it follows that K is a smooth triangulation of M and the first vertex, p_1, of K is the origin. □

Lemma 5.1.2. *Let $\widehat{K}$ be a smooth triangulation of M contained in $\mathbb{R}^{\widehat{k}}$ for some $\widehat{k} \in \mathbb{N}$ and let K be the modified smooth triangulation of M contained in $\mathbb{R}^k$ where $k = \widehat{k} + \left|\widehat{K}^0\right| - 1$ as per Lemma 2.2.14. If $\widehat{K}$ has barycentric coordinates $\widehat{\mu}$, then*

$$\widehat{\mu} \circ \mathrm{pr} \, , \tag{5.1.2}$$

are barycentric coordinates for K, where $\mathrm{pr} : \mathbb{R}^k \to \mathbb{R}^{\widehat{k}}$ is the projection onto the first $\widehat{k}$ components.

Proof. This result follows immediately from the definition of barycentric coordinates in Definition 2.2.8. □

5.2 Barycentric Coordinates

This section shows that the barycentric coordinates of a simplicial complex can be represented in a matrix equation.

Consider the set of vertices $K_n^0 = \left\{p_1, p_2, \ldots, p_{|K_n^0|}\right\}$. By Definition 2.2.15, K_n sits inside $\mathbb{R}^m$ for some $m \geq \left|K_n^0\right|$, and by hypotheses of Theorem 4.4.1, p_1 is the origin, and $\left\{p_2, p_3, \ldots, p_{|K_n^0|}\right\}$ is a linearly independent set when each point is viewed as a vector. For each $j = 1, 2, \ldots, \left|K_n^0\right|$, let

$$p_j = \left(x_j^1, x_j^2, \ldots, x_j^m\right) , \tag{5.2.1}$$

and consider any point $p \in K_n$ where

$$p = \left(y^1, y^2, \ldots, y^m\right) . \tag{5.2.2}$$

Let $\mu_1, \mu_2, \ldots, \mu_{|K_n^0|}$ be the barycentric coordinates corresponding to the vertices $p_1, p_2, \ldots, p_{|K_n^0|}$ respectively. Hence for $k = 1, 2, \ldots, m$, by the definition of barycentric coordinates, every component of p can be written in the form

$$y^k = \mu_1 x_1^k + \mu_2 x_2^k + \cdots + \mu_{|K_n^0|} x_{|K_n^0|}^k , \tag{5.2.3}$$

where $\mu_1 + \mu_2 + \cdots + \mu_{|K_n^0|} = 1$ and for all $j = 1, 2, \ldots, \left|K_n^0\right|$, it holds that $\mu_j \in [0, 1]$. As p_1 is the origin, it follows that (5.2.3) becomes

$$y^k = \mu_2 x_2^k + \mu_3 x_3^k + \cdots + \mu_{|K_n^0|} x_{|K_n^0|}^k . \tag{5.2.4}$$

This is a system of equations where each equation is given by a choice of $k = 1, 2, \ldots, m$. In matrix notation, this system of equations is

$$\begin{bmatrix} x_2^1 & x_3^1 & \cdots & x_{|K_n^0|}^1 \\ x_2^2 & x_3^2 & \cdots & x_{|K_n^0|}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^m & x_3^m & \cdots & x_{|K_n^0|}^m \end{bmatrix} \begin{bmatrix} \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{|K_n^0|} \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{bmatrix} . \tag{5.2.5}$$

Define the $m \times \left(\left|K_n^0\right| - 1\right)$ matrix $D_{\widehat{\mu}}$ as well as the vectors $\widehat{\boldsymbol{\mu}}$ and $\mathbf{y}$ to be the corresponding matrix and vectors in (5.2.5). That is

$$D_{\widehat{\mu}} = \begin{bmatrix} x_2^1 & x_3^1 & \cdots & x_{|K_n^0|}^1 \\ x_2^2 & x_3^2 & \cdots & x_{|K_n^0|}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^m & x_3^m & \cdots & x_{|K_n^0|}^m \end{bmatrix} , \tag{5.2.6}$$

$$\widehat{\boldsymbol{\mu}} = \begin{bmatrix} \mu_2 & \mu_3 & \cdots & \mu_{\left|K_n^0\right|} \end{bmatrix}^{\mathtt{T}}, \tag{5.2.7}$$

and

$$\mathbf{y} = \begin{bmatrix} y^1 & y^2 & \cdots & y^m \end{bmatrix}^{\mathtt{T}}. \tag{5.2.8}$$

Hence

$$D_{\widehat{\mu}} \widehat{\boldsymbol{\mu}} = \mathbf{y}. \tag{5.2.9}$$

Lemma 5.2.1. *The matrix $D_{\widehat{\mu}}$ is injective as a linear map.*

Proof. This result follows immediately by the hypotheses of the initial triangulation in Theorem 4.4.1. □

5.3 Barycentric Coordinates of Standard Subdivision

This section constructs the same matrix equation for the barycentric coordinates of a simplicial complex as Section 5.2 did but this time for the standard subdivision of the simplicial complex. Through algebraic manipulation of these matrix equations, the relationship between the barycentric coordinates of a simplicial complex and its subdivision is established.

Consider the standard subdivision, K_{n+1}, of K_n where the vertex set comprises of points of the form

$$\begin{aligned} p_{r,s} &= \frac{1}{2}\left(p_r + p_s\right) \\ &= \left(x_{r,s}^1, x_{r,s}^2, \ldots, x_{r,s}^m\right), \end{aligned} \tag{5.3.1}$$

and hence for $k = 1, 2, \ldots, m$

$$x_{r,s}^k = \frac{1}{2} x_r^k + \frac{1}{2} x_s^k. \tag{5.3.2}$$

Definition 5.3.1. *For all $r = 1, 2, \ldots, \left|K_n^0\right|$ and $s = 1, 2, \ldots, \left|K_n^0\right|$ where $r \leq s$, define the indicator function $\mathbb{I}_n^{r,s}$ to be*

$$\mathbb{I}_n^{r,s} = \begin{cases} 1 & \text{if } \exists e \in K_n^1 \left[(\partial_0 e = p_r \text{ and } \partial_1 e = p_s) \text{ or } (\partial_0 e = p_s \text{ and } \partial_1 e = p_r) \right] \\ 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}. \tag{5.3.3}$$

In essence, the indicator function indicates if the point $p_{r,s}$ is included in the standard subdivision or not. The first case of the indicator function indicates if $p_{r,s}$ is the midpoint between two vertices and hence is included in the standard subdivision. Further, it is trivially the case that an existing vertex is included in the standard subdivision and hence the second case.

For ease of reading the subsequent equations, define the following parameter.

Definition 5.3.2. *Define $\varkappa$ to be the following parameter of the triangulation*

$$\varkappa = \begin{pmatrix} \left|K_n^0\right| + 1 \\ 2 \end{pmatrix}, \tag{5.3.4}$$

where $\begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$ is the binomial coefficient.

Chapter Definition 5.3.3. *Define ν to be the pair of indexing variables such that for all $j = 0, 1, \ldots, \left|K_n^0\right|$ and for all $k = 1, 2, \ldots, \left|K_n^0\right| - j$ it is defined to be*

$$\nu_{j\left|K_n^0\right|+k} = (j+1, j+k) \,. \tag{5.3.5}$$

For example, $\nu_1 = (1,1)\,, \nu_2 = (1,2)\,, \nu_{\left|K_n^0\right|} = \left(1, \left|K_n^0\right|\right)$ and $\nu_{\left|K_n^0\right|+1} = (2,2)$.

Notationally, for $j = 1, 2, \ldots, \varkappa$, as ν_j is a pair of numbers, say (r, s), let $\nu_j^1 = r$ be the first number and $\nu_j^2 = s$ be the second number.

For each $p_{r,s} \in K_{n+1}^0$, let $\xi_{r,s}$ be the barycentric coordinates of K_{n+1} corresponding to $p_{r,s}$. In a similar process as was the case for K_n, let $p \in K_{n+1}$ and by definition of barycentric coordinates, every component y^k of p can be written in the form

$$y^k = \sum_{s=1}^{\left|K_n^0\right|} \sum_{r=1}^{s} \mathbb{I}_n^{r,s} \xi_{r,s} x_{r,s}^k \,, \tag{5.3.6}$$

for $k = 1, 2, \ldots, m$, where it can be shown that

$$\sum_{s\,=1}^{\left|K_n^0\right|} \sum_{r\,=1}^{s} \mathbb{I}_n^{r,s} \xi_{r,s} = 1 \,, \tag{5.3.7}$$

and for all $\xi_{r,s}$, it holds that $\xi_{r,s} \in [0,1]$. It should be noted that this is a slight abuse of notation as strictly speaking there may be some combinations of r and s for which $\xi_{r,s}$ is not defined as $p_{r,s} \notin K_{n+1}^0$. However this is not an actual issue as the indicator function was introduced to alleviate this problem. Specifically by definition, $\mathbb{I}_n^{r,s} = 0$ whenever $\xi_{r,s}$ is not defined.

As $p_{1,1} = p_1$ is the origin, it follows that (5.3.6) becomes

$$y^k = \sum_{s=2}^{\left|K_n^0\right|} \sum_{r=1}^{s} \mathbb{I}_n^{r,s} \xi_{r,s} x_{r,s}^k \,. \tag{5.3.8}$$

This is a system of equations where each equation is given by a choice of $k = 1, 2, \ldots, m$. In matrix notation, this system of equations is

$$\begin{bmatrix} \mathbb{I}_n^{\nu_2} x_{\nu_2}^1 & \mathbb{I}_n^{\nu_3} x_{\nu_3}^1 & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^1 \\ \mathbb{I}_n^{\nu_2} x_{\nu_2}^2 & \mathbb{I}_n^{\nu_3} x_{\nu_3}^2 & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{I}_n^{\nu_2} x_{\nu_2}^m & \mathbb{I}_n^{\nu_3} x_{\nu_3}^m & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^m \end{bmatrix} \begin{bmatrix} \xi_{\nu_2} \\ \xi_{\nu_3} \\ \vdots \\ \xi_{\nu_\varkappa} \end{bmatrix} = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{bmatrix} . \tag{5.3.9}$$

Define the $m \times (\varkappa - 1)$ matrix $D_{\widehat{\xi}}$ as well as the vector $\widehat{\boldsymbol{\xi}}$ to be the corresponding matrix and vector in (5.3.9). That is

$$D_{\widehat{\xi}} = \begin{bmatrix} \mathbb{I}_n^{\nu_2} x_{\nu_2}^1 & \mathbb{I}_n^{\nu_3} x_{\nu_3}^1 & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^1 \\ \mathbb{I}_n^{\nu_2} x_{\nu_2}^2 & \mathbb{I}_n^{\nu_3} x_{\nu_3}^2 & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{I}_n^{\nu_2} x_{\nu_2}^m & \mathbb{I}_n^{\nu_3} x_{\nu_3}^m & \cdots & \mathbb{I}_n^{\nu_\varkappa} x_{\nu_\varkappa}^m \end{bmatrix} , \tag{5.3.10}$$

and

$$\widehat{\boldsymbol{\xi}} = \begin{bmatrix} \xi_{\nu_2} & \xi_{\nu_3} & \dots & \xi_{\nu_\varkappa} \end{bmatrix}^{\mathtt{T}}. \tag{5.3.11}$$

Hence

$$D_{\widehat{\xi}} \widehat{\boldsymbol{\xi}} = \mathbf{y}\,. \tag{5.3.12}$$

5.4 Relationship of Barycentric Coordinates

Consider the j-th component of the vector $D_{\widehat{\xi}} \widehat{\boldsymbol{\xi}}$ after substituting (5.3.2) into $D_{\widehat{\xi}}$, which is

$$\begin{aligned}
\left(D_{\widehat{\xi}} \widehat{\boldsymbol{\xi}}\right)_j &= \frac{1}{2}\left(\sum_{s=2}^{|K_n^0|}\sum_{r=1}^{s} \mathbb{I}_n^{r,s} x_r^j \xi_{r,s} + \sum_{s=2}^{|K_n^0|}\sum_{r=1}^{s} \mathbb{I}_n^{r,s} x_s^j \xi_{r,s}\right) \\
&= \frac{1}{2}\left(\sum_{r=2}^{|K_n^0|} \mathbb{I}_n^{1,r} x_1^j \xi_{1,r} + \sum_{s=2}^{|K_n^0|}\sum_{r=s}^{|K_n^0|} \mathbb{I}_n^{s,r} x_s^j \xi_{s,r} + \sum_{s=2}^{|K_n^0|}\sum_{r=1}^{s} \mathbb{I}_n^{r,s} x_s^j \xi_{r,s}\right) \quad \text{(rearranging sums)} \\
&= \frac{1}{2}\sum_{s=2}^{|K_n^0|} x_s^j \left(\sum_{r=s}^{|K_n^0|} \mathbb{I}_n^{s,r} \xi_{s,r} + \sum_{r=1}^{s} \mathbb{I}_n^{r,s} \xi_{r,s}\right) \quad (\text{as } p_1 \text{ is the origin})\,.
\end{aligned} \tag{5.4.1}$$

Hence

$$D_{\widehat{\xi}} \widehat{\boldsymbol{\xi}} = \frac{1}{2} D_{\widehat{\mu}} \begin{bmatrix}
\sum_{r=2}^{|K_n^0|} \mathbb{I}_n^{2,r} \xi_{2,r} + \sum_{r=1}^{2} \mathbb{I}_n^{r,2} \xi_{r,2} \\
\sum_{r=3}^{|K_n^0|} \mathbb{I}_n^{3,r} \xi_{3,r} + \sum_{r=1}^{3} \mathbb{I}_n^{r,3} \xi_{r,3} \\
\vdots \\
\sum_{r=|K_n^0|-1}^{|K_n^0|} \mathbb{I}_n^{|K_n^0|-1,r} \xi_{|K_n^0|-1,r} + \sum_{r=1}^{|K_n^0|-1} \mathbb{I}_n^{r,|K_n^0|-1} \xi_{r,|K_n^0|-1} \\
\mathbb{I}_n^{|K_n^0|,|K_n^0|} \xi_{|K_n^0|,|K_n^0|} + \sum_{r=1}^{|K_n^0|} \mathbb{I}_n^{r,|K_n^0|} \xi_{r,|K_n^0|}
\end{bmatrix}. \tag{5.4.2}$$

Combining (5.2.9) and (5.3.12) gives

$$D_{\widehat{\mu}} \widehat{\boldsymbol{\mu}} = D_{\widehat{\xi}} \widehat{\boldsymbol{\xi}}$$

$$= D_{\widehat{\mu}} \begin{bmatrix} \frac{1}{2}\sum_{r=2}^{\left|K_n^0\right|} \mathbb{I}_n^{2,r}\xi_{2,r} + \frac{1}{2}\sum_{r=1}^{2} \mathbb{I}_n^{r,2}\xi_{r,2} \\ \frac{1}{2}\sum_{r=3}^{\left|K_n^0\right|} \mathbb{I}_n^{3,r}\xi_{3,r} + \frac{1}{2}\sum_{r=1}^{3} \mathbb{I}_n^{r,3}\xi_{r,3} \\ \vdots \\ \frac{1}{2}\sum_{r=\left|K_n^0\right|-1}^{\left|K_n^0\right|} \mathbb{I}_n^{\left|K_n^0\right|-1,r}\xi_{\left|K_n^0\right|-1,r} + \frac{1}{2}\sum_{r=1}^{\left|K_n^0\right|-1} \mathbb{I}_n^{r,\left|K_n^0\right|-1}\xi_{r,\left|K_n^0\right|-1} \\ \frac{1}{2}\mathbb{I}_n^{\left|K_n^0\right|,\left|K_n^0\right|}\xi_{\left|K_n^0\right|,\left|K_n^0\right|} + \frac{1}{2}\sum_{r=1}^{\left|K_n^0\right|} \mathbb{I}_n^{r,\left|K_n^0\right|}\xi_{r,\left|K_n^0\right|} \end{bmatrix} \quad \text{(by (5.4.2))}\,. \tag{5.4.3}$$

As $D_{\widehat{\mu}}$ is injective by Lemma 5.2.1, it follows that

$$\begin{aligned} (\widehat{\boldsymbol{\mu}})_j &= \frac{1}{2}\left(\sum_{r=j+1}^{\left|K_n^0\right|} \mathbb{I}_n^{j+1,r}\xi_{j+1,r} + \sum_{r=1}^{j+1} \mathbb{I}_n^{r,j+1}\xi_{r,j+1}\right) \\ &= \frac{1}{2}\left(2\xi_{j+1,j+1} + \sum_{r=j+2}^{\left|K_n^0\right|} \mathbb{I}_n^{j+1,r}\xi_{j+1,r} + \sum_{r=1}^{j} \mathbb{I}_n^{r,j+1}\xi_{r,j+1}\right) \quad \text{(by (5.3.3))}\,. \end{aligned} \tag{5.4.4}$$

Therefore it is clear that each entry of $\widehat{\boldsymbol{\mu}}$ is simply a linear combination of the entries of $\widehat{\boldsymbol{\xi}}$. Define $\widehat{\Xi}$ to be the matrix with integer entries ($\widehat{\Xi}_{j,k} \in \mathbb{Z}$) of this linear combination such that (5.4.4) is equivalent to

$$\widehat{\boldsymbol{\mu}} = \frac{1}{2}\widehat{\Xi}\widehat{\boldsymbol{\xi}}\,, \tag{5.4.5}$$

where $\widehat{\boldsymbol{\xi}}$ is given by (5.3.11). As the matrix $\widehat{\Xi}$ requires significant physical space to print, an explicit expression can be found in Appendix B. However a particular entry is given by

$$\widehat{\Xi}_{j,k} = \begin{cases} 2 & \text{if } j+1 = \nu_{k+1}^1 = \nu_{k+1}^2 \\ \mathbb{I}_n^{\nu_k} & \text{if } \left[j+1 = \nu_{k+1}^1 \text{ or } j+1 = \nu_{k+1}^2\right] \text{ and } \nu_{k+1}^1 \neq \nu_{k+1}^2 \\ 0 & \text{otherwise} \end{cases}\,. \tag{5.4.6}$$

Hence it is clear that all the entries of Ξ are either a 0 or 1 except for the single entry of a 2 in each row corresponding to the coefficient of a term of the form $\xi_{r,r}$.

From (5.4.5) it is known that for $j = 2, 3, \ldots, \left|K_n^0\right|$

$$\mu_j = \frac{1}{2}\sum_{k=2}^{\varkappa} \widehat{\Xi}_{j-1,k-1}\xi_{\nu_k}\,. \tag{5.4.7}$$

Applying the property that the sum of barycentric coordinates is one gives

$$
\begin{aligned}
\mu_1 &= 1 - \sum_{j=2}^{\left|K_n^0\right|} \mu_j \\
&= 1 - \frac{1}{2} \sum_{j=2}^{\left|K_n^0\right|} \sum_{k=2}^{\varkappa} \widehat{\Xi}_{j-1,k-1} \xi_{\nu_k} \quad \text{(by (5.4.7))} \\
&= \sum_{k=1}^{\varkappa} \mathbb{I}_n^{\nu_k} \xi_{\nu_k} - \frac{1}{2} \sum_{j=2}^{\left|K_n^0\right|} \sum_{k=2}^{\varkappa} \widehat{\Xi}_{j-1,k-1} \xi_{\nu_k} \quad \text{(by (5.3.7))} \\
&= \frac{1}{2} \left(2\xi_{\nu_1} + \sum_{k=2}^{\varkappa} \left(2\mathbb{I}_n^{\nu_k} - \sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,k-1} \right) \xi_{\nu_k} \right) \quad \text{(by (5.3.3))} \,.
\end{aligned}
\tag{5.4.8}
$$

Using (5.4.5) and (5.4.8) gives the following block matrix equation

$$
\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{\left|K_n^0\right|} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & \begin{bmatrix} 2\mathbb{I}_n^{\nu_2} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,1} & 2\mathbb{I}_n^{\nu_3} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,2} & \cdots & 2\mathbb{I}_n^{\nu_\varkappa} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,\varkappa-1} \end{bmatrix} \\ 0 & \widehat{\Xi} \end{bmatrix} \begin{bmatrix} \xi_{\nu_1} \\ \xi_{\nu_2} \\ \vdots \\ \xi_{\nu_\varkappa} \end{bmatrix},
\tag{5.4.9}
$$

which is (5.4.5) but includes all barycentric coordinates. Define enlarged $\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\xi}}$ and $\widehat{\Xi}$ such that (5.4.9) is

$$
\boldsymbol{\mu} = \frac{1}{2} \Xi \boldsymbol{\xi} \,,
\tag{5.4.10}
$$

where

$$
\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_{\left|K_n^0\right|} \end{bmatrix}^{\mathtt{T}},
\tag{5.4.11}
$$

$$
\Xi = \begin{bmatrix} 2 & \begin{bmatrix} 2\mathbb{I}_n^{\nu_2} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,1} & 2\mathbb{I}_n^{\nu_3} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,2} & \cdots & 2\mathbb{I}_n^{\nu_\varkappa} - \displaystyle\sum_{j=2}^{\left|K_n^0\right|} \widehat{\Xi}_{j-1,\varkappa-1} \end{bmatrix} \\ 0 & \widehat{\Xi} \end{bmatrix},
\tag{5.4.12}
$$

and

$$
\boldsymbol{\xi} = \begin{bmatrix} \xi_{\nu_1} & \xi_{\nu_2} & \cdots & \xi_{\nu_\varkappa} \end{bmatrix}^{\mathtt{T}}.
\tag{5.4.13}
$$

Using (5.4.12) and an explicit expression for $\widehat{\Xi}$ allows Ξ to be computed. As the matrix Ξ requires significant physical space to print, an explicit expression can be found in Appendix B.

However a particular entry is given by

$$\Xi_{j,k} = \begin{cases} 2 & \text{if } j = \nu_k^1 = \nu_k^2 \\ \mathbb{I}_n^{\nu_k} & \text{if } \left[j = \nu_k^1 \text{ or } j = \nu_k^2 \right] \text{ and } \nu_k^1 \neq \nu_k^2 \\ 0 & \text{otherwise} \end{cases} . \tag{5.4.14}$$

Of note is the fact that every entry in $\widehat{\Xi}$ is either a 0 or 1 except for the single entry of a 2 in each row corresponding to the coefficient of a term of the form $\xi_{r,r}$.

This working proves the following proposition.

Proposition 5.4.1. *The barycentric coordinates $\boldsymbol{\mu}$ of a simplicial complex K_n are a linear combination of the barycentric coordinates $\boldsymbol{\xi}$ of its standard subdivision K_{n+1}. Specifically*

$$\boldsymbol{\mu} = \frac{1}{2} \Xi \boldsymbol{\xi} \,, \tag{5.4.15}$$

where the matrix Ξ is given by (5.4.14).

CHAPTER 6

1887 – 1961 — ERWIN SCHRÖDINGER *(Austrian)*

GROWTH RATE OF THE WHITNEY MAP

In this chapter, the growth rate of the Whitney map is calculated using the relationship of the barycentric coordinates between subdivisions. After some technical results are established, the relationship of the Whitney form of a single simplex and its standard subdivision is examined. This relationship of Whitney forms is subsequently generalised to an arbitrary cochain.

The main result of this chapter is Proposition 6.3.4 which, given a $\rho \in K_n^N$, provides a bound for $\|W_n c\|_1^\rho$ in terms of $\|c\|_{C,2}$ and a growth rate dependent on n. This bound is used in subsequent chapters to determine commutativity properties of the Whitney map. Ultimately this leads to a lower bound on the eigenvalues.

The notation and Chapter Definitions used in Chapter 5 will continue to be used in this chapter.

6.1 Preliminaries

This section establishes some notation and a helper lemma that will be required in subsequent sections.

Chapter Definition 6.1.1. *Let* $K_n^0 = \left\{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_{\left|K_n^0\right|}\right\}$. *Then for a given simplex* $\sigma \in K_n^q$ *where* $\sigma = \left[p_0, p_1, \ldots, p_q\right]$, *define the indexing variable* $\chi^\sigma : \{0, 1, \ldots, q\} \to \left\{1, 2, \ldots, \left|K_n^0\right|\right\}$ *such that for all* $j = 0, 1, \ldots, q$ *it holds that*

$$p_j = \widehat{p}_{\chi^\sigma(j)} \,. \tag{6.1.1}$$

To aid with notational conciseness, χ_j^σ will be used to notate $\chi^\sigma(j)$. Where clear from context χ_j is used.

Lemma 6.1.2. *For all* $q = 0, 1, \ldots, N$, *consider a collection of distinct vertices* $\left\{p_0, p_1, \ldots, p_q\right\}$ *of the simplicial complex* K *and corresponding barycentric coordinates* $\left\{\mu_0, \mu_1, \ldots, \mu_q\right\}$. *If*

$$\bigcap_{j=0}^{q} \left\{p \in K \middle| \mu_j(p) \neq 0\right\} \neq \emptyset \,, \tag{6.1.2}$$

then $\left[p_0, p_1, \ldots, p_q\right]$ *is a* q*-dimensional simplex in the simplicial complex* K.

Proof. Let q be given and suppose that $\bigcap_{j=0}^{q}\left\{p\in K\middle|\mu_j\left(p\right)\neq 0\right\}\neq\emptyset$. Hence there exists a point $p\in K$ such that

$$\mu_j\left(p\right)>0\,, \tag{6.1.3}$$

for all $j=0,1,\ldots,q$.

As $p\in K$, it is clear that there exists an N-simplex $\sigma\in K^N$ such that $p\in\sigma$. Let $\sigma=\left[p_{\alpha_0},p_{\alpha_1},\ldots,p_{\alpha_N}\right]$ and hence $p\in\left[p_{\alpha_0},p_{\alpha_1},\ldots,p_{\alpha_N}\right]$. Then Property 2.2.7 gives

$$\mu_k\left(p\right)=0\,, \tag{6.1.4}$$

for $k\in\left\{1,2,\ldots,\left|K^0\right|\right\}\setminus\left\{\alpha_0,\alpha_1,\ldots,\alpha_N\right\}$. Hence combining (6.1.3) and (6.1.4) implies that

$$\left\{p_0,p_1,\ldots,p_q\right\}\subseteq\left\{p_{\alpha_0},p_{\alpha_1},\ldots,p_{\alpha_N}\right\}\,. \tag{6.1.5}$$

However, by definition of the simplex σ, a $q+1$ subset of vertices of σ is a q-dimensional face of σ and hence the $q+1$ vertices define a q-dimensional simplex $\left[p_0,p_1,\ldots,p_q\right]$ as required. □

6.2 Relationship for a Simplex

For a given $\sigma\in K_n^q$, this section develops an expression for $W_n\sigma$ in terms of $W_{n+1}\tau$ where $\tau\in K_{n+1}^q$.

Proposition 5.4.1 says that $\boldsymbol{\mu}=\frac{1}{2}\Xi\boldsymbol{\xi}$. Hence, given a simplex $\sigma=\left[p_0,p_1,\ldots,p_q\right]\in K_n^q$, for $j=0,1,\ldots,q$ it holds that

$$\mu_j=\frac{1}{2}\sum_{k=1}^{\varkappa}\Xi_{\chi_j,k}\xi_{\nu_k}\,, \tag{6.2.1}$$

and hence

$$d\mu_j=\frac{1}{2}\sum_{k=1}^{\varkappa}\Xi_{\chi_j,k}\,d\xi_{\nu_k}\,. \tag{6.2.2}$$

Therefore by definition

$$\begin{aligned}W_n\sigma&=q!\sum_{j=0}^{q}\left(-1\right)^j\mu_j\,d\mu_0\wedge d\mu_1\wedge\cdots\wedge d\mu_{j-1}\wedge d\mu_{j+1}\wedge\cdots\wedge d\mu_q\\&=q!\left(\frac{1}{2}\right)^{q+1}\sum_{j=0}^{q}\left(-1\right)^j\sum_{s=1}^{\varkappa}\Xi_{\chi_j,s}\xi_{\nu_s}\sum_{k^1,k^2,\ldots,k^q=1}^{\varkappa}\\&\qquad\left(\prod_{r=0}^{j-1}\Xi_{\chi_r,k^{r+1}}\right)\left(\prod_{r=j+1}^{q}\Xi_{\chi_r,k^r}\right)d\xi_{\nu_{k^1}}\wedge d\xi_{\nu_{k^2}}\wedge\cdots\wedge d\xi_{\nu_{k^q}}\\&\qquad\text{(by (6.2.1) and (6.2.2))}\end{aligned} \tag{6.2.3}$$

Definition 6.2.1. *For all $q=1,2,\ldots,N$, for all simplices $\sigma\in K_n^q$ and for all distinct $s,k^1,k^2,\ldots,k^q=1,2,\ldots,\varkappa$, define $\Xi\left(s,k^1,k^2,\ldots,k^q\right)$ to be the $(q+1)\times(q+1)$ matrix constructed from Ξ as follows. Staring with the matrix Ξ, remove all of the rows with index $\left\{1,2,\ldots,\left|K_n^0\right|\right\}\setminus\left\{\chi_0^\sigma,\chi_1^\sigma,\ldots,\chi_q^\sigma\right\}$. Denote the resultant $(q+1)\times\varkappa$ matrix by $\widetilde{\Xi}$ and denote the s-th column of $\widetilde{\Xi}$ by $\widetilde{\Xi}_s$. Then define $\Xi\left(s,k^1,k^2,\ldots,k^q\right)$ to be the following block matrix*

$$\Xi\left(s,k^1,k^2,\ldots,k^q\right)=\begin{bmatrix}\widetilde{\Xi}_s&\widetilde{\Xi}_{k^1}&\widetilde{\Xi}_{k^2}&\cdots&\widetilde{\Xi}_{k^q}\end{bmatrix}\,. \tag{6.2.4}$$

Definition 6.2.2. *For all $q = 1, 2, \ldots, N$, for all $\sigma \in K_n^q$, for all $j = 0, 1, \ldots, q$ and for all distinct $k^1, k^2, \ldots, k^q = 1, 2, \ldots, \varkappa$, define $\Xi'\left(\chi_j^\sigma; k^1, k^2, \ldots, k^q\right)$ to be the following $q \times q$ matrix minor of $\Xi\left(1, k^1, k^2, \ldots, k^q\right)$. Starting with $\Xi\left(1, k^1, k^2, \ldots, k^q\right)$, remove the row with index χ_j^σ and remove the first column. The resultant $q \times q$ matrix is by definition $\Xi'\left(\chi_j^\sigma; k^1, k^2, \ldots, k^q\right)$.*

With this notation, (6.2.3) becomes

$$\begin{aligned}
W_n\sigma &= q!\left(\frac{1}{2}\right)^{q+1}\sum_{j=0}^{q}(-1)^j\sum_{s=1}^{\varkappa}\Xi_{\chi_j,s}\xi_{\nu_s}\sum_{1\le k^1<k^2<\cdots<k^q\le\varkappa} \\
&\qquad \det\left(\Xi'\left(\chi_j; k^1, k^2, \ldots, k^q\right)\right) d\xi_{\nu_{k^1}} \wedge d\xi_{\nu_{k^2}} \wedge \cdots \wedge d\xi_{\nu_{k^q}} \\
&= q!\left(\frac{1}{2}\right)^{q+1}\sum_{1\le k^1<k^2<\cdots<k^q\le\varkappa}\ \sum_{\substack{s=1\\ s\neq k^1,k^2,\ldots,k^q}}^{\varkappa} \\
&\qquad \det\left(\Xi\left(s, k^1, k^2, \ldots, k^q\right)\right)\xi_{\nu_s}\, d\xi_{\nu_{k^1}} \wedge d\xi_{\nu_{k^2}} \wedge \cdots \wedge d\xi_{\nu_{k^q}} \\
&\qquad \text{(where } s \neq k^1, k^2, \ldots, k^q \text{ as otherwise the determinant is zero)} \\
&= q!\left(\frac{1}{2}\right)^{q+1}\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}\ \sum_{j=0}^{q}\det\left(\Xi\left(k^j, k^0, k^1, \ldots, k^{j-1}, k^{j+1}, \ldots, k^q\right)\right) \\
&\qquad \xi_{\nu_{k^j}}\, d\xi_{\nu_{k^0}} \wedge d\xi_{\nu_{k^1}} \wedge \cdots \wedge d\xi_{\nu_{k^{j-1}}} \wedge d\xi_{\nu_{k^{j+1}}} \wedge \cdots \wedge d\xi_{\nu_{k^q}} \\
&\qquad \text{(by rearrangement of sums)} \\
&= \left(\frac{1}{2}\right)^{q+1}\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}\det\left(\Xi\left(k^0, k^1, \ldots, k^q\right)\right) \\
&\qquad q!\sum_{j=0}^{q}(-1)^j\,\xi_{\nu_{k^j}}\, d\xi_{\nu_{k^0}} \wedge d\xi_{\nu_{k^1}} \wedge \cdots \wedge d\xi_{\nu_{k^{j-1}}} \wedge d\xi_{\nu_{k^{j+1}}} \wedge \cdots \wedge d\xi_{\nu_{k^q}}\,. \qquad (6.2.5)
\end{aligned}$$

Chapter Definition 6.2.3. *For all $\sigma \in K_n^q$ and for all $\tau \in K_{n+1}^q$, define the constant $\eta_{\tau,\sigma}$ as follows. Let $\sigma = \left[p_0, p_1, \ldots, p_q\right]$, let $\tau = \left[p'_0, p'_1, \ldots, p'_q\right]$, and consider the set of points $\left\{\widehat{p}_{\nu_1}, \widehat{p}_{\nu_2}, \ldots, \widehat{p}_{\nu_\varkappa}\right\} \supseteq K_{n+1}^0$. Pick $k^0, k^1, \ldots, k^q \in \{1, 2, \ldots, \varkappa\}$ such that for all $r = 0, 1, \ldots, q$, it holds that $p'_r = \widehat{p}_{\nu_{k^r}}$, then define*

$$\eta_{\tau,\sigma} = \det\left(\Xi\left(k^0, k^1, \ldots, k^q\right)\right), \qquad (6.2.6)$$

where $\det(\cdot)$ is the determinant of a matrix.

Consider a barycentric coordinate on its region of support. On this region, it is a non-constant piecewise linear function and hence it is clear that the support of the barycentric coordinate and its derivative are identical. Therefore, for given values of j and $k^0, k^1, \ldots, k^q$, examining (6.2.5), it is clear to see that if the barycentric coordinates that make up the differential form

$$\xi_{\nu_{k^j}}\, d\xi_{\nu_{k^0}} \wedge d\xi_{\nu_{k^1}} \wedge \cdots \wedge d\xi_{\nu_{k^{j-1}}} \wedge d\xi_{\nu_{k^{j+1}}} \wedge \cdots \wedge d\xi_{\nu_{k^q}}\,, \qquad (6.2.7)$$

do not have any support in common, then the differential form in (6.2.7) is zero.

If the barycentric coordinates do have a region of common support, then by Lemma 6.1.2 the corresponding collection of vertices form a simplex. Specifically, $\left[p_{\nu_{k^0}}, p_{\nu_{k^1}}, \ldots, p_{\nu_{k^q}}\right]$ is a simplex in K^q_{n+1} with an orientation compatible with the orientation of K^q_{n+1} by construction. Let $\tau = \left[p_{\nu_{k^0}}, p_{\nu_{k^1}}, \ldots, p_{\nu_{k^q}}\right]$ be that simplex. As a result of the compatible orientation, for this simplex τ, the expression

$$q! \sum_{j=0}^{q} (-1)^j \, \xi_{\nu_{k^j}} \, d\xi_{\nu_{k^0}} \wedge d\xi_{\nu_{k^1}} \wedge \cdots \wedge d\xi_{\nu_{k^{j-1}}} \wedge d\xi_{\nu_{k^{j+1}}} \wedge \cdots \wedge d\xi_{\nu_{k^q}} , \tag{6.2.8}$$

in (6.2.5) is equal to the Whitney map applied to this simplex, $W\tau$.

As $\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}$ iterates over all possible combinations of $q+1$ many distinct ordered vertices of K_{n+1}, every q-dimensional simplex in K_{n+1} corresponds to some term in $\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}$. Additionally by the earlier explanation, all the terms in $\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}$ which do not correspond to a q-dimensional simplex in K_{n+1} are zero. It is also clear that there are no repeated simplices in the sum. Hence the sum $\sum_{1\le k^0<k^1<\cdots<k^q\le\varkappa}$ reduces to the sum $\sum_{\tau\in K^q_{n+1}}$.

Using Chapter Definition 6.2.3 (which was constructed specifically so that this next simplification holds), it follows that (6.2.5) can be simplified and hence proves the following proposition.

Proposition 6.2.4. *For all $\sigma \in K^q_n$ it holds that*

$$W_n\sigma = \left(\frac{1}{2}\right)^{q+1} \sum_{\tau\in K^q_{n+1}} \eta_{\tau,\sigma} W_{n+1}\tau \,. \tag{6.2.9}$$

Lemma 6.2.5. *For all $\sigma \in K^q_n$ and for all $\tau \in K^q_{n+1}$, it holds that $\eta_{\tau,\sigma} \in \mathbb{Z}$ and additionally the set*

$$\left\{\eta_{\tau,\sigma} \middle| \sigma \in K^q_n, \tau \in K^q_{n+1}\right\} , \tag{6.2.10}$$

is finite and not equal to $\{0\}$.

Proof. From Chapter Definition 6.2.3 and by using the fact that the entries of $\widehat{\Xi}$ are either $0, 1$ or 2 from (5.4.14), it is clear that $\eta_{\tau,\sigma} \in \mathbb{Z}$.

Further, by Lemma 2.2.3, K_n and K_{n+1} are finite simplicial complexes and hence the set

$$\left\{\eta_{\tau,\sigma} \middle| \sigma \in K^q_n, \tau \in K^q_{n+1}\right\} , \tag{6.2.11}$$

is finite. Further, it is not equal to $\{0\}$ as by Corollary 2.3.4 and Proposition 6.2.4 there must be at least one nonzero $\eta_{\tau,\sigma}$. □

The next result, Proposition 6.2.10 requires considering different cases for the dimension of a simplex. First consider the case when $q = N$.

Lemma 6.2.6. *For all $n = 0, 1, \ldots,$ and for all $\tau \in K^N_{n+1}$, there exists a $\sigma \in K^N_n$ such that*

$$\eta_{\tau,\sigma} \neq 0 \,. \tag{6.2.12}$$

Proof. Let n and $\tau \in K_{n+1}^N$ be given. Pick the unique $\sigma \in K_n^N$ such that $\tau \subseteq \sigma$. Then by Proposition 6.2.4

$$\begin{aligned} W_n\sigma|_{\operatorname{int}(\tau)} &= \left(\frac{1}{2}\right)^{N+1} \sum_{\rho \in K_{n+1}^N} \eta_{\rho,\sigma}\, W_{n+1}\rho|_{\operatorname{int}(\tau)} \\ &= \left(\frac{1}{2}\right)^{N+1} \eta_{\tau,\sigma}\, W_{n+1}\tau|_{\operatorname{int}(\tau)} \quad \text{(by Property 2.2.7)}\,. \end{aligned} \tag{6.2.13}$$

Hence by Definition 2.2.10 as both the left and right and hand side are positive multiples of the volume form, $\eta_{\tau,\sigma} \neq 0$ as required. $\square$

Then consider the case when $q < N$, which requires some initial technical results to be established.

Lemma 6.2.7. *For all $n = 0, 1, \ldots$, let μ be the barycentric coordinates of K_n and let ξ be the barycentric coordinates of K_{n+1}, then for all integers $0 \leq j \leq k \leq \left|K_n^0\right|$ it holds that*

$$\operatorname{supp}\left(\xi_{j,k}\right) \subseteq \operatorname{supp}\left(\mu_j\right) \cap \operatorname{supp}\left(\mu_k\right), \tag{6.2.14}$$

where $\operatorname{supp}(\cdot)$ is the support of a function.

Proof. Let $0 \leq j \leq k \leq \left|K_n^0\right|$ be given integers and let $p \in \operatorname{supp}\left(\xi_{j,k}\right)$. Then consider the following cases.

Case 1 Consider the case where $p \in K_{n+1}^0$.

Case 1a Consider the case where $j = k$. Then $p = p_{j,j} = p_j$ and hence by Property 2.2.7 it follows that

$$\mu_j(p) = 1\,. \tag{6.2.15}$$

Case 1b Consider the case where $j \neq k$. Then there exists a unique edge $e_{j,k} \in K_n^1$ such that $p \in e_{j,k}$, and p_j and p_k are the end points of $e_{j,k}$. Hence by Property 2.2.7 it follows that

$$\mu_j(p) + \mu_k(p) = 1\,, \tag{6.2.16}$$

and $\mu_j(p) \neq 1$ as otherwise $j = k$. Therefore

$$\mu_j(p) \neq 0\,, \tag{6.2.17}$$

and

$$\mu_k(p) \neq 0\,. \tag{6.2.18}$$

Case 2 Consider the case where $p \notin K^0_{n+1}$. Then there exists a unique q and a unique $\sigma \in K^q_n$ such that $p \in \operatorname{int}(\sigma)$. Let $\sigma = \left[p_{\nu_0}, p_{\nu_1}, \ldots, p_{\nu_q}\right]$, then by Property 2.2.7 it follows that

$$\sum_{r=0}^{q} \mu_{\nu_r}(p) = 1\,, \tag{6.2.19}$$

and

$$\mu_{\nu_r}(p) > 0\,, \tag{6.2.20}$$

for all $r = 0, 1, \ldots, q$.

Similarly there exists a unique $\tau \in K^q_{n+1}$ such that $p \in \operatorname{int}(\tau)$. In fact it must be the case that $\tau \in \mathfrak{S}\sigma$. Let $\tau = \left[p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_q}\right]$, where each α_r is a pair of numbers. Then by Property 2.2.7 it follows that

$$\sum_{r=0}^{q} \xi_{\alpha_r}(p) = 1\,, \tag{6.2.21}$$

and

$$\xi_{\alpha_r}(p) > 0\,, \tag{6.2.22}$$

for all $r = 0, 1, \ldots, q$.

As $p \in \operatorname{supp}\left(\xi_{j,k}\right)$, it follows from (6.2.22) that there exists a unique s such that $(j,k) = \alpha_s$. Hence as $\tau \in \mathfrak{S}\sigma$ it must be the case that $j \in \left\{\nu_0, \nu_1, \ldots, \nu_q\right\}$ and $k \in \left\{\nu_0, \nu_1, \ldots, \nu_q\right\}$. Therefore from (6.2.20) it follows that

$$\mu_j(p) \neq 0\,, \tag{6.2.23}$$

and

$$\mu_k(p) \neq 0\,. \tag{6.2.24}$$

Hence in all cases it is clear that

$$p \in \operatorname{supp}\left(\mu_j\right) \cap \operatorname{supp}\left(\mu_k\right)\,, \tag{6.2.25}$$

as required. $\square$

Lemma 6.2.8. *For all $n = 0, 1, \ldots,$ for all $q = 0, 1, \ldots, N-1$ and for all $\tau \in K^q_{n+1}$, there exists a $\sigma \in K^q_n$ such that*

$$\operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right)\right) \subseteq \operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_n\sigma\right)\right). \tag{6.2.26}$$

Proof. Let n and $\tau \in K^q_{n+1}$ be given. By the definition of the standard subdivision, let $\tau = \left[p_{r_0,s_0}, p_{r_1,s_1}, \ldots, p_{r_q,s_q}\right]$ where $r_j \in \{0, 1, \ldots, N\}$, $s_j \in \{0, 1, \ldots, N\}$ and $r_j < s_j$ (except possibly for the case $r_0 = s_0$) for all $j = 0, 1, \ldots, q$. For this proof alone, it is assumed that the points in $\left[p_{r_0,s_0}, p_{r_1,s_1}, \ldots, p_{r_q,s_q}\right]$ are listed in ascending order according to the partial ordering given by (2.2.15). In fact, the points are strictly increasing as otherwise τ is not a simplex.

By definition of the standard subdivision, τ is an upward travelling path with $q+1$ points in the Hasse diagram of the partial ordering. Hence let $j = 1, 2, \ldots, q$ and consider the following three cases.

Case 1 Consider the case where $r_{j-1} = r_j$, then it has to hold that $s_{j-1} < s_j$.

Case 2 Consider the case where $s_{j-1} = s_j$, then it has to hold that $r_{j-1} > r_j$.

Case 3 Consider the case where $r_{j-1} \neq r_j$ and $s_{j-1} \neq s_j$, then it has to hold that $r_{j-1} > r_j$ and $s_{j-1} < s_j$.

Hence that it is clear that given a pair of numbers $\left(r_j, s_j\right)$, there always exists a number $m_j \in \left\{r_{j-1}, s_{j-1}\right\}$ such that $m_j \neq r_j$ or $m_j \neq s_j$ (including possibly both).

From this construct a set of points as follows. For all $j = 1, 2, \ldots, q$, include the number m_j as per above in the set. By construction it is clear that this set contains q distinct points and either r_q or s_q (including possibly both) is not contained in this set. Add either r_q or s_q to the set such that it now contains $q+1$ distinct points. As these points all come from the standard subdivision of a simplex and hence this set of points defines a q-simplex which will be denoted $\sigma \in K^q_n$.

By construction it holds that for all $j = 0, 1, \ldots, q$, either p_{r_j} or p_{s_j} (including possibly both) is a vertex of σ. Hence by Lemma 6.2.7, the support of every barycentric coordinate of τ is contained in the support of a corresponding barycentric coordinate of σ. Considering that the Whitney map is purely defined in terms of the barycentric coordinates, it follows that

$$\operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right)\right) \subseteq \operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_n\sigma\right)\right), \tag{6.2.27}$$

as required. □

Lemma 6.2.9. *For all $n = 0, 1, \ldots,$ for all $q = 0, 1, \ldots, N-1$ and for all $\tau \in K^q_{n+1}$, there exists a $\sigma \in K^q_n$ such that*

$$\eta_{\tau,\sigma} \neq 0\,. \tag{6.2.28}$$

Proof. Let n and $\tau \in K^q_{n+1}$ be given. Then by Lemma 6.2.8 pick a simplex $\sigma \in K^q_n$ such that $\operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right)\right) \subseteq \operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_n\sigma\right)\right)$. Applying Proposition 6.2.4 to σ gives

$$\begin{aligned}
\iota^*_{\operatorname{int}(\tau)}\left(W_n\sigma\right) &= \left(\frac{1}{2}\right)^{q+1} \sum_{\rho\in K^q_{n+1}} \eta_{\rho,\sigma}\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\rho\right) \\
&= \left(\frac{1}{2}\right)^{q+1} \eta_{\tau,\sigma}\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right),
\end{aligned} \tag{6.2.29}$$

where the last equality comes from the fact that Property 2.2.7 implies that the differential form in the sum is zero for all simplices ρ that do not share a face with τ and the pullback annihilates all of the remaining $\rho \neq \tau$ as there is at least one barycentric coordinate that is different between ρ and τ.

As $\operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right)\right) \subseteq \operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_n\sigma\right)\right)$, it follows that both the left and right and hand side are nonzero for any point $p \in \operatorname{supp}\left(\iota^*_{\operatorname{int}(\tau)}\left(W_{n+1}\tau\right)\right)$ (which is nonempty by Definition 2.2.11) and hence $\eta_{\tau,\sigma} \neq 0$ as required. □

Proposition 6.2.10. *For all $n = 0, 1, \ldots,$ for all $q = 0, 1, \ldots, N$ and for all $\tau \in K^q_{n+1}$, there exists a $\sigma \in K^q_n$ such that*

$$\eta_{\tau,\sigma} \neq 0\,. \tag{6.2.30}$$

Proof. This result follows immediately from Lemmas 6.2.6 and 6.2.9. □

6.3 Relationship for a Cochain and Growth Rate

This section generalises the work of Section 6.2 to an arbitrary cochain as well as establishes the growth rate of the Whitney map under standard subdivisions.

Consider $c \in C^q(K_{n+1}, \mathbb{C})$ and hence

$$W_{n+1}c = \sum_{\tau \in K^q_{n+1}} c_\tau W_{n+1}\tau\,. \tag{6.3.1}$$

In order to apply Proposition 6.2.4, the coefficients c_τ need to be decomposed into the form

$$c_\tau = \sum_{\sigma \in K^q_n} \zeta_{\tau,\sigma}\eta_{\tau,\sigma}\,, \tag{6.3.2}$$

for a collection of constants $\zeta_{\tau,\sigma}$. This is done as follows. For each $\tau \in K^q_{n+1}$, select a single $\sigma \in K^q_n$ such that $\eta_{\tau,\sigma} \neq 0$ which is always possible by Proposition 6.2.10. Hence set $\zeta_{\tau,\sigma}$ such that

$$c_\tau = \zeta_{\tau,\sigma}\eta_{\tau,\sigma}\,, \tag{6.3.3}$$

and set

$$\zeta_{\tau,\sigma'} = 0\,, \tag{6.3.4}$$

for all $\sigma' \in K^q_n \setminus \{\sigma\}$.

Lemma 6.3.1. *For all $c \in C^q(K_{n+1}, \mathbb{C})$, the constants $\zeta_{\tau,\sigma}$ constructed in (6.3.3) and (6.3.4) satisfy*

$$\max_{\sigma\, \in K^q_n, \tau \in K^q_{n+1}} \left\{\left|\zeta_{\tau,\sigma}\right|\right\} \leq \|c\|_{C,\infty}\,. \tag{6.3.5}$$

Proof. Let $c \in C^q(K_{n+1}, \mathbb{C})$, let $\sigma \in K^q_n$ and let $\tau \in K^q_{n+1}$.

Case 1 The value of $\zeta_{\tau,\sigma}$ is set to 0. It is then trivially the case that

$$\left|\zeta_{\tau,\sigma}\right| \leq \|c\|_{C,\infty}\,. \tag{6.3.6}$$

Case 2 The value of $\zeta_{\tau,\sigma}$ is set to a nonzero value. Specifically, by (6.3.3) it is set to be

$$\zeta_{\tau,\sigma} = \frac{c_\tau}{\eta_{\tau,\sigma}}\,, \tag{6.3.7}$$

which is well defined as $\eta_{\tau,\sigma} \neq 0$. Hence

$$\begin{aligned}\left|\zeta_{\tau,\sigma}\right| &= \left|\frac{c_\tau}{\eta_{\tau,\sigma}}\right| \\ &\leq |c_\tau| \quad \text{(by Lemma 6.2.5)} \\ &\leq \|c\|_{C,\infty}\,.\end{aligned} \tag{6.3.8}$$

□

Lemma 6.3.2. *For all $\rho \in K^N_{n+1}$ and for all $c \in C^q(K_{n+1}, \mathbb{C})$, it holds that*

$$\left(\|W_{n+1}c\|^\rho_1\right)^2 \leq 2^{2q+2}\left(\|c\|_{C,\infty}\right)^2\left(\left\|W_n\mathbb{1}_{K^q_n}\right\|^\rho_1\right)^2\,, \tag{6.3.9}$$

where $\mathbb{1}_{K^q}$ is the unity cochain given by Definition 2.2.4.

Proof. Let $\rho \in K^N_{n+1}$ and let $c \in C^q(K_{n+1}, \mathbb{C})$. Then by (6.3.3) and (6.3.4) gives

$$\begin{aligned} W_{n+1}c &= \sum_{\tau \in K^q_{n+1}} c_\tau W_{n+1}\tau \\ &= \sum_{\sigma \in K^q_n} \sum_{\tau \in K^q_{n+1}} \zeta_{\tau,\sigma}\eta_{\tau,\sigma} W_{n+1}\tau \,. \end{aligned} \tag{6.3.10}$$

Hence

$$\begin{aligned} \left(\|W_{n+1}c\|^\rho_1\right)^2 &= \left(\left\| \sum_{\sigma \in K^q_n} \sum_{\tau \in K^q_{n+1}} \zeta_{\tau,\sigma}\eta_{\tau,\sigma} W_{n+1}\tau \right\|^\rho_1\right)^2 \\ &\leq \max_{\sigma \in K^q_n,\, \tau \in K^q_{n+1}} \left\{\left|\zeta_{\tau,\sigma}\right|^2\right\} \left(\left\| \sum_{\sigma' \in K^q_n} 2^{q+1} W_n \sigma' \right\|^\rho_1\right)^2 \quad \text{(by Proposition 6.2.4)} \\ &= 2^{2q+2} \max_{\sigma \in K^q_n,\, \tau \in K^q_{n+1}} \left\{\left|\zeta_{\tau,\sigma}\right|^2\right\} \left(\left\| W_n \mathbb{1}_{K^q_n} \right\|^\rho_1\right)^2 \\ &\leq 2^{2q+2} \left(\|c\|_{C,\infty}\right)^2 \left(\left\| W_n \mathbb{1}_{K^q_n} \right\|^\rho_1\right)^2 \quad \text{(by Lemma 6.3.1)}\,. \qquad \square \end{aligned} \tag{6.3.11}$$

Corollary 6.3.3. *For all $\rho \in K^N_n$ and for all $q = 0, 1, \ldots, N$, it holds that*

$$\left(\left\| W_n \mathbb{1}_{K^q_n} \right\|^\rho_1\right)^2 \leq 2^{2nq+2n} \left(\left\| W_0 \mathbb{1}_{K^q_0} \right\|^\rho_1\right)^2 . \tag{6.3.12}$$

Proof. As a result of Lemma 5.1.2, the modification of the triangulation at each subdivision step does not change the barycentric coordinates and hence does not alter the Whitney form. Hence the result is easily shown by induction on n applied to Lemma 6.3.2 and using the fact that trivially $\left\|\mathbb{1}_{K^q_n}\right\|_{C,\infty} = 1$. $\square$

Proposition 6.3.4. *For all $\rho \in K^N_n$ and for all $c \in C^q(K_n, \mathbb{C})$, it holds that*

$$\left(\|W_n c\|^\rho_1\right)^2 \leq 2^{2nq+2n} \left(\|c\|_{C,2}\right)^2 \left(\left\| W_0 \mathbb{1}_{K^q_0} \right\|^\rho_1\right)^2 . \tag{6.3.13}$$

Proof. Let $\rho \in K^N_n$ and let $c \in C^q(K_n, \mathbb{C})$. Then by Lemma 6.3.2 and Corollary 6.3.3

$$\begin{aligned} \left(\|W_n c\|^\rho_1\right)^2 &\leq 2^{2nq+2n} \left(\|c\|_{C,\infty}\right)^2 \left(\left\| W_0 \mathbb{1}_{K^q_0} \right\|^\rho_1\right)^2 \\ &\leq 2^{2nq+2n} \left(\|c\|_{C,2}\right)^2 \left(\left\| W_0 \mathbb{1}_{K^q_0} \right\|^\rho_1\right)^2 . \qquad \square \end{aligned} \tag{6.3.14}$$

CHAPTER 7

1901 — 1976 — WERNER KARL HEISENBERG (German)

COMMUTATIVITY ESTIMATES

This chapter examines the interplay of the Whitney map with the twisted exterior derivative and its combinatorial counterpart. In particular estimates for the following commutators are derived, $\|W_n\delta_a c - d_A W_n c\|_0$ and $\left\|W_n(\delta_a)^* c - (d_A)^* W_n c\right\|_0$, as well as an estimate of $\left\|W_n(\delta_a)^* R_n\omega - (d_A)^*\omega\right\|_0$.

7.1 Estimate for Twisted Derivatives

An estimate on the commutativity of the exterior derivative and the Whitney map is developed in this section. Such an estimate is required if Zahariev's method is to be generalised to higher degrees.

In the following results, given a $c \in C^q(K_n, \mathbb{C})$ and $\sigma \in K_n^N$, the norm $\|W_n c\|_1^\sigma$ is well defined even though in general $\|W_n c\|_1$ is not defined.

Lemma 7.1.1. *There exists a positive constant κ independent of n and A such that for all $c \in C^q(K_n, \mathbb{C})$, for all $\sigma \in K_n^N$ and for all $p \in \sigma$, it holds that*

$$|W_n\delta_a c - d_A W_n c|_p \leq \kappa\kappa_{A,0} h_n \|W_n c\|_1^\sigma , \tag{7.1.1}$$

where $\kappa_{A,0}$ is a positive constant dependent on A given by Definition 2.3.12.

Proof. This proof is based on Zahariev's proof of Proposition 4.3c [30]. Let $c \in C^q(K_n, \mathbb{C})$, let $\sigma \in K_n^N$, let $p \in \sigma$, let $\omega = W_n c$, let $r > \frac{N}{2} + 1$ be an integer, let κ_1 be the constant κ as per Proposition 2.3.15 and let κ_2 be the constant κ as per Theorem 2.3.10. Then by Proposition 2.3.2

$$\begin{aligned}
|W_n\delta_a c - d_A W_n c|_p &= |W_n\delta_a R_n\omega - d_A W_n R_n\omega|_p \\
&\leq \kappa_1\kappa_{A,0}h_n\left(\sup_{p'\in\sigma}|\omega|_{p'} + \max_{j=1,2,\ldots,N}\ \sup_{p'\in\sigma}\left|\frac{\partial\omega}{\partial x^j}\right|_{p'}\right) \quad \text{(by Proposition 2.3.15)} \\
&= \kappa_1\kappa_{A,0}h_n\left(\sup_{p\in\sigma}|W_n c|_p + \max_{j=1,2,\ldots,N}\ \sup_{p\in\sigma}\left|\frac{\partial W_n c}{\partial x^j}\right|_p\right) \\
&\leq \kappa_1\kappa_{A,0}h_n\left(\kappa_2\|W_n c\|_r^\sigma + \kappa_2\|W_n c\|_r^\sigma\right) \quad \text{(by Theorem 2.3.10)} \\
&= 2\kappa_1\kappa_2\kappa_{A,0}h_n\|W_n c\|_r^\sigma \\
&= 2\kappa_1\kappa_2\kappa_{A,0}h_n\|W_n c\|_1^\sigma ,
\end{aligned} \tag{7.1.2}$$

where the last step follows from the fact that $W_n c$ is linear on each σ. Then a simple substitution of $\kappa = 2\kappa_1\kappa_2$ yields the desired result. □

Lemma 7.1.2. *There exists a positive constant κ independent of n and A such that for all $c \in C^q(K_n, \mathbb{C})$ it holds that*

$$\left(\|W_n\delta_a c - d_A W_n c\|_0\right)^2 \leq \kappa^2 \left(\kappa_{A,0}\right)^2 (h_n)^{N+2} \left(\|W_n c\|_1^{K_n}\right)^2 . \tag{7.1.3}$$

Proof. This proof is based on Zahariev's proof of Proposition 4.3c [30]. Let $c \in C^q(K_n, \mathbb{C})$ and let κ be the constant κ as per Lemma 7.1.1. Then

$$\begin{aligned}
\left(\|W_n\delta_a c - d_A W_n c\|_0\right)^2 &= \sum_{\sigma\in K_n^N} \int_\sigma \left(|W_n\delta_a c - d_A W_n c|_p\right)^2 \, d\,\mathrm{Vol}_g \\
&\leq \sum_{\sigma\in K_n^N} \int_\sigma \left(\kappa\kappa_{A,0} h_n \|W_n c\|_1^\sigma\right)^2 \, d\,\mathrm{Vol}_g \quad \text{(by Lemma 7.1.1)} \\
&= \kappa^2 \left(\kappa_{A,0}\right)^2 (h_n)^2 \sum_{\sigma\in K_n^N} \mathrm{vol}\,(\sigma) \left(\|W_n c\|_1^\sigma\right)^2 \\
&\leq \kappa^2 \left(\kappa_{A,0}\right)^2 (h_n)^{N+2} \left(\|W_n c\|_1^{K_n}\right)^2 \quad \text{(by Proposition 2.2.33)}. \quad \square
\end{aligned} \tag{7.1.4}$$

Lemma 7.1.3. *For all $r = 0, 1, \ldots$, for all $\omega \in W_n\left(C^q(K_n, \mathbb{C})\right)$, for all $n = 0, 1, \ldots$ and for all $j = 1, 2, \ldots$, it holds that*

$$\|\omega\|_r^{K_n} = \|\omega\|_r^{K_{n+j}} . \tag{7.1.5}$$

Proof. Let r be given, let $\omega \in W_n\left(C^q(K_n, \mathbb{C})\right)$ and let n and j be given. Then by definition

$$\begin{aligned}
\left(\|\omega\|_r^{K_n}\right)^2 &= \sum_{\sigma\in K_n^N} \sum_{j=0}^{r} \int_\sigma \left(\left|(\nabla^{g_\sigma})^j \omega\right|_{g_\sigma}\right)^2 d\,\mathrm{Vol}_g \\
&= \sum_{\sigma\in K_n^N} \sum_{\substack{\rho\in K_{n+j}^N \\ \rho\subseteq\sigma}} \sum_{j=0}^{r} \int_\rho \left(\left|(\nabla^{g_\sigma})^j \omega\right|_{g_\sigma}\right)^2 d\,\mathrm{Vol}_g \\
&= \sum_{\rho\in K_{n+j}^N} \sum_{j=0}^{r} \int_\rho \left(\left|(\nabla^{g_\sigma})^j \omega\right|_{g_\sigma}\right)^2 d\,\mathrm{Vol}_g \\
&= \left(\|\omega\|_r^{K_{n+j}}\right)^2 . \qquad \square
\end{aligned} \tag{7.1.6}$$

Proposition 7.1.4. *There exists a positive constant κ independent of n and A such that for all $c \in C^q(K_n, \mathbb{C})$, it holds that*

$$\|W_n\delta_a c - d_A W_n c\|_0 \leq 2^{nq+n} (\beta_n)^{q+1} \kappa\kappa_{A,0} \|W_n c\|_0 , \tag{7.1.7}$$

where β_n is the mesh reduction parameter of (2.2.27).

Proof. Let $c \in C^q(K_n, \mathbb{C})$, let κ_1 be the constant κ as per Lemma 7.1.2 and let $\frac{1}{\kappa_2}$ be the constant κ_1 as per Lemma 2.3.8. Then Lemma 7.1.2 gives

$$\begin{aligned}
\left(\|W_n\delta_a c - d_A W_n c\|_0\right)^2 &\le (\kappa_1)^2 (\kappa_{A,0})^2 (h_n)^{N+2} \left(\|W_n c\|_1^{K_n}\right)^2 \\
&= (\kappa_1)^2 (\kappa_{A,0})^2 (h_n)^{N+2} \sum_{\rho \in K_n^N} \left(\|W_n c\|_1^\rho\right)^2 \\
&\le 2^{2nq+2n} (\kappa_1)^2 (\kappa_{A,0})^2 (h_n)^{N+2} \left(\|c\|_{C,2}\right)^2 \sum_{\rho \in K_n^N} \left(\left\|W_0 \mathbb{1}_{K_0^q}\right\|_1^\rho\right)^2 \quad \text{(by Proposition 6.3.4)} \\
&= 2^{2nq+2n} (\kappa_1)^2 (\kappa_{A,0})^2 (h_n)^{N+2} \left(\|c\|_{C,2}\right)^2 \left(\left\|W_0 \mathbb{1}_{K_0^q}\right\|_1^{K_n}\right)^2 \\
&= 2^{2nq+2n} (\beta_n)^{N+2} (\kappa_1)^2 (\kappa_{A,0})^2 \left(\|c\|_{C,2}\right)^2 (h_0)^{N+2} \left(\left\|W_0 \mathbb{1}_{K_0^q}\right\|_1^{K_0}\right)^2 \quad \text{(by Lemma 7.1.3)} \\
&\le 2^{2nq+2n} (\beta_n)^{N+2} (\kappa_1)^2 (\kappa_2)^2 (\kappa_{A,0})^2 (h_n)^{2q-N} \left(\|W_n c\|_0\right)^2 (h_0)^{N+2} \left(\left\|W_0 \mathbb{1}_{K_0^q}\right\|_1^{K_0}\right)^2 \\
&\qquad \text{(by Lemma 2.3.8)} \\
&= 2^{2nq+2n} (\beta_n)^{2q+2} \kappa^2 (\kappa_{A,0})^2 \left(\|W_n c\|_0\right)^2, \qquad (7.1.8)
\end{aligned}$$

where $\kappa = \kappa_1 \kappa_2 (h_0)^{q+1} \left\|W_0 \mathbb{1}_{K_0^q}\right\|_1^{K_0}$ is a constant based on the initial triangulation. □

Corollary 7.1.5. *There exists a positive constant κ independent of n and A such that for all $c \in C^q(K_n, \mathbb{C})$, it holds that*

$$\left\|\Sigma_n(c)\right\|_0 \le 2^{nq+n} (\beta_n)^{q+1} \kappa \kappa_{A,0} \|W_n c\|_0, \qquad (7.1.9)$$

where Σ_n is a map on cochains given by Definition 2.4.1.

Proof. Let $c \in C^q(K_n, \mathbb{C})$ and let κ be the constant κ as per Proposition 7.1.4. Then by Proposition 2.3.7

$$\begin{aligned}
\left\|\Sigma_n(c)\right\|_0 &= \left\|W_n \delta c + W_n (ia \cup c) - dW_n c - iA \wedge W_n c\right\|_0 \\
&= \|W_n \delta_a c - d_A W_n c\|_0 \\
&\le 2^{nq+n} (\beta_n)^{q+1} \kappa \kappa_{A,0} \|W_n c\|_0 \quad \text{(by Proposition 7.1.4)}. \qquad \square \qquad (7.1.10)
\end{aligned}$$

Corollary 7.1.6. *There exists a positive constant κ independent of n and A such that for all $c \in C^q(K_n, \mathbb{C})$, it holds that*

$$\begin{aligned}
\left(\frac{\|W_n \delta_a c\|_0}{\|W_n c\|_0}\right)^2 \ge \left(\frac{\|d_A W_n c\|_0}{\|W_n c\|_0}\right)^2 &- 2^{nq+n+1} (\beta_n)^{q+1} \widetilde{\vartheta}_n^1(c) \kappa \kappa_{A,0} \qquad (7.1.11) \\
&+ 2^{2nq+2n} (\beta_n)^{2q+2} (\kappa)^2 (\kappa_{A,0})^2.
\end{aligned}$$

Proof. Let $c \in C^q(K_n, \mathbb{C})$ and let κ be the constant κ as per Proposition 7.1.4. Then by applying the triangle inequality to Proposition 7.1.4 gives

$$\begin{aligned}\left(\frac{\|W_n\delta_a c\|_0}{\|W_n c\|_0}\right)^2 &\geq \left(\frac{\|d_A W_n c\|_0 - 2^{nq+n}(\beta_n)^{q+1}\kappa\kappa_{A,0}\|W_n c\|_0}{\|W_n c\|_0}\right)^2 \\ &= \left(\frac{\|d_A W_n c\|_0}{\|W_n c\|_0}\right)^2 - \frac{2^{nq+n+1}(\beta_n)^{q+1}\kappa\kappa_{A,0}\|d_A W_n c\|_0}{\|W_n c\|_0} \\ &\quad + 2^{2nq+2n}(\beta_n)^{2q+2}(\kappa)^2(\kappa_{A,0})^2 \\ &= \left(\frac{\|d_A W_n c\|_0}{\|W_n c\|_0}\right)^2 - 2^{nq+n+1}(\beta_n)^{q+1}\widetilde{\vartheta}_n^1(c)\,\kappa\kappa_{A,0} \\ &\quad + 2^{2nq+2n}(\beta_n)^{2q+2}(\kappa)^2(\kappa_{A,0})^2 . \qquad \square\end{aligned} \tag{7.1.12}$$

7.2 Expression for the Combinatorial Adjoint

This section develops an expression for the adjoint of the combinatorial twisted derivative.

Lemma 7.2.1. *For all $c_1 \in C^{q+1}(K, \mathbb{C})$ and for all $c_2 \in C^q(K, \mathbb{C})$, it holds that*

$$\int_M (d\star W c_1) \wedge W c_2 = (-1)^{N-q} \int_M (\star W c_1) \wedge dW c_2 . \tag{7.2.1}$$

Proof. Let $c_1 \in C^{q+1}(K, \mathbb{C})$ and let $c_2 \in C^q(K, \mathbb{C})$. Then

$$\begin{aligned}\int_M (d\star W c_1) \wedge W c_2 + (-1)^{N-q-1} \int_M (\star W c_1) \wedge dW c_2 &= \int_M d\left((\star W c_1) \wedge W c_2\right) \\ &= \int_{\partial M} (\star W c_1) \wedge W c_2 \\ &\qquad \text{(by Stokes' Theorem)} \\ &= 0 . \qquad \square\end{aligned} \tag{7.2.2}$$

Lemma 7.2.2. *For all $c_1 \in C^{q+1}(K, \mathbb{C})$ and for all $c_2 \in C^q(K, \mathbb{C})$, it holds that*

$$\int_M W c_1 \wedge \star W \delta c_2 = (-1)^{qN+1} \int_M (\star d\star W c_1) \wedge \star W c_2 . \tag{7.2.3}$$

Proof. Let $c_1 \in C^{q+1}(K, \mathbb{C})$ and let $c_2 \in C^q(K, \mathbb{C})$ then by Proposition 2.3.7

$$\begin{aligned}\int_M W c_1 \wedge \star W \delta c_2 &= \int_M W c_1 \wedge \star dW c_2 \\ &= (-1)^{(q+1)(N-q-1)} \int_M (\star W c_1) \wedge dW c_2 \quad \text{(by Lemma 2.1.3)} \\ &= (-1)^{qN+1} \int_M (d\star W c_1) \wedge W c_2 \quad \text{(by Lemma 7.2.1)} \\ &= (-1)^{qN+1} \int_M (\star d\star W c_1) \wedge \star W c_2 \\ &\qquad \text{(by Lemma 2.1.3 and Theorem 2.1.2)} . \qquad \square\end{aligned} \tag{7.2.4}$$

Lemma 7.2.3. *For all $c_1 \in C^{q+1}(K, \mathbb{C})$ and for all $c_2 \in C^q(K, \mathbb{C})$, it holds that*

$$\int_M Wc_1 \wedge \star W(ia \cup c_2) = (-1)^{qN} \int_M \big(\star(iA \wedge \star Wc_1)\big) \wedge \star Wc_2 + \int_M Wc_1 \wedge \star\Sigma(c_2) \,. \tag{7.2.5}$$

Proof. Let $c_1 \in C^{q+1}(K, \mathbb{C})$ and let $c_2 \in C^q(K, \mathbb{C})$ then by Definition 2.4.1

$$\begin{aligned}
\int_M Wc_1 \wedge \star W(ia \cup c_2) &= \int_M Wc_1 \wedge \star(iA \wedge Wc_2) + \int_M Wc_1 \wedge \star\Sigma(c_2) \\
&= (-1)^{(q+1)(N-q-1)} \int_M (\star Wc_1) \wedge (iA \wedge Wc_2) + \int_M Wc_1 \wedge \star\Sigma(c_2) \\
&\qquad \text{(by Lemma 2.1.3)} \\
&= (-1)^{qN} \int_M (iA \wedge \star Wc_1) \wedge Wc_2 + \int_M Wc_1 \wedge \star\Sigma(c_2) \\
&= (-1)^{q} \int_M (iA \wedge \star Wc_1) \wedge \star(\star Wc_2) + \int_M Wc_1 \wedge \star\Sigma(c_2) \\
&\qquad \text{(by Theorem 2.1.2)} \\
&= (-1)^{q} \int_M (\star Wc_2) \wedge \star(iA \wedge \star Wc_1) + \int_M Wc_1 \wedge \star\Sigma(c_2) \\
&\qquad \text{(by Lemma 2.1.3)} \\
&= (-1)^{qN} \int_M \big(\star(iA \wedge \star Wc_1)\big) \wedge \star Wc_2 + \int_M Wc_1 \wedge \star\Sigma(c_2) \,. \quad \square
\end{aligned} \tag{7.2.6}$$

Lemma 7.2.4. *For all $c_1 \in C^{q+1}(K, \mathbb{C})$ and for all $c_2 \in C^q(K, \mathbb{C})$, it holds that*

$$\langle c_1, \delta_a c_2 \rangle_W = \big\langle (d_A)^* Wc_1, Wc_2 \big\rangle_0 + \int_M W\bar{c}_1 \wedge \star\Sigma(c_2) \,. \tag{7.2.7}$$

Proof. Let $c_1 \in C^{q+1}(K, \mathbb{C})$ and let $c_2 \in C^q(K, \mathbb{C})$ then

$$\begin{aligned}
\langle c_1, \delta_a c_2 \rangle_W &= \int_M W\bar{c}_1 \wedge \star W\delta_a c_2 \\
&= \int_M W\bar{c}_1 \wedge \star W\delta c_2 + \int_M W\bar{c}_1 \wedge \star W(ia \cup c_2) \\
&= (-1)^{qN+1} \int_M (\star d \star W\bar{c}_1) \wedge \star Wc_2 + (-1)^{qN} \int_M \big(\star(iA \wedge \star W\bar{c}_1)\big) \wedge \star Wc_2 \\
&\quad + \int_M W\bar{c}_1 \wedge \star\Sigma(c_2) \quad \text{(by Lemmas 7.2.2 and 7.2.3)} \\
&= (-1)^{qN+1} \big\langle \star(d + iA\,\wedge) \star Wc_1, Wc_2 \big\rangle_0 + \int_M W\bar{c}_1 \wedge \star\Sigma(c_2) \\
&= \big\langle (d_A)^* Wc_1, Wc_2 \big\rangle_0 + \int_M W\bar{c}_1 \wedge \star\Sigma(c_2) \quad \text{(by Theorem 2.1.10)}. \quad \square
\end{aligned} \tag{7.2.8}$$

Lemma 7.2.5. *For all $\omega_1 \in \Omega^{q+1}(M, \mathbb{C})$ and for all $\omega_2 \in \Omega^q(M, \mathbb{C})$, it holds that*

$$\begin{aligned}
\big\langle (d_A)^* WR\omega_1, WR\omega_2 \big\rangle_0 = \big\langle R(d_A)^*\omega_1, R\omega_2 \big\rangle_W - \Big\langle \Theta\left((d_A)^*\omega_1\right), WR\omega_2 \Big\rangle_0 \\
+ \big\langle (d_A)^* \Theta(\omega_1), WR\omega_2 \big\rangle_0 \,,
\end{aligned} \tag{7.2.9}$$

where Θ is a map on differential forms given by Definition 2.4.2.

Proof. Let $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and let $\omega_2 \in \Omega^q(M,\mathbb{C})$ then

$$\begin{aligned}\left\langle (d_A)^* W R\omega_1, W R\omega_2\right\rangle_0 &= \left\langle (d_A)^*\omega_1, W R\omega_2\right\rangle_0 + \left\langle (d_A)^*\Theta(\omega_1), W R\omega_2\right\rangle_0 \\ &= \left\langle R(d_A)^*\omega_1, R\omega_2\right\rangle_W - \left\langle \Theta\left((d_A)^*\omega_1\right), W R\omega_2\right\rangle_0 \\ &\quad + \left\langle (d_A)^*\Theta(\omega_1), W R\omega_2\right\rangle_0 . \end{aligned} \tag{7.2.10}$$

□

Lemma 7.2.6. *For all $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and for all $\omega_2 \in \Omega^q(M,\mathbb{C})$, it holds that*

$$\begin{aligned}\left\langle (\delta_a)^* R\omega_1, R\omega_2\right\rangle_W &= \left\langle R(d_A)^*\omega_1, R\omega_2\right\rangle_W - \left\langle \Theta\left((d_A)^*\omega_1\right), W R\omega_2\right\rangle_0 \\ &\quad + \left\langle (d_A)^*\Theta(\omega_1), W R\omega_2\right\rangle_0 + \int_M W R\overline{\omega}_1 \wedge \star\Sigma(R\omega_2) . \end{aligned} \tag{7.2.11}$$

Proof. Let $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and let $\omega_2 \in \Omega^q(M,\mathbb{C})$ then by Lemma 7.2.4

$$\begin{aligned}\left\langle R\omega_1, \delta_a R\omega_2\right\rangle_W &= \left\langle (d_A)^* W R\omega_1, W R\omega_2\right\rangle_0 + \int_M W R\overline{\omega}_1 \wedge \star\Sigma(R\omega_2) \\ &= \left\langle R(d_A)^*\omega_1, R\omega_2\right\rangle_W - \left\langle \Theta\left((d_A)^*\omega_1\right), W R\omega_2\right\rangle_0 \\ &\quad + \left\langle (d_A)^*\Theta(\omega_1), W R\omega_2\right\rangle_0 + \int_M W R\overline{\omega}_1 \wedge \star\Sigma(R\omega_2) \\ &\qquad \text{(by Lemma 7.2.5)} . \end{aligned} \tag{7.2.12}$$

□

7.3 Bounds for the Combinatorial Adjoint

In this section, bounds for the error terms in the expression of the combinatorial adjoint are developed.

Lemma 7.3.1. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2}+1$, for all $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and for all $\omega_2 \in \Omega^q(M,\mathbb{C})$, if $\beta_n h_0 \leq 1$ then*

$$\left|\int_M W_n R_n\overline{\omega}_1 \wedge \star\Sigma_n(R_n\omega_2)\right| \leq 2^{nq+n}(\beta_n)^{q+1}\kappa\kappa_{A,0}\|\omega_1\|_r \|W_n R_n\omega_2\|_0 . \tag{7.3.1}$$

Proof. Let $r > \frac{N}{2}+1$ be an integer, let $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$, let $\omega_2 \in \Omega^q(M,\mathbb{C})$, let κ_1 be the constant κ as per Corollary 7.1.5 and let κ_2 be the constant κ as per Corollary 2.3.11. Then

$$\begin{aligned}\left|\int_M W_n R_n\overline{\omega}_1 \wedge \star\Sigma_n(R_n\omega_2)\right| &= \left|\left\langle W_n R_n\omega_1, \Sigma_n(R_n\omega_2)\right\rangle_0\right| \\ &\leq \left|\left\langle \omega_1, \Sigma_n(R_n\omega_2)\right\rangle_0\right| + \left|\left\langle \Theta_n(\omega_1), \Sigma_n(R_n\omega_2)\right\rangle_0\right| \\ &\leq \|\omega_1\|_0 \left\|\Sigma_n(R_n\omega_2)\right\|_0 + \left\|\Theta_n(\omega_1)\right\|_0 \left\|\Sigma_n(R_n\omega_2)\right\|_0 \\ &\leq 2^{nq+n}(\beta_n)^{q+1}\kappa_1\kappa_{A,0}\|\omega_1\|_0 \|W_n R_n\omega_2\|_0 \\ &\quad + 2^{nq+n}(\beta_n)^{q+1}\kappa_1\kappa_2\kappa_{A,0}\|\omega_1\|_r \|W_n R_n\omega_2\|_0 \\ &\qquad \text{(by Corollary 2.3.11 and Corollary 7.1.5)} \\ &\leq 2^{nq+n}(\beta_n)^{q+1}\kappa_1\kappa_{A,0}\|\omega_1\|_r \|W_n R_n\omega_2\|_0 \\ &\quad + 2^{nq+n}(\beta_n)^{q+1}\kappa_1\kappa_2\kappa_{A,0}\|\omega_1\|_r \|W_n R_n\omega_1\|_0 \\ &\qquad \text{(by Lemma 2.1.8)} \\ &\leq 2^{nq+n}(\beta_n)^{q+1}\kappa\kappa_{A,0}\|\omega_1\|_r \|W_n R_n\omega_2\|_0 , \end{aligned} \tag{7.3.2}$$

where $\kappa = 2\max\{\kappa_1, \kappa_1\kappa_2\}$. □

Lemma 7.3.2. *For all $\omega \in \Omega^q(M,\mathbb{C})$ it holds that the adjoint of the Hodge star operator is given by*

$$\star^*\omega = (-1)^{q(N-q)} \star\omega\,. \tag{7.3.3}$$

Proof. Let $\omega_1 \in \Omega^q(M,\mathbb{C})$, $\omega_2 \in \Omega^{N-q}(M,\mathbb{C})$ and $u = (-1)^{q(N-q)}\star$ and hence

$$\begin{aligned}\langle u\omega_1,\omega_2\rangle_0 - \langle\omega_1,\star\omega_2\rangle_0 &= (-1)^{q(N-q)}\int_M \star\overline{\omega}_1 \wedge \star\omega_2 - \int_M \overline{\omega}_1 \wedge \star\star\omega_2 \\ &= (-1)^{q(N-q)}\int_M \star\overline{\omega}_1 \wedge \star\omega_2 - \int_M \star\omega_2 \wedge \star\overline{\omega}_1 \quad \text{(by Lemma 2.1.3)} \\ &= 0\,. \qquad\square\end{aligned} \tag{7.3.4}$$

Lemma 7.3.3. *It holds that*

$$\|\star\|_{\mathrm{op}} = 1\,. \tag{7.3.5}$$

Proof.

$$\begin{aligned}\left(\|\star\|_{\mathrm{op}}\right)^2 &= \|\star^*\star\|_{\mathrm{op}} \\ &= \|\star\star\|_{\mathrm{op}} \quad \text{(by Lemma 7.3.2)} \\ &= \|\mathrm{Id}\|_{\mathrm{op}} \quad \text{(by Theorem 2.1.2)} \\ &= 1\,. \qquad\square\end{aligned} \tag{7.3.6}$$

The following assumption is a critical ingredient in proving Theorem 4.4.1 as previously discussed in Section 4.5.

Assumption 7.3.4. *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M,\mathbb{C})$ it holds that*

$$\left\|d\star(W_nR_n\omega - \omega)\right\|_0 \leq \kappa h_n \|\omega\|_r\,. \tag{7.3.7}$$

The following lemma may be of use in proving Assumption 7.3.4.

Lemma 7.3.5. *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M,\mathbb{C})$ it holds that*

$$\|\star W_nR_n\omega - W_nR_n\star\omega\|_0 \leq \kappa h_n\left(\|\omega\|_r + \|\star\omega\|_r\right)\,. \tag{7.3.8}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M,\mathbb{C})$, let $r > \frac{N}{2} + 1$ be an integer and let κ be the constant κ as per Corollary 2.3.11. Then by Lemma 7.3.3

$$\begin{aligned}\|\star W_nR_n\omega - W_nR_n\star\omega\|_0 &\leq \|W_nR_n\omega - \omega\|_0 + \|W_nR_n\star\omega - \star\omega\|_0 \\ &\leq \kappa h_n\left(\|\omega\|_r + \|\star\omega\|_r\right) \quad \text{(by Corollary 2.3.11)}\,. \qquad\square\end{aligned} \tag{7.3.9}$$

Lemma 7.3.6. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M,\mathbb{C})$, if Assumption 7.3.4 holds then*

$$\left\|(d_A)^*\Theta_n(\omega)\right\|_0 \leq \kappa\kappa_{A,0}h_n\|\omega\|_r\,. \tag{7.3.10}$$

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $\omega \in \Omega^q(M,\mathbb{C})$, let κ_1 be the constant κ as per Corollary 2.3.11 and let κ_2 be the constant κ as per Assumption 7.3.4. Then by Theorem 2.1.10 and Lemma 7.3.3

$$\begin{aligned}\left\|(d_A)^*\Theta_n(\omega)\right\|_0 &\le \left\|(d + iA\wedge)\star\Theta_n(\omega)\right\|_0 \\ &\le \left\|d\star\Theta_n(\omega)\right\|_0 + \kappa_{A,0}\left\|\Theta_n(\omega)\right\|_0 \\ &\le \left\|d\star(W_nR_n\omega - \omega)\right\|_0 + \kappa_1\kappa_{A,0}h_n\|\omega\|_r \quad \text{(by Corollary 2.3.11)} \\ &\le \kappa\kappa_{A,0}h_n\|\omega\|_r \quad \text{(by Assumption 7.3.4)}, \end{aligned} \tag{7.3.11}$$

where $\kappa = 2\max\{\kappa_1,\kappa_2\}$. □

Lemma 7.3.7. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$, for all $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and for all $\omega_2 \in \Omega^q(M,\mathbb{C})$, if Assumption 7.3.4 holds and $h_n \le 1$ then*

$$\left|\left\langle(d_A)^*\Theta_n(\omega_1), W_nR_n\omega_2\right\rangle_0\right| \le \kappa\kappa_{A,0}h_n\|\omega_1\|_r\|\omega_2\|_r. \tag{7.3.12}$$

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$, let $\omega_2 \in \Omega^q(M,\mathbb{C})$, let κ_1 be the constant κ as per Lemma 7.3.6 and let κ_2 be the constant κ as per Corollary 2.3.11. Then

$$\begin{aligned}\left|\left\langle(d_A)^*\Theta_n(\omega_1), W_nR_n\omega_2\right\rangle_0\right| &\le \left\|(d_A)^*\Theta_n(\omega_1)\right\|_0\left\|\omega_2 + \Theta_n(\omega_2)\right\|_0 \\ &\le \kappa_1\kappa_{A,0}h_n\|\omega_1\|_r\|\omega_2\|_0 + \kappa_1\kappa_{A,0}h_n\|\omega_1\|_r\left\|\Theta_n(\omega_2)\right\|_0 \\ &\qquad \text{(by Lemma 7.3.6)} \\ &\le \kappa_1\kappa_{A,0}h_n\|\omega_1\|_r\|\omega_2\|_0 + \kappa_1\kappa_2\kappa_{A,0}h_n\|\omega_1\|_r\|\omega_2\|_r \\ &\qquad \text{(by Corollary 2.3.11)} \\ &\le \kappa\kappa_{A,0}h_n\|\omega_1\|_r\|\omega_2\|_r \quad \text{(by Lemma 2.1.8)}, \end{aligned} \tag{7.3.13}$$

where $\kappa = 2\max\{\kappa_1,\kappa_1\kappa_2\}$ □

Lemma 7.3.8. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$, for all $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$ and for all $\omega_2 \in \Omega^q(M,\mathbb{C})$, if $h_n \le 1$ then*

$$\left|\left\langle\Theta_n\left((d_A)^*\omega_1\right), W_nR_n\omega_2\right\rangle_0\right| \le \kappa h_n\left\|(d_A)^*\omega_1\right\|_r\|\omega_2\|_r. \tag{7.3.14}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega_1 \in \Omega^{q+1}(M,\mathbb{C})$, let $\omega_2 \in \Omega^q(M,\mathbb{C})$ and let $\widehat{\kappa}$ be the constant κ as per Corollary 2.3.11. Then

$$\begin{aligned}\left|\left\langle\Theta_n\left((d_A)^*\omega_1\right), W_nR_n\omega_2\right\rangle_0\right| &\le \left\|\Theta_n\left((d_A)^*\omega_1\right)\right\|_0\|\omega_2\|_0 + \left\|\Theta_n\left((d_A)^*\omega_1\right)\right\|_0\left\|\Theta_n(\omega_2)\right\|_0 \\ &\le \widehat{\kappa}h_n\left\|(d_A)^*\omega_1\right\|_r\|\omega_2\|_0 + (\widehat{\kappa})^2 h_n\left\|(d_A)^*\omega_1\right\|_r\|\omega_2\|_r \\ &\qquad \text{(by Corollary 2.3.11)} \\ &\le \kappa h_n\left\|(d_A)^*\omega_1\right\|_r\|\omega_2\|_r \quad \text{(by Lemma 2.1.8)}, \end{aligned} \tag{7.3.15}$$

where $\kappa = 2\max\left\{\widehat{\kappa},(\widehat{\kappa})^2\right\}$. □

7.4 Estimates for the Adjoints of Twisted Derivatives

Two estimates, Propositions 7.4.8 and 7.4.10, on the commutativity of the adjoint of the exterior derivative and the Whitney map are developed in this section.

Lemma 7.4.1. *There exists a positive constant κ independent of A such that for all integers $r \geq 0$, for all $\omega \in \Omega^q(M, \mathbb{C})$ it holds that*

$$\left\|(d_A)^* \omega\right\|_r \leq \kappa \kappa_{A,r} \|\omega\|_{r+1} . \tag{7.4.1}$$

Proof. As M is compact and $(d_A)^*$ is a first order differential operator, it can be shown that the operator defines a bounded map from $\mathcal{H}^q_r(M, \mathbb{C})$ to $\mathcal{H}^q_{r+1}(M, \mathbb{C})$. $\square$

Lemma 7.4.2. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\left(\left\|(\delta_a)^* R_n \omega - R_n (d_A)^* \omega\right\|_W\right)^2 \leq \kappa \left(\kappa_{A,r}\right)^2 \beta_n \|\omega\|_r \|P_n \omega\|_r + 2^{nq} \left(\beta_n\right)^q \kappa \kappa_{A,r} \|\omega\|_r \|W_n R_n P_n \omega\|_0 , \tag{7.4.2}$$

where $P^q_n : \Omega^{q+1}(M, \mathbb{C}) \to \Omega^q(M, \mathbb{C})$ is given by Definition 2.3.6.

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C})$, let $r' = r - 1$, let κ_1 be the constant κ as per Lemma 7.3.8, let κ_2 be the constant κ as per Lemma 7.3.7 and let κ_3 be the constant κ as per Lemma 7.3.1. Then as $R_n (R_n)^{-1} = \mathrm{Id}$, it follows that

$$\begin{aligned}
\left(\left\|(\delta_a)^* R_n \omega - R_n (d_A)^* \omega\right\|_W\right)^2 &= \left\langle (\delta_a)^* R_n \omega, R_n P_n \omega \right\rangle_W - \left\langle R_n (d_A)^* \omega, R_n P_n \omega \right\rangle_W \\
&= \left\langle (d_A)^* \Theta_n(\omega), W_n R_n P_n \omega \right\rangle_0 - \left\langle \Theta_n\left((d_A)^* \omega\right), W_n R_n P_n \omega \right\rangle_0 \\
&\quad + \int_M W_n R_n \overline{\omega} \wedge \star \Sigma_n\left(R_n P_n \omega\right) \quad \text{(by Lemma 7.2.6)} \\
&\leq \kappa_1 h_n \left\|(d_A)^* \omega\right\|_{r'} \|P_n \omega\|_{r'} + \kappa_2 \kappa_{A,0} h_n \|\omega\|_r \|P_n \omega\|_r \\
&\quad + 2^{nq} \left(\beta_n\right)^q \kappa_3 \kappa_{A,0} \|\omega\|_{r'} \|W_n R_n P_n \omega\|_0 \\
&\qquad\qquad \text{(by Lemmas 7.3.1, 7.3.7 and 7.3.8)} \\
&\leq \kappa_1 \kappa_3 \kappa_{A,r} h_n \|\omega\|_r \|P_n \omega\|_{r'} + \kappa_2 \kappa_{A,0} h_n \|\omega\|_r \|P_n \omega\|_r \\
&\quad + 2^{nq} \left(\beta_n\right)^q \kappa_3 \kappa_{A,0} \|\omega\|_{r'} \|W_n R_n P_n \omega\|_0 \quad \text{(by Lemma 7.4.1)} \\
&\leq \kappa \left(\kappa_{A,r}\right)^2 \beta_n \|\omega\|_r \|P_n \omega\|_r + 2^{nq} \left(\beta_n\right)^q \kappa \kappa_{A,r} \|\omega\|_r \|W_n R_n P_n \omega\|_0 \\
&\qquad\qquad \text{(by Lemmas 2.1.8 and 2.3.13)} ,
\end{aligned} \tag{7.4.3}$$

where $\kappa = \max\left\{\kappa_1 \kappa_3 h_0 + \kappa_2 h_0, \kappa_3\right\}$. $\square$

Lemma 7.4.3. *For all $n = 0, 1, \ldots,$ and for all $q = 0, 1, \ldots, N$ it holds that*

$$\beta_n \leq 2^{nq} \left(\beta_n\right)^q . \tag{7.4.4}$$

Proof. Let n and q be given. Then by Corollary 2.2.26

$$\begin{aligned}
2^{nq} \left(\beta_n\right)^q &\geq 1 \\
&\geq \beta_n .
\end{aligned} \quad \square \tag{7.4.5}$$

Corollary 7.4.4. *For all $n = 0, 1, \ldots,$ and for all $q = 0, 1, \ldots, N$ it holds that*

$$\beta_n \leq 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} . \tag{7.4.6}$$

Proof. This result follows immediately as $\beta_n \leq 1$ and hence $\beta_n \leq (\beta_n)^{\frac{1}{2}}$. □

Lemma 7.4.5. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if $\beta_n h_0 \leq 1$ then*

$$\left(\kappa_{A,0}\right)^2 \beta_n \|\omega\|_r \|P_n\omega\|_r + 2^{nq} (\beta_n)^q \kappa_{A,0} \|\omega\|_r \|W_n R_n P_n \omega\|_0 \leq 2^{nq} (\beta_n)^q \kappa \left(\kappa_{A,0}\right)^2 \|\omega\|_r \|P_n\omega\|_r . \tag{7.4.7}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C})$ and let $\widehat{\kappa}$ be the constant κ as per Corollary 2.3.11. Then by Lemma 7.4.3 and as $\kappa_{A,0} \geq 1$

$$\begin{aligned}
&\left(\kappa_{A,0}\right)^2 \beta_n \|\omega\|_r \|P_n\omega\|_r + 2^{nq} (\beta_n)^q \kappa_{A,0} \|\omega\|_r \|W_n R_n P_n \omega\|_0 \\
&\quad \leq 2^{nq} (\beta_n)^q \left(\kappa_{A,0}\right)^2 \|\omega\|_r \left(\|P_n\omega\|_r + \|R_n P_n \omega\|_W\right) \\
&\quad \leq 2^{nq} (\beta_n)^q \left(\kappa_{A,0}\right)^2 \|\omega\|_r \left(\|P_n\omega\|_r + \|P_n\omega\|_0 + \left\|\Theta_n (P_n\omega)\right\|_0\right) \\
&\quad \leq 2^{nq} (\beta_n)^q \left(\kappa_{A,0}\right)^2 \|\omega\|_r \left(\|P_n\omega\|_r + \|P_n\omega\|_0 + \widehat{\kappa} h_n \|P_n\omega\|_r\right) \quad \text{(by Corollary 2.3.11)} \\
&\quad \leq 2^{nq} (\beta_n)^q \kappa \left(\kappa_{A,0}\right)^2 \|\omega\|_r \|P_n\omega\|_r \quad \text{(by Lemma 2.1.8)} ,
\end{aligned} \tag{7.4.8}$$

where $\kappa = 2 + \widehat{\kappa}$. □

Corollary 7.4.6. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if $\beta_n h_0 \leq 1$ then*

$$\begin{aligned}
&\kappa_{A,0} \left(\beta_n \|\omega\|_r \|P_n\omega\|_r + 2^{nq} (\beta_n)^q \|\omega\|_r \|W_n R_n P_n \omega\|_0\right)^{\frac{1}{2}} + \beta_n \left\|(d_A)^*\omega\right\|_0 \\
&\quad \leq 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \kappa_{A,0} \left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}}\right) .
\end{aligned} \tag{7.4.9}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C})$ and let $\widehat{\kappa}$ be the constant κ as per Lemma 7.4.5. Then by Lemma 7.4.5

$$\begin{aligned}
&\kappa_{A,0} \left(\beta_n \|\omega\|_r \|P_n\omega\|_r + 2^{nq} (\beta_n)^q \|\omega\|_r \|W_n R_n P_n \omega\|_0\right)^{\frac{1}{2}} + \beta_n \left\|(d_A)^*\omega\right\|_0 \\
&\quad \leq 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \sqrt{\widehat{\kappa}} \kappa_{A,0} \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}} + \beta_n \left\|(d_A)^*\omega\right\|_0 \\
&\quad \leq 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \kappa_{A,0} \left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}}\right) \quad \text{(by Corollary 7.4.4)} ,
\end{aligned} \tag{7.4.10}$$

where $\kappa = \max\left\{1, \sqrt{\widehat{\kappa}}\right\}$. □

Corollary 7.4.7. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if $\beta_n h_0 \leq 1$ then*

$$\begin{aligned}
&\kappa_{A,0} \left(\beta_n \|\omega\|_r \|P_n\omega\|_r + 2^{nq} (\beta_n)^q \|\omega\|_r \|W_n R_n P_n \omega\|_0\right)^{\frac{1}{2}} + \left(\kappa_{A,0}\right)^2 \beta_n \|\omega\|_r \\
&\quad \leq 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \left(\kappa_{A,0}\right)^2 \left(\|\omega\|_r + \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}}\right) .
\end{aligned} \tag{7.4.11}$$

Proof. This follows from Lemma 7.4.5 and Corollary 7.4.4 in the same way as Corollary 7.4.6. □

Proposition 7.4.8. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\begin{aligned}\left\|W_n(\delta_a)^* R_n\omega - (d_A)^*\omega\right\|_0 \leq \kappa\kappa_{A,r}\left(\beta_n\|\omega\|_r\|P_n\omega\|_r + 2^{nq}(\beta_n)^q\|\omega\|_r\|W_nR_nP_n\omega\|_0\right)^{\frac{1}{2}} \\ + \kappa\beta_n\left\|(d_A)^*\omega\right\|_0.\end{aligned} \tag{7.4.12}$$

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C})$, let κ_1 be the constant κ as per Corollary 2.3.11 and let κ_2 be the constant κ as per Lemma 7.4.2. Then

$$\begin{aligned}\left\|(\delta_a)^*R_n\omega - R_n(d_A)^*\omega\right\|_W &= \left\|W_n(\delta_a)^*R_n\omega - (d_A)^*\omega - \Theta_n\left((d_A)^*\omega\right)\right\|_0 \\ &\geq \left\|W_n(\delta_a)^*R_n\omega - (d_A)^*\omega\right\|_0 - \left\|\Theta_n\left((d_A)^*\omega\right)\right\|_0.\end{aligned} \tag{7.4.13}$$

Hence by Corollary 2.3.11

$$\begin{aligned}\left\|W_n(\delta_a)^*R_n\omega - (d_A)^*\omega\right\|_0 &\leq \left\|(\delta_a)^*R_n\omega - R_n(d_A)^*\omega\right\|_W + \kappa_1 h_n\left\|(d_A)^*\omega\right\|_0 \\ &\leq \sqrt{\kappa_2}\kappa_{A,r}\left(\beta_n\|\omega\|_r\|P_n\omega\|_r + 2^{nq}(\beta_n)^q\|\omega\|_r\|W_nR_nP_n\omega\|_0\right)^{\frac{1}{2}} \\ &\quad + \kappa_1\beta_n h_0\left\|(d_A)^*\omega\right\|_0 \quad \text{(by Lemma 7.4.2)} \\ &\leq \kappa\kappa_{A,r}\left(\beta_n\|\omega\|_r\|P_n\omega\|_r + 2^{nq}(\beta_n)^q\|\omega\|_r\|W_nR_nP_n\omega\|_0\right)^{\frac{1}{2}} \\ &\quad + \kappa\beta_n\left\|(d_A)^*\omega\right\|_0,\end{aligned} \tag{7.4.14}$$

where $\kappa = \max\left\{\kappa_1 h_0, \sqrt{\kappa_2}\right\}$. □

Corollary 7.4.9. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\left\|W_n(\delta_a)^*R_n\omega - (d_A)^*\omega\right\|_0 \leq 2^{\frac{nq}{2}}(\beta_n)^{\frac{q}{2}}\kappa\kappa_{A,r}\left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r\|P_n\omega\|_r\right)^{\frac{1}{2}}\right). \tag{7.4.15}$$

Proof. This result follows immediately from Proposition 7.4.8 by Corollary 7.4.6. □

Proposition 7.4.10. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $c \in C^q(K_n, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\begin{aligned}&\left\|W_n(\delta_a)^*c - (d_A)^*W_nc\right\|_0 \\ &\leq \kappa\kappa_{A,r}\left(\beta_n\left\|(R_n)^{-1}c\right\|_r\left\|P_n(R_n)^{-1}c\right\|_r + 2^{nq}(\beta_n)^q\left\|(R_n)^{-1}c\right\|_r\left\|W_nR_nP_n(R_n)^{-1}c\right\|_0\right)^{\frac{1}{2}} \\ &\quad + \kappa\left(\kappa_{A,r}\right)^2\beta_n\left\|(R_n)^{-1}c\right\|_r.\end{aligned} \tag{7.4.16}$$

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $c \in C^q(K_n, \mathbb{C})$, let κ_1 be the constant κ as per Proposition 7.4.8, let κ_2 be the constant κ as per Lemma 7.3.6 and let κ_3 be the constant κ as per Lemma 7.4.1. As $R_n(R_n)^{-1}c = c$, it follows that $(R_n)^{-1}c = W_n c - \Theta_n\left((R_n)^{-1}c\right)$. Hence by Proposition 7.4.8, with $\omega = (R_n)^{-1}c$

$$\begin{aligned}
&\left\|W_n(\delta_a)^*c - (d_A)^*W_n c\right\|_0 - \left\|(d_A)^*\Theta_n\left((R_n)^{-1}c\right)\right\|_0 \\
&\leq \kappa_1\Big(\left(\kappa_{A,r}\right)^2\beta_n\left\|(R_n)^{-1}c\right\|_r\left\|P_n(R_n)^{-1}c\right\|_r + 2^{nq}\left(\beta_n\right)^q\kappa_{A,r}\left\|(R_n)^{-1}c\right\|_r\left\|W_nR_nP_n(R_n)^{-1}c\right\|_0\Big)^{\frac{1}{2}} \\
&\quad + \kappa_1\beta_n\left\|(d_A)^*(R_n)^{-1}c\right\|_0 .
\end{aligned} \tag{7.4.17}$$

Therefore by Lemma 7.3.6 it follows that

$$\begin{aligned}
&\left\|W_n(\delta_a)^*c - (d_A)^*W_n c\right\|_0 \\
&\quad\leq \kappa_1\kappa_{A,r}\left(\beta_n\left\|(R_n)^{-1}c\right\|_r\left\|P_n(R_n)^{-1}c\right\|_r + 2^{nq}\left(\beta_n\right)^q\left\|(R_n)^{-1}c\right\|_r\left\|W_nR_nP_n(R_n)^{-1}c\right\|_0\right)^{\frac{1}{2}} \\
&\qquad + \kappa_1\beta_n\left\|(d_A)^*(R_n)^{-1}c\right\|_0 + \kappa_2\kappa_{A,r}h_n\left\|(R_n)^{-1}c\right\|_r \\
&\quad\leq \kappa\kappa_{A,r}\left(\beta_n\left\|(R_n)^{-1}c\right\|_r\left\|P_n(R_n)^{-1}c\right\|_r + 2^{nq}\left(\beta_n\right)^q\left\|(R_n)^{-1}c\right\|_r\left\|W_nR_nP_n(R_n)^{-1}c\right\|_0\right)^{\frac{1}{2}} \\
&\qquad + \kappa\kappa_{A,r}\beta_n\left\|(R_n)^{-1}c\right\|_r \quad \text{(by Lemmas 2.1.8 and 7.4.1)},
\end{aligned} \tag{7.4.18}$$

where $\kappa = \max\{\kappa_1, \kappa_1\kappa_3 + \kappa_2 h_0\}$. □

Corollary 7.4.11. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $c \in C^q(K_n, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\begin{aligned}
&\left\|W_n(\delta_a)^*c - (d_A)^*W_n c\right\|_0 \\
&\quad\leq 2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}}\kappa\left(\kappa_{A,r}\right)^2\left(\left\|(R_n)^{-1}c\right\|_r + \left(\left\|(R_n)^{-1}c\right\|_r\left\|P_n(R_n)^{-1}c\right\|_r\right)^{\frac{1}{2}}\right).
\end{aligned} \tag{7.4.19}$$

Proof. This result follows immediately from Proposition 7.4.10 by Corollary 7.4.7. □

Corollary 7.4.12. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $c \in C^q(K_n, \mathbb{C})$, if Assumption 7.3.4 holds and $\beta_n h_0 \leq 1$ then*

$$\left(\frac{\left\|W_n(\delta_a)^*c\right\|_0}{\left\|W_nc\right\|_0}\right)^2 \geq \left(\frac{\left\|(d_A)^*W_nc\right\|_0}{\left\|W_nc\right\|_0}\right)^2 - 2^{\frac{nq}{2}+1}\left(\beta_n\right)^{\frac{q}{2}}\widetilde{\vartheta}_n^2(c)\,\kappa\left(\kappa_{A,r}\right)^2 + 2^{nq}\left(\beta_n\right)^q\widetilde{\vartheta}_n^3\kappa^2\left(\kappa_{A,r}\right)^4, \tag{7.4.20}$$

where $\widetilde{\vartheta}_n^2$ and $\widetilde{\vartheta}_n^3$ depend on r.

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $c \in C^q(K_n, \mathbb{C})$ then by applying the triangle inequality to Corollary 7.4.11 gives the following for some positive constant κ

$$\begin{aligned}
&\left(\frac{\left\|W_n(\delta_a)^* c\right\|_0}{\|W_n c\|_0}\right)^2 \\
&\geq \left(\frac{\left\|(d_A)^* W_n c\right\|_0 - 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \left(\kappa_{A,r}\right)^2 \left(\left\|(R_n)^{-1} c\right\|_r + \left(\left\|(R_n)^{-1} c\right\|_r \left\|P_n (R_n)^{-1} c\right\|_r\right)^{\frac{1}{2}}\right)}{\|W_n c\|_0}\right)^2 \\
&= \left(\frac{\left\|(d_A)^* W_n c\right\|_0}{\|W_n c\|_0}\right)^2 \\
&\quad - \frac{2^{\frac{nq}{2}+1} (\beta_n)^{\frac{q}{2}} \kappa \left(\kappa_{A,r}\right)^2 \left\|(d_A)^* W_n c\right\|_0 \left(\left\|(R_n)^{-1} c\right\|_r + \left(\left\|(R_n)^{-1} c\right\|_r \left\|P_n (R_n)^{-1} c\right\|_r\right)^{\frac{1}{2}}\right)}{\left(\|W_n c\|_0\right)^2} \\
&\quad + \frac{2^{nq} (\beta_n)^q \kappa^2 \left(\kappa_{A,r}\right)^4 \left(\left\|(R_n)^{-1} c\right\|_r + \left(\left\|(R_n)^{-1} c\right\|_r \left\|P_n (R_n)^{-1} c\right\|_r\right)^{\frac{1}{2}}\right)^2}{\left(\|W_n c\|_0\right)^2} \\
&= \left(\frac{\left\|(d_A)^* W_n c\right\|_0}{\|W_n c\|_0}\right)^2 - 2^{\frac{nq}{2}+1} (\beta_n)^{\frac{q}{2}} \widetilde{\vartheta}_n^2(c) \kappa \left(\kappa_{A,r}\right)^2 + 2^{nq} (\beta_n)^q \widetilde{\vartheta}_n^3 \kappa^2 \left(\kappa_{A,r}\right)^4 . \quad \square \qquad (7.4.21)
\end{aligned}$$

CHAPTER 8

1826 — 1866 — GEORG RIEMANN (German)

Lower Bound

This chapter establishes a large chain of technical estimates required for the lower bound (Theorem 4.4.1), first in the non-adjoint setting (Corollary 8.1.15) and then in the adjoint setting (Corollary 8.2.7) after which the lower bound is proved.

8.1 Estimate for Twisted Derivatives

Initially, for $\omega \in \Omega^q(M,\mathbb{C})$, an estimate for

$$\left| \frac{\langle W\delta_a R\omega, W\delta_a R\omega\rangle_0}{\langle WR\omega, WR\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega,\omega\rangle_0} \right| \tag{8.1.1}$$

is derived depending on ω. Subsequently, it is shown that this bound can be made independent of ω.

8.1.1 Estimate Dependent on Differential Form

Lemma 8.1.1. *For all $\omega \in \Omega^q(M,\mathbb{C})$ let $v = W\delta_a R\omega - d_A\omega$. Then*

$$\langle W\delta_a R\omega, W\delta_a R\omega\rangle_0 = \langle d_A\omega, d_A\omega\rangle_0 + \langle v,v\rangle_0 + 2\mathrm{Re}\left(\langle v, d_A\omega\rangle_0\right). \tag{8.1.2}$$

Proof. Let $\omega \in \Omega^q(M,\mathbb{C})$ and let $v = W\delta_a R\omega - d_A\omega$ then

$$\begin{aligned}\langle W\delta_a R\omega, W\delta_a R\omega\rangle_0 &= \langle d_A\omega + v, d_A\omega + v\rangle_0\\ &= \langle d_A\omega, d_A\omega\rangle_0 + \langle v,v\rangle_0 + \langle v, d_A\omega\rangle_0 + \langle d_A\omega, v\rangle_0\\ &= \langle d_A\omega, d_A\omega\rangle_0 + \langle v,v\rangle_0 + 2\mathrm{Re}\left(\langle v, d_A\omega\rangle_0\right). \qquad \square\end{aligned} \tag{8.1.3}$$

Lemma 8.1.2. *For all $\omega \in \Omega^q(M,\mathbb{C})\setminus\{0\}$ let $v = W\delta_a R\omega - d_A\omega$ then*

$$\begin{aligned}\left| \frac{\langle W\delta_a R\omega, W\delta_a R\omega\rangle_0}{\langle WR\omega, WR\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega,\omega\rangle_0} \right| \le \left(\|d_A\omega\|_0\right)^2 &\frac{\left|\left(\|WR\omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2\right|}{\left(\|WR\omega\|_0\right)^2\left(\|\omega\|_0\right)^2}\\ + \|v\|_0 &\frac{2\|d_A\omega\|_0 + \|v\|_0}{\left(\|WR\omega\|_0\right)^2}\end{aligned} \tag{8.1.4}$$

Proof. Let $\omega \in \Omega^q(M,\mathbb{C}) \setminus \{0\}$ and let $v = W\delta_a R\omega - d_A\omega$. Then by Lemma 8.1.1

$$\begin{aligned}
&\left| \frac{\langle W\delta_a R\omega, W\delta_a R\omega\rangle_0}{\langle WR\omega, WR\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega,\omega\rangle_0} \right| \\
&\quad = \left| \frac{\langle d_A\omega, d_A\omega\rangle_0 + \langle v,v\rangle_0 + 2\mathrm{Re}\left(\langle v, d_A\omega\rangle_0\right)}{\langle WR\omega, WR\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega,\omega\rangle_0} \right| \\
&\quad \leq \left| \frac{\left(\|d_A\omega\|_0\right)^2}{\left(\|WR\omega\|_0\right)^2} - \frac{\left(\|d_A\omega\|_0\right)^2}{\left(\|\omega\|_0\right)^2} \right| + \left| \frac{\left(\|v\|_0\right)^2 + 2\mathrm{Re}\left(\langle v, d_A\omega\rangle_0\right)}{\left(\|WR\omega\|_0\right)^2} \right| \\
&\quad \leq \left(\|d_A\omega\|_0\right)^2 \frac{\left|\left(\|WR\omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2\right|}{\left(\|WR\omega\|_0\right)^2\left(\|\omega\|_0\right)^2} + \left| \frac{\left(\|v\|_0\right)^2 + 2\mathrm{Re}\left(\langle v, d_A\omega\rangle_0\right)}{\left(\|WR\omega\|_0\right)^2} \right| \\
&\quad \leq \left(\|d_A\omega\|_0\right)^2 \frac{\left|\left(\|WR\omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2\right|}{\left(\|WR\omega\|_0\right)^2\left(\|\omega\|_0\right)^2} + \|v\|_0 \frac{2\|d_A\omega\|_0 + \|v\|_0}{\left(\|WR\omega\|_0\right)^2}\,. \qquad \square
\end{aligned} \tag{8.1.5}$$

Lemma 8.1.3. *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M,\mathbb{C})$ it holds that*

$$\left(\|W_n R_n\omega\|_0\right)^2 \geq \left(\|\omega\|_0\right)^2 - 2\kappa h_n \|\omega\|_0 \|\omega\|_r\,. \tag{8.1.6}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer and let $\omega \in \Omega^q(M,\mathbb{C})$. Then by Corollary 2.3.11 for some positive constant κ

$$\begin{aligned}
\kappa h_n \|\omega\|_r &\geq \|W_n R_n\omega - \omega\|_0 \\
&\geq \|\omega\|_0 - \|W_n R_n\omega\|_0\,.
\end{aligned} \tag{8.1.7}$$

Hence

$$\|W_n R_n\omega\|_0 \geq \|\omega\|_0 - \kappa h_n \|\omega\|_r\,, \tag{8.1.8}$$

and therefore

$$\begin{aligned}
\left(\|W_n R_n\omega\|_0\right)^2 &\geq \left(\|\omega\|_0 - \kappa h_n\|\omega\|_r\right)^2 \\
&= \left(\|\omega\|_0\right)^2 - 2\kappa h_n \|\omega\|_0\|\omega\|_r + \kappa^2 \left(h_n\right)^2 \left(\|\omega\|_r\right)^2 \\
&\geq \left(\|\omega\|_0\right)^2 - 2\kappa h_n \|\omega\|_0\|\omega\|_r\,. \qquad \square
\end{aligned} \tag{8.1.9}$$

The later lemmas frequently require an upper bound on the size of h_n which allows the following lemmas to be applied.

Lemma 8.1.4. *For all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M,\mathbb{C}) \setminus \{0\}$, if*

$$h_n \leq \frac{\|\omega\|_0}{4\kappa\|\omega\|_r}\,, \tag{8.1.10}$$

where κ is defined as per Lemma 8.1.3, then

$$\left(\|W_n R_n\omega\|_0\right)^2 \geq \frac{1}{2}\left(\|\omega\|_0\right)^2\,. \tag{8.1.11}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer and let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$. Then by the hypothesis

$$2\kappa h_n \|\omega\|_r \leq \frac{1}{2}\|\omega\|_0 . \tag{8.1.12}$$

Lemma 8.1.3 gives

$$\begin{aligned}\left(\|W_n R_n \omega\|_0\right)^2 &\geq \left(\|\omega\|_0\right)^2 - 2\kappa h_n \|\omega\|_0 \|\omega\|_r \\ &\geq \frac{1}{2}\left(\|\omega\|_0\right)^2 \quad (\text{by } (8.1.12)) . \qquad \square\end{aligned} \tag{8.1.13}$$

Lemma 8.1.5. *For all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, if*

$$h_n \leq \frac{\|d_A \omega\|_0}{\kappa \kappa_{A,0}\left(\|\omega\|_r + \|d\omega\|_r\right)} , \tag{8.1.14}$$

where κ is defined as per Proposition 2.3.16, then

$$\|W_n \delta_a R_n \omega - d_A \omega\|_0 \leq \|d_A \omega\|_0 . \tag{8.1.15}$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer and let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$. Then by the hypothesis

$$\kappa \kappa_{A,0} h_n \left(\|\omega\|_r + \|d\omega\|_r\right) \leq \|d_A \omega\|_0 . \tag{8.1.16}$$

Proposition 2.3.16 gives

$$\begin{aligned}\|W_n \delta_a R_n \omega - d_A \omega\|_0 &\leq \kappa \kappa_{A,0} h_n \left(\|\omega\|_r + \|d\omega\|_r\right) \\ &\leq \|d_A \omega\|_0 \quad (\text{by } (8.1.16)) . \qquad \square\end{aligned} \tag{8.1.17}$$

Lemma 8.1.6. *For all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$ it holds that*

$$\|d_A \omega\|_0 \leq \kappa_{A,0}\left(\|d\omega\|_0 + \|\omega\|_0\right) . \tag{8.1.18}$$

Proof. Let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$. Then

$$\begin{aligned}\|d_A \omega\|_0 &\leq \|d\omega\|_0 + \|iA \wedge\|_{\mathrm{op}} \|\omega\|_0 \\ &\leq \kappa_{A,0}\left(\|d\omega\|_0 + \|\omega\|_0\right) . \qquad \square\end{aligned} \tag{8.1.19}$$

Lemma 8.1.7. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, if $\widehat{\kappa} = \max\{\kappa_1, \kappa_2\}$ where κ_1 is defined to be the constant κ in Proposition 2.3.16 and κ_2 is defined to be the constant κ in Lemma 8.1.3, and*

$$h_n \leq \min\left\{\frac{\|d_A \omega\|_0}{\widehat{\kappa} \kappa_{A,0}\left(\|\omega\|_r + \|d\omega\|_r\right)}, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\} , \tag{8.1.20}$$

then

$$\|v\|_0 \frac{2\|d_A \omega\|_0 + \|v\|_0}{\left(\|W_n R_n \omega\|_0\right)^2} \leq 6\kappa \left(\kappa_{A,0}\right)^2 h_n \left(\|\omega\|_r + \|d\omega\|_r\right) \frac{\|d\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2} , \tag{8.1.21}$$

where $v = W_n \delta_a R_n \omega - d_A \omega$.

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, let κ be the constant κ as per Proposition 2.3.16 and let $v = W\delta_a R\omega - d_A\omega$. Then by Lemma 8.1.5

$$\begin{aligned}
\|v\|_0 \frac{2\|d_A\omega\|_0 + \|v\|_0}{\left(\|W_n R_n \omega\|_0\right)^2} &\le 3\|v\|_0 \frac{\|d_A\omega\|_0}{\left(\|W_n R_n \omega\|_0\right)^2} \\
&\le 6\|v\|_0 \frac{\|d_A\omega\|_0}{\left(\|\omega\|_0\right)^2} \quad \text{(by Lemma 8.1.4)} \\
&\le 6\kappa_{A,0}\|v\|_0 \frac{\|d\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2} \quad \text{(by Lemma 8.1.6)} \\
&\le 6\kappa \left(\kappa_{A,0}\right)^2 h_n \left(\|\omega\|_r + \|d\omega\|_r\right) \frac{\|d\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2} \qquad (8.1.22) \\
&\qquad \text{(by Proposition 2.3.16)}\,.
\end{aligned}$$

□

Lemma 8.1.8. *There exists a positive constant κ independent of n such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C})$, if $h_n \le 1$ then*

$$\left| \left(\|W_n R_n \omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2 \right| \le \kappa h_n \left(\|\omega\|_r\right)^2 . \qquad (8.1.23)$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$ and let $\widehat{\kappa}$ be the constant κ as per Corollary 2.3.11. Then by bilinearity of the inner product

$$\begin{aligned}
\left| \left(\|W_n R_n \omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2 \right| &= \left| \left(\|W_n R_n \omega - \omega\|_0\right)^2 + 2\mathrm{Re}\left(\langle W_n R_n \omega - \omega, \omega \rangle_0\right) \right| \\
&\le \left(\|W_n R_n \omega - \omega\|_0\right)^2 + 2\|W_n R_n \omega - \omega\|_0 \|\omega\|_0 \\
&\le \left(\widehat{\kappa} h_n \|\omega\|_r\right)^2 + 2\widehat{\kappa} h_n \|\omega\|_0 \|\omega\|_r \quad \text{(by Corollary 2.3.11)} \\
&\le h_n \left(\widehat{\kappa}^2 \left(\|\omega\|_r\right)^2 + 2\widehat{\kappa}\|\omega\|_0 \|\omega\|_r\right) \\
&\le h_n \left(\widehat{\kappa}^2 \left(\|\omega\|_r\right)^2 + 2\widehat{\kappa} \left(\|\omega\|_r\right)^2\right) \quad \text{(by Lemma 2.1.8)} \\
&= \kappa h_n \left(\|\omega\|_r\right)^2 , \qquad (8.1.24)
\end{aligned}$$

where $\kappa = \widehat{\kappa}^2 + 2\widehat{\kappa}$. □

Lemma 8.1.9. *For all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, if κ is defined as per Lemma 8.1.3 and*

$$h_n \le \min\left\{1, \frac{\|\omega\|_0}{4\kappa\|\omega\|_r}\right\} , \qquad (8.1.25)$$

then

$$\left(\|d_A\omega\|_0\right)^2 \frac{\left| \left(\|W_n R_n \omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2 \right|}{\left(\|W_n R_n \omega\|_0\right)^2 \left(\|\omega\|_0\right)^2} \le 2\kappa \left(\kappa_{A,0}\right)^2 h_n \left(\|\omega\|_r\right)^2 \left(\frac{\|d\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2}\right)^2 . \qquad (8.1.26)$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer and let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$. Then by Lemma 8.1.8

$$\begin{aligned}
(\|d_A\omega\|_0)^2 \frac{\left|(\|W_nR_n\omega\|_0)^2 - (\|\omega\|_0)^2\right|}{(\|W_nR_n\omega\|_0)^2(\|\omega\|_0)^2} &\leq (\|d_A\omega\|_0)^2 \frac{\kappa h_n(\|\omega\|_r)^2}{(\|W_nR_n\omega\|_0)^2(\|\omega\|_0)^2} \\
&\leq 2\kappa h_n(\|\omega\|_r)^2 \left(\frac{\|d_A\omega\|_0}{(\|\omega\|_0)^2}\right)^2 \qquad \text{(by Lemma 8.1.4)} \\
&\leq 2\kappa(\kappa_{A,0})^2 h_n(\|\omega\|_r)^2 \left(\frac{\|d\omega\|_0 + \|\omega\|_0}{(\|\omega\|_0)^2}\right)^2 \qquad (8.1.27) \\
&\qquad \text{(by Lemma 8.1.6)}. \qquad \square
\end{aligned}$$

Proposition 8.1.10. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, if $\widehat{\kappa} = \max\{\kappa_1, \kappa_2\}$ where κ_1 is defined to be the constant κ in Proposition 2.3.16 and κ_2 is defined to be the constant κ in Lemma 8.1.3, and*

$$h_n \leq \min\left\{1, \frac{\|d_A\omega\|_0}{\widehat{\kappa}\kappa_{A,0}(\|\omega\|_r + \|d\omega\|_r)}, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \qquad (8.1.28)$$

then

$$\left|\frac{\langle W_n\delta_aR_n\omega, W_n\delta_aR_n\omega\rangle_0}{\langle W_nR_n\omega, W_nR_n\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega, \omega\rangle_0}\right| \leq \kappa(\kappa_{A,0})^2 h_n \left(\frac{\|d\omega\|_r + \|\omega\|_r}{\|\omega\|_0}\right)^2 \left(\left(\frac{\|\omega\|_r}{\|\omega\|_0}\right)^2 + 1\right). \qquad (8.1.29)$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, let κ_1 be the constant κ as per Lemma 8.1.3 and let κ_2 be the constant κ as per Lemma 8.1.7 and let $v = W_n\delta_aR_n\omega - d_A\omega$. Then by Lemma 8.1.2

$$\begin{aligned}
&\left|\frac{\langle W_n\delta_aR_n\omega, W_n\delta_aR_n\omega\rangle_0}{\langle W_nR_n\omega, W_nR_n\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle \omega, \omega\rangle_0}\right| \leq (\|d_A\omega\|_0)^2 \frac{\left|(\|W_nR_n\omega\|_0)^2 - (\|\omega\|_0)^2\right|}{(\|W_nR_n\omega\|_0)^2(\|\omega\|_0)^2} \\
&\qquad + \|v\|_0 \frac{2\|d_A\omega\|_0 + \|v\|_0}{(\|W_nR_n\omega\|_0)^2} \\
&\leq 2\kappa_1(\kappa_{A,0})^2 h_n(\|\omega\|_r)^2 \left(\frac{\|d\omega\|_0 + \|\omega\|_0}{(\|\omega\|_0)^2}\right)^2 + 6\kappa_2(\kappa_{A,0})^2 h_n(\|\omega\|_r + \|d\omega\|_r) \frac{\|d\omega\|_0 + \|\omega\|_0}{(\|\omega\|_0)^2} \\
&\qquad \text{(by Lemmas 8.1.7 and 8.1.9)} \\
&\leq \kappa(\kappa_{A,0})^2 h_n \frac{\|d\omega\|_0 + \|\omega\|_0}{(\|\omega\|_0)^2} \left((\|\omega\|_r)^2 \frac{\|d\omega\|_0 + \|\omega\|_0}{(\|\omega\|_0)^2} + \|\omega\|_r + \|d\omega\|_r\right) \\
&\qquad \text{(where } \kappa = \max\{2\kappa_1, 6\kappa_2\}) \\
&\leq \kappa(\kappa_{A,0})^2 h_n \left(\frac{\|d\omega\|_r + \|\omega\|_r}{\|\omega\|_0}\right)^2 \left(\left(\frac{\|\omega\|_r}{\|\omega\|_0}\right)^2 + 1\right) \qquad \text{(by Lemma 2.1.8)}. \qquad \square \qquad (8.1.30)
\end{aligned}$$

8.1.2 Estimate Independent of Differential Form

By Theorem 2.1.12, there exists a Hilbert basis of $\mathcal{H}_0^q(M,\mathbb{C})$ of eigenforms of Δ_A.

Chapter Definition 8.1.11. *Fix such a Hilbert basis $\left\{\varpi_j^q\right\}_{j=1}^{\infty}$ such that ϖ_j^q is an eigenform of λ_j^q.*

Context permitting, ϖ_j is used.

Definition 8.1.12. *For a given q, define $V_m^q \subseteq \Omega^q(M,\mathbb{C})$ to be the vector space spanned by the first m eigenforms ϖ_j^q of Δ_A.*

Hence

$$V_m^q = \operatorname{span}\left\{\varpi_1^q, \varpi_2^q, \ldots, \varpi_m^q\right\}\,, \tag{8.1.31}$$

and

$$\left\{\varpi_1^q, \varpi_2^q, \ldots, \varpi_m^q\right\}\,, \tag{8.1.32}$$

is an orthonormal basis for V_m^q with respect to $\langle\cdot,\cdot\rangle_0$.

Chapter Definition 8.1.13. *For a given $\omega \in V_m^q$, define constants η_j such that*

$$\omega = \sum_{k=1}^{m} \eta_j \varpi_j\,. \tag{8.1.33}$$

Proposition 8.1.14. *There exists a positive constant κ dependent of m such that for all integers $r > \frac{N}{2}+1$ and for all $\omega \in V_m^q \setminus \{0\}$ it holds that*

$$\left(\frac{\|d\omega\|_r + \|\omega\|_r}{\|\omega\|_0}\right)^2 \left(\left(\frac{\|\omega\|_r}{\|\omega\|_0}\right)^2 + 1\right) \le \kappa\,. \tag{8.1.34}$$

Proof. Let $r > \frac{N}{2}+1$ be an integer and let $\omega \in V_m^q \setminus \{0\}$. Then by Chapter Definition 8.1.11

$$\begin{aligned}
&\left(\frac{\|d\omega\|_r + \|\omega\|_r}{\|\omega\|_0}\right)^2 \left(\left(\frac{\|\omega\|_r}{\|\omega\|_0}\right)^2 + 1\right) \\
&\quad= \left(\frac{\left\|d\sum_{j=1}^m \eta_j\varpi_j\right\|_r + \left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_r}{\left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_0}\right)^2 \left(\left(\frac{\left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_r}{\left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_0}\right)^2 + 1\right) \\
&\quad\le \frac{\left(\sum_{j=1}^m \left|\eta_j\right| \left(\left\|d\varpi_j\right\|_r + \left\|\varpi_j\right\|_r\right)\right)^2}{\left(\left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_0\right)^2} \left(\frac{\left(\sum_{j=1}^m \left|\eta_j\right| \left\|\varpi_j\right\|_r\right)^2}{\left(\left\|\sum_{j=1}^m \eta_j\varpi_j\right\|_0\right)^2} + 1\right) \\
&\quad= \frac{\left(\sum_{j=1}^m \left|\eta_j\right| \left(\left\|d\varpi_j\right\|_r + \left\|\varpi_j\right\|_r\right)\right)^2}{\sum_{j=1}^m \left(\left\|\eta_j\varpi_j\right\|_0\right)^2} \left(\frac{\left(\sum_{j=1}^m \left|\eta_j\right| \left\|\varpi_j\right\|_r\right)^2}{\sum_{j=1}^m \left(\left\|\eta_j\varpi_j\right\|_0\right)^2} + 1\right) \qquad \text{(by Theorem 2.1.7)}
\end{aligned}$$

$$= \frac{\left(\sum_{j=1}^{m}\left|\eta_j\right|\left(\left\|d\varpi_j\right\|_r + \left\|\varpi_j\right\|_r\right)\right)^2}{\sum_{j=1}^{m}\left|\eta_j\right|^2}\left(\frac{\left(\sum_{j=1}^{m}\left|\eta_j\right|\left\|\varpi_j\right\|_r\right)^2}{\sum_{j=1}^{m}\left|\eta_j\right|^2} + 1\right) \tag{8.1.35}$$

(by orthonormal basis) .

Let $\widehat{\kappa} = \max_{j=1,2,\ldots,m}\left\{\left\|\varpi_j\right\|_r, \left\|d\varpi_j\right\|_r + \left\|\varpi_j\right\|_r\right\}$ and hence

$$\begin{aligned}\left(\frac{\|d\omega\|_r + \|\omega\|_r}{\|\omega\|_0}\right)^2\left(\left(\frac{\|\omega\|_r}{\|\omega\|_0}\right)^2 + 1\right) &\leq \frac{\widehat{\kappa}^2\left(\sum_{j=1}^{m}\left|\eta_j\right|\right)^2}{\sum_{j=1}^{m}\left|\eta_j\right|^2}\left(\frac{\widehat{\kappa}^2\left(\sum_{j=1}^{m}\left|\eta_j\right|\right)^2}{\sum_{j=1}^{m}\left|\eta_j\right|^2} + 1\right)\\ &\leq m\widehat{\kappa}^2\left(m\widehat{\kappa}^2 + 1\right)\\ &\qquad\text{(quadratic mean bounds arithmetic mean)}\\ &= \kappa\,, \end{aligned} \tag{8.1.36}$$

where $\kappa = m\widehat{\kappa}^2\left(m\widehat{\kappa}^2 + 1\right)$. □

Corollary 8.1.15. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in V_m^q \setminus \{0\}$, if $\widehat{\kappa} = \max\{\kappa_1, \kappa_2\}$ where κ_1 is defined to be the constant κ in Proposition 2.3.16 and κ_2 is defined to be the constant κ in Lemma 8.1.3, and*

$$h_n \leq \min\left\{1, \frac{\|d_A\omega\|_0}{\widehat{\kappa}\kappa_{A,0}\left(\|\omega\|_r + \|d\omega\|_r\right)}, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \tag{8.1.37}$$

then

$$\left|\frac{\langle W_n\delta_a R_n\omega, W_n\delta_a R_n\omega\rangle_0}{\langle W_n R_n\omega, W_n R_n\omega\rangle_0} - \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle\omega,\omega\rangle_0}\right| \leq \kappa\left(\kappa_{A,0}\right)^2 h_n\,. \tag{8.1.38}$$

Proof. This result follows immediately by applying Proposition 8.1.14 to Proposition 8.1.10. □

Corollary 8.1.16. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in V_m^q \setminus \{0\}$, if $\widehat{\kappa} = \max\{\kappa_1, \kappa_2\}$ where κ_1 is defined to be the constant κ in Proposition 2.3.16 and κ_2 is defined to be the constant κ in Lemma 8.1.3, and*

$$h_n \leq \min\left\{1, \frac{\|d_A\omega\|_0}{\widehat{\kappa}\kappa_{A,0}\left(\|\omega\|_r + \|d\omega\|_r\right)}, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \tag{8.1.39}$$

then

$$\frac{\langle W_n\delta_a R_n\omega, W_n\delta_a R_n\omega\rangle_0}{\langle W_n R_n\omega, W_n R_n\omega\rangle_0} \leq \kappa\left(\kappa_{A,0}\right)^2 h_n + \frac{\langle d_A\omega, d_A\omega\rangle_0}{\langle\omega,\omega\rangle_0}\,. \tag{8.1.40}$$

Proof. This result follows immediately from Corollary 8.1.15 by the triangle inequality. □

8.2 Estimate for the Adjoint of Twisted Derivatives

8.2.1 Estimate Dependent on Differential Form

Lemma 8.2.1. *For all* $\omega \in \Omega^q(M, \mathbb{C})$ *let* $v = W(\delta_a)^* R\omega - (d_A)^*\omega$. *Then*

$$\left\langle W(\delta_a)^* R\omega, W(\delta_a)^* R\omega \right\rangle_0 = \left\langle (d_A)^*\omega, (d_A)^*\omega \right\rangle_0 + \langle v, v \rangle_0 + 2\mathrm{Re}\left(\left\langle v, (d_A)^*\omega \right\rangle_0 \right). \quad (8.2.1)$$

Proof. This result can be proven using the same method as used in Lemma 8.1.1. □

Lemma 8.2.2. *For all* $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$ *let* $v = W(\delta_a)^* R\omega - (d_A)^*\omega$. *Then*

$$\begin{aligned} &\left| \frac{\left\langle W(\delta_a)^* R\omega, W(\delta_a)^* R\omega \right\rangle_0}{\langle WR\omega, WR\omega \rangle_0} - \frac{\left\langle (d_A)^*\omega, (d_A)^*\omega \right\rangle_0}{\langle \omega, \omega \rangle_0} \right| \\ &\quad \le \left(\left\| (d_A)^*\omega \right\|_0 \right)^2 \frac{\left| \left(\|WR\omega\|_0 \right)^2 - \left(\|\omega\|_0 \right)^2 \right|}{\left(\|WR\omega\|_0 \right)^2 \left(\|\omega\|_0 \right)^2} + \|v\|_0 \frac{2\left\| (d_A)^*\omega \right\|_0 + \|v\|_0}{\left(\|WR\omega\|_0 \right)^2}. \end{aligned} \quad (8.2.2)$$

Proof. This result can be proven using the same method as used in Lemma 8.1.2. □

Lemma 8.2.3. *For all integers* $r > \frac{N}{2} + 1$ *and for all differential forms* $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$ *let* $v = W_n(\delta_a)^* R_n\omega - (d_A)^*\omega$, *if*

$$h_n \le \frac{\|\omega\|_0}{4\kappa \|\omega\|_r}, \quad (8.2.3)$$

where κ *is defined as per Lemma 8.1.3 then*

$$\|v\|_0 \frac{2\left\| (d_A)^*\omega \right\|_0 + \|v\|_0}{\left(\|W_n R_n\omega\|_0 \right)^2} \le 2\|v\|_0 \frac{2\left\| (d_A)^*\omega \right\|_0 + \|v\|_0}{\left(\|\omega\|_0 \right)^2}. \quad (8.2.4)$$

Proof. This result follows immediately from Lemma 8.1.4. □

Lemma 8.2.4. *There exists a positive constant* κ *independent of* n *such that for all integers* $r > \frac{N}{2} + 1$ *and for all* $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, *if*

$$h_n \le \frac{\|\omega\|_0}{4\widehat{\kappa} \|\omega\|_r}, \quad (8.2.5)$$

where $\widehat{\kappa}$ *is defined as per Lemma 8.1.3 then*

$$\left(\left\| (d_A)^*\omega \right\|_0 \right)^2 \frac{\left| \left(\|W_n R_n\omega\|_0 \right)^2 - \left(\|\omega\|_0 \right)^2 \right|}{\left(\|W_n R_n\omega\|_0 \right)^2 \left(\|\omega\|_0 \right)^2} \le 2\kappa h_n \left(\|\omega\|_r \right)^2 \left(\frac{\left\| (d_A)^*\omega \right\|_0}{\left(\|\omega\|_0 \right)^2} \right)^2. \quad (8.2.6)$$

Proof. Let $r > \frac{N}{2} + 1$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, and let κ be the constant κ as per Lemma 8.1.8. Then by Lemma 8.1.8

$$\begin{aligned} \left(\left\| (d_A)^*\omega \right\|_0 \right)^2 \frac{\left| \left(\|W_n R_n\omega\|_0 \right)^2 - \left(\|\omega\|_0 \right)^2 \right|}{\left(\|W_n R_n\omega\|_0 \right)^2 \left(\|\omega\|_0 \right)^2} &\le \left(\left\| (d_A)^*\omega \right\|_0 \right)^2 \frac{\kappa h_n \left(\|\omega\|_r \right)^2}{\left(\|W_n R_n\omega\|_0 \right)^2 \left(\|\omega\|_0 \right)^2} \\ &\le 2\kappa h_n \left(\|\omega\|_r \right)^2 \left(\frac{\left\| (d_A)^*\omega \right\|_0}{\left(\|\omega\|_0 \right)^2} \right)^2 \quad (8.2.7) \\ &\quad (\text{by Lemma 8.1.4}). \end{aligned}$$

□

Proposition 8.2.5. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, if Assumption 7.3.4 holds and*

$$\beta_n h_0 \le \min\left\{1, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \tag{8.2.8}$$

where $\widehat{\kappa}$ is defined to be the constant κ in Lemma 8.1.3 then

$$\begin{aligned}
&\left|\frac{\langle W_n(\delta_a)^* R_n\omega, W_n(\delta_a)^* R_n\omega\rangle_0}{\langle W_n R_n\omega, W_n R_n\omega\rangle_0} - \frac{\langle (d_A)^*\omega, (d_A)^*\omega\rangle_0}{\langle\omega,\omega\rangle_0}\right| \\
&\quad\le \kappa\left(\kappa_{A,r}\right)^2 \beta_n \left(\|\omega\|_r\right)^2 \left(\frac{\|d\star\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2}\right)^2 \\
&\qquad + \frac{2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}} \kappa\kappa_{A,r}\left\|(d_A)^*\omega\right\|_0\left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r\|P_n\omega\|_r\right)^{\frac{1}{2}}\right)}{\left(\|\omega\|_0\right)^2} \\
&\qquad + \frac{2^{nq}\left(\beta_n\right)^q \kappa\left(\kappa_{A,r}\right)^2\left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r\|P_n\omega\|_r\right)^{\frac{1}{2}}\right)^2}{\left(\|\omega\|_0\right)^2}.
\end{aligned} \tag{8.2.9}$$

Proof. Let $r > \frac{N}{2} + 2$ be an integer, let $\omega \in \Omega^q(M, \mathbb{C}) \setminus \{0\}$, let κ_1 be the constant κ as per Lemma 8.2.4, let κ_2 be the constant κ as per Corollary 7.4.9 and let $v = W_n(\delta_a)^* R_n\omega - (d_A)^*\omega$. Then by Lemma 8.2.2

$$\begin{aligned}
&\left|\frac{\langle W_n(\delta_a)^* R_n\omega, W_n(\delta_a)^* R_n\omega\rangle_0}{\langle W_n R_n\omega, W_n R_n\omega\rangle_0} - \frac{\langle (d_A)^*\omega, (d_A)^*\omega\rangle_0}{\langle\omega,\omega\rangle_0}\right| \\
&\le \left(\left\|(d_A)^*\omega\right\|_0\right)^2 \frac{\left|\left(\|W_n R_n\omega\|_0\right)^2 - \left(\|\omega\|_0\right)^2\right|}{\left(\|W_n R_n\omega\|_0\right)^2\left(\|\omega\|_0\right)^2} + \|v\|_0 \frac{2\left\|(d_A)^*\omega\right\|_0 + \|v\|_0}{\left(\|W R_n\omega\|_0\right)^2} \\
&\le 2\kappa_1\left(\kappa_{A,r}\right)^2 h_n\left(\|\omega\|_r\right)^2\left(\frac{\|d\star\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2}\right)^2 + 2\|v\|_0\frac{2\left\|(d_A)^*\omega\right\|_0 + \|v\|_0}{\left(\|\omega\|_0\right)^2} \\
&\hspace{20em}\text{(by Lemmas 8.2.3 and 8.2.4)} \\
&\le \kappa\left(\kappa_{A,r}\right)^2 \beta_n\left(\|\omega\|_r\right)^2\left(\frac{\|d\star\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2}\right)^2 \\
&\quad + \frac{2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}}\kappa\kappa_{A,r}\left\|(d_A)^*\omega\right\|_0\left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r\|P_n\omega\|_r\right)^{\frac{1}{2}}\right)}{\left(\|\omega\|_0\right)^2} \\
&\quad + \frac{2^{nq}\left(\beta_n\right)^q\kappa\left(\kappa_{A,r}\right)^2\left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r\|P_n\omega\|_r\right)^{\frac{1}{2}}\right)^2}{\left(\|\omega\|_0\right)^2} \quad \text{(by Corollary 7.4.9)},
\end{aligned} \tag{8.2.10}$$

where $\kappa = \max\left\{2\kappa_1 h_0, 4\kappa_2, 2\left(\kappa_2\right)^2\right\}$ and κ_2 is defined as per Corollary 7.4.9. □

8.2.2 Estimate Independent of Differential Form

Proposition 8.2.6. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 1$ and for all $\omega \in V_m^q \setminus \{0\}$ it holds that*

$$\begin{aligned} &\widehat{\kappa}\left(\kappa_{A,r}\right)^2 \beta_n \left(\|\omega\|_r\right)^2 \left(\frac{\|d\star\omega\|_0 + \|\omega\|_0}{\left(\|\omega\|_0\right)^2}\right)^2 \\ &\quad + \frac{2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}} \widehat{\kappa}\kappa_{A,r}\left\|(d_A)^*\omega\right\|_0 \left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}}\right)}{\left(\|\omega\|_0\right)^2} \\ &\quad + \frac{2^{nq}\left(\beta_n\right)^q \widehat{\kappa}\left(\kappa_{A,r}\right)^2 \left(\left\|(d_A)^*\omega\right\|_0 + \left(\|\omega\|_r \|P_n\omega\|_r\right)^{\frac{1}{2}}\right)^2}{\left(\|\omega\|_0\right)^2} \\ &\quad \le \kappa\left(\kappa_{A,r}\right)^2 \beta_n + 2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}} \kappa\kappa_{A,r} + 2^{nq}\left(\beta_n\right)^q \kappa\left(\kappa_{A,r}\right)^2 . \end{aligned} \tag{8.2.11}$$

Proof. This result can be proven using the same method as used in Proposition 8.1.14. □

Corollary 8.2.7. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in V_m^q \setminus \{0\}$, if Assumption 7.3.4 holds and*

$$\beta_n h_0 \le \min\left\{1, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \tag{8.2.12}$$

where $\widehat{\kappa}$ is defined to be the constant κ in Lemma 8.1.3 then

$$\begin{aligned} &\left|\frac{\left\langle W_n(\delta_a)^* R_n\omega, W_n(\delta_a)^* R_n\omega\right\rangle_0}{\left\langle W_n R_n\omega, W_n R_n\omega\right\rangle_0} - \frac{\left\langle (d_A)^*\omega, (d_A)^*\omega\right\rangle_0}{\langle\omega,\omega\rangle_0}\right| \\ &\quad \le \kappa\left(\kappa_{A,r}\right)^2 \beta_n + 2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}} \kappa\kappa_{A,r} + 2^{nq}\left(\beta_n\right)^q \kappa\left(\kappa_{A,r}\right)^2 . \end{aligned} \tag{8.2.13}$$

Proof. This result follows immediately by applying Proposition 8.2.6 to Proposition 8.2.5. □

Corollary 8.2.8. *There exists a positive constant κ independent of n and A such that for all integers $r > \frac{N}{2} + 2$ and for all $\omega \in V_m^q \setminus \{0\}$, if Assumption 7.3.4 holds and*

$$\beta_n h_0 \le \min\left\{1, \frac{\|\omega\|_0}{4\widehat{\kappa}\|\omega\|_r}\right\}, \tag{8.2.14}$$

where $\widehat{\kappa}$ is defined to be the constant κ in Lemma 8.1.3 then

$$\begin{aligned} \frac{\left\langle W_n(\delta_a)^* R_n\omega, W_n(\delta_a)^* R_n\omega\right\rangle_0}{\left\langle W_n R_n\omega, W_n R_n\omega\right\rangle_0} &\le \frac{\left\langle (d_A)^*\omega, (d_A)^*\omega\right\rangle_0}{\langle\omega,\omega\rangle_0} + \kappa\left(\kappa_{A,r}\right)^2 \beta_n \\ &\quad + 2^{\frac{nq}{2}}\left(\beta_n\right)^{\frac{q}{2}} \kappa\kappa_{A,r} + 2^{nq}\left(\beta_n\right)^q \kappa\left(\kappa_{A,r}\right)^2 . \end{aligned} \tag{8.2.15}$$

Proof. This result follows immediately from Corollary 8.2.7 by the triangle inequality. □

8.3 Proof of Theorem 4.4.1

Lemma 8.3.1. *For all $\omega \in V^q_m$ it holds that*

$$\frac{\langle \Delta_A \omega, \omega \rangle_0}{\langle \omega, \omega \rangle_0} \leq \lambda_m \,. \tag{8.3.1}$$

Proof. Let $\omega \in V^q_m$, then by Chapter Definition 8.1.13

$$\begin{aligned}
\frac{\langle \Delta_A \omega, \omega \rangle_0}{\langle \omega, \omega \rangle_0} &= \frac{\left\langle \Delta_A \sum_{j=1}^m \eta_j \varpi_j, \sum_{k=1}^m \eta_k \varpi_k \right\rangle_0}{\left\langle \sum_{j=1}^m \eta_j \varpi_j, \sum_{k=1}^m \eta_k \varpi_k \right\rangle_0} && \text{(8.3.2a)}\\
&= \frac{\left\langle \sum_{j=1}^m \lambda_j \eta_j \varpi_j, \sum_{k=1}^m \eta_k \varpi_k \right\rangle_0}{\left\langle \sum_{j=1}^m \eta_j \varpi_j, \sum_{k=1}^m \eta_k \varpi_k \right\rangle_0} && \text{(8.3.2b)}\\
&= \frac{\sum_{j=1}^m \lambda_j \left|\eta_j\right|^2}{\sum_{j=1}^m \left|\eta_j\right|^2} \quad \text{(by orthonormal basis)} && \text{(8.3.2c)}\\
&\leq \max_j \left\{\lambda_j\right\} && \text{(8.3.2d)}\\
&= \lambda_m \,. && \square \quad \text{(8.3.2e)}
\end{aligned}$$

The following proof proves Theorem 4.4.1 and shows that the analytic eigenvalues are bounded below by the combinatorial eigenvalues of the corresponding combinatorial Laplacian operator.

Proof of Theorem 4.4.1. This proof is based on an argument used by Zahariev [30] to prove his Theorem 4.6. Let r equal to $\frac{N}{2} + 3$, rounded up if N is odd, let κ_1 be the constant κ as per Corollary 8.1.16, let κ_2 be the constant κ as per Corollary 8.2.8. Then by Theorem 2.1.13, for a given q, the m-th eigenvalue is given by

$$\begin{aligned}
\gamma_m &= \sup_{\substack{c_j \in C^q(K_n,\mathbb{C})\\ j=1,2,\dots,m-1}} \inf_{\substack{c \in C^q(K_n,\mathbb{C})\setminus\{0\}\\ \langle c, c_j\rangle_W = 0\\ j=1,2,\dots,m-1}} \frac{\langle \blacktriangle_a c, c\rangle_W}{\langle c, c\rangle_W}\\
&\leq \sup_{\substack{c_j \in C^q(K_n,\mathbb{C})\\ j=1,2,\dots,m-1}} \inf_{\substack{c \in R_n V^q_m\setminus\{0\}\\ \langle c, c_j\rangle_W = 0\\ j=1,2,\dots,m-1}} \frac{\langle \blacktriangle_a c, c\rangle_W}{\langle c, c\rangle_W} \quad \text{(as infimum over a smaller set is larger)}\\
&\leq \sup_{c \in R_n V^q_m\setminus\{0\}} \frac{\langle \blacktriangle_a c, c\rangle_W}{\langle c, c\rangle_W} \quad \text{(as infimum} \leq \text{supremum)}\\
&= \sup_{\omega \in V^q_m\setminus\{0\}} \frac{\langle \blacktriangle_a R_n\omega, R_n\omega\rangle_W}{\langle R_n\omega, R_n\omega\rangle_W}\\
&= \sup_{\omega \in V^q_m\setminus\{0\}} \left(\frac{\langle W_n \delta_a R_n \omega, W_n \delta_a R_n \omega\rangle_0}{\langle W_n R_n \omega, W_n R_n \omega\rangle_0} + \frac{\langle W_n (\delta_a)^* R_n \omega, W_n (\delta_a)^* R_n \omega\rangle_0}{\langle W_n R_n \omega, W_n R_n \omega\rangle_0} \right)\\
&\leq \sup_{\omega \in V^q_m\setminus\{0\}} \left(\frac{\langle d_A \omega, d_A \omega\rangle_0}{\langle \omega, \omega\rangle_0} + \frac{\langle (d_A)^* \omega, (d_A)^* \omega\rangle_0}{\langle \omega, \omega\rangle_0} \right) + \kappa_1 (\kappa_A)^2 h_n + \kappa_2 (\kappa_A)^2 \beta_n\\
&\quad + 2^{\frac{nq+n}{2}} (\beta_n)^{\frac{q+1}{2}} \kappa_2 \kappa_A + 2^{nq+n} (\beta_n)^{q+1} \kappa_2 (\kappa_A)^2 \quad \text{(by Corollaries 8.1.16 and 8.2.8)}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\omega \in V_m^q \setminus \{0\}} \frac{\langle \Delta_A \omega, \omega \rangle_0}{\langle \omega, \omega \rangle_0} + \kappa_1 (\kappa_A)^2 h_n + \kappa_2 (\kappa_A)^2 \beta_n + 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa_2 \kappa_A \\
&\quad + 2^{nq} (\beta_n)^q \kappa_2 (\kappa_A)^2 \ .
\end{aligned}
\tag{8.3.3}
$$

As V_m^q is a finite dimensional space, there exists an $\widehat{\omega} \in V_m^q \setminus \{0\}$ for which the supremum is attained and hence by applying Lemma 8.3.1 to $\widehat{\omega}$ it follows that

$$
\gamma_m \leq \lambda_m + \kappa (\kappa_A)^2 \beta_n + 2^{\frac{nq}{2}} (\beta_n)^{\frac{q}{2}} \kappa \kappa_A + 2^{nq} (\beta_n)^q \kappa (\kappa_A)^2 \ , \tag{8.3.4}
$$

where $\kappa = \kappa_1 h_0 + \kappa_2$. $\square$

Part III
Concluding Remarks

Chapter 9

1815 – 1897 — Karl Weierstrass *(German)*

Upper Bound

This chapter discusses why it appears difficult to generalise Zahariev's methods [30] to derive an upper bound for the eigenvalues of the Laplacian operator.

9.1 Outline of Method for Upper Bound

Lemma 9.1.1. *For all inner product spaces H, for all closed subspaces $V \subseteq H$, for all functions $f : H \to \mathbb{R}$, and for all $m = 1, 2, \ldots, \dim(V)$, it holds that*

$$\sup_{\substack{x_j \in V \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, x_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) = \sup_{\substack{y_j \in H \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, y_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) \,. \tag{9.1.1}$$

Proof. Let $V \subseteq H$ be a closed subspace, let $f : H \to \mathbb{R}$, and let m be given. Consider $x \in V \setminus \{0\}$ and $y_j \in H$. Then y_j can be decomposed into the component $y_j^V \in V$ in the subspace V and the component $y_j^\perp \in V^\perp$ in the orthogonal complement of the subspace V. Hence let

$$y_j = y_j^V + y_j^\perp \,. \tag{9.1.2}$$

Therefore

$$\begin{aligned} \left\langle x, y_j \right\rangle_H &= \left\langle x, y_j^V \right\rangle_H + \left\langle x, y_j^\perp \right\rangle_H \\ &= \left\langle x, y_j^V \right\rangle_H , \end{aligned} \tag{9.1.3}$$

as $x \in V$ and $y_j^\perp \in V^\perp$. Therefore

$$\left\langle x, y_j \right\rangle_H = 0 \iff \left\langle x, y_j^V \right\rangle_H = 0 \,. \tag{9.1.4}$$

Hence

$$\begin{aligned} \sup_{\substack{y_j \in H \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, y_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) &= \sup_{\substack{y_j \in H \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, y_j^V \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) \\ &= \sup_{\substack{x_j \in V \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, x_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) \,. \end{aligned} \tag{9.1.5}$$

$\square$

Corollary 9.1.2. *For all inner product spaces H, for all closed subspaces $V \subseteq H$, for all functions $f : H \to \mathbb{R}$, and for all $m = 1, 2, \ldots, \dim(V)$, it holds that*

$$\sup_{\substack{x_j \in V \\ j=1,2,\ldots,m}} \inf_{\substack{x \in V \setminus \{0\} \\ \langle x, x_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) \geq \sup_{\substack{y_j \in H \\ j=1,2,\ldots,m}} \inf_{\substack{x \in H \setminus \{0\} \\ \langle x, y_j \rangle_H = 0 \\ j=1,2,\ldots,m}} f(x) . \tag{9.1.6}$$

Proof. This result follows immediately as the infimum over a larger set is smaller. □

The following working is the extent to which the techniques of this thesis can be combined with the argument used by Zahariev [30] to prove his Theorem 4.6.

Let $\widehat{\kappa}_1$ be the constant κ as per Corollary 7.1.6 and let $\widehat{\kappa}_2$ be the constant κ as per Corollary 7.4.12. Then from Theorem 2.1.13 it is known that

$$\begin{aligned}
\gamma_m &= \sup_{\substack{c_j \in C^q(K_n,\mathbb{C}) \\ j=1,2,\ldots,m-1}} \inf_{\substack{c \in C^q(K_n,\mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j=1,2,\ldots,m-1}} \frac{\langle \blacktriangle_a c, c \rangle_W}{\langle c, c \rangle_W} \\
&= \sup_{\substack{c_j \in C^q(K_n,\mathbb{C}) \\ j=1,2,\ldots,m-1}} \inf_{\substack{c \in C^q(K_n,\mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j=1,2,\ldots,m-1}} \left(\frac{\langle \delta_a c, \delta_a c \rangle_W}{\langle c, c \rangle_W} + \frac{\langle (\delta_a)^* c, (\delta_a)^* c \rangle_W}{\langle c, c \rangle_W} \right) . \\
&\geq \sup_{\substack{c_j \in C^q(K_n,\mathbb{C}) \\ j=1,2,\ldots,m-1}} \inf_{\substack{c \in C^q(K_n,\mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j=1,2,\ldots,m-1}} \frac{\left(\| d_A W_n c \|_0 \right)^2 + \left(\| (d_A)^* W_n c \|_0 \right)^2}{\langle c, c \rangle_W} \\
&\quad - 2^{nq+n+1} (\beta_n)^{q+1} \widehat{\vartheta}_n^1 \widehat{\kappa}_1 \kappa_{A,r} - 2^{\frac{nq}{2}+1} (\beta_n)^{\frac{q}{2}} \widehat{\vartheta}_n^2 \widehat{\kappa}_2 \left(\kappa_{A,r} \right)^2 \\
&\quad + 2^{2nq+2n} (\beta_n)^{2q+2} (\widehat{\kappa}_1)^2 \left(\kappa_{A,r} \right)^2 + 2^{nq} (\beta_n)^q \widehat{\vartheta}_n^3 (\widehat{\kappa}_2)^2 \left(\kappa_{A,r} \right)^4 \\
&\qquad\qquad\qquad\qquad \text{(by Corollaries 7.1.6 and 7.4.12)}
\end{aligned} \tag{9.1.7}$$

Let $\kappa_1 = 2\max\{\widehat{\kappa}_1, \widehat{\kappa}_2\}$ and $(\kappa_2)^2 = \min\left\{ (\widehat{\kappa}_1)^2, (\widehat{\kappa}_2)^2 \right\}$. Hence by Corollary 2.2.26

$$\begin{aligned}
\gamma_m &\geq \sup_{\substack{c_j \in C^q(K_n,\mathbb{C}) \\ j=1,2,\ldots,m-1}} \inf_{\substack{c \in C^q(K_n,\mathbb{C}) \setminus \{0\} \\ \langle c, c_j \rangle_W = 0 \\ j=1,2,\ldots,m-1}} \frac{\langle \Delta_A W_n c, W_n c \rangle_0}{\langle W_n c, W_n c \rangle_0} \\
&\quad - 2^{nq+n} (\beta_n)^{q+1} \vartheta_n^{1,2} \kappa_1 \left(\kappa_{A,r} \right)^2 + 2^{nq} (\beta_n)^q \vartheta_n^3 (\kappa_2)^2 \left(\kappa_{A,r} \right)^2 \\
&= \sup_{\substack{\omega_j \in W_n\left(C^q(K_n,\mathbb{C})\right) \\ j=1,2,\ldots,m-1}} \inf_{\substack{\omega \in W_n\left(C^q(K_n,\mathbb{C}) \setminus \{0\}\right) \\ \langle \omega, \omega_j \rangle_0 = 0 \\ j=1,2,\ldots,m-1}} \frac{\langle \Delta_A \omega, \omega \rangle_0}{\langle \omega, \omega \rangle_0} \\
&\quad - 2^{nq+n} (\beta_n)^{q+1} \vartheta_n^{1,2} \kappa_1 \left(\kappa_{A,r} \right)^2 + 2^{nq} (\beta_n)^q \vartheta_n^3 (\kappa_2)^2 \left(\kappa_{A,r} \right)^2 .
\end{aligned} \tag{9.1.8}$$

This is as far as Zahariev's method works in conjunction with the results of this thesis. The next step would likely involve applying Corollary 9.1.2 but this is not possible as $W_n\left(C^q(K_n, \mathbb{C})\right)$ is not a subset of $\mathcal{H}_1^q(M,\mathbb{C})$ as pointed out by Remark 4.4 of Zahariev's thesis [30]. The final step of the proof would be using the fact that by Theorem 2.1.13

$$\begin{aligned}
&\sup_{\substack{\omega_j\in\mathcal{H}_0^q(M,\mathbb{C})\\ j=1,2,\ldots,m-1}} \inf_{\substack{\omega\in\mathcal{H}_1^q(M,\mathbb{C})\setminus\{0\}\\ \langle\omega,\omega_j\rangle_0=0\\ j=1,2,\ldots,m-1}} \frac{\langle\Delta_A\omega,\omega\rangle_0}{\langle\omega,\omega\rangle_0} - 2^{nq+n}(\beta_n)^{q+1}\vartheta_n^{1,2}\kappa_1(\kappa_{A,r})^2 + 2^{nq}(\beta_n)^q\vartheta_n^3(\kappa_2)^2(\kappa_{A,r})^2\\
&= \lambda_m - 2^{nq+n}(\beta_n)^{q+1}\vartheta_n^{1,2}\kappa_1(\kappa_{A,r})^2 + 2^{nq}(\beta_n)^q\vartheta_n^3(\kappa_2)^2(\kappa_{A,r})^2 . \qquad (9.1.9)
\end{aligned}$$

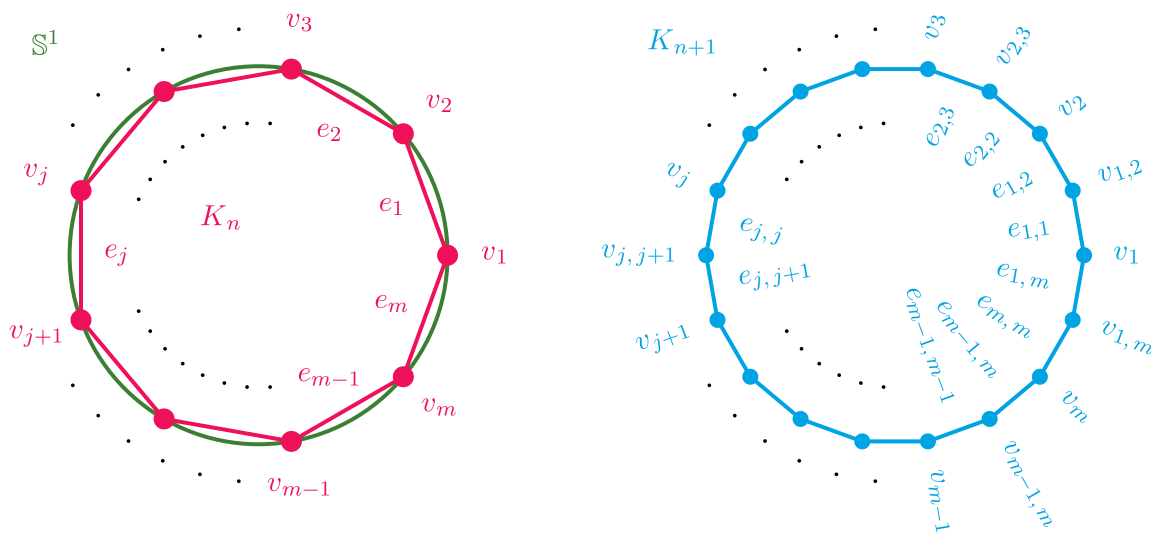
The use of the min-max principle for the Laplacian operator requires special care as the Laplacian operator has a domain that is a dense subspace of the respective Hilbert space. For more details concerning the analysis of densely-defined operators, see Chapter 13 of Rudin's book [22].

There does not appear to be an obvious way to apply the methods of this thesis to complete the proof of the upper bound.

9.2 Example Calculation

This section provides some concrete calculations in the case of $M = \mathbb{S}^1$. The barycentric coordinates as well as the Whitney map applied to edges are derived.

Without loss of generality up to affine homeomorphism, any triangulation of the circle is determined by the number of vertices in the triangulation. Let the number of vertices and edges be m (that is $m = \left|K_n^0\right| = \left|K_n^1\right|$) and adopt the following labelling of vertices and edges:



All edges are oriented in a counter clockwise direction and hence a choice of K_n is equivalent to choosing a value for m.

Let θ be the angle of a point on the circle, then the barycentric coordinates for $j = 2,3,\ldots,m$ are given by

$$\mu_j(\theta) = \begin{cases} -\frac{m}{2\pi}\theta + j & \text{if } \theta \in \left[\frac{2(j-1)\pi}{m},\ \frac{2j\pi}{m}\right] \\ \frac{m}{2\pi}\theta - j + 2 & \text{if } \theta \in \left[\frac{2(j-2)\pi}{m},\ \frac{2(j-1)\pi}{m}\right] \\ 0 & \text{otherwise} \end{cases} , \qquad (9.2.1)$$

and

$$\mu_1(\theta) = \begin{cases} -\frac{m}{2\pi}\theta + 1 & \text{if } \theta \in \left[0,\ \frac{2\pi}{m}\right] \\ \frac{m}{2\pi}\theta - m + 1 & \text{if } \theta \in \left[\frac{2(m-1)\pi}{m},\ 2\pi\right] \\ 0 & \text{otherwise} \end{cases}. \tag{9.2.2}$$

By definition $Wv_j = \mu_j$ as $q = 0$ and hence it follows that $Wc \in \mathcal{H}_1^0(\mathbb{S}^1, \mathbb{C})$. This is holds as Wc is a linear combination of continuous piecewise linear functions and all continuous piecewise linear functions have weak derivatives.

Compare this with the case of $q = 1$ where

$$We_j(\theta) = \begin{cases} \frac{m}{2\pi}\, d\theta & \text{if } \theta \in \left[\frac{2(j-1)\pi}{m},\ \frac{2j\pi}{m}\right] \\ 0 & \text{otherwise} \end{cases}, \tag{9.2.3}$$

which is not a continuous function even though it is piecewise linear. Hence it does not have a weak derivative and therefore $We_j \notin \mathcal{H}_1^1(\mathbb{S}^1, \mathbb{C})$.

As a result, the method outlined in Section 9.1 cannot be used as the image of the Whiney map does not lie in $\mathcal{H}_1^1(\mathbb{S}^1, \mathbb{C})$, verifying Remark 4.4 of Zahariev's thesis [30], and hence no upper bound can be derived using this method for even the simplest of manifold.

Chapter 10

1810 1893 — Ernst Eduard KUMMER (German)

Generalised Problem Setup

This chapter defines the additional notation required to present possible future paths of investigation for research in this area. In essence it generalises the problem setup of Chapter 2 from complex valued differential forms and cochains to ones with values in a Hermitian complex vector bundle.

To aid with reading, instead of introducing new notation, some of the notation used in Chapter 2 is redefined for this more general setting. For example Definition 2.1.1 is generalised to Definition 10.1.1 whilst the notation $\langle\cdot,\cdot\rangle_0$ remains the same for both definitions. The context, specifically if the differential forms and cochains are complex or vector valued allows for the distinguishing of the two definitions.

10.1 Analytic Setting

For a Hermitian complex vector bundle E of rank ε over M with connection ∇^E, let $\Omega^q(M,E)$ be the set of smooth sections of $(\Lambda^q T^*M)\otimes E$. That is, the vector space of q-th degree smooth differential forms on M with values in E and let $\langle\cdot,\cdot\rangle_V$ be the inner product for the vector space that is each fibre, with associated norm $|\cdot|_V$.

The Hodge star operator is generalised to the map $\star : \Omega^q(M,E) \to \Omega^{N-q}(M,E)$ where

$$\star(\varpi\otimes v) = (\star\varpi)\otimes v\,. \tag{10.1.1}$$

The exterior product is extended to the map $\wedge : \Omega^{q_1}(M,E)\times\Omega^{q_2}(M,E) \to \Omega^{q_1+q_2}(M,\mathbb{C})$ where

$$\varpi_1\otimes v_1\wedge\varpi_2\otimes v_2 = \langle v_1,v_2\rangle_V\ \varpi_1\wedge\varpi_2\,. \tag{10.1.2}$$

Definition 10.1.1. *Define the inner product $\langle\cdot,\cdot\rangle_0$ on $\Omega^q(M,E)$ by*

$$\langle\omega_1,\omega_2\rangle_0 = \int_M \overline{\omega}_1\wedge\star\omega_2\,. \tag{10.1.3}$$

For a given Hermitian connection ∇^E on E, let $\nabla^q : \Omega^q(M,E) \to \Omega^{q+1}(M,E)$ be the induced covariant differential where

$$\nabla^q(\varpi\otimes v) = (d\varpi)\otimes v + (-1)^q\,\varpi\wedge\left(\nabla^E v\right)\,. \tag{10.1.4}$$

This connection is notated ∇ for short and is the associated covariant derivative. With respect to the inner product $\langle\cdot,\cdot\rangle_0$, denote the (formal) adjoint of ∇^q by $(\nabla^q)^* : \Omega^{q+1}(M,E) \to \Omega^q(M,E)$.

Consider a coordinate chart $\left(U, x^1, x^2, \ldots, x^N\right)$, a point $p \in U$ and a differential form $\omega \in \Omega^q(M,E)$. Let $\Upsilon^1, \Upsilon^2, \ldots, \Upsilon^N$ be a local orthonormal basis (smooth 1-forms) for the cotangent space with respect to the metric g. Then locally

$$\omega = \sum_{1\leq\nu_1<\nu_2<\cdots<\nu_q\leq N} f_{\nu_1,\nu_2,\ldots,\nu_q}\, \Upsilon^{\nu_1} \wedge \Upsilon^{\nu_2} \wedge \cdots \wedge \Upsilon^{\nu_q} \otimes \upsilon_{\nu_1,\nu_2,\ldots,\nu_q}\,, \tag{10.1.5}$$

for some collection of smooth functions $f_{\nu_1,\nu_2,\ldots,\nu_q} : U \to \mathbb{C}$ and local smooth sections $\upsilon_{\nu_1,\nu_2,\ldots,\nu_q}$ of E.

Definition 10.1.2. *Define the pointwise norm $|\cdot|_p$ of a differential form $\omega \in \Omega^q(M,E)$ at a point p to be*

$$|\omega|_p = \left(\sum_{1\leq\nu_1<\nu_2<\cdots<\nu_q\leq N} \left(\left|f_{\nu_1,\nu_2,\ldots,\nu_q}(p)\right|\left|\upsilon_{\nu_1,\nu_2,\ldots,\nu_q}\right|_V\right)^2\right)^{\frac{1}{2}}. \tag{10.1.6}$$

Definition 10.1.3. *Define $\|\cdot\|_r$ to be the Sobolev norm on $\Omega^q(M,E)$ with parameter $r \in \mathbb{N}$ to be*

$$\|\omega\|_r = \left(\sum_{j=0}^{r} \int_M \left(\left|(\nabla^q)^j\,\omega\right|_p\right)^2 d\,\mathrm{Vol}_g\right)^{\frac{1}{2}}, \tag{10.1.7}$$

where $(\nabla^q)^j$ is the covariant derivative on $\Omega^q(M,E)$ as defined in (10.1.4) applied j times.

By a short computation, for $\Omega^q(M,E)$, the norm associated to $\langle\cdot,\cdot\rangle_0$ is $\|\cdot\|_0$.

Definition 10.1.4. *Define the Sobolev space $\mathcal{H}^q_r(M,E)$ to be the completion of $\Omega^q(M,E)$ with respect to the norm $\|\cdot\|_r$.*

Hence $\mathcal{H}^q_0(M,E)$ is the L^2 completion of $\Omega^q(M,E)$ with respect to the inner product $\langle\cdot,\cdot\rangle_0$. That is, the space of square integrable differential q-forms on M.

Definition 10.1.5. *Define the Laplacian $\Delta^q : \Omega^q(M,E) \to \Omega^q(M,E)$ to be*

$$\Delta^q = (\nabla^q)^*\,\nabla^q + \nabla^{q-1}(\nabla^{q-1})^*\,. \tag{10.1.8}$$

Where clear from context, the Laplace operator (Laplacian) is also denoted by Δ.

This general definition of the Laplacian encapsulates many common variants of the Laplacian through a suitable choice of E and connection on E. For example, $\Delta = d^*d + dd^*$ can be recovered by letting $E = \mathbb{C}$ be the trivial line bundle with connection d.

The eigenvalues of Δ^q are denoted by[†]

$$\lambda^q_1 \leq \lambda^q_2 \leq \cdots \leq \lambda^q_j \leq \cdots, \tag{10.1.9}$$

where eigenvalues are repeated according to their multiplicity.

[†]A priori it may not be clear that the eigenvalues are real, bounded below and that there are infinitely many of them, however this property holds by Theorem 2.1.12 and hence the notation makes sense.

10.2 Combinatorial Setting

10.2.1 Cochains

Let $C^q(K,E)$ be the vector space of q-th degree cochains on K with values in E. For each simplex σ, fix a reference point p_σ in the interior of σ. A q-cochain c can be defined of as a function $c : K^q \to E$ that assigns to each q-simplex a vector in the bundle with the property that for all $\sigma \in K^q$, $c(\sigma)$ is in E_{p_σ}, the fibre above the reference point of σ. Alternatively, it can be defined of as a formal sum of q-simplices

$$c = \sum_{\sigma\in K^q} c_\sigma \sigma \tag{10.2.1}$$

for some vector coefficients c_σ in the fibre at the reference point of σ. The translation between the two approaches is given by $c(\sigma) = c_\sigma$ for all $\sigma \in K^q$.

Definition 10.2.1. *Define the inner product $\langle\cdot,\cdot\rangle_C$ on $C^q(K,E)$ by*

$$\langle c_1, c_2\rangle_C = \sum_{\sigma\in K^q} \langle c_1(\sigma), c_2(\sigma)\rangle_V . \tag{10.2.2}$$

This gives an associated norm, $\|\cdot\|_{C,2}$, on cochains.

Definition 10.2.2. *Define the supremum norm, $\|\cdot\|_{C,\infty}$ on cochains to be*

$$\|c\|_{C,\infty} = \max_{\sigma\in K^q} |c(\sigma)|_V . \tag{10.2.3}$$

Definition 10.2.3. *Define the simplicial coboundary operator $\delta^q : C^q(K,E) \to C^{q+1}(K,E)$ to be*

$$((\delta^q c))(\sigma) = \sum_{j=0}^{q} (-1)^j\, c\left(\partial_j^{q+1}\sigma\right) , \tag{10.2.4}$$

where $c \in C^q(K,E)$ and $\sigma \in K^{q+1}$.

10.2.2 The de Rham and Whitney Maps

Definition 10.2.4. *The de Rham map, $R_n^q : \Omega^q(M,E) \to C^q(K_n,E)$, is defined by de Rham [7] and generalised by Dodziuk and Patodi [10, Page 4] as*

$$\left(R_n^q(\varpi \otimes v)\right)(\sigma) = \left(\int_\sigma \varpi\right) v|_{p_\sigma} , \tag{10.2.5}$$

where ϖ is a smooth section of $\Lambda^q T^*M$, v is a smooth section of E and $q > 0$. When $q = 0$, the de Rham map is defined as the evaluation of the differential form at the vertices.

Consider a q-simplex $\sigma = [p_0, p_1, \ldots, p_q]$ and let μ_j be the barycentric coordinate function corresponding to the vertex at p_j for $j = 0, 1, \ldots, q$.

Definition 10.2.5. *The Dodziuk and Patodi [10, Page 5] generalised Whitney map [28, Chapter VII.11 (16)] $W_n^q : C^q(K_n,E) \to \mathcal{H}_0^q(M,E)$ is defined to be*

$$W_n^q c_\sigma \sigma = q! \sum_{j=0}^{q} (-1)^j \mu_j\, d\mu_0 \wedge d\mu_1 \wedge \cdots \wedge d\mu_{j-1} \wedge d\mu_{j+1} \wedge \cdots \wedge d\mu_q \otimes c_\sigma , \tag{10.2.6}$$

for $q > 0$. When $q = 0$, define

$$W_n^0 c_\sigma \sigma = \mu_0 \otimes c_\sigma . \tag{10.2.7}$$

This definition extends linearly to any cochain by viewing the cochain as a formal sum.

10.2.3 Construction of the Combinatorial Laplacian

Definition 10.2.6. *Define the Whitney inner product $\langle\cdot,\cdot\rangle_W$ on $C^q(K,E)$ by*

$$\langle c_1, c_2\rangle_W = \langle Wc_1, Wc_2\rangle_0 . \tag{10.2.8}$$

This inner product makes sense as W is injective by Proposition 2.3.2. The associated norm is denoted by $\|\cdot\|_W$.

Using Definition 2.2.19 as a starting point for the finite open cover $\{U_j\}_{j=1}^m$ of M, the combinatorial construction of the Laplace operator assumes the following additional properties of this open cover.

Assumption 10.2.7. *For a given K and for all $j = 1, 2, \ldots, m$, it holds that $\operatorname{cl}(U_j)$ is a subsimplicial complex of K, denoted by L_j*[‡]*.*

In other words, $\operatorname{cl}(U_j)$ is the union of some collection of N-simplices of K.

Assumption 10.2.8. *For all $j = 1, 2, \ldots, m$, the restriction of the vector bundle E to $\operatorname{cl}(U_j)$, $E|_{\operatorname{cl}(U_j)}$, is trivial.*

It can be seen that given $\mathcal{U}$ and trivialisations of E that the required finite open cover $\{U_j\}_{j=1}^m$ can be constructed through suitable refinements of $\mathcal{U}$ and the given triangulation K.

The following construction of the combinatorial Laplacian is due to Zahariev [30, Chapter 5]. It differs slightly from the combinatorial Laplacian considered by Dodziuk and Patodi [10, Page 6] as detailed in Section 10.2.4.

The key ingredient for constructing the combinatorial Laplacian is the following theorem by Narasimhan and Ramanan [16], but in the form developed by Quillen [18, Appendix].

Theorem 10.2.9. *Let E be a vector bundle of rank ε over M equipped with an inner product and ∇ a connection that preserves this inner product. Then there exists a trivial Hermitian vector bundle B of rank b and an isometric embedding of bundles $\iota : E \to B$ such that $\nabla = \iota^* d\iota$, where ι^* is the fibrewise adjoint taken with respect to the inner products on the fibres of E and B respectively.*

An outline of some of key steps of the construction of B and ι will help introduce the notation required for the combinatorial Laplacian.

Notation 10.2.10. *Let $\{\psi_j\}_{j=1}^m$ to be a modified partition of unity subordinate to $\{U_j\}_{j=1}^m$ such that $\psi_j : M \to [0,1]$ is a smooth function whose support is compact and contained in U_j, and*

$$\sum_{j=1}^{m} (\psi_j)^2 = 1 . \tag{10.2.9}$$

Quillen [18, Appendix] shows such a partition of unity can easily be constructed from a conventional partition of unity $\left\{\widehat{\psi}_j\right\}_{j=1}^m$ with $\sum_{j=1}^m \widehat{\psi}_j = 1$ by defining

$$\psi_j = \widehat{\psi}_j \left(\sum_{k=1}^{m} \left(\widehat{\psi}_k\right)^2 \right)^{-\frac{1}{2}} . \tag{10.2.10}$$

[‡]Care must be taken with the notation of the subcomplex as the subscript j indexes which open set U_j is associated to L_j. This is in contrast to subscripts on K_n where n indexes the number of subdivisions that have been performed. The dependence of n on L_j is implied and is not notated.

Technically speaking for each $j = 1, 2, \ldots, m$, $\psi_j : M \to [0,1] \subseteq \mathbb{C}$ and by definition ψ_j is smooth and hence $\psi_j \in \Omega^0(M, \mathbb{C})$. This viewpoint will be important later.

For each $j = 1, 2, \ldots, m$, let $B_j = M \times \mathbb{C}^\varepsilon$ be the trivial vector bundle with the standard flat metric. Then by Assumption 10.2.8, $E|_{U_j}$ is trivial and hence by the use of an orthonormal frame of sections for E over U_j, let $\iota_j : E|_{U_j} \to B_j\big|_{U_j}$ be a specific isometric embedding satisfying some additional conditions. These additional conditions are required in order for $\nabla = \iota^* d\iota$ to hold but these are not relevant for the construction of the combinatorial Laplacian and hence not mentioned here. For the full details of the construction of this embedding, see Quillen [18, Appendix]. From this define

$$B = \bigoplus_{j=1}^{m} B_j \,, \tag{10.2.11}$$

and $\iota : E \to \bigoplus_{j=1}^{m} B_j$ where

$$\iota = (\iota_1\psi_1, \iota_2\psi_2, \ldots, \iota_m\psi_m) \,. \tag{10.2.12}$$

It is this construction that necessities the condition imposed by (10.2.9) as it ensures $\iota^*\iota = \mathrm{Id}$ and subsequently $\nabla = \iota^* d\iota$ [18, Appendix].

Chapter Definition 10.2.11. *For a given $j = 1, 2, \ldots, m$ and $p \in \mathrm{cl}\left(U_j\right)$, consider the local trivialisation of the bundles E and B on $\mathrm{cl}\left(U_j\right)$ and define the maps $\mathrm{pr}_2, \iota_j^p, \chi$ and φ_j^E by means of the following diagram:*

$$\begin{tikzcd} E|_{\mathrm{cl}(U_j)} \arrow[r, "\varphi_j^E", "\simeq"'] \arrow[d, "\iota|_{\mathrm{cl}(U_j)}"] & \mathrm{cl}\left(U_j\right) \times \mathbb{C}^\varepsilon \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[d, "\mathrm{Id}\times\chi"] & \mathbb{C}^\varepsilon \arrow[l, shift left, "\iota_j^p"] \arrow[d, "\chi"] \\ B|_{\mathrm{cl}(U_j)} \arrow[r, "\mathrm{Id}", "="'] & \mathrm{cl}\left(U_j\right) \times \mathbb{C}^b \arrow[r, shift left, "\mathrm{pr}_2"] & \mathbb{C}^b \arrow[l, shift left, "\iota_j^p"] \end{tikzcd} \tag{10.2.13}$$

Here pr_2 is the canonical projection map and ι_j^p is the canonical inclusion maps where for a fixed $p \in \mathrm{cl}\left(U_j\right)$, by definition $\iota_j^p(v) = (p, v)$. The map χ is constructed in the same manner as transition functions are constructed for vector bundles. For a given $p \in \mathrm{cl}\left(U_j\right)$, it holds that χ^p is a $b \times \varepsilon$ matrix of complex numbers. Notationally, $\chi_{r,s}^p$ will denote the (r, s)-th entry of this matrix. Separately, for fixed r and s but varying p, the map $\chi_{r,s}^p : \mathrm{cl}\left(U_j\right) \to \mathbb{C}$ can be viewed as a smooth function and hence $\chi_{r,s}^p \in \Omega^0\left(\mathrm{cl}\left(U_j\right), \mathbb{C}\right)$, an important viewpoint for later.

It should be noted that the dependence on j of the map χ is implied and not notated.

The embedding map $\iota : E \to B$ induces the maps $\iota^q : \Omega^q(M, E) \to \Omega^q(M, B)$ and $\iota^q : C^q(K, E) \to C^q(K, B)$ in the following ways. For a given $\varpi \otimes v \in \Omega^q(M, E)$, define

$$\iota^q(\varpi \otimes v) = \varpi \otimes (\iota v) \,. \tag{10.2.14}$$

Recall that for each $\sigma \in K$ a reference point p_σ was fixed such that the coefficient of the simplex, $c_\sigma \in E_{p_\sigma}$, is a value in the fibre of the reference point. Hence given a $c \in C^q(K, E)$ define

$$\iota^q\left(\sum_{\sigma \in K^q} c_\sigma \sigma\right) = \sum_{\sigma \in K^q} \left(\iota|_{E_{p_\sigma}} c_\sigma\right) \sigma \,. \tag{10.2.15}$$

Where clear from context, the superscript is omitted.

Definition 10.2.12. *For a given* $j = 1, 2, \ldots, m$, *define the map* $\Phi_j^E : C^q(K, E) \to C^q\left(L_j, \mathbb{C}^\varepsilon\right)$ *as follows. Given a* $c \in C^q(K, E)$ *define*

$$\Phi_j^E\left(\sum_{\sigma \in K^q} c_\sigma \sigma\right) = \sum_{\sigma \in L_j^q}\left(\mathrm{pr}_2 \circ \varphi_j^E(c_\sigma)\right)\sigma\,. \tag{10.2.16}$$

In the above definition, the dependence on q is implied and not notated.

Definition 10.2.13. *For a given* $j = 1, 2, \ldots, m$ *and using the appropriate reference points for each simplex, define the map* $\Phi_j^B : C^q\left(L_j, \mathbb{C}^b\right) \to C^q(K, B)$ *as follows. Given a* $c \in C^q\left(L_j, \mathbb{C}^b\right)$ *define*

$$\Phi_j^B\left(\sum_{\sigma \in L_j^q} c_\sigma \sigma\right) = \sum_{\sigma \in L_j^q}\left(\iota_j^{p_\sigma} c_\sigma\right)\sigma + \sum_{\sigma \in K^q \setminus L_j^q} 0\sigma\,, \tag{10.2.17}$$

where the second sum assigns the zero vector in the fibre of the corresponding reference point for each simplex as the coefficient for all simplices not contained in L_j^q.

In the above definition, the dependence on q is implied and not notated.

For a cochain $c \in C^q\left(L_j, \mathbb{C}^\varepsilon\right)$, each coefficient c_σ is a vector in $\mathbb{C}^\varepsilon$ and hence define the notation $(c_\sigma)_k$ to be the k-th component of the vector c_σ. Similarly, define

$$(c)_k = \sum_{\sigma \in L_j^q} (c_\sigma)_k\, \sigma\,, \tag{10.2.18}$$

which is an element of $C^q(L_j, \mathbb{C})$.

Definition 10.2.14. *For a given* $j = 1, 2, \ldots, m$, *define the map* $I_j^q : C^q\left(L_j, \mathbb{C}^\varepsilon\right) \to C^q\left(L_j, \mathbb{C}^b\right)$ *as follows*

$$\left(I_j^q c\right)_r = \sum_{s=1}^{\varepsilon}\left(R^0 \chi_{r,s}\right) \cup (c)_s\,. \tag{10.2.19}$$

This is where the viewpoint of $\chi_{r,s}$ as an element of $\Omega^0\left(\mathrm{cl}\left(U_j\right), \mathbb{C}\right)$ is required in order to apply de Rham map and cup product.

Definition 10.2.15. *For a given* $j = 1, 2, \ldots, m$, *define the map* $\Psi_j^q : C^q(K, E) \to C^q\left(L_j, \mathbb{C}^\varepsilon\right)$ *as follows*

$$\left(\Psi_j^q c\right)_s = \left(R^0 \psi_j\right) \cup \left(\Phi_j^E c\right)_s\,. \tag{10.2.20}$$

This is where the viewpoint of ψ_j as an element of $\Omega^0\left(\mathrm{cl}\left(U_j\right), \mathbb{C}\right)$ is required in order to apply de Rham map and cup product.

Definition 10.2.16. *Define the map* $I^q : C^q(K, E) \to C^q(K, B)$ *as follows*

$$I^q c = \sum_{j=1}^{m} \Phi_j^B I_j^q \Psi_j^q c\,. \tag{10.2.21}$$

Where clear from context, this map is notated as I.

Let I^* be the adjoint of I with respect to the inner products on the fibres of E and B respectively.

These maps allow the definition of the twisted versions of the de Rham and Whitney maps.

Definition 10.2.17. *Define the twisted de Rham map, $\widetilde{R}_n^q : \Omega^q(M,E) \to C^q(K_n,E)$, to be*

$$\widetilde{R}_n^q = (I^q)^* R_n^q \iota^q \,. \tag{10.2.22}$$

Definition 10.2.18. *Define the twisted Whitney map, $\widetilde{W}_n^q : C^q(K_n,E) \to \mathcal{H}_0^q(M,E)$, to be*

$$\widetilde{W}_n^q = (\iota^q)^* W_n^q I^q \,. \tag{10.2.23}$$

Definition 10.2.19. *Define the twisted Whitney inner product $\langle\cdot,\cdot\rangle_{\widetilde{W}}$ on $C^q(K,E)$ by*

$$\langle c_1, c_2\rangle_{\widetilde{W}} = \left\langle \widetilde{W} c_1, \widetilde{W} c_2 \right\rangle_0 . \tag{10.2.24}$$

This inner product is makes sense as $\widetilde{W}$ is injective for sufficiently fine triangulation by Zahariev's Lemma 5.3 [30].

Definition 10.2.20. *Define the combinatorial covariant derivative $\blacktriangledown^q : C^q(K,E) \to C^{q+1}(K,E)$ to be*

$$\blacktriangledown^q = (I^q)^* \delta^q I^q \,. \tag{10.2.25}$$

Denote the adjoint of $\blacktriangledown^q$ by $(\blacktriangledown^q)^* : C^{q+1}(K,E) \to C^q(K,E)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\widetilde{W}}$.

The interplay of all of these various maps is best visualised via the following not necessarily commutative diagram:

$$\begin{tikzcd}
\Omega^q(M,E) \arrow[rr, shift left, "\nabla"] \arrow[dr, shift left, "\iota"] \arrow[dd, shift left, "{R,\widetilde{R}}"] & & \Omega^{q+1}(M,E) \arrow[ll, shift left, "\nabla^*"] \arrow[dr, shift left, "\iota"] \arrow[dd, shift left, "{R,\widetilde{R}}"] & \\
& \Omega^q(M,B) \arrow[ul, shift left, "\iota^*"] \arrow[rr, shift left, "d"] \arrow[dd, shift left, "R"] & & \Omega^{q+1}(M,B) \arrow[ul, shift left, "\iota^*"] \arrow[ll, shift left, "d^*"] \arrow[dd, shift left, "R"] \\
C^q(K,E) \arrow[uu, shift left, "{W,\widetilde{W}}"] \arrow[rr, shift left, "\blacktriangledown"] \arrow[dr, shift left, "{\iota,I}"] & & C^{q+1}(K,E) \arrow[uu, shift left, "{W,\widetilde{W}}"] \arrow[ll, shift left, "\blacktriangledown^*"] \arrow[dr, shift left, "{\iota,I}"] & \\
& C^q(K,B) \arrow[ul, shift left, "{\iota^*,I^*}"] \arrow[uu, shift left, "W"] \arrow[rr, shift left, "\delta"] & & C^{q+1}(K,B) \arrow[ul, shift left, "{\iota^*,I^*}"] \arrow[uu, shift left, "W"] \arrow[ll, shift left, "\delta^*"]
\end{tikzcd} \tag{10.2.26}$$

Definition 10.2.21. *Define the combinatorial Laplacian $\blacktriangle^q : C^q(K,E) \to C^q(K,E)$ to be*

$$\blacktriangle^q = (\blacktriangledown^q)^* \blacktriangledown^q + \blacktriangledown^{q-1} (\blacktriangledown^{q-1})^* \,. \tag{10.2.27}$$

This definition of the Laplacian does not encapsulate all combinatorial versions of the various analytic variants of the Laplacian considered in this thesis. The details of these discrepancies are discussed in Section 10.2.4.

The eigenvalues of $\blacktriangle^q$ are denoted by

$$\gamma_1^q \leq \gamma_2^q \leq \cdots \leq {\gamma_j}^q \leq \cdots \leq \gamma^q_{\dim\left(C^q(K,E)\right)}, \tag{10.2.28}$$

where eigenvalues are repeated according to their multiplicity.

This formal construction allows Theorem 3.4.1 to be stated precisely and leads to the natural question of what analogous conclusions can be drawn in the positive degree setting. This is discussed in Section 11.4.

10.2.4 Comparison of the Combinatorial Laplacians

Dodziuk and Patodi [10] also constructed a combinatorial Laplacian for the case of a flat vector bundle. This combinatorial Laplacian is not dependent on an embedding and hence the two combinatorial Laplacians are not equivalent in the flat vector bundle setting as a different embedding results in a different combinatorial Laplacian. The difference between the combinatorial Laplacian Zahariev uses and the one used by Dodziuk and Patodi lies in the fact that the partition of unity $\{\psi_j\}$ used by Zahariev satisfies

$$\sum_j \left(\psi_j\right)^2 = 1\,. \tag{10.2.29}$$

Chasing the definitions reveals that the two combinatorial Laplacians would be identical if the partition of unity used by Zahariev is a conventional one[§]. In such a case on a trivial vector bundle $I = \mathrm{Id}$ and hence it is clear that Zahariev's construction reduces to that of the one used by Dodziuk and Patodi, but this is not the case.

[§]A conventional partition of unity $\left\{\widehat{\psi}_j\right\}$ shares all of the properties of the partition of unity used by Zahariev except that $\sum_j \widehat{\psi}_j = 1$ compared with (10.2.29).

CHAPTER 11

1811 – 1832 — ÉVARISTE GALOIS (French)

FUTURE WORK

This chapter provides an overview of the types of results that could be achieved using this thesis as a basis. It includes a discussion of the excepted difficulties and possible methods to overcome these.

11.1 Dependence on Choices

As was discussed in Section 4.7, there are some choices that are made in order to derive Theorem 4.4.1. It would be of interest to understand exactly how these choices affect Theorem 4.4.1. It is also anticipated that understanding this dependence in detail will aid with Section 11.2.

11.2 Path to Theorem 3.3.1 for Arbitrary Degree

The main aim of this thesis is to prove a statement analogous to Theorem 3.3.1 for arbitrary degree. Completing this aim is the logical next step for this area, requiring work in a number of areas as detailed in Sections 11.2.1 to 11.2.3. A possible alternative method for achieving this aim is presented in Section 11.2.4.

11.2.1 Upper Bound

As outlined in Chapter 9, the methods of this thesis are unable to derive an upper bound for the eigenvalues. The key obstacle being the fact that $W_n\left(C^q\left(K_n,\mathbb{C}\right)\right)$ is not contained in $\mathcal{H}_1^q\left(M,\mathbb{C}\right)$. There does not seem to be an obvious approach to overcome this obstacle. The key missing ingredient appears to be a space that contains both $W_n\left(C^q\left(K_n,\mathbb{C}\right)\right)$ and $\mathcal{H}_1^q\left(M,\mathbb{C}\right)$ and allows the use of the min-max principle.

11.2.2 Assumption 7.3.4

At its core, the missing link of this thesis is the need for Assumption 7.3.4. In order to be able to remove this assumption, there would appear to be three paths of investigation to explore.

11.2.2.1 Direct Approach

The obvious approach is to try and prove Assumption 7.3.4 directly. If this estimate is provable or at least an estimate of this form would allow Assumption 7.3.4 to be removed from Theorem 4.4.1. Obtaining this result would appear to be a careful analysis problem dealing with Sobolev spaces and when the operator d is bounded.

11.2.2.2 Adjoint Approximation

Alternatively, the work of Smits [24] and later Arnold, Falk, Guzman and Tsogtgerel [3] could provide a different path to proving Assumption 7.3.4. They established that the adjoint of the coboundary operator approximates the adjoint of the exterior derivative as required but only for degree one using the regular standard subdivision. However Arnold, Falk, Guzman and Tsogtgerel also provided counter examples when convergence failed in the setting of other subdivision schemes such as the standard subdivision and separately, in the case of higher degree differential forms and cochains [3, Page 5489].

Using these results it is possible to derive the lower bound without Assumption 7.3.4. However the upper bound still requires the assumption and assumes that the Whitney map growth rate calculations can be adapted to the setting of the subdivision scheme considered. It may well be the case that through careful examination of the proves used by Arnold, Falk, Guzman and Tsogtgerel that they can be adapted to produce the estimates required for the upper bound and hence remove Assumption 7.3.4 entirely.

11.2.2.3 Combinatorial Hodge Star

Finally, the work by Wilson [29] and later Tanabe [26] who showed that the combinatorial Hodge star operator approximates the analytic Hodge star operator in a number of ways that could offer a path to proving Assumption 7.3.4. As the Hodge star is included of the definition of the adjoint of the exterior derivative, it is possible that using the combinatorial Hodge star for defining the adjoint of the coboundary operator allows for the required estimates to be derived.

11.2.3 Convergence

In order for the eigenvalues to converge, the error terms in both the upper and lower bounds must converge to zero as $n \to \infty$.

11.2.3.1 Lower Bound

As a consequence of Corollary 2.2.26, in the best case scenario ($\beta_n = 2^{-n}$), the error term approaches a constant and in all other cases the error term grows in size. This growth in error is a direct result of the growth rate of the Whitney map. Therefore, the only way to reduce the growth rate is to consider a different subdivision scheme for which the growth rate of the Whitney map is reduced.

This problem has many aspects to it considering the complicated and extensive relationship between the subdivision scheme and Whitney map. Hence it is difficult to say if it is possible to reduce the growth rate of the Whitney map but worth investigating.

11.2.3.2 Upper Bound

A first step for the upper bound is to establish the behaviour of the ϑ_n terms. Whilst it is strongly suspected that the ϑ_n terms grow in size as $n \to \infty$, it may be the case that they do not grow. In this case, the problem of having the error terms converge to zero is identical to the lower bound setting and hence already discussed (in Section 11.2.3.1).

Should the ϑ_n terms grow in size, the exact growth rate will determine if the error term goes to zero or not. Of course any improvement in the rate of convergence from the lower bound will aid in the convergence of the upper bound.

In the case where the growth rate of ϑ_n terms is too large, tighter estimates for Propositions 7.1.4 and 7.4.10 will be required. This would be a fairly major undertaking and likely will require a significantly different approach to the one used by this thesis.

11.2.4 Application of Theorem 3.4.1

It is of note that even though Zahariev's Theorem 3.4.1 is for the case of $q = 0$, by the following construction, it may be able to provide an approximation for the eigenvalue problem in the trivial line bundle case for $q = 0, 1, \ldots, N$.

In Theorem 3.4.1, suppose $E = \mathbb{C} \otimes \Lambda^q T^*M$, where $\mathbb{C}$ is viewed as the trivial line bundle and the connection ∇ on this vector bundle $\mathbb{C} \otimes \Lambda^q T^*M$ is that which is induced by the given connection d_A on $\mathbb{C}$ and the Levi-Civita connection on $\Lambda^q T^*M$ with respect to the metric g, then the Bochner Laplacian operator $\nabla^*\nabla$ acting on differential forms with values in $\mathbb{C} \otimes \Lambda^q T^*M$ has the same symbol (up to a fixed constant) as the Hodge Laplacian operator $(d_A)^* d_A + d_A(d_A)^*$ considered throughout this thesis. The two operators thus differ by a first order operator (by Weitzenböck formulas), which can be viewed as a compact operator when acting on $\mathcal{H}_2^q(M, \mathbb{C})$ sections. Hence using Zahariev's Theorem 3.4.1, the eigenvalues of this Bochner Laplacian, $\nabla^*\nabla$, can be approximated by Zahariev's discrete methods, and therefore, in some sense, so too can the eigenvalues of the Hodge Laplacian considered in this thesis, $(d_A)^* d_A + d_A(d_A)^*$. However, the precise nature of that latter approximation is by no means straightforward to determine nor to estimate by discrete methods and for this reason, this approach was not pursued but offers a possible future path of investigation.

11.3 Path to Trivial Vector Bundle

To generalise the work of Section 11.2 to trivial vector bundles of arbitrary rank in the case of degree zero requires the generalisation of the cup product to matrix valued cochains and letting A be a matrix of 1-forms according to Zahariev [30, Remark 4.7].

Whilst not investigated for any obstacles, it is excepted that this approach can also be employed for arbitrary degree.

11.4 Path to Theorem 3.4.1 for Arbitrary Degree

In the setting of degree zero, Zahariev [30] constructs a very specific combinatorial Laplacian for the case of any arbitrary vector bundle. This construction is presented in detail in Chapter 10.

Generalising Zahariev's proof of Theorem 3.4.1 to higher degrees may be possible from Section 11.3 and is the ultimate goal of this path of investigation.

Considering some of Zahariev's intermediate results are only for degree zero (such as his Lemma 6.1 [30]), there would appear to be obstacles for higher degrees that need to be overcome. These obstacles have not been investigated and hence no comment can be made as to the feasibility of overcoming them.

11.5 New Paths of Investigation

Once Theorem 3.4.1 for arbitrary degree has been proved, it concludes a logical sequence of generalisation of convergence of eigenvalues from the Laplacian action on functions to the Laplacian action on any degree differential forms with values in an arbitrary vector bundle. From this point, there are at least two logical paths of investigation to continue on.

11.5.1 Universal Combinatorial Laplacian

As was discussed in Section 10.2.4, the combinatorial Laplacian considered by Dodziuk and Patodi [10] and the one considered by Zahariev [30] are not equivalent. Therefore it would be of interest if it is possible to construct a combinatorial Laplacian for the setting of an arbitrary vector bundle that coincides with the Dodziuk and Patodi combinatorial Laplacian on flat bundles and converges in eigenvalues to its analytic counterpart.

Alternatively, it may prove beneficial to define a more general combinatorial Laplacian or family of such combinatorial Laplacians that encapsulate both the Dodziuk and Patodi, and Zahariev constructions as special cases whilst at the same time having eigenvalues that converge to the eigenvalues of the analytic Laplacian.

11.5.2 Expanding Candidate Manifolds

So far, this sequence of converge results has been for manifolds that are connected and closed amongst other conditions. Naturally one might wonder if these conditions can be relaxed.

The non-compact setting is of particular interest as Sunada [25] studies the twisted Laplace operator, or discrete magnetic Laplacian (DML), acting on infinite graphs. The discrete magnetic Laplacian is the combinatorial analogue to the analytic magnetic Schrödinger operator. Therefore the non-compact setting as is of importance for physicists. Zahariev [30] provides some suggestions into paths of investigation in his Chapters 7 and 8.

Part IV

Appendix

APPENDIX A — JULES HENRI POINCARÉ *(French)*

Hasse Diagram for Partial Ordering

Figure A.1 provides a large Hasse diagram for the partial order with an explicit ordering for up to $q = 16$.

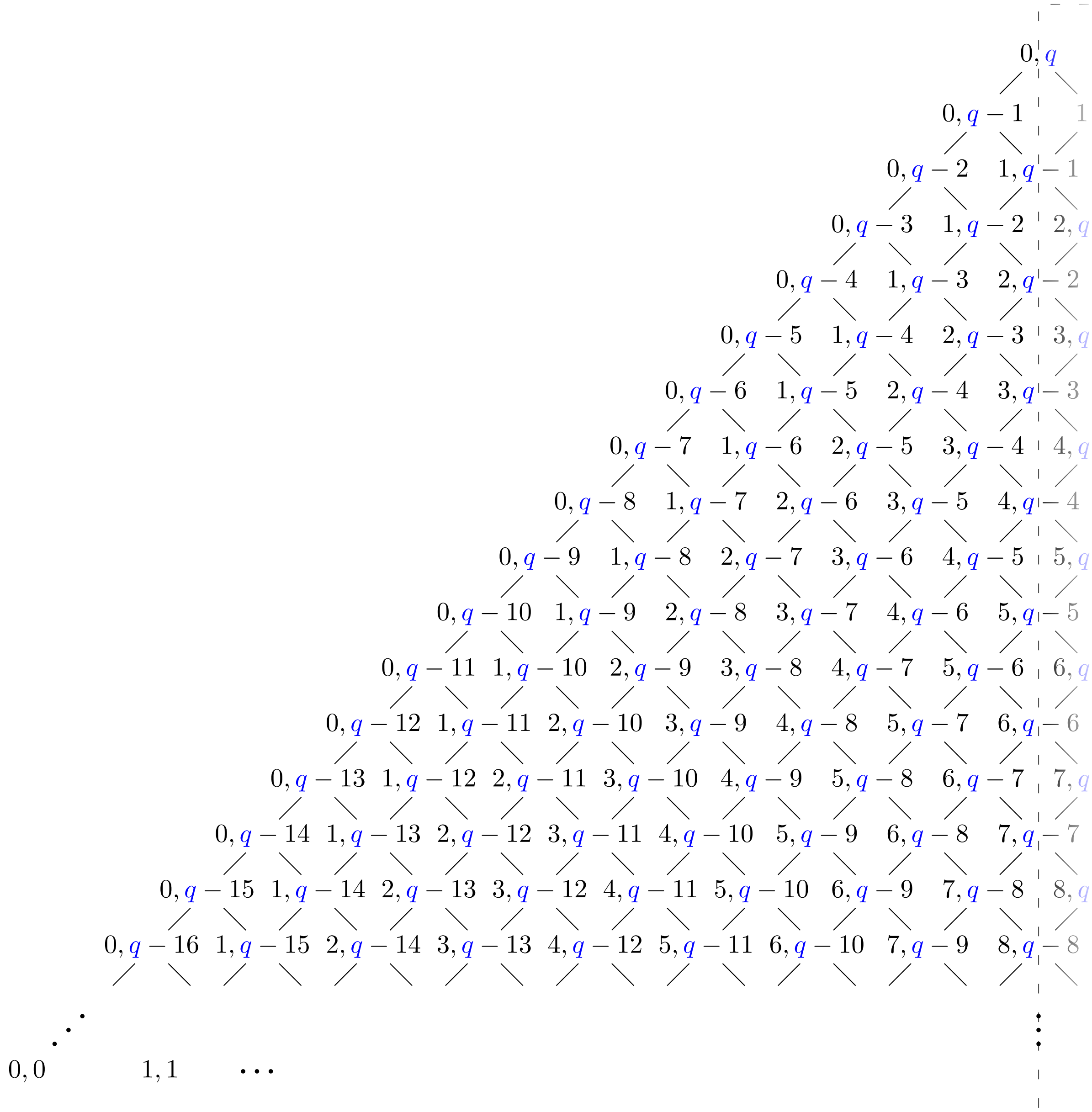
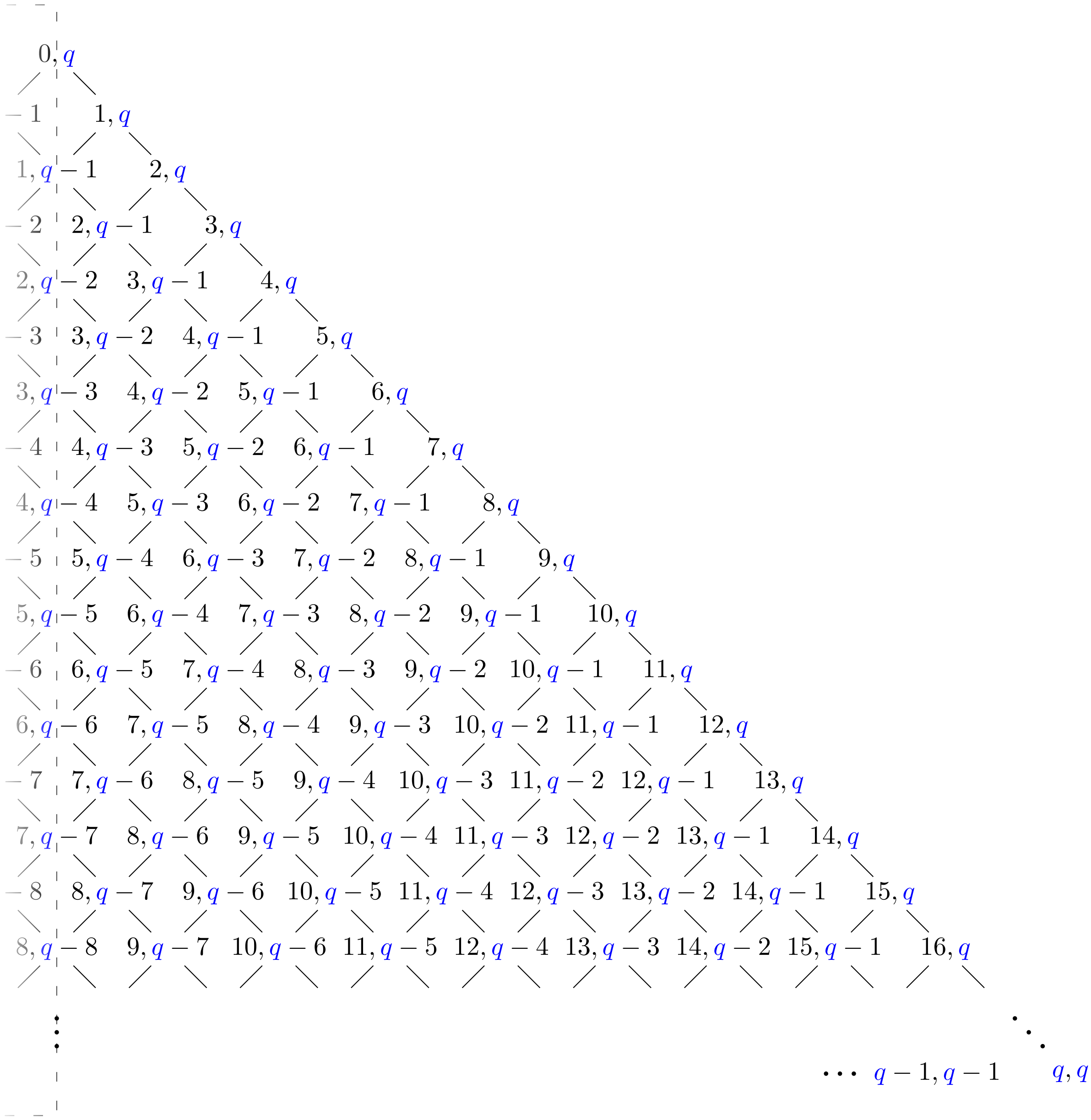


Figure A.1: Hasse diagram for partial ordering of vertices of $\mathfrak{S}\sigma$ where σ is a q-simplex. Note the small overlap of the figure across both pages to aid with reading.



0, q
−1 1, q
1, q − 1 2, q
−2 2, q − 1 3, q
2, q − 2 3, q − 1 4, q
−3 3, q − 2 4, q − 1 5, q
3, q − 3 4, q − 2 5, q − 1 6, q
−4 4, q − 3 5, q − 2 6, q − 1 7, q
4, q − 4 5, q − 3 6, q − 2 7, q − 1 8, q
−5 5, q − 4 6, q − 3 7, q − 2 8, q − 1 9, q
5, q − 5 6, q − 4 7, q − 3 8, q − 2 9, q − 1 10, q
−6 6, q − 5 7, q − 4 8, q − 3 9, q − 2 10, q − 1 11, q
6, q − 6 7, q − 5 8, q − 4 9, q − 3 10, q − 2 11, q − 1 12, q
−7 7, q − 6 8, q − 5 9, q − 4 10, q − 3 11, q − 2 12, q − 1 13, q
7, q − 7 8, q − 6 9, q − 5 10, q − 4 11, q − 3 12, q − 2 13, q − 1 14, q
−8 8, q − 7 9, q − 6 10, q − 5 11, q − 4 12, q − 3 13, q − 2 14, q − 1 15, q
8, q − 8 9, q − 7 10, q − 6 11, q − 5 12, q − 4 13, q − 3 14, q − 2 15, q − 1 16, q
··· q − 1, q − 1 q, q

Appendix B

Joseph-Louis Lagrange *(Italian)*

Matrices of Barycentric Coordinates

On the following pages are explicit expressions for the matrices $\widehat{\Xi}$ and Ξ from (5.4.6) and (5.4.14) respectively.

The matrices Ξ and $\widehat{\Xi}$ have the following form printed across two pages

$$
\widehat{\Xi} = \begin{array}{c}
\\
2 \\ 3 \\ 4 \\ \vdots \\ |K_n^0|-2 \\ |K_n^0|-1 \\ |K_n^0|
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccccc}
1,2 & 1,3 & 1,4 & 1,5 & \ldots & 1,|K_n^0|-3 & 1,|K_n^0|-2 & 1,|K_n^0|-1 & 1,|K_n^0| & 2,2 & 2,3 & 2,4 & 2,5 & \ldots & 2,|K_n^0|-3 & 2,|K_n^0|-2 & 2,|K_n^0|-1 & 2,|K_n^0|
\end{array} \\
\left[\begin{array}{cccccccccccccccccc}
\mathbb{I}_n & 0 & \cdots & & & & & & 0 & 2 & \mathbb{I}_n & \cdots & & & & & & \mathbb{I}_n \\
0 & \mathbb{I}_n & 0 & \cdots & & & & & 0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & & 0 \\
0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & 0 & 0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & 0 \\
\vdots & & & \ddots & \ddots & \ddots & & & \vdots & \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\
0 & \cdots & & & & 0 & \mathbb{I}_n & 0 & 0 & 0 & \cdots & & & & 0 & \mathbb{I}_n & 0 & 0 \\
0 & \cdots & & & & & 0 & \mathbb{I}_n & 0 & 0 & \cdots & & & & & 0 & \mathbb{I}_n & 0 \\
0 & \cdots & & & & & & 0 & \mathbb{I}_n & 0 & \cdots & & & & & & 0 & \mathbb{I}_n
\end{array}\right]
\end{array}
$$

and

$$
\Xi = \begin{array}{c}
\\
1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ |K_n^0|-2 \\ |K_n^0|-1 \\ |K_n^0|
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccc}
1,1 & 1,2 & 1,3 & 1,4 & 1,5 & \ldots & 1,|K_n^0|-3 & 1,|K_n^0|-2 & 1,|K_n^0|-1 & 1,|K_n^0| & 2,2 & 2,3 & 2,4 & 2,5 & \ldots & 2,|K_n^0|-3 & 2,|K_n^0|-2 & 2,|K_n^0|-1 & 2,|K_n^0|
\end{array} \\
\left[\begin{array}{ccccccccccccccccccc}
2 & \mathbb{I}_n & \cdots & & & & & & & \mathbb{I}_n & 0 & \cdots & & & & & & & 0 \\
0 & \mathbb{I}_n & 0 & \cdots & & & & & & 0 & 2 & \mathbb{I}_n & \cdots & & & & & & \mathbb{I}_n \\
0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & & 0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & & 0 \\
0 & \ldots & 0 & \mathbb{I}_n & 0 & \cdots & & & & 0 & 0 & 0 & \mathbb{I}_n & 0 & \cdots & & & & 0 \\
\vdots & & & & \ddots & \ddots & \ddots & & & \vdots & \vdots & & & \ddots & \ddots & \ddots & & & \vdots \\
0 & \cdots & & & & & 0 & \mathbb{I}_n & 0 & 0 & 0 & \cdots & & & & 0 & \mathbb{I}_n & 0 & 0 \\
0 & \cdots & & & & & & 0 & \mathbb{I}_n & 0 & 0 & \cdots & & & & & 0 & \mathbb{I}_n & 0 \\
0 & \cdots & & & & & & & 0 & \mathbb{I}_n & 0 & \cdots & & & & & & 0 & \mathbb{I}_n
\end{array}\right]
\end{array}
$$

Due to space constraints, the superscripts of the indicator functions $\mathbb{I}_n^{r,s}$ have been omitted. The superscripts of $\mathbb{I}_n$ are given by the column number. For example the $\mathbb{I}_n$ in $\widehat{\Xi}$ denotes $\mathbb{I}_n^{2,4}$ as it is in the $(2, 4)$ column.

It should be noted that the dots iterate over two variables in a nested fashion. For a given column number (r, s), the inner iteration is over the variable s and the outer iteration is over the variable r. The black dots indicate the inner iteration over s whilst the red dots indicate the outer iteration over r which in turn contains the inner iterations over s. The terms encircled in red correspond to the 'diagonal elements' if for each r, the iteration over s is viewed as a block matrix.

$$
\begin{array}{cccccccccccccccccccccc}
{\scriptstyle 3,3} & {\scriptstyle 3,4} & {\scriptstyle 3,5} & {\scriptstyle \ldots} & {\scriptstyle 3,|K_n^0|-3} & {\scriptstyle 3,|K_n^0|-2} & {\scriptstyle 3,|K_n^0|-1} & {\scriptstyle 3,|K_n^0|} & {\scriptstyle 4,4} & {\scriptstyle 4,5} & {\scriptstyle \ldots} & {\scriptstyle 4,|K_n^0|-3} & {\scriptstyle 4,|K_n^0|-2} & {\scriptstyle 4,|K_n^0|-1} & {\scriptstyle 4,|K_n^0|} & \cdots & {\scriptstyle |K_n^0|-2,|K_n^0|-2} & {\scriptstyle |K_n^0|-2,|K_n^0|-1} & {\scriptstyle |K_n^0|-2,|K_n^0|} & {\scriptstyle |K_n^0|-1,|K_n^0|-1} & {\scriptstyle |K_n^0|-1,|K_n^0|} & {\scriptstyle |K_n^0|,|K_n^0|}
\end{array}
$$

$$
\left.\begin{array}{cccccccccccccccccccccc}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
2 & \mathbb{I}_n & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2 & \mathbb{I}_n & \cdots & \cdots & \cdots & \cdots & \mathbb{I}_n & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots & & & & & & & & \ddots & & & & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & 0 & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & \cdots & 2 & \mathbb{I}_n & \mathbb{I}_n & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & \cdots & 0 & \mathbb{I}_n & 0 & 2 & \mathbb{I}_n & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & \cdots & 0 & 0 & \mathbb{I}_n & 0 & \mathbb{I}_n & 2
\end{array}\right], \qquad \text{(B.0.1)}
$$

$$
\begin{array}{cccccccccccccccccccccc}
{\scriptstyle 3,3} & {\scriptstyle 3,4} & {\scriptstyle 3,5} & {\scriptstyle \ldots} & {\scriptstyle 3,|K_n^0|-3} & {\scriptstyle 3,|K_n^0|-2} & {\scriptstyle 3,|K_n^0|-1} & {\scriptstyle 3,|K_n^0|} & {\scriptstyle 4,4} & {\scriptstyle 4,5} & {\scriptstyle \ldots} & {\scriptstyle 4,|K_n^0|-3} & {\scriptstyle 4,|K_n^0|-2} & {\scriptstyle 4,|K_n^0|-1} & {\scriptstyle 4,|K_n^0|} & \cdots & {\scriptstyle |K_n^0|-2,|K_n^0|-2} & {\scriptstyle |K_n^0|-2,|K_n^0|-1} & {\scriptstyle |K_n^0|-2,|K_n^0|} & {\scriptstyle |K_n^0|-1,|K_n^0|-1} & {\scriptstyle |K_n^0|-1,|K_n^0|} & {\scriptstyle |K_n^0|,|K_n^0|}
\end{array}
$$

$$
\left.\begin{array}{cccccccccccccccccccccc}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
2 & \mathbb{I}_n & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 2 & \mathbb{I}_n & \cdots & \cdots & \cdots & \cdots & \mathbb{I}_n & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots & & & & & & & & \ddots & & & & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & 0 & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & \cdots & 2 & \mathbb{I}_n & \mathbb{I}_n & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & 0 & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & \cdots & 0 & \mathbb{I}_n & 0 & 2 & \mathbb{I}_n & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \mathbb{I}_n & \cdots & 0 & 0 & \mathbb{I}_n & 0 & \mathbb{I}_n & 2
\end{array}\right]. \qquad \text{(B.0.2)}
$$

Appendix C — Nicolaus Copernicus *(Polish)*

Symbols

C.1 Greek Symbols

α	set of indexing variables	17
β	mesh reduction parameter	19
γ	eigenvalue of $\blacktriangle$	22, 94
Δ	Laplacian operator	9, 88
δ	combinatorial exterior derivative	19, 21, 89
δ^*	adjoint of δ	21
ε	rank of E	28, 87
ζ	collection of constants	52
η	collection of constants	47, 74
Θ	map on differential forms	24
θ	fullness, angle	20, 85
ϑ	constant dependent on n	25, 97
$\widehat{\vartheta}$	constant dependent on n	25
$\widetilde{\vartheta}$	function on cochains dependent on n	25, 57, 66
ι	inclusion, embedding	14, 90
ι^*	adjoint of embedding	90
κ	arbitrary constant	20
κ_A	constant dependent on A	24
$\kappa_{A,r}$	constant dependent on A and r	24
$\varkappa$	parameter of triangulation	39
λ	eigenvalue of Δ	9, 88
μ	barycentric coordinate function	11
$\boldsymbol{\mu}$	vector of barycentric coordinate functions	38, 43

ν set of indexing variables . . . 8, 40

Ξ barycentric coordinate function matrix . . . 42, 43
$\boldsymbol{\xi}$ vector of barycentric coordinate functions . . . 40, 43

π circle constant . . . 85
$\pi_1(\cdot)$ fundamental group . . . 28
ϖ differential form . . . 74, 87

ρ simplex . . . 45

Σ map on cochains . . . 24, 57
σ simplex . . . 10

τ simplex . . . 18

Υ orthonormal basis for T^*M . . . 8, 88

Φ vector bundle trivialisation . . . 92
φ coordinate chart, vector bundle trivialisation . . . 10, 91

χ set of indexing variables, matrix . . . 45, 91

Ψ combinatorial partition of unity . . . 92
ψ partition of unity . . . 90

$\Omega(\cdot,\cdot)$ smooth differential forms . . . 7, 87
ω differential form . . . 7, 87

C.2 Latin Symbols

A fixed real smooth 1-form . . . 9
a fixed 1-cochain . . . 21

B trivial vector bundle . . . 90
b rank of B . . . 90

$\mathbb{C}$ set of complex numbers . . . 7
$C^q(\cdot,\cdot)$ cochains . . . 10, 89
c cochain . . . 10
$\mathrm{cl}(\cdot)$ closure . . . 18

D matrix . . . 38, 40
d exterior derivative . . . 7, 9
d^* adjoint of d . . . 7, 9
$d_{g_\sigma}(\cdot,\cdot)$ local Euclidean distance on M . . . 19
$\det(\cdot)$ determinant . . . 47
$\dim(\cdot)$ dimension . . . 22
$d\,\mathrm{Vol}_g$ Riemannian volume form . . . 8

E	vector bundle over M	87, 90
e	edge	19
F	homeomorphism of triangulation	10
f	function	8
g	Riemannian metric on M, group element	7, 28
g_σ	local Euclidian metric on M	19
H	Hilbert space	9
$\mathcal{H}^q_r(\cdot,\cdot)$	Sobolev space of differential forms	8, 88
h	mesh of triangulation	19
I	combinatorial embedding	92
I^*	adjoint of combinatorial embedding	93
Id	identity operator	22
$\mathbb{I}$	indicator function	39
i	imaginary unit	9
$\inf$	infimum	9
$\mathrm{int}(\cdot)$	interior	13
j	indexing variable	9
K	triangulation of M	10
k	indexing variable	9
L	subcomplex of K	90
$\log(\cdot)$	natural logarithm	28
m	indexing variable	10
M	manifold	7
$\widetilde{M}$	universal cover of M	28
$\max$	maximum	11
$\min$	minimum	25
N	dimension of M	7
$\mathbb{N}$	set of natural numbers	8
n	iteration of subdivision of K_0	17
$\mathcal{O}$	representation of fundamental group	28
P	quasi projection map	22, 63
p	point in M	8, 15, 89
pr	projection operator	37
Q	domain of associated quadratic form	9
q	order of differential forms or cochains	7
R	de Rham map	21, 89, 93
R^{-1}	linear right inverse of R	22
$\mathrm{Re}(\cdot)$	real part	69

$\mathbb{R}$	set of real numbers	9
r	parameter of norm, indexing variable	8, 15
$\mathbb{S}^1$	circle	85
$\mathfrak{S}$	standard subdivision	15
s	indexing variable	9
span	span of vectors	74
sup	supremum	9
supp $(\cdot)$	support	49
TM	tangent space of M	8
T^*M	cotangent space of M	8, 87
$^\mathsf{T}$	transpose	39
U	local neighbourhood of M	8
$\mathcal{U}$	finite set of coordinate charts of M	18
u	linear operator	9
V	vector space	74
v	vertex, vector, differential form	11, 28, 69
vol $(\cdot)$	Riemannian volume	20
W	Whitney map	13, 89, 93
x	local coordinate, coordinate, vector	8, 38, 83
y	local coordinate, coordinate, vector	14, 38, 83
$\mathbf{y}$	vector of coordinates	38
$\mathbb{Z}$	set of integers	42

C.3 Other Symbols

$\mathbb{1}_{K^q}$	unity cochain	10, 52
∞	infinity	11
$\wedge$	exterior product	7, 87
$\otimes$	tensor product	87
$\cup$	cup product	21
$\subseteq$	subset (including equality)	11
$\supseteq$	superset (including equality)	47
$\sim$	equivalence relation	28
$\star$	Hodge star operator	7, 87
$\blacktriangle$	combinatorial Laplacian operator	21, 93
∇	covariant derivative	8, 87
∇^*	adjoint of ∇	88
$\blacktriangledown$	combinatorial covariant derivative	93
$\blacktriangledown^*$	adjoint of $\blacktriangledown$	93

∂	face map, partial derivate, boundary of space	18, 23, 58
$\emptyset$	empty set	45
$\overline{\cdot}$	complex conjugate	7
$!$	factorial	13
$\lvert\cdot\rvert$	absolute value, cardinality of set	8
$\lvert\cdot\rvert_p$	pointwise norm on $\Omega^q(\cdot,\cdot)$	8, 88
$\lvert\cdot\rvert_V$	Hermitian norm on E	87
$\lVert\cdot\rVert_r$	Sobolev norm	8, 88
$\lVert\cdot\rVert_r^\sigma$	local seminorm on differential forms depended on K	20
$\lVert\cdot\rVert_r^K$	global norm on differential forms depended on K	20, 56
$\lVert\cdot\rVert_{C,r}$	norm on $C^q(\cdot,\cdot)$	11, 89
$\lVert\cdot\rVert_W$	Whitney norm on $C^q(\cdot,\cdot)$	21
$\lVert\cdot\rVert_{\mathrm{op}}$	operator norm	24
$\langle\cdot,\cdot\rangle_0$	inner product on $\Omega^q(\cdot,\cdot)$	7, 87
$\langle\cdot,\cdot\rangle_C$	inner product on $C^q(\cdot,\cdot)$	10, 89
$\langle\cdot,\cdot\rangle_W$	Whitney inner product on $C^q(\cdot,\cdot)$	21, 90
$\langle\cdot,\cdot\rangle_V$	inner product on V	87
$\langle\cdot,\cdot\rangle_{\widetilde{W}}$	twisted Whitney inner product on $C^q(\cdot,\cdot)$	93

Appendix D — George Boole (English)

Index

For efficient finding of each term, in the thesis the referenced term is written in red.

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