AN INTRODUCTION TO DIRAC GEOMETRY

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Abstract. These notes are based on the introductory mini-course on Dirac Geometry given at the *10th International Young Researcher Workshop on Geometry, Mechanics and Control*, Institut Henri Poincaré, Paris, January 13-15, 2016. The notes are focused on canonical operations with Dirac structures, but they do not cover some important topics in Dirac geometry, e.g. twisted Dirac structures, cohomology and deformations, coisotropic submanifolds, representation by spinors, generalized complex structures, Lie algebroids, integration by presymplectic Lie groupoids etc. Also, many proofs are just sketched and left as exercises, or are even completely omitted. For more on Dirac Geometry, I warmly recommend [2, 4, 9].

Contents

1. Lagrangian subspaces in $(V \oplus V^*, (\cdot,\cdot))$

1.1. Let V be a real vector space of dimension n. We denote

$$\mathbb{V} := V \oplus V^* = \{v + \alpha \ : \ v \in V, \ \alpha \in V^*\}.$$

This space carries a bilinear symmetric 2-form of signature (n, n):

$$(\mathbb{V}, (\cdot,\cdot)), \quad (v + \alpha, w + \beta) := \alpha(w) + \beta(v).$$

1.2. A linear subspace $L \subset \mathbb{V}$ is said to be a **Lagrangian subspace** if it satisfies:
(a) $(L, L) = 0$;
(b) $\dim L = n$.

1.3. *Exercise.* Show that a subspace $L \subset \mathbb{V}$ is Lagrangian if and only if it satisfies the following maximality condition:

$$\forall \ \ v + \alpha \in \mathbb{V} \ : \ (L, v + \alpha) = 0 \iff v + \alpha \in L.$$

1.4. We denote the space of all Lagrangian subspaces of $\mathbb{V}$ by $\mathrm{Lag}(\mathbb{V})$. Then $\mathrm{Lag}(\mathbb{V})$ is a compact submanifold of the Grassmanian of n-subspaces in $\mathbb{V}$: $\mathrm{Gr}(n; \mathbb{V})$ of

$$\dim \mathrm{Lag}(\mathbb{V}) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Charts on $\mathrm{Lag}(\mathbb{V})$ will be described below.

1.5. *Examples.* $V \in \mathrm{Lag}(\mathbb{V})$ and $V^* \in \mathrm{Lag}(\mathbb{V})$. The **annihilator** of a linear subspace $W \subset V$ is denoted:

$$W^\circ := \{\alpha \in V^* \ : \ \alpha|_W = 0\}.$$

For every linear subspace $W \subset V$, we have that

$$L_W := W \oplus W^\circ \in \mathrm{Lag}(\mathbb{V}).$$

1.6. *Lemma.* Denote by $p_V : \mathbb{V} \to V$ the projection $p_V(v + \alpha) := v$. Then, for every $L \in \mathrm{Lag}(\mathbb{V})$, we have that

$$\ker p_V|_L = L \cap V^* = (p_V(L))^\circ.$$

Proof. The first equality is obvious. Denote $W := p_V(L)$. Note that:

$$(L, W^\circ) = (p_V(L), W^\circ) = (W, W^\circ) = 0.$$

By 1.3, we have that $W^\circ \subset L$; thus $W^\circ \subset L \cap V^* = \ker p_V|_L$. The other inclusion follows from the dimension count:

$$\dim(W^\circ) = n - \dim(W) = \dim(L) - \dim(p_V(L)) = \dim(\ker p_V|_L).$$

1.7. The Lagrangian subspaces $L_1, L_2 \in \mathrm{Lag}(\mathbb{V})$ are said to be **transverse**, denoted by $L_1 \pitchfork L_2$, if $L_1 + L_2 = \mathbb{V}$. This condition is equivalent $L_1 \cap L_2 = 0$, and also to $\mathbb{V} = L_1 \oplus L_2$. Such a decomposition of $\mathbb{V}$ is called a **Lagrangian splitting**, and L_2 is called a **Lagrangian complement** of L_1. Note that a Lagrangian splitting $\mathbb{V} = L_1 \oplus L_2$ gives a linear isomorphism

$$L_2 \xrightarrow{\sim} L_1^*,$$

$$v + \alpha \in L_2 \mapsto (v + \alpha, \cdot)|_{L_1} : L_1 \to \mathbb{R}.$$

1.8. *Exercise.* Prove that for every $L \in \mathrm{Lag}(\mathbb{V})$ has a Lagrangian complement.

1.9. *Example.* Let us determine the set of Lagrangian subspaces $L \in \mathrm{Lag}(\mathbb{V})$ transverse to V^*, i.e. $L \cap V^* = 0$. By Lemma 1.6, also equivalent to

$$p_V|_L : L \xrightarrow{\sim} V \quad \text{being a linear isomorphism.}$$

The inverse of this map has the form $v \mapsto v + Av$, where $A : V \to V^*$ is a linear map. Thus, L must be of the form:

$$L = \{v + Av \ : \ v \in V\}.$$

Condition (a), $(L, L) = 0$ gives:

$$Av(w) + Aw(v) = 0 \quad \forall\ v, w \in V;$$

which is equivalent to A being skew symmetric: $A^* = -A$, where $A^* : V \to V^*$. Such maps are in 1-1 correspondence with the space $\bigwedge^2 V^*$ of skew-symmetric 2-forms on V; the correspondence is given by:

$$\bigwedge^2 V^* \xrightarrow{\sim} \{A : V \to V^* \ : \ A^* = -A\},$$

$$\omega \mapsto A \iff Av = \iota_v \omega,$$

where $\iota_v : \bigwedge^2 V^* \to V^*$ denotes interior product with $v \in V$. Hence, L is of the form

$$V^\omega := \{v + \iota_v \omega \ : \ v \in V\} \in \mathrm{Lag}(\mathbb{V}).$$

We conclude that the map

$$\bigwedge^2 V^* \longrightarrow \mathrm{Lag}(\mathbb{V}), \quad \omega \mapsto V^\omega$$

gives a local parameterization of the open subset of $\mathrm{Lag}(\mathbb{V})$ consisting of Lagrangian subspaces L so that $L \pitchfork V^*$.

1.10. *Example.* Interchanging the roles of V and V^* in 9, we conclude that $\bigwedge^2 V$ parameterizes the open neighborhood of V^* in $\mathrm{Lag}(\mathbb{V})$ consisting of Lagrangian subspaces transverse to V. Namely, for $\pi \in \bigwedge^2 V$, we denote the induced skew-symmetric map by

$$\pi^\sharp : V^* \longrightarrow V, \quad \pi^\sharp(\alpha)(\beta) := \pi(\alpha, \beta),$$

and denote the corresponding Lagrangian subspace by:

$$V^*_\pi := \{\pi^\sharp(\alpha) + \alpha \ : \ \alpha \in V^*\}.$$

Then, these Lagrangian subspaces transverse to V are parameterized by:

$$\bigwedge^2 V \longrightarrow \mathrm{Lag}(\mathbb{V}), \quad \pi \mapsto V^*_\pi.$$

1.11. By Exercise 1.8, any Lagrangian $L \in \mathrm{Lag}(\mathbb{V})$ has a Lagrangian complement C, and by 1.7, $C \cong L^*$. The analysis from 9 can be applied to parameterize an open neighborhood of L by the vector space $\bigwedge^2 L^*$. This produces charts on $\mathrm{Lag}(\mathbb{V})$, which can be used to construct a **smooth atlas** on the manifold $\mathrm{Lag}(\mathbb{V})$.

1.12. *Main Example.* This is a combination of 1.5 and 1.9. Consider a linear subspace $W \subset L$ and a 2-form $\omega \in \bigwedge^2 W^*$. Define:

$$L(W, \omega) := \{v + \iota_v\widetilde{\omega} + \alpha \ : \ v \in W \ \ \alpha \in W^\circ\},$$

where $\widetilde{\omega} \in \bigwedge^2 V^*$ is any 2-form extending ω:

$$\widetilde{\omega}|_W = \omega.$$

Note that the construction is independent of the chosen extension: If $\widetilde{\omega}' \in \bigwedge^2 V^*$ is a second such extension, the for any $v \in V$ we have that

$$\eta(v) := \iota_v(\widetilde{\omega}' - \widetilde{\omega}) \in W^\circ,$$

and therefore

$$v + \iota_v\widetilde{\omega}' + \alpha = (v + \iota_v\omega) + (\eta(v) + \alpha) \in L(W, \omega).$$

It is easy to see that

$$L(W, \omega) \in \mathrm{Lag}(\mathbb{V}).$$

1.13. *Theorem.* For every $L \in \mathrm{Lag}(\mathbb{V})$ there exists a unique subspace $W \subset V$ and a unique 2-form $\omega \in \bigwedge^2 W^*$ so that $L = L(W, \omega)$. These are defined as follows:

$$W := p_V(L), \quad \omega(v_1, v_2) := \alpha_1(v_2) = -\alpha_2(v_1), \quad v_1 + \alpha_1, \ v_2 + \alpha_2 \in L.$$

Proof. Let $W := p_V(L)$. By Lemma 1.6, we have the exact sequence of vector spaces:

$$0 \longrightarrow W^\circ \longrightarrow L \xrightarrow{p_V} W \longrightarrow 0.$$

A splitting of the surjection $p_V|_L : L \to W$ has the form

$$W \longrightarrow L, \quad v \mapsto v + Av,$$

where $A : W \to V^*$. Fixing such a splitting, L can be written as:

$$L = \{v + Av + \eta \ : \ v \in W, \ \ \eta \in W^\circ\}.$$

Condition $(L, L) = 0$ is equivalent to $(Av)(w) + (Aw)(v) = 0$ for all $v, w \in W$. Therefore, the map

$$W \longrightarrow W^*, \quad v \mapsto Av|_W$$

is skew-symmetric, and so, it is given by a 2-form $\omega \in \bigwedge^2 W^*$:

$$Av|_W = \iota_v\omega.$$

If $\widetilde{\omega} \in \bigwedge^2 V^*$ an extension of ω, then, $Av - \widetilde{\omega}_v \in W^\circ$ for all $v \in W$. This implies that:

$$L = \{v + Av + \eta \ : \ v \in W, \ \ \eta \in W^\circ\} = \{v + \iota_v\widetilde{\omega} + \eta \ : \ v \in W, \ \ \eta \in W^\circ\} = L(W, \omega).$$

Finally, consider $v_1 + \alpha_1, v_2 + \alpha_2 \in L$. Then $\alpha_i = \iota_{v_i}\omega + \eta_i$, with $\eta_i \in W^\circ$; and therefore:

$$\alpha_1(v_2) = \iota_{v_1}\omega(v_2) = \omega(v_1, v_2) = \ldots = -\alpha_2(v_1).$$

1.14. *Exercise.* Prove that $\mathrm{Lag}(\mathbb{V})$ has two connected components which can be described by:

$$\mathrm{Lag}(\mathbb{V})^{\mathrm{od}} = \{L(W, \omega) \ : \ \dim(W) \equiv 1 \bmod 2\}$$
$$\mathrm{Lag}(\mathbb{V})^{\mathrm{ev}} = \{L(W, \omega) \ : \ \dim(W) \equiv 0 \bmod 2\}.$$

1.15. *Scalar multiplication.* We define an action of $(\mathbb{R}\backslash\{0\}, \cdot)$ on $\mathrm{Lag}(\mathbb{V})$:

$$t \cdot L := \{v + t\alpha \ : \ v + \alpha \in L\}, \quad t \neq 0, \ L \in \mathrm{Lag}(\mathbb{V}).$$

In fact this action is generated by a vector field $X \in \mathfrak{X}(\mathrm{Lag}(\mathbb{V}))$, whose flow is

$$\phi^t_X(L) = e^t \cdot L.$$

Writing $L = L^\omega_W$, as in Theorem 1.13, scalar multiplication becomes just rescaling of the 2-form ω:

$$t \cdot L(W, \omega) = L(W, t\omega).$$

1.16. *Exercise.* Prove that the zeroes of the vector field X from 1.15 are given by

$$X_L = 0 \iff L \in \{L_W \ : \ W \subset V\}.$$

1.17. *Product of Lagrangian subspaces.* One can define a product on $\mathrm{Lag}(\mathbb{V})$:

$$\star : \mathrm{Lag}(\mathbb{V}) \times \mathrm{Lag}(\mathbb{V}) \longrightarrow \mathrm{Lag}(\mathbb{V}),$$

$$L_1 \star L_2 = \{v + \alpha_1 + \alpha_2 \ : \ v + \alpha_1 \in L_1, \ v + \alpha_2 \in L_2\}.$$

Let us check that $L_1 \star L_2 \in \mathrm{Lag}(\mathbb{V})$. Let $v + \alpha_1$, $w + \beta_1 \in L_1$ and $v + \alpha_2$, $w + \beta_2 \in L_2$. Then we have that:

$$(v + \alpha_1 + \alpha_2, w + \beta_1 + \beta_2) = (v + \alpha_1, w + \beta_1) + (v + \alpha_2, w + \beta_2) = 0;$$

therefore $(L_1 \star L_2, L_1 \star L_2) = 0$. Denote $W_i := p_V(L_i)$. By the definition of the product, we have that:

$$p_V(L_1 \star L_2) = W_1 \cap W_2.$$

On the other hand $\ker(p_V|_{L_1 \star L_2})$ consists of elements $\alpha_1 + \alpha_2$, with $\alpha_i \in L_i \cap V^*$, which, by Lemma 6, is equivalent to $\alpha_i \in W_i^\circ$. We conclude that $\ker(p_V|_{L_1 \star L_2}) = W_1^\circ + W_2^\circ$, hence, $L_1 \star L_2$ fits in the short exact sequence:

$$0 \longrightarrow W_1^\circ + W_2^\circ \longrightarrow L_1 \star L_2 \longrightarrow W_1 \cap W_2 \longrightarrow 0.$$

Since $W_1^\circ + W_2^\circ = (W_1 \cap W_2)^\circ$, we obtain that:

$$\dim(L_1 \star L_2) = \dim(W_1 \cap W_2) + \dim((W_1 \cap W_2)^\circ) = n.$$

Hence, $L_1 \star L_2 \in \mathrm{Lag}(\mathbb{V})$.

1.18. *Exercise.* Prove that, in the description of 1.13:

$$L_1 = L(W_1, \omega_1) \quad \text{and} \quad L_2 = L(W_2, \omega_2),$$

the product has the following geometric description:

$$L(W_1, \omega_1) \star L(W_2, \omega_2) = L(W_1 \cap W_2, \omega_1 + \omega_2).$$

1.19. *Properties of the product.* Exercise 16 can be used to prove that the product satisfies the following algebraic properties:

$$\begin{aligned} &L_1 \star L_2 = L_2 \star L_1 \\ &(L_1 \star L_2) \star L_3 = L_1 \star (L_2 \star L_3) \\ &L \star V = L = V \star L \\ &L \star V^* = V^* = V^* \star L \\ &t \cdot (L_1 \star L_2) = (t \cdot L_1) \star (t \cdot L_2) \end{aligned}$$

for all $L_1, L_2, L_3 \in \mathrm{Lag}(\mathbb{V})$ and $t \neq 0$.

1.20. The product $\star$ is **not a smooth map** on the entire $\mathrm{Lag}(\mathbb{V}) \times \mathrm{Lag}(\mathbb{V})$, but it is smooth when restricted to the open set:

$$\mathcal{U} := \{(L_1, L_2) \ : \ p_V(L_1) + p_V(L_2) = V\} \subset \mathrm{Lag}(\mathbb{V}) \times \mathrm{Lag}(\mathbb{V}).$$

Two Lagrangian subspaces L_1, L_2 so that $p_V(L_1) + p_V(L_2) = V$ (i.e. $(L_1, L_2) \in \mathcal{U}$) will be called **tangentially-transverse**.

1.21. *Exercise.* Prove that the product restricted to the set $\mathcal{U}$, defined in 1.20, is smooth.

1.22. *Gauge transformations.* Note that for any $\omega \in \bigwedge^2 V^*$, and any $L \in \mathrm{Lag}(\mathbb{V})$, we have that V^ω and L are t-transverse. Their product is denoted by:

$$L^\omega := V^\omega \star L = \{v + \iota_v \omega + \alpha \ : \ v + \alpha \in L\},$$

and is called the **gauge transformation** of L by ω. Note that the elements V^ω are precisely the invertible elements for the product $\star$.

1.23. Consider $W \subset V$ and $\omega \in \bigwedge^2 W^*$. If $\widetilde{\omega} \in \bigwedge^2 V^*$ is an extension of ω, then we have the decomposition:

$$L(W, \omega) = L_W^{\widetilde{\omega}}.$$

1.24. *Push forward and pull back operations.* Consider now a second vector space U and denote $\mathbb{U} := U \oplus U^*$. Let $A : V \to U$ be a linear map. The **push forward** along A is the map:

$$\begin{aligned} &A_* : \mathrm{Lag}(\mathbb{V}) \longrightarrow \mathrm{Lag}(\mathbb{U}), \\ &A_*(L) := \{Av + \beta \ : \ v + A^*\beta \in L\}. \end{aligned}$$

The **pull back** along A is the map:

$$\begin{aligned} &A^* : \mathrm{Lag}(\mathbb{U}) \longrightarrow \mathrm{Lag}(\mathbb{V}), \\ &A^*(L) := \{v + A^*\beta \ : \ Av + \beta \in L\}. \end{aligned}$$

1.25. *Exercise.* The following items give a proof that the push forward and pull-back maps are well-defined:

(a) For $W \subset \mathrm{Lag}(\mathbb{U})$ and $\omega \in \bigwedge^2 W^*$, denote

$$A^*\omega := (A|_{A^{-1}W})^*\omega \in \bigwedge^2 (A^{-1}W)^*.$$

Prove that

$$A^*(L(W, \omega)) = L(A^{-1}W, A^*\omega).$$

The following relation might be useful: $(A^{-1}W)^\circ = A^*(W^\circ)$.
Conclude that $A^*(L) \in \mathrm{Lag}(\mathbb{V})$ for all $L \in \mathrm{Lag}(\mathbb{U})$.

(b) By exchanging the roles of V and V^*, and of U and U^*, conclude also that $A_*(L) \in \mathrm{Lag}(\mathbb{U})$ for all $L \in \mathrm{Lag}(\mathbb{V})$.

1.26. A linear map $A : V \to U$ is said to be **transverse** to a linear subspace $W \subset U$, denoted $A \pitchfork W$, if

$$AV + W = U.$$

The pull back operation is not smooth globally (in A and L); an open set on which it is smooth is given by pairs of transverse maps and Lagrangian subspaces:

$$\mathcal{O} := \{(A, L) \ : \ A \pitchfork p_U(L)\} \subset \mathrm{Lin}(V, U) \times \mathrm{Lag}(\mathbb{U}).$$

1.27. *Exercise.* Prove that the pull back restricts to a smooth map on open set $\mathcal{O}$, defined in 1.26:

$$\mathcal{O} \longrightarrow \mathrm{Lag}(\mathbb{V}), \quad (A, L) \mapsto A^*(L).$$

1.28. *Exercise.* Let $A : V \to U$ be a linear map. Prove the following relations for the pullback along A:

$$\begin{aligned} &A^*(U) = V, && A^*(U^*) = L_{\ker(A)}, \\ &A^*(U^\omega) = V^{A^*(\omega)}, && A^*(L_W) = L_{A^{-1}W}, \\ &A^*(L_1 \star L_2) = A^*(L_1) \star A^*(L_2). \end{aligned}$$

where $L_1, L_2 \in \mathrm{Lag}(\mathbb{V})$, $\omega \in \bigwedge^2 U^*$, and $W \subset U$.

Prove the following relations for the push forward along A:

$$A_*(V) = L_{AV}, \quad A_*(V^*) = U^*, \quad A_*(L_W) = L_{AW},$$

where $W \subset V$ is a linear subspace. Show that in general

$$A_*(L_1 \star L_2) \neq A_*(L_1) \star A_*(L_2)$$

Hint: use Lagrangian spaces of the form L_W.

1.29. *Duality.* Note that V and V^* play symmetric roles; in fact for any construction there is a dual construction, in which V and V^* are interchanged. For example, a dual version of Theorem 1.13 is: for every Lagrangian subspace $L \in \mathrm{Lag}(\mathbb{V})$ there exists a unique subspace $W \subset V$ and an unique element $\pi \in \bigwedge^2 V/W$ so that

$$L = L(W^\circ, \pi) = \{v + \widetilde{\pi}^\sharp(\alpha) + \alpha \ : \ \alpha \in W^\circ, \ v \in W\}.$$

Also, one can define a dual product

$$\circledast : \mathrm{Lag}(\mathbb{V}) \times \mathrm{Lag}(\mathbb{V}) \longrightarrow \mathrm{Lag}(\mathbb{V})$$

$$L_1 \circledast L_2 = \{v_1 + v_2 + \alpha \ : \ v_1 + \alpha \in L_1, \ v_2 + \alpha \in L_2\},$$

for which the dual versions of 20-27 hold; in particular

$$A_*(L_1 \circledast L_2) = A_*(L_1) \circledast A_*(L_2).$$

2. Dirac structures: definition and examples

2.1. Let M be a smooth manifold of $\dim(M) = n$. Denote the direct sum of the tangent and cotangent bundle of M by

$$\mathbb{T}M := TM \oplus T^*M = \{v + \alpha \ : \ v \in T_pM, \ \alpha \in T^*_pM, \ p \in M\}.$$

This vector bundle carries a nondegenerate symmetric bilinear 2-form of signature (n, n)

$$(\mathbb{T}M, (\cdot, \cdot)),$$

defined on each fiber $\mathbb{T}_pM = T_pM \oplus T^*_pM$ as in the previous section.

2.2. On the space of sections of $\mathbb{T}M$ we define the so-called *Dorfman bracket*, which extends the Lie bracket of vector fields:

$$[\cdot,\cdot] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \longrightarrow \Gamma(\mathbb{T}M),$$
$$[v+\alpha, w+\beta] := [v,w] + \mathscr{L}_v\beta - \iota_w \mathrm{d}\alpha,$$

for

$$v, w \in \mathfrak{X}(M) := \Gamma(TM) \quad \text{and} \quad \alpha, \beta \in \Omega^1(M) := \Gamma(T^*M).$$

Using *Cartan's magic formula* for the Lie derivative:

$$\mathscr{L}_x = \mathrm{d} \circ \iota_x + \iota_x \circ \mathrm{d}, \quad x \in \mathfrak{X}(M),$$

we can rewrite the Dorfman bracket in the following ways

$$\begin{aligned} [v+\alpha, w+\beta] =& [v,w] + \mathscr{L}_v\beta - \iota_w\mathrm{d}\alpha = \\ =& [v,w] + \mathscr{L}_v\beta - \mathscr{L}_w\alpha + \mathrm{d}(\alpha, w) = \\ =& [v,w] + \iota_v\mathrm{d}\beta - \iota_w\mathrm{d}\alpha + \mathrm{d}(\beta, v), \end{aligned}$$

each expression being useful in computations.

2.3. The Dorfman bracket satisfies the following relations:

(a) The Leibniz type relation:

$$[a_1, fa_2] = \mathscr{L}_{p_T(a_1)}(f)a_2 + f[a_1, a_2].$$

(b) It is skew-symmetric up to an exact form:

$$[a_1, a_2] + [a_2, a_1] = \mathrm{d}(a_1, a_2).$$

(c) It satisfies the Jacobi identity written in the following form:

$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]].$$

(d) It preserves the metric $(\cdot,\cdot)$ in the following sense:

$$\mathscr{L}_{p_T(a_1)}(a_2, a_3) = ([a_1, a_2], a_3) + (a_2, [a_1, a_3]),$$

for all $a_1, a_2, a_3 \in \Gamma(\mathbb{T}M)$ and all $f \in C^\infty(M)$, where we have denoted

$$p_T : \mathbb{T}M \longrightarrow TM, \quad p_T(v+\alpha) = v.$$

The Dorfman bracket is **not a Lie bracket**, in the sense that $(\Gamma(\mathbb{T}M), [\cdot,\cdot])$ is not a Lie algebra, because the bracket is not skew-symmetric (b). The relations above are precisely the axioms of a so-called *Courant Lie algebroid*, but this discussion is outside of the scope of this lectures.

2.4. *Exercise.* Prove the relations from 2.3.

2.5. A vector subbundle $L \subset \mathbb{T}M$ is called a **Lagrangian subbundle** if for every $p \in M$, we have that L_p is a Lagrangian subspace of $\mathbb{T}_pM$; equivalently, L satisfies

(a) $(L, L) = 0$;

(b) $\mathrm{rank}(L) = n = \dim(M)$.

A Lagrangian subbundle is also called an **almost Dirac structure** on M.

2.6. A Lagrangian subbundle $L \subset \mathbb{T}M$ is called a **Dirac structure** on M, if it is **involutive**:

$$[a_1, a_2] \in \Gamma(L), \quad \text{for all } a_1, a_2 \in \Gamma(L).$$

We denote the set of all Dirac structures on M by $\mathrm{Dir}(M)$.

2.7. If $L \in \mathrm{Dir}(M)$, then for all $a_1, a_2 \in \Gamma(L)$, we have that $(a_1, a_2) = 0$; therefore, by 2.3 (b), we have that the bracket is skew-symmetric on $\Gamma(L)$: $[a_1, a_2] = -[a_2, a_1]$. Thus, we have that $(\Gamma(L), [\cdot,\cdot])$ is a Lie algebra.

2.8. *Exercise.* Let $L \subset \mathbb{T}M$ be a Lagrangian subbundle. Prove that the map

$$\Gamma(L) \times \Gamma(L) \times \Gamma(L) \longrightarrow C^{\infty}(M), \quad (a_1, a_2, a_3) \mapsto ([a_1, a_2], a_3)$$

is $C^{\infty}(M)$-multi-linear and alternating. Conclude that there is unique 3-form on L

$$\Upsilon_L \in \Gamma\Big(\bigwedge^3 L^*\Big)$$

so that for all $a_1, a_2, a_3 \in \Gamma(L)$ and all $p \in M$ we have that:

$$(\Upsilon_L)_p(a_{1,p}, a_{2,p}, a_{3,p}) = ([a_1, a_2], a_3)_p.$$

2.9. Recall that, by 1.2, an element $b \in \mathbb{T}_pM$ we have that

$$b \in L_p \iff (b, L_p) = 0.$$

Therefore, by the previous exercise, we have that:

Proposition: A Lagrangian subspace $L \subset \mathbb{T}M$ is a Dirac structure on M if and only if $\Upsilon_L = 0$.

This has the immediate consequence:

Corollary: Let $L \subset \mathbb{T}M$ be a Lagrangian subspace. If there exists an open dense subset $U \subset M$ so that $L|_U \subset \mathbb{T}U$ is a Dirac structure, then L is a Dirac structure.

2.10. *Examples.* We have that $TM \in \mathrm{Dir}(M)$, and the bracket on $\mathfrak{X}(M) = \Gamma(TM)$ is the usual bracket on vector fields. Also $T^*M \in \mathrm{Dir}(M)$, and the bracket on $\Omega^1(M) = \Gamma(T^*M)$ is trivial.

2.11. *Presymplectic forms.* Consider Lagrangian subspaces $L \subset \mathbb{T}M$ so that $L \pitchfork T^*M$, i.e. $L_p \cap T^*_pM = 0$ for all $p \in M$. By example 1.9, there exists a unique 2-form

$$\omega \in \Omega^2(M) = \Gamma(\bigwedge^2 T^*M)$$

so that

$$L = TM^{\omega} = \{v + \iota_v\omega \;:\; v \in TM\}.$$

Let us prove that integrability of TM^{ω} is equivalent to ω being closed:

$$TM^{\omega} \in \mathrm{Dir}(M) \iff \mathrm{d}\omega = 0.$$

For all $v, w \in \mathfrak{X}(M)$ we have that:

$$\begin{aligned}[v + \iota_v\omega, w + \iota_w\omega] &= [v, w] + \mathscr{L}_v\iota_w\omega - \iota_w \mathrm{d}\iota_v\omega = \\ &= [v, w] + (\mathscr{L}_v\iota_w - \iota_w\mathscr{L}_v)\omega + \iota_w\iota_v\mathrm{d}\omega = \\ &= [v, w] + \iota_{[v,w]}\omega + \iota_w\iota_v\mathrm{d}\omega,\end{aligned}$$

where we have used the relations

$$\mathscr{L}_w = \mathrm{d} \circ \iota_w + \iota_w \circ \mathrm{d} \quad \text{and} \quad \mathscr{L}_v \circ \iota_w - \iota_w \circ \mathscr{L}_v = \iota_{[v,w]}.$$

The last term belongs to TM^{ω} iff $\iota_w\iota_v\mathrm{d}\omega = 0$ for all $v, w \in \mathfrak{X}(M)$, which is equivalent to $\mathrm{d}\omega = 0$.

Thus, every closed 2-form $\omega \in \Omega^2_{\mathrm{cl}}(M)$ can be regarded as a Dirac structure on M. The pair (M, ω) is also called a **presymplectic manifold**, and ω a **presymplectic form**; and if ω is nondegenerate, then it called a **symplectic** form, and (M, ω) a **symplectic manifold**.

2.12. *Foliations.* Recall that a (regular) **foliation** $\mathcal{F}$ of dimension r on M is a decomposition

$$M = \bigsqcup_{\lambda \in \Lambda} F_\lambda,$$

where each $F_\lambda \subset M$ is a connected, regularly immersed submanifold of M with $\dim(F_\lambda) = r$, and around every point in M there is a chart

$$(U, \chi), \quad \chi : U \xrightarrow{\sim} \mathbb{R}^r \times \mathbb{R}^{n-r}, \quad \chi(p) = (x(p), y(p))$$

so that the connected components of $F_\lambda \cap U$ correspond under χ to the parallel r-planes $\mathbb{R}^r \times \{y_0\}$, with $y_0 \in \mathbb{R}^{n-r}$. The submanifold F_λ are called the **leaves** of the foliation.

A smooth subbundle $W \subset TM$ (also called a distribution on M) is said to be **involutive**, if

$$[\Gamma(W), \Gamma(W)] \subset \Gamma(W).$$

Recall: **Frobenius Theorem**. Involutive subbundles $W \subset TM$ of $\mathrm{rank}(W) = r$ are in 1-1 correspondence with r-dimensional foliations $\mathcal{F}$ on M. Namely if $\mathcal{F} = \{F_\lambda\}_{\lambda \in \Lambda}$ is a foliation, then the corresponding involutive distribution is given by

$$T\mathcal{F} := \bigsqcup_{\lambda \in \Lambda} TF_\lambda \subset TM.$$

Conversely, let $W \subset TM$ be an involutive distribution. Then the corresponding foliation $\mathcal{F}$ with leaves $\{F_\lambda\}_{\lambda \in \Lambda}$ can be described as follows: two point $p, q \in M$ belong to the same leaf iff there exists a smooth path $\gamma : [0, 1] \to M$ so that:

$$\gamma(0) = p, \quad \gamma(1) = q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in W_{\gamma(t)}, \ \forall \ t \in [0, 1].$$

To any distribution $W \subset TM$ one can associate the Lagrangian subbundle

$$L_W := W \oplus W^\circ \subset \mathbb{T}M.$$

We have that L_W is a Dirac structure on M if and only if W is involutive. Thus, the class of Dirac structures on M includes foliations on M.

2.13. *Exercise.* For a distribution $W \subset TM$, prove that $L_W \in \mathrm{Dir}(M)$ iff W is involutive.

2.14. *Poisson structures.* A **Poisson structure** on M is a bilinear operation

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M),$$

satisfying:

(a) skew-symmetry:

$$\{f, g\} = -\{g, f\},$$

(b) the Leibniz-type relation:

$$\{f, gh\} = g\{f, h\} + h\{f, g\},$$

(c) and the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Condition (b) says that the map

$$X_f : C^\infty(M) \longrightarrow C^\infty(M),$$
$$g \mapsto X_f(g) = \{f, g\}$$

is a vector field on M (i.e. a **derivation** of $C^\infty(M)$); this vector field is called the **Hamiltonian vector field** of f:

$$X_f \in \mathfrak{X}(M).$$

Conditions (a) and (c) say that $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra.

Operations satisfying (a) and (b) are in 1-1 correspondence with **bivector fields** on M, i.e. sections of the bundle $\bigwedge^2 TM$; this correspondence is given by

$$\pi \in \mathfrak{X}^2(M) \longleftrightarrow \{\cdot,\cdot\},$$
$$\{f,g\} := \pi(\mathrm{d}f, \mathrm{d}g),$$

where we denote by $\mathfrak{X}^k(M) := \Gamma(\bigwedge^k TM)$. There is a natural extension of the Lie bracket of vector fields to an operation

$$[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{k+l-1}(M),$$

called the **Schouten bracket**. The Jacobi identity (c) can be written in terms of the bivector field π using this operation as

$$[\pi, \pi] = 0.$$

The Hamiltonian of $f \in C^\infty(M)$ can be written in terms of the bivector as:

$$X_f = \pi^\sharp(\mathrm{d}f),$$

where $\pi^\sharp : T^*M \to TM$ is π viewed as a skew-symmetric map.

2.15. *Poisson structures as Dirac structures.* By 1.10, for any Lagrangian subbundle $L \subset \mathbb{T}M$ satisfying $L \pitchfork TM$ there is a unique bivector field $\pi \in \mathfrak{X}^2(M)$ so that

$$L = T^*M_\pi = \{\pi^\sharp(\alpha) + \alpha \ : \ \alpha \in T^*M\}.$$

Let us prove that $T^*M_\pi \in \mathrm{Dir}(M)$ iff π is a Poisson structure. For this, we show that the 3-tensor $\Upsilon_{T^*M_\pi}$ from 2.8 vanishes iff π is Poisson. Since T^*M_π is spanned by elements of the form $X_f + \mathrm{d}f$, it suffices calculate $\Upsilon_{T^*M_\pi}$ on such sections. Using the third formula for the Dorfman bracket in 2.2, we obtain:

$$\begin{aligned}\Upsilon_{T^*M_\pi}(X_f + \mathrm{d}f, X_g + \mathrm{d}g, X_h + \mathrm{d}h) &= ([X_f + \mathrm{d}f, X_g + \mathrm{d}g], X_h + \mathrm{d}h) = \\ &= ([X_f, X_g] + \mathrm{d}\iota_{X_f}\mathrm{d}g, X_h + \mathrm{d}h) = \\ &= [X_f, X_g](h) + X_h(X_f(g)) = \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} + \{h, \{f, g\}\}.\end{aligned}$$

This shows that $\Upsilon_{T^*M_\pi} = 0$ iff $\{\cdot,\cdot\}$ satisfies the Jacobi identity, which is equivalent to π being a Poisson structure. We conclude the Dirac structures includes also the class of Poisson structures.

2.16. *Nondegenerate Poisson structures and symplectic structures.* Let $\omega \in \Omega^2(M)$ be a symplectic structure. Then $\pi := \omega^{-1} \in \mathfrak{X}^2(M)$ is a **nondegenerate Poisson structure**; and conversely the inverse of a non-degenerate Poisson structure is a symplectic structure. This follows because both types of structures are encoded by Dirac structures $L \in \mathrm{Dir}(M)$ so that $L \pitchfork TM$ and $L \pitchfork T^*M$.

2.17. *Regular Dirac structures.* The **rank** of a Lagrangian subbundle $L \subset \mathbb{T}M$ at $p \in M$ is defined as:

$$\mathrm{rank}(L, p) := \dim(p_T(L_p));$$

and L is said to be **regular** is its rank is constant.

To a regular Lagrangian subbundle $L \subset \mathbb{T}M$ one can associate the (smooth) distribution

$$W := p_T(L) \subset TM,$$

and the (smooth) 2-form on the vector bundle W:

$$\omega \in \Omega^2(W) := \Gamma\Big(\bigwedge^2 W^*\Big),$$
$$\omega_p(v_1, v_2) := \alpha_1(v_2) = -\alpha_2(v_1), \quad v_1 + \alpha_1,\ v_2 + \alpha_2 \in L_p.$$

Using Theorem 1.13, one can prove that L is given by:

$$L = L(W,\omega) = \{v + \iota_v\widetilde{\omega} + \alpha \ : \ \alpha \in W_p^\circ,\ v \in W_p,\ p \in M\},$$

where $\widetilde{\omega} \in \Omega^2(M)$ is a 2-form extending ω:

$$\widetilde{\omega}_p(v_1, v_2) = \omega_p(v_1, v_2), \quad \forall\ v_1, v_2 \in W_p.$$

Then we have that $L(W,\omega) \in \mathrm{Dir}(M)$ iff

(a) W is an involutive distribution:

$$[v_1, v_2] \in \Gamma(W) \ \ \forall\ v_1, v_2 \in \Gamma(W)$$

(b) and $\widetilde{\omega}$ is closed on W, i.e.

$$\mathrm{d}\widetilde{\omega}(v_0, v_1, v_2) = 0$$

for all $v_0, v_1, v_2 \in \Gamma(W)$.

If (a) holds, by using the standard formula for the exterior derivative, we see that (b) is independent of the extension $\widetilde{\omega}$:

$$\begin{aligned} \mathrm{d}\widetilde{\omega}(v_0, v_1, v_2) =& \mathscr{L}_{v_0}(\omega(v_1, v_2)) - \mathscr{L}_{v_1}(\omega(v_0, v_2)) + \mathscr{L}_{v_2}(\omega(v_0, v_1)) \\ & - \omega([v_0, v_1], v_2) + \omega([v_0, v_2], v_1) - \omega([v_1, v_2], v_0). \end{aligned} \tag{*}$$

By Frobenius Theorem, W comes from a regular foliation $\mathcal{F} = \{F_\lambda\}_{\lambda\in\Lambda}$. Since, for $p \in F_\lambda$, we have that $W_p = T_pF_\lambda$, we can regard the 2-form ω as a family of 2-forms on the leaves of $\mathcal{F}$; namely, define:

$$\omega_\lambda \in \Omega^2(F_\lambda), \quad \omega_\lambda(v_1, v_2) := \omega(v_1, v_2), \quad v_1, v_2 \in W|_{F_\lambda} = TF_\lambda.$$

Then condition (b) is equivalent to $\mathrm{d}\omega_\lambda = 0$ for all $\lambda \in \Lambda$. Thus, we can regard a regular Dirac structure L on M as a decomposition of M into presymplectic manifolds:

$$(M, L) = \bigsqcup_{\lambda\in\Lambda} (F_\lambda, \omega_\lambda).$$

2.18. *Local description of regular Dirac structures.* Let $L = L(W,\omega)$ be a regular Dirac structure, and let $\mathcal{F} = \{F_\lambda\}_{\lambda\in\Lambda}$ denote the associated foliation. Consider a chart on M adapted to the foliation $\mathcal{F}$:

$$\chi : U \xrightarrow{\sim} \mathbb{R}^r \times \mathbb{R}^{n-r}, \quad \chi(p) = (x_1(p), \ldots, x_r(p), y_1(p), \ldots, y_{n-r}(p));$$

thus the leaves of $\mathcal{F}$ correspond to the parallel r-planes $\mathbb{R}^r \times \{y\}$, with $y \in \mathbb{R}^{n-r}$. Then the 2-form ω is given by a smooth family

$$\{\omega_y \in \Omega^2(\mathbb{R}^r)\}_{y\in\mathbb{R}^{n-r}}, \quad \omega_y = \frac{1}{2}\sum_{i,j} \omega_{i,j}(x, y)\mathrm{d}x_i \wedge \mathrm{d}x_j,$$

so that for each $y \in \mathbb{R}^{n-r}$ we have that $\mathrm{d}^x\omega_y = 0$, where d^x denotes the exterior derivative with respect to the x-coordinates.

By the Poincaré Lemma the forms ω_y are exact; and by applying the standard construction of homotopy operators, we can find a smooth family of 1-forms

$$\{\eta_y \in \Omega^1(\mathbb{R}^r)\}_{y\in\mathbb{R}^{n-r}}$$

so that $\mathrm{d}^x\eta_y = \omega_y$; e.g. one can define:

$$\eta_y := \sum_{i,j} \Big(\int_0^1 t\omega_{i,j}(tx, y)\mathrm{d}t\Big) x_i \mathrm{d}x_j.$$

Consider a 1-form $\widetilde{\eta} \in \Omega^1(\mathbb{R}^r \times \mathbb{R}^{n-r})$ which extends η, i.e. $\widetilde{\eta}$ satisfies:

$$\widetilde{\eta}|_{\mathbb{R}^r\times\{y\}} = \eta_y, \quad \forall\ y \in \mathbb{R}^{n-r}.$$

Then $\widetilde{\omega} := \mathrm{d}\widetilde{\eta}$ is an extension of ω:

$$\widetilde{\omega}|_{\mathbb{R}^r\times\{y\}} = \omega_y, \quad \forall\, y \in \mathbb{R}^{n-r}.$$

This argument proves the following:

Lemma. Let $L = L(W, \omega)$ be a regular Dirac structure on M. Then, for every point in M there exists an open set U and a closed 2-form $\widetilde{\omega} \in \Omega^2(U)$ so that $\widetilde{\omega}|_W = \omega$; i.e.

$$L|_U = \{v + \iota_v\widetilde{\omega} + \alpha \;:\; v \in W|_U, \quad \alpha \in W^\circ|_U\}.$$

2.19. *Regular points.* Let L be a Dirac structure on M. A point $p \in M$ is called a **regular point** of L, if the rank of L is constant in a neighborhood of p. If p is not regular, then p is called a **singular point**.

Note that the rank map $p \mapsto \mathrm{rank}(L, p)$ is a lower semi-continuous, i.e. every $p \in M$ has a neighborhood U so that

$$\mathrm{rank}(L, q) \geq \mathrm{rank}(L, p), \quad \forall q \in U.$$

This implies that the set of regular points on M is **open and dense**.

2.20. *General structure.* Any Dirac structure L on M, can be described geometrically as a **singular presymplectic foliation**:

$$(M, L) = \bigsqcup_{\lambda\in\Lambda}(F_\lambda, \omega_\lambda).$$

More precisely, M comes with a partition into connected, regularly immersed submanifolds F_λ, called the leaves of the foliation; each leaf F_λ comes with a closed 2-form $\omega_\lambda \in \Omega^2(F_\lambda)$. The Dirac structure L can be reconstructed from these data as follows: if $p \in M$ belongs to the leaf F_λ, then, in the notation of Example 1.12:

$$L_p = L(T_pF_\lambda, \omega_{\lambda,p}).$$

Conversely, given L, then: the singular foliation $\mathcal{F}$ can be described as follows: two points $p, q \in M$ are in the same leaf iff there exists a smooth path $\gamma : [0, 1] \to M$ so that

$$\gamma(0) = p, \quad \gamma(1) = q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in p_T(L_{\gamma(t)}), \;\forall\, t \in [0, 1];$$

if F_λ is the leaf through $p \in M$ then

$$T_pF_\lambda = p_T(L_p),$$

and the 2-form ω_λ at p is given as above by

$$\omega_{\lambda,p}(v_1, v_2) := \alpha_1(v_2) = -\alpha_2(v_1), \quad v_1 + \alpha_1,\; v_2 + \alpha_2 \in L_p.$$

The proof of this correspondence is based on a version of the Frobenius Theorem for singular foliations.

2.21. Thinking of a Dirac structure L on a connected manifold M as a singular presymplectic foliation $\{(F_\lambda, \omega_\lambda)\}_{\lambda\in\Lambda}$, we have that:

(a) L is corresponds to a closed 2-form iff the foliation consists only of one leaf;
(b) L is corresponds to a foliation iff $\omega_\lambda = 0$ for all λ, and this conditions implies also that $\{F_\lambda\}_{\lambda\in\Lambda}$ is a regular foliation;
(c) L is corresponds to a Poisson structure iff each ω_λ is a symplectic structure on F_λ, i.e. ω_λ is nondegenerate.

2.22. *Exercise.* Let L be a Dirac structure on a connected manifold M. Prove that either all presymplectic leaves of L are even dimensional, or all are odd-dimensional. (*Hint: Use Exercise 1.14.*)

2.23. *Linear Poisson structures and Lie algebras.* A Poisson structure π on $\mathbb{R}^n$ is called a **linear Poisson structure**, if the coefficients of the bivector π are linear functions on $\mathbb{R}^n$, i.e.

$$\pi = \sum_{i,j,k} \frac{1}{2} C^k_{i,j} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

A linear bivector as above is a Poisson structure if and only if the number $\{C^k_{i,j}\}$ are the **structure constants of a Lie algebra** i.e. if the bilinear operation:

$$[\cdot,\cdot] : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad [e_i, e_j] = \sum_k C^k_{i,j} e_k$$

satisfies the Jacobi identity (here $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$). In fact, linear Poisson structures are in 1-1 correspondence with Lie algebra structure. Coordinate-free, this correspondence works as follows. Consider a Lie algebra

$$(\mathfrak{g}, [\cdot,\cdot]).$$

The corresponding linear Poisson structure, lives on the dual space

$$\pi_{\mathfrak{g}} \in \mathfrak{X}^2(\mathfrak{g}^*),$$

and it is given by the Lie bracket under the obvious identifications:

$$\pi_{\mathfrak{g},\xi} = \xi \circ [\cdot,\cdot] \in \bigwedge^2 \mathfrak{g}^* = \bigwedge^2 T_\xi \mathfrak{g}^*, \quad \xi \in \mathfrak{g}^*.$$

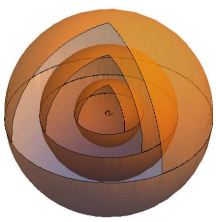
The symplectic leaves of the Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ are the so-called **coadjoint orbits**, i.e. the orbits of canonical the action of a connected Lie group G integrating $\mathfrak{g}$ on $\mathfrak{g}^*$.

2.24. *Examples: 3-dimensional Lie algebras.* To illustrate how singular the underlying foliation of a Dirac structure can be, we will consider the linear Poisson structures corresponding to 3-dimensional Lie algebras. It is in fact not very difficult to classify all real 3-dimensional Lie algebras; but we will discuss only some of them.

(a) The linear Poisson structure corresponding to the Lie algebra $\mathfrak{so}(3)$ is given by:

$$\pi_{\mathfrak{so}(3)} = x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \in \mathfrak{X}^2(\mathbb{R}^3).$$

Its symplectic leaves (i.e. coadjoint orbits) are: the origin of $\mathbb{R}^3$ and all the spheres centered at the origin:



In standard spherical coordinates,

$$x = r\cos(\theta)\sin(\phi), \quad y = r\sin(\theta)\sin(\phi), \quad z = r\cos(\phi),$$

$$0 \le r, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi,$$

the Poisson tensor becomes

$$\pi_{\mathfrak{so}(3)} = \frac{1}{r\sin(\phi)} \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial \theta},$$

and the symplectic structure on the leaf

$$S_r^2 := \{(x, y, z) \ : \ x^2 + y^2 + z^2 = r^2\}$$

is given by

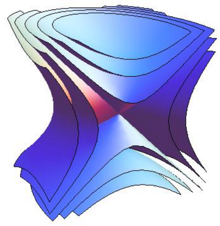
$$\omega_{S_r^2} = r\mathrm{d}(\cos(\phi)) \wedge \mathrm{d}\theta.$$

Note that the following holds: if (F_p, ω_p) is the leaf through a point $p \in \mathfrak{g}^*$, then for $s \neq 0$, the leaf through the point $s \cdot p$ is given by

$$(F_{sp}, \omega_{sp}) = (sF_p, s\omega_p),$$

where $sF_p = \{sq : q \in F_p\}$ and we regard ω_p as a 2-form on sF_p via pullback by the obvious diffeomorphism $sF_p \xrightarrow{\sim} F_p$, $q \mapsto 1/sq$. So both the foliation and the closed 2-forms vary linearly. It is easy to check that this property is shared by all linear Poisson structures, and in fact this property can be used to characterize linear Poisson structures.

(b) By using suitable coordinates, the coadjoint orbits of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ are given by the connected components of the fibers of the map $x^2 + y^2 - z^2 \in C^\infty(\mathbb{R}^3)$:



More precisely: the level set $x^2 + y^2 - z^2 = 0$ is composed of three leaves (the origin, the upper part and the lower part of the cone), each level set $x^2+y^2-z^2 = c > 0$ is a leaf which is a one sheeted hyperboloids, and the level sets $x^2 + y^2 - z^2 = c < 0$ are each decompose into two leaves, which are the two sheets of a hyperboliod (one with $z > 0$ and the other with $z < 0$).

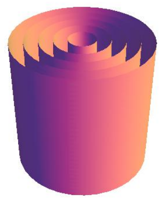
(c) Besides the $\mathfrak{so}(3)$ and $\mathfrak{sl}_2(\mathbb{R})$ all other linear Poisson structures can be decomposed as follows:

$$\pi_{\mathfrak{g}} = X \wedge \frac{\partial}{\partial z},$$

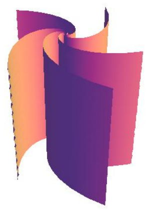
where X is a linear vector field on $\mathbb{R}^2$, i.e.

$$X = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y}.$$

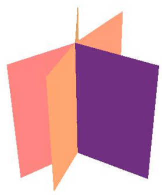
Such vector fields are easily classified; namely, they correspond to the conjugacy class of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Rescaling of this matrix by a non-negative number is equivalent to rescaling the z-direction, so one also obtains an isomorphic Poisson structure (hence Lie algebra). The 2-dimensional leaves of the Poisson structure are the $C \times \mathbb{R}$, for $C \subset \mathbb{R}^2$ a (nontrivial) flow line of X, and points of the form (p, z) where p is a zero of X. Let us discuss some examples. First, if A has purely imaginary eigenvalues, e.g. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the leaves will be concetric cylinders and each point on the z-axis:



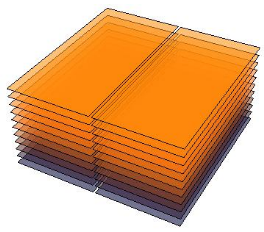
When the eigenvalues of A are neither real nor purely imaginary, the leaves spiral towards the z-axis (and each point on the z-axis is a 0-dimensional leaf):



If A is the idendity matrix, the symplectic foliation looks like an open book:



If A has eigenvalues 1 and 0, then the Lie algebra is the direct product $\mathfrak{aff}(1)\times\mathbb{R}$; its Poisson bivector becomes $x\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}$; its symplectic foliations looks as follows (note that the y- and z-axis were interchanged):



i.e. the leaves are the points on the plane $x = 0$, and parallel half-planes, all arriving tranversely at this plane.

Quite different looks the symplectic foliation corresponding to the Heisenberg algebra $x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$ (i.e. when A is nilpotent); in this case the leaves are the points on the plane $x = 0$ and all the planes parallel to this plane i.e. $x = c$, for $c \neq 0$.

3. The product of Dirac structures

3.1. Let $L_1, L_2 \subset \mathbb{T}M$ be two Lagrangian distributions. At every point $p \in M$ we can define the Lagrangian subspace:

$$(L_1 \star L_2)_p := L_{1,p} \star L_{2,p} \subset \mathbb{T}_pM.$$

In general, $L_1 \star L_2 \subset \mathbb{T}M$ might fail to be a smooth distribution.

3.2. *Theorem.* Consider two Dirac structures $L_1, L_2 \in \mathrm{Dir}(M)$. We have that

(a) If $L_1 \star L_2 \subset \mathbb{T}M$ is a smooth subbundle, then $L_1 \star L_2$ is a Dirac structure on M.

(b) If the vector spaces
$$p_T(L_{1,p}) + p_T(L_{2,p}), \quad p \in M$$
have constant dimension, then $L_1 \star L_2$ is smooth. In particular, if L_1 and L_2 are **tangentially transverse**:
$$p_T(L_{1,p}) + p_T(L_{2,p}) = T_pM, \quad \forall\, p \in M,$$
then $L_1 \star L_2$ is smooth.

Remark. Note that annihilator of the space $p_T(L_{1,p}) + p_T(L_{2,p})$ is given by
$$(p_T(L_{1,p}) + p_T(L_{2,p}))^\circ = L_{1,p} \cap L_{2,p} \cap T_p^*M.$$
Therefore, the first condition in (b) is equivalent to this bundle being of constant dimension, and the second is equivalent to this space being trivial.

Proof. (a) Consider the set U consisting of points $p \in M$ so that L_1, L_2 and $L_1 \star L_2$ all have constant rank around p. Then U is the intersection of three open and dense sets 2.19; therefore U is itself open and dense. Since $L_1 \star L_2$ is smooth, by Corollary 2.9, it suffices to check that $L_1 \star L_2$ is involutive on U. So, by replacing M by U, we can assume that L_1, L_2 and $L_1 \star L_2$ are all three regular Lagrangian subbundles of $\mathbb{T}M$. The Dirac structures L_1 and L_2 are determines by the smooth distributions $W_1 := p_T(L_1)$, $W_2 := p_T(L_2)$ and corresponding smooth 2-forms $\omega_1 \in \Omega^2(W_1)$ and $\omega_2 \in \Omega^2(W_2)$; i.e. $L_1 = L(W_1, \omega_1)$ and $L_2 = L(W_2, \omega_2)$. Integrability of L_1 and L_2 is equivalent to W_1 and W_2 being involutive, and to ω_1 and ω_2 being closed on W_1 and W_2. We have that $L_1 \star L_2 = L(W_1 \cap W_2, (\omega_1 + \omega_2)|_{W_1 \cap W_2})$, and we are assuming that $W_1 \cap W_2$ has constant rank. Since W_1 and W_2 are involutive, it follows that also $W_1 \cap W_2$ is involutive; since ω_1 (resp. ω_2) is closed on W_1 (resp. W_2) so is the restriction of ω_1 (resp. ω_2) to $W_1 \cap W_2$: the closeness condition can be expressed by formula (*) in 2.17; therefore if it holds for $v_0, v_1, v_2 \in \Gamma(W_1)$ (resp. $\Gamma(W_2)$) it certainly also holds for $v_0, v_1, v_2 \in \Gamma(W_1 \cap W_2)$. This proves (a).

(b). The assumption implies that the vector bundle map
$$L_1 \oplus L_2 \longrightarrow TM, \quad (v + \alpha, w + \beta) \mapsto v - w$$
has constant rank; therefore its kernel is a smooth vector bundle:
$$K := \{(v + \alpha, v + \beta) \ : \ v + \alpha \in L_1, v + \beta \in L_2\} \subset L_1 \oplus L_2$$
Note that $L_1 \star L_2$ is the image of K under the smooth vector bundle map
$$K \longrightarrow \mathbb{T}M, \quad (v + \alpha, v + \beta) \mapsto v + \alpha + \beta.$$
Since this map has constant rank (we know that $n = \dim(L_1 \star L_2)_p$ for all $p \in M$), it follows that its image, which is $L_1 \star L_2$, is smooth. This proves (b).

3.3. Let L_1 and L_2 be two Dirac structures so that $L_1 \star L_2$ is smooth. Let (F_1, ω_1) and (F_1, ω_2) be presymplectic leaves of L_1 and L_2, respectively. Let $p \in F_1 \cap F_2$, and consider the presymplectic leaf (F, ω) of $L_1 \star L_2$ through p. Then
$$F \subset F_1 \cap F_2, \quad T_pF = T_pF_1 \cap T_pF_2, \quad \omega = \omega_1|_F + \omega_2|_F.$$
The first two conditions mean that F_1 and F_2 intersect cleanly at p in F. In particular, if $F_1 \cap F_2$ is a connected submanifold, then
$$(F, \omega) = (F_1 \cap F_2, \omega_1|_{F_1 \cap F_2} + \omega_2|_{F_1 \cap F_2}).$$
In general $F_1 \cap F_2$ might have several connected components of different dimensions; but one can show that the leaves of $L_1 \star L_2$ which intersect $F_1 \cap F_2$ are the smooth path-connected components of $F_1 \cap F_2$.

The condition that L_1 and L_2 be **tangentially transverse**:

$$TM = p_T(L_1) + p_T(L_2),$$

is equivalent to any leaf of F_1 of L_1 be transverse to any leaf F_2 of L_2:

$$F_1 \pitchfork F_2, \quad \text{i.e. } T_pF_1 + T_pF_2 = T_pM, \quad \forall\, p \in F_1 \cap F_2.$$

This implies automatically that $F_1 \cap F_2$ is a smooth submanifold of

$$\dim(F_1 \cap F_2) = \dim(F_1) + \dim(F_2) - \dim(M).$$

In this case, the leaves of $L_1 \star L_2$ are the connected components of the leaves of the intersections $F_1 \cap F_2$, where F_1 and F_2 are leaves of L_1 and L_2, respectively.

3.4. *Intersections of foliations.* Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations, and assume they are transverse: $T\mathcal{F}_1 + T\mathcal{F}_2 = TM$. This is equivalent to their corresponding Dirac structures $L_{T\mathcal{F}_1}$ and $L_{T\mathcal{F}_2}$ be tangentially transverse. The product $L_{T\mathcal{F}_1} \star L_{T\mathcal{F}_2}$ corresponds to the foliation $\mathcal{F}_1 \cap \mathcal{F}_2$ with leaves the intersection of the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$.

3.5. *Gauge transformations.* Let $\omega \in \Omega^2(M)$ be a closed 2-form on M. Then

$$TM^\omega := \{v + \iota_v\omega \ : \ v \in TM\}$$

is tangentially transverse to any Dirac structure $L \in \mathrm{Dir}(M)$. The product

$$L^\omega := L \star TM^\omega = \{v + \iota_v\omega + \alpha \ : \ v + \alpha \in L\}$$

is called the **gauge transformation** of L by ω. If $\{(F_\lambda, \omega_\lambda)\}_{\lambda\in\Lambda}$ is the singular presymplectic foliation of L, then the singular presymplectic foliation of L^ω is given by

$$\{(F_\lambda, \omega_\lambda + \omega|_{F_\lambda})\}_{\lambda\in\Lambda}.$$

3.6. Let π be a Poisson structure, with corresponding Dirac structure

$$T^*M_\pi = \{\pi^\sharp(\alpha) + \alpha \ : \ \alpha \in T^*M\}.$$

The gauge transformation of T^*M_π by a closed 2-form $\omega \in \Omega^2(M)$ is

$$(T^*M_\pi)^\omega = \{\pi^\sharp(\alpha) + (\mathrm{id} + \omega \circ \pi^\sharp)\alpha \ : \ \alpha \in T^*M\}.$$

We have that $(T^*M_\pi)^\omega$ is a Poisson structure iff the map:

$$\mathrm{id} + \omega \circ \pi : T^*M \longrightarrow T^*M, \quad \alpha \mapsto \alpha + \iota_{\pi^\sharp\alpha}\omega$$

is a linear isomorphism; and in this case the corresponding Poisson bivector is given by:

$$\pi^\omega \in \mathfrak{X}^2(M), \quad \pi^\omega = \pi \circ (\mathrm{id} + \omega \circ \pi)^{-1} = (\mathrm{id} + \pi \circ \omega)^{-1} \circ \pi.$$

If π is nondegenerate, i.e. $\pi = \eta^{-1}$, where $\eta \in \Omega^2(M)$ is a symplectic structure, then $(T^*M_\pi)^\omega$ is Poisson iff $\eta + \omega$ is again a symplectic structure, and in this case:

$$\pi^\omega = (\eta + \omega)^{-1}.$$

3.7. Lemma 2.18 implies that around every regular point of a Dirac structure L there is an open set U, a foliation $\mathcal{F}$ on U and a closed 2-form $\widetilde{\omega}$ on U so that

$$L|_U = L(T\mathcal{F})^{\widetilde{\omega}}.$$

3.8. *Exercise.*

(a) Let L_1 and L_2 be two Dirac structures. If L_1 and $-L_2$ are transverse:

$$L_1 \oplus (-L_2) = \mathbb{T}M,$$

prove that $L_1 \star L_2$ is a Poisson structure.

(b) Let π_1 and π_2 be two Poisson structures so that $\pi_1+\pi_2$ is nondegenerate. Prove that the bivector

$$\pi_1 \star \pi_2 = \pi_1 \circ (\pi_1 + \pi_2)^{-1} \circ \pi_2 \in \mathfrak{X}^2(M)$$

is a Poisson structure on M (*Hint: consider the product of the corresponding Dirac structures*).

3.9. *The dual product.* By duality, one can consider the also the **dual product** of two Lagrangian subbundles $L_1, L_2 \subset \mathbb{T}M$, which is defines as in 1.29,

$$(L_1 \circledast L_2)_p := L_{1,p} \circledast L_{2,p}.$$

In general, if $L_1 \circledast L_2$ is not a smooth subbundle; a sufficient condition for smoothness is that L_1 and L_2 are **co-tangentially transverse**:

$$p_{T^*}(L_1) + p_{T^*}(L_1) = T^*M,$$

where p_{T^*} denotes the projection map

$$p_{T^*} : \mathbb{T}M \longrightarrow T^*M, \quad p_{T^*}(v + \alpha) = \alpha.$$

Even if L_1 and L_2 are Dirac structures and $L_1 \circledast L_2$ is smooth, we don't necessarily have that $L_1 \circledast L_2$ is again a Dirac structure. Nevertheless the involutivity condition on $L_1 \circledast L_2$ appears in various geometric settings where it plays an important role.

3.10. *Bi-Hamiltonian systems.* Let π_1 and π_2 be two Poisson structures. Then, the dual product of their Dirac structures is given by:

$$T^*M_{\pi_1} \circledast T^*M_{\pi_2} = T^*M_{\pi_1+\pi_2}.$$

Thus the integrability condition on their product is equivalent to $\pi_1 + \pi_2$ be also a Poisson structure. Two Poisson structures π_1 and π_2 which have this property are called **compatible**; i.e. they satisfy $[\pi_1, \pi_2] = 0$. A **bi-Hamiltonian system** (M, π_1, π_2, f) consists of compatible Poisson structures π_1 and π_2 and a smooth function $f \in C^\infty(M)$ which has the same Hamiltonian vector field for both Poisson structures:

$$X_f^{\pi_1} = X_f^{\pi_2}, \quad X_f^{\pi_i} := \pi_i^\sharp \mathrm{d}f.$$

4. The pullback of Dirac structures

4.1. Let $f : N \to M$ be a smooth map and let L be a Dirac structure on M. The **pullback** of L along f is the family of Lagrangian subspaces of $\mathbb{T}N$ given by:

$$f^*(L)_p := (\mathrm{d}_p f)^*(L_{f(p)}) = \{v + f^*(\alpha) \in \mathbb{T}_p N : \ \mathrm{d}_p f(v) + \alpha \in L_{f(p)}\},$$

for all $p \in N$. In general, $f^*(L)$ is not a smooth subbundle.

4.2. *Theorem.* Let $f : N \to M$ be a smooth map and let $L \in \mathrm{Dir}(M)$.

(a) If $f^*(L)$ is a smooth subbundle of $\mathbb{T}N$, then $f^*(L) \in \mathrm{Dir}(N)$.

(b) If the family of vector spaces

$$\mathrm{d}_p f(T_pN) + p_T(L_{f(p)}) \subset T_pM, \quad p \in N$$

have constant dimension, then $f^*(L)$ is smooth; hence $f^*(L) \in \mathrm{Dir}(N)$. In particular, this condition is satisfied if f is **transverse** to L, $f \pitchfork L$, i.e.

$$\mathrm{d}_p f(T_pN) + p_T(L_{f(p)}) = T_{f(p)}M, \quad \forall\ p \in N.$$

Proof. The proof is similar to that of Theorem 3.2; see [2]. Note that the annihilator of the space $\mathrm{d}_p f(T_pN) + p_T(L_{f(p)})$ is given by:

$$\left(\mathrm{d}_p f(T_pN) + p_T(L_{f(p)})\right)^\circ = \ker((\mathrm{d}_p f)^*) \cap L_{f(p)},$$

so the first condition in (b) is equivalent to the condition appearing in [2].

4.3. If $\omega \in \Omega^2(M)$ is a closed 2-form, then any map $f : N \to M$ is transverse to TN^ω, and the pullback generalizes the pullback of closed 2-forms:

$$f^*(TN^\omega) = TM^{f^*(\omega)}.$$

4.4. Let $f : N \to M$ be a smooth map and let $L \in \mathrm{Dir}(M)$ be so that $f^*(L)$ is smooth. Let $p \in N$. Denote by (F_N, ω_N) the presymplectic leaf of $f^*(L)$ through p, and by (F_M, ω_M) the presymplectic leaf of L through $f(p)$. Then, by Exercise 1.25, we have that:

$$T_pF_M = (\mathrm{d}_p f)^{-1}(T_{f(p)}F_N).$$

Note that this implies that, if $\gamma : [0,1] \to M$ is a smooth path in F_M, then $f \circ \gamma : [0,1] \to N$ is a smooth path in F_N. Therefore:

$$F_M \subset f^{-1}(F_N),$$

and also by Exercise 1.25, we have that

$$\omega_M = (f|_{F_M})^*(\omega_N).$$

In general, equality does not need to hold; $f^{-1}(F_N)$ can be have several connected components; and can be made out of leaves of various dimensions. However, the above argument can be used to show that F_M is the smooth-path connected component of $f^{-1}(F_N)$.

The map f satisfies $f \pitchfork L$ if and only if f is transverse to any leaf (F_N, ω_N) of L, i.e.

$$\mathrm{d}_p f(T_pN) + T_{f(p)}F_N = T_{f(p)}N,$$

for all $p \in f^{-1}(F_N)$ and all presymplectic leaves $F_N \subset N$. In this case, $f^{-1}(F_N)$ is a smooth manifold of the same codimension as F_N, and its connected components are precisely the leaves of $f^*(L)$ which intersect $f^{-1}(F_N)$.

4.5. Assume that $f : M \to N$ is a **submersion**. Then

$$f \pitchfork L \quad \forall\ L \in \mathrm{Dir}(M),$$

and therefore the pullback is a well-defined map

$$f^* : \mathrm{Dir}(M) \longrightarrow \mathrm{Dir}(N).$$

If f has connected fibers then, on the presymplectic foliations, f^* acts as:

$$f^*\{(F_\lambda, \omega_\lambda)\}_{\lambda\in\Lambda} = \{(f^{-1}(F_\lambda), f|^*_{f^{-1}(F_\lambda)}\omega_\lambda)\}_{\lambda\in\Lambda}.$$

4.6. Let $L_1, L_2 \in \mathrm{Dir}(N)$, and let $f : M \to N$ be a smooth map. If $L_1 \star L_2$, $f^*(L_1)$, $f^*(L_2)$ and $f^*(L_1 \star L_2)$ are smooth, then also $f^*(L_1) \star f^*(L_2)$ is smooth, because:

$$f^*(L_1) \star f^*(L_2) = f^*(L_1 \star L_2).$$

This follows from Exercise 1.28. In particular, if $L \in \mathrm{Dir}(N)$ and $f^*(L)$ is smooth, and $\omega \in \Omega^2(N)$ is a closed 2-form, we have that

$$f^*(L^{\omega}) = f^*(L)^{f^*(\omega)}.$$

4.7. *Exercise.* Let $f : N \to M$ be a smooth map, where N is connected. Show that

$$f^*(T^*M) = \ker(\mathrm{d}f) \oplus \ker(\mathrm{d}f)^{\circ},$$

and conclude that $f^*(T^*M)$ is smooth if and only if f is a constant rank map.

4.8. *Exercise.* Let $f : N \to M$ be a smooth map and $L \in \mathrm{Dir}(M)$ be so that $f^*(L)$ is a Poisson structure on N. Prove that f is an **immersion**.

5. Push forward of Dirac structures

5.1. Let $f : N \to M$ be a smooth, and assume that L is Lagrangian distribution. Then for any $p \in N$, we can define the Lagrangian subspace:

$$f_*(L_p) := \{\mathrm{d}_p f(v) + \alpha \ : \ v + f^*(\alpha) \in L_p\} \subset \mathbb{T}_{f(p)}N.$$

Clearly, this does not define a Lagrangian subbundle of $\mathbb{T}N$ (unless f is a diffeomorphism).

Given Dirac structures $L_N \in \mathrm{Dir}(N)$ and $L_M \in \mathrm{Dir}(M)$, the map $f : N \to M$ is said to be a **forward Dirac map** if

$$f_*(L_{N,p}) = L_{M,f(p)}, \quad \forall\, p \in N.$$

5.2. If (N, π_N) and (M, π_M) are two Poisson manifolds, a smooth map $f : N \to M$ is said to be a **Poisson map**, if

$$\{a \circ f, b \circ f\}_{\pi_N} = \{a, b\}_{\pi_M} \circ f, \quad \forall\, a, b \in C^{\infty}(M);$$

in other words, the pullback map along f is a Lie algebra morphism:

$$f^* : (C^{\infty}(M), \{\cdot,\cdot\}_{\pi_M}) \longrightarrow (C^{\infty}(N), \{\cdot,\cdot\}_{\pi_N}).$$

In terms of the bivector fields, this condition means that π_N is f-related to π_M:

$$\mathrm{d}_p f(\pi_{N,p}) = \pi_{M,f(p)}, \ \ \forall\, p \in N.$$

In terms of the Dirac structures, the condition is equivalent to f being a forward Dirac map:

$$f_*(T^*_p N_{\pi_N}) = T^*_{f(p)} M_{\pi_M}, \ \ \forall\, p \in N.$$

5.3. Let (P, π) be a Poisson manifold. Consider the action of a Lie group G by Poisson diffeomorphisms on (P, π). Assume that the action is so that the quotient P/G is a smooth manifold, for which the projection map $\mathrm{pr} : P \to P/G$ is a smooth submersion (e.g. one can assume that the action is free and proper so that P is a principal G-bundle over P/G). We claim that there is a unique Poisson structure $\pi_{P/G}$ on P/G so that

$$\mathrm{pr} : (P, \pi) \longrightarrow (P/G, \pi_{P/G}).$$

is a Poisson map. Since pr is a submersion, note that we can identify smooth functions on the quotient P/G with G-invariant smooth functions on P, i.e.

$$C^{\infty}(P/G) \xrightarrow{\sim} C^{\infty}(P)^G, \quad a \mapsto a \circ \mathrm{pr}.$$

Since the Poisson bracket is G-invariant, $C^{\infty}(P)^G$ is a Lie subalgebra of

$$(C^{\infty}(P), \{\cdot,\cdot\}_{\pi});$$

thus $C^\infty(P)^G$ has an induced Poisson bracket, and therefore there is an induced bracket $\{\cdot,\cdot\}_{P/G}$ on $C^\infty(P/G)$ so that pr is a Poisson map.

5.4. A particular case of the above is when π is nondegenerate, i.e. $\pi = \omega^{-1}$, where ω is a symplectic structure. Moreover, assume that the action is **Hamiltonian**, i.e. there exists a smooth G-equivalent map (called the **moment map**)

$$\mu : (P, \omega) \longrightarrow \mathfrak{g}^*,$$

which satisfies

$$\iota_{\overline{\xi}}\omega = \mathrm{d}\big(\mu(\xi)\big), \quad \forall\, \xi \in \mathfrak{g},$$

where $\mathfrak{g}$ denotes the Lie algebra of G, and for $\xi \in \mathfrak{g}$ we have denoted by $\overline{\xi} \in \mathfrak{X}(P)$ the infinitesimal action on P:

$$\overline{\xi}_p := \frac{\mathrm{d}}{\mathrm{d}\epsilon} p \cdot \exp(\epsilon\xi)\big|_{\epsilon=0}, \quad p \in P.$$

Then the symplectic leaves of $(P/G, \pi_{P/G})$ are the quotients $\mu^{-1}(\mathcal{O})/G$, where $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit of G.

5.5. The following gives a characterization of which Dirac structures can be pushed down via a submersion:

Theorem. Let $f : N \to M$ be a surjective submersion with connected fibers, and let $L_N \in \mathrm{Dir}(N)$. Then there exists a (necessarily unique) Dirac structure $L_M \in \mathrm{Dir}(M)$ so that $f : (N, L_N) \to (M, L_M)$ is a forward Dirac map **if and only if** the following is a (smooth) Dirac structure on N:

$$L_N^f := L_N \circledast \big(\ker(\mathrm{d}f) \oplus \ker(\mathrm{d}f)^\circ\big) \subset \mathbb{T}N.$$

The following exercise constitutes a proof of this Theorem.

5.6. *Exercise.* Let $f : N \to M$ be a surjective submersion with connected fibers, and let $L_N \in \mathrm{Dir}(N)$.

(a) Prove the following relation:

$$f^* f_*(L_{N,p}) = L^f_{N,p}, \quad p \in N.$$

If there exists $L_M \in \mathrm{Dir}(M)$ so that $f : (N, L_N) \to (M, L_M)$ is a forward Dirac map, prove that L_N^f is a Dirac structure on N.

(b) If $\ker(\mathrm{d}f) \subset L_N$, prove that there exists a Dirac structure L_M on M so that $f : (N, L_N) \to (M, L_M)$ is a forward Dirac map. *Hint: By using flows of vector fields in* $\ker(\mathrm{d}f)$*, show first that* $f_*(L_{N,p}) = f_*(L_{N,q})$ *for all* $p, q \in N$ *so that* $f(p) = f(q)$*. To show involutivity, see* [2].

(c) Show that

$$f_*(L^f_{N,p}) = f_*(L_{N,p}).$$

Using this relation and (b), prove the following: if L_N^f is a smooth Dirac structure on N then there exists a Dirac structure $L_M \in \mathrm{Dir}(M)$ so that $f : (N, L_N) \to (M, L_M)$ is a forward Dirac map.

5.7. Let $f : (N, L_N) \to (M, L_M)$ be a forward Dirac map, which is a surjective submersion. If L_N corresponds to a Poisson structure, prove that also L_M corresponds to a Poisson structure.

5.8. *Libermann's Theorem.* Let (N,ω) be a symplectic manifold, and let $f : N \to M$ be a surjective submersion with connected fibers. There exists a Poisson structure π on M so that $f : (N,\omega) \to (M,\pi)$ is a Poisson map if and only if the symplectic orthogonal to the fibers of f,

$$\ker(\mathrm{d}f)^{\perp} := \{v \in TM \ : \ \omega(v,w) = 0 \ \forall \ w \in \ker(\mathrm{d}f)\} \subset TN$$

is an involutive subbundle.

Let us show that this classical result fits into the framework of Theorem 5.5. Denote

$$W := \ker(\mathrm{d}f), \quad W^{\perp} := \ker(\mathrm{d}f)^{\perp}.$$

It is easy to check that

$$TN^{\omega} \circledast (W \oplus W^{\circ}) = W \oplus (W^{\perp})^{\omega},$$

where we have denoted

$$(W^{\perp})^{\omega} = \{v + \iota_v\omega \ : \ v \in W^{\perp}\}.$$

Since ω is nondegenerate, note that the above is indeed a direct sum decomposition, because:

$$(W^{\perp})^{\omega} \cap TM = 0.$$

By Theorem 5.5, and Exercise 5.7, it suffices to show that $W \oplus (W^{\perp})^{\omega} \in \mathrm{Dir}(N)$ if and only if $W^{\perp} \subset TN$ is an integrable distribution.

First assume that $W \oplus (W^{\perp})^{\omega} \in \mathrm{Dir}(N)$. Consider $v_1, v_2 \in \Gamma(W^{\perp})$. Since $\mathrm{d}\omega = 0$, by 2.11, we have that

$$[v_1 + \iota_{v_1}\omega, v_2 + \iota_{v_2}\omega] = [v_1, v_2] + \iota_{[v_1,v_2]}\omega.$$

By assumption, this element is in $W \oplus (W^{\perp})^{\omega}$, so it can be written uniquely as:

$$[v_1, v_2] + \iota_{[v_1,v_2]}\omega = w + v + \iota_v\omega,$$

with $w \in \Gamma(W)$ and $v \in \Gamma(W^{\perp})$. Since ω is non-degenerate, it follows that $v = [v_1, v_2]$ and so $w = 0$. Thus, we have that $W^{\perp}$ is involutive:

$$[v_1, v_2] \in \Gamma(W^{\perp}), \quad \forall \ v_1, v_2 \in \Gamma(W^{\perp}).$$

Conversely, assume that $W^{\perp}$ is involutive. The above argument shows that

$$[\Gamma((W^{\perp})^{\omega}, (W^{\perp})^{\omega})] \subset \Gamma((W^{\perp})^{\omega}),$$

and clearly we also have that:

$$[\Gamma(W), \Gamma(W)] \subset \Gamma(W).$$

Let us check that also:

$$[\Gamma(W), \Gamma((W^{\perp})^{\omega})] \subset \Gamma(W) \oplus \Gamma((W^{\perp})^{\omega}).$$

For this, let $w \in \Gamma(W)$ and $v \in \Gamma(W^{\perp})$. We have that

$$[w, v + \iota_v\omega] = [w, v] + \mathscr{L}_w(\iota_v\omega).$$

By Exercise 1.3, to show that $[w, v + \iota_v\omega] \in W \oplus (W^{\perp})^{\omega}$, it suffices to show that for all $w_1 \in \Gamma(W)$ and for all $v_1 \in \Gamma(W^{\perp})$, we have that

$$(w_1, [w, v + \iota_v\omega]) = 0, \quad (v_1 + \iota_{v_1}\omega, [w, v + \iota_v\omega]) = 0.$$

Using that $\mathscr{L}_w \circ \iota_{w_1} - \iota_{w_1} \circ \mathscr{L}_w = \iota_{[w,w_1]}$, the first relation is equivalent to:

$$\iota_{w_1}\mathscr{L}_w\iota_v\omega = \iota_{[w_1,w]}\iota_v\omega + \mathscr{L}_w\iota_{w_1}\iota_v\omega = \omega([w_1,w],v) + \mathscr{L}_w(\omega(w_1,v)) = 0,$$

and the right hand side is zero because $[w_1, w], w \in \Gamma(W)$ and $v \in \Gamma(W^\perp)$. The second relation holds as well:

$$\iota_{[w,v]}\iota_{v_1}\omega + \iota_{v_1}\mathscr{L}_w\iota_v\omega = \iota_{v_1}\left(\mathscr{L}_w\iota_v - \iota_{[w,v]}\right)\omega = \iota_{v_1}\iota_v\mathscr{L}_w\omega = \iota_{v_1}\iota_v \mathrm{d}\iota_w\omega =$$
$$= (\mathrm{d}\iota_w\omega)(v, v_1) = \mathscr{L}_v(\omega(w, v_1)) - \mathscr{L}_{v_1}(\omega(w, v)) - \omega(w, [v, v_1]) = 0,$$

where in the last equality we have used that $v, v_1, [v, v_1] \in \Gamma(W^\perp)$ and $w \in \Gamma(W)$, and before that, we have used the classical formula:

$$\mathrm{d}\eta(x, y) = \mathscr{L}_x(\eta(y)) - \mathscr{L}_y(\eta(x)) - \eta([x, y]),$$

for $\eta = \iota_w\omega$. This ends the proof.

6. Poisson transversals

6.1. By Theorem 4.2, a submanifold $N \subset M$ of a Dirac manifold (M, L) has an induced Dirac structure, provided that $i^*(L)$ is a smooth subbundle of $\mathbb{T}N$, where $i : N \to M$ is the inclusion map. This condition is insured by certain transversality conditions of the bundle $L|_N \subset \mathbb{T}M|_N$ and the bundles TN and TN°; and corresponding to these conditions there are several classes of interesting submanifolds in Dirac geometry: e.g. *Dirac submanifold*, *Poisson-Dirac submanifold*, *co-isotropic submanifold*, *cosymplectic submanifold* etc. Here, we will only discuss the class of *Poisson transversals*; these other types of submanifolds are discussed in [2, 5, 13].

6.2. A **Poisson transversal** of a Dirac manifold (M, L) is an embedded submanifold $i : N \to M$ which satisfies any of the equivalent conditions:

(a)
$$(T_pN \oplus TN_p^\circ)^\circ \cap L_p = \{0\}, \quad \forall\, p \in N;$$

(b) $N \pitchfork p_T(L)$, i.e.
$$T_pN + p_T(L_p) = T_pM,$$
and $i^*(L)$ is given by a Poisson structure π_N on N;

(c) For every presymplectic leaf (F, ω) of L, we have that $F \pitchfork N$, i.e.
$$T_pF + T_pN = T_pM, \quad \forall\, p \in F \cap N,$$
and $\omega|_{F\cap N}$ is a symplectic structure on $F \cap N$.

In other words, N is a Poisson transversal if and only if N intersects all presymplectic leaves of L transversally and symplectically. The symplectic leaves of (N, π_N) are the connected components of the intersections $(F \cap N, \omega|_{F\cap N})$, where (F, ω) is a presymplectic leaf of L which hits N.

6.3. *Exercise.* Check that the conditions above are indeed equivalent.

6.4. *Example.* Let (F, ω) be a presymplectic of (M, L). Consider a complementary transversal submanifold $N \subset M$ through a point $p \in L$, i.e.

$$T_pF \oplus T_pN = T_pM.$$

Then there is an open neighborhood U of p in N which is a Poisson transversal. To see this, consider assume that $(v + \alpha) \in (T_pN \oplus T_pN^\circ) \cap L_p$. Since $T_pF = p_T(L_p)$, note that $v = 0$, because:

$$v \in T_pF \cap T_pN = 0.$$

So $\alpha \in L_p$, and therefore:

$$0 = (\alpha, L_p) = \alpha(p_T(L_p)),$$

hence $\alpha \in p_T(L_p)^\circ$. We conclude that $\alpha = 0$, because

$$\alpha \in T_pN^\circ \cap p_T(L_p)^\circ = \left(T_pN + p_T(L_p)\right)^\circ = T_pM^\circ = 0.$$

We have shown that
$$(T_pN \oplus T_pN^\circ) \cap L_p = 0.$$
Note that this is an open condition on the points of N, so if it holds for p, it also holds for q in an open neighborhood $U \subset N$ of p.

6.5. A Poisson transversal N inside a symplectic manifold (M, ω) is the same as a symplectic submanifold.

6.6. In Poisson geometry, Poisson transversals are also called **cosymplectic submanifolds** (e.g. [2]; we prefer not to use this terminology, since the word *cosymplectic* has a well-established meaning in symplectic geometry); and in mechanics, Poisson transversals are the same as *second-class constraints*. Let us explain this terminology. Consider a Poisson transversal N of a Poisson manifold (M, π). That N is a Poisson transversal is equivalent to the following direct sum decomposition:
$$T_pM = T_pN \oplus \pi^\sharp(T_pN^\circ), \quad \forall\, p \in N.$$
This also equivalent to the fact that π restricts to a nondegenerate pairing on the conormal bundle:
$$\pi|_{T_pN^\circ} : T_pN^\circ \times T_pN^\circ \longrightarrow \mathbb{R},$$
and this is the origin of the term *cosymplectic*.

Let us think about the submanifold N as being a *constraint*, which in local coordinated it is given by the vanishing of the independent functions $y_1, \ldots, y_k$, i.e. $N = \{y_1 = \ldots = y_k = 0\}$. That N is a Poisson transversal is equivalent to non-degeneracy along N of the matrix:
$$c_{i,j} := \{y_i, y_j\}_\pi.$$
In mechanics terminology, N is also called a *second-class constraint*. The induced Poisson bracket on N corresponding to π_N is given by the **Dirac formula**:
$$\{f, g\}_{\pi_N} = \big(\{\widetilde{f}, \widetilde{g}\}_\pi - \sum_{i,j} \{\widetilde{f}, y_i\}_\pi c^{i,j} \{y_j, \widetilde{g}\}_\pi\big)|_N,$$
where $c^{i,j}$ denotes the entries of the inverse of the matrix $c_{i,j}$, and for $f, g \in C^\infty(N)$ we have denoted by $\widetilde{f}, \widetilde{g}$ smooth extensions of the functions f and g to a neighborhood of N.

6.7. Around a Poisson transversal, the Dirac structure can be written in special coordinates:

Theorem[1] Let N be a Poisson transversal in (M, L), with induced Poisson structure π_N. Around every point $p \in N$ there exists:

(a) an open neighborhood U of p, and a diffeomorphism: $\varphi : U \xrightarrow{\sim} O \times V$, where $V := U \cap N$ is an open neighborhood of p in N,
(b) a closed 2-form $\omega \in \Omega^2(O \times V)$

so that the restriction of L to U corresponds under φ to
$$\varphi : (U, L|_U) \xrightarrow{\sim} (TO \times T^*V_{\pi_N})^\omega.$$

6.8. In the previous theorem, if L is corresponds to a Poisson structure π on M, then the closed 2-form ω can be replaced by the standard symplectic structure on O; in this case we obtain *Weinstein's splitting theorem*:

Theorem[12] Let (N, π_N) be a Poisson transversal in the Poisson manifold (M, π). Then, around every point in N there is an open set U with coordinates
$$(x_1, \ldots, x_n, p_1, \ldots, p_k, q_1, \ldots, q_k) : U \longrightarrow \mathbb{R}^{n+2k},$$
so that
$$N \cap U = \{p_i = q_j = 0\},$$

and in these coordinates the Poisson structure becomes:

$$\pi|_U = \sum_{u,v=1}^{n} \frac{1}{2}\pi_N^{u,v}(x)\frac{\partial}{\partial x_u}\wedge\frac{\partial}{\partial x_v} + \sum_{i=1}^{k}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i},$$

where the first sum is the Poisson structure $\pi_N|_{N\cap U}$.

6.9. Here is a global version of theorem 6.8:

Theorem[8] Let (N,π_N) be an embedded Poisson transversal of the Poisson manifold (M,π). There exists a tubular neighborhood $\varphi : E \hookrightarrow M$ of N in M with projection $\mathrm{p} : E \to N$[1] and there exists a closed 2-form ω on E, so that

$$\varphi^*(T^*M_\pi) = (\mathrm{p}^*(T^*N_{\pi_N}))^{\omega}.$$

7. The linearization problem around presymplectic leaves

7.1. In this section we will discuss the linearization problem for Dirac structures around presymplectic leaves. We will construct a first order linear model for the Dirac structure around a leaf (actually, it is an “affine” model, instead of a linear one).

As an analogy, which will be useful to have in mind in the next subsection, let us first discuss a simpler problem, namely that of linearizing a smooth map

$$f : \mathcal{M} \longrightarrow \mathbb{R},$$

around a point p on a manifold $\mathcal{M}$. Consider a chart around p with coordinates $x = (x_1,\ldots,x_m)$ for which p corresponds to $0 \in \mathbb{R}^m$. The **first order approximation** of f around p can be defined in this chart as follows:

$$f_{\mathrm{lin},p} = f(0) + \lim_{t\to 0}\frac{1}{t}\big(f(tx) - f(0)\big) = m_0^*(f) + \lim_{t\to 0}\frac{1}{t}\big(m_t^*(f) - m_0^*(f)\big),$$

where m_t denotes rescaling by t, i.e. $m_t(x) = tx$. We say that f is **linearizable around** p, if there is a chart centered at p, $\varphi : \mathbb{R}^m \xrightarrow{\sim} U \subset \mathcal{M}$, so that $f\circ\varphi = f_{\mathrm{lin},p}$. For example, if p is a regular point of f, then we know that f is linearizable around p.

7.2. Consider a Dirac manifold (M,L), and let (F,ω) denote a compact[2] presymplectic leaf of L. We are interested in the local structure of L around F, and therefore we will replace M by a tubular neighborhood of F; i.e. we will assume that L is defined on the total space E of a vector bundle

$$\mathrm{p} : E \longrightarrow F,$$

for which the zero-section is the presymplectic leaf (F,ω). In analogy with the discussion of linearization of maps, this passage to a tubular neighborhood, corresponds to taking a chart around p. Denote by $m_t : E \to E$ the fiberwise multiplication by $t \in \mathbb{R}$. In the definition of linearization of maps from the previous subsection, let us replace the map f by the Dirac structure L, and the operations $+$ and $t\cdot$ with the product $*$ of Dirac structures and the product of a Dirac structure by a scalar (see 1.15) respectively. We obtain the following path of Dirac structure:

$$L_t := m_0^*(L) \star \frac{1}{t}\big(m_t^*(L) \star m_0^*(-L)\big), \quad t \neq 0.$$

To see that these form indeed a smooth path of Dirac structures, note first that:

$$m_0^*(L) = TE^{\widetilde{\omega}},$$

[1] i.e. $\mathrm{p} : E \to N$ is a vector bundle and $\varphi : E \hookrightarrow M$ is an open embedding which sends the zero-section of E to N

[2] For most of the discussion, it would be enough to assume that F is an embedded submanifold; but for simplicity, we will also assume compactness.

where we have denoted by $\widetilde{\omega}$ the pullback of ω to E:

$$\widetilde{\omega} = \mathrm{p}^*(\omega) \in \Omega^2_{\mathrm{closed}}(E).$$

Therefore, L_t can be also written as:

$$L_t := \Big(\frac{1}{t}\big(m_t^*(L)^{-\widetilde{\omega}}\big)\Big)^{\widetilde{\omega}} = \Big(\frac{1}{t}m_t^*(L)\Big)^{\frac{t-1}{t}\widetilde{\omega}}.$$

The following Lemma can be proven for example by using local coordinates (for more conceptual proofs see e.g. [10] [7])

Lemma. The following limit exists as a Dirac structure on E

$$L_{\mathrm{lin},F} := \lim_{t\to 0} L_t \in \mathrm{Dir}(E).$$

The Dirac structure $L_{\mathrm{lin},F}$ is called the **linearization** of L at the presymplectic leaf F. The Dirac structure L is called **linearizable** around F if there is a diffeomorphism $\varphi : U \to V$, where U and V are open neighborhoods of F in E, so that

$$\varphi^*(L|_V) = L_{\mathrm{lin},F}|_U, \ \text{ and } \ \varphi(p) = p, \ \forall\, p \in F.$$

The linearization problem around leaves in Dirac geometry is to find sufficient conditions on L at F (preferably on the first jet of L at F) which ensure that L is linearizable at F. These conditions turn out to be quite strong, they usually involve certain "compactness assumptions"; this will become clearer in the examples below.

7.3. Let us mention that the natural isomorphisms between Dirac structures are not just diffeomorphisms, but are made out of a diffeomorphism and a 1-form. Namely, we say that two Dirac manifolds (N, L_N) and (M, L_M) are **isomorphic** if there exists a pair (φ, α)

$$\varphi : N \xrightarrow{\sim} M, \quad \alpha \in \Omega^1(N),$$

where φ is a diffeomorphism and α is a 1-form so that

$$\big(\varphi^* L_M\big)^{\mathrm{d}\alpha} = L_N.$$

The linearization problem should be stated using these isomorphisms; but, for simplicity, we will consider just diffeomorphisms.

7.4. Consider the family L_t appearing in the linearization problem 7.2. Then we have that

$$L_t^{-\widetilde{\omega}} = 1/t m_t^*(L^{-\widetilde{\omega}}).$$

Thus, for all $s \neq 0$, we have that:

$$m_s^*(L_t^{-\widetilde{\omega}}) = 1/t m_{st}^*(L^{-\widetilde{\omega}}) = sL_{st}^{-\widetilde{\omega}}.$$

Taking the limit at $t = 0$, we obtain that:

$$m_s^*(L_{\mathrm{lin},F}^{-\widetilde{\omega}}) = sL_{\mathrm{lin},F}^{-\widetilde{\omega}}.$$

This translates into the following geometric property of the linearization (which we have already observed for linear Poisson structures 2.24): Let (F_e, ω_e) be the presymplectic leaf of $L_{\mathrm{lin},F}$ through a point $e \in E$. Then the leaf through $m_s(e) = se$ is given by

$$(F_{se}, \omega_{se}) = (sF_e, \widetilde{\omega} + s(\omega_e - \widetilde{\omega})),$$

where, to be precise, the notation means: $sF_e := m_s(F_e)$ and

$$\widetilde{\omega} + s(\omega_e - \widetilde{\omega}) := \widetilde{\omega}|_{sF_e} + s(m_s^*(\omega_e) - \widetilde{\omega}|_{sF_e}) \in \Omega^2(sF_e).$$

This shows that the foliation varies linearly on the fibers, and the closed 2-forms on the leaves vary in an affine fashion, with $\omega_0 = \omega$.

7.5. *Linearization of Foliations.* Consider the case when L is given by a foliation $\mathcal{F}$, i.e.

$$L = T\mathcal{F} \oplus T\mathcal{F}^{\circ}.$$

In this situation, it is not difficult to prove that:

Lemma. The foliation $\mathcal{F}$ is linearizable at the compact leaf F if and only if the holonomy action is linearizable at F.

Let us sketch the definition of the holonomy action; for more details see [11]. Fix a point $p \in F$, and fix a complementary transverse submanifold $N \subset F$ through p, i.e.

$$T_pF \oplus T_pN = T_pM.$$

Consider a loop $\gamma : [0,1] \to F$ based at p, i.e. $\gamma(0) = \gamma(1) = p$. For every $q \in N$, which is close enough to p, we associate a point $\mathrm{hol}(\gamma)(q) \in N$ as follows. Let F_q denote the leaf through q. If q is close enough to p then there exists a path $\widetilde{\gamma}_q : [0,1] \to F_q$ which is close enough to γ, which starts at q, i.e. $\widetilde{\gamma}_q(0) = q$, and which ends at a point in N, i.e. $\widetilde{\gamma}_q(1) \in N$. Such a path can be constructed by covering $\gamma([0,1])$ with a finite number of foliated charts. Define

$$\mathrm{hol}(\gamma)(q) := \widetilde{\gamma}_q(1).$$

If we fix the cover of $\gamma([0,1])$, this operation can be made into a diffeomorphism

$$\mathrm{hol}(\gamma) : U \xrightarrow{\sim} V,$$

where U, V are open neighborhoods of p in N, and which fixes p. It can be shown that the germ of this map at p is independent on the chosen charts, and depends only on the homotopy class $[\gamma] \in \pi_1(F,p)$ of the loop γ (i.e. if we change all these data, the final diffeomorphisms will agree on a small neighborhood of p in N). The **holonomy action** is defined as the resulting group homomorphism:

$$\mathrm{hol} : \pi_1(F,p) \longrightarrow \Gamma_p(N),$$

where we have denoted by $\Gamma_p(N)$ the group of diffeomorphisms between neighborhoods of p which fix p, modulo the equivalence relation that two such diffeomorphisms are the same iff they coincide on a small neighborhood of p; $\Gamma_p(N)$ is called the group of **germs** at p of diffeomorphisms of N. The holonomy action is said to be **linearizable** if there is a chart on N centered at p, so that for the resulting coordinates, we have that $\mathrm{hol}[\gamma]$ is a linear map (around p) for every $[\gamma] \in \pi_1(F,p)$. The **holonomy group** of F at p, it the quotient:

$$\mathrm{Hol}(F,p) := \pi_1(F,p)/(\ker \mathrm{hol}),$$

which is also isomorphic to the image of the holonomy action. In this setting, we have:

Theorem (Reeb-Thurston)[11]. If F is compact and the holonomy group $\mathrm{Hol}(F,p)$ is finite, then the foliation $\mathcal{F}$ is linearizable around F.

Notice the "compactness assumptions" in the statement: F and $\mathrm{Hol}(F,p)$ are compact.

7.6. *Linearization around fixed points.* Consider the linearization problem in the case when the leaf F is just a point $F = \{p\}$. In this case, $p_T(L_p) = 0$, hence $L_p \cap T_pM = 0$. This condition is open on $p \in M$, therefore, after restricting to a neighborhood of p, we may assume that $L \cap TM = 0$, i.e. L is given by a Poisson structure π. Let us write the bivector in a chart centered at p:

$$\pi = \frac{1}{2}\sum_{i,j} \pi_{i,j}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The point 0 being a leaf is equivalent to $\pi_{i,j}(0) = 0$ for all i, j. The Dirac structure L_t corresponding to this chart given by the bivector:

$$\pi_t = \frac{1}{2}\sum_{i,j}\frac{1}{t}\pi_{i,j}(tx)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}.$$

Using that $\pi_{i,j}(0) = 0$, we obtain that the linearization $L_{\text{lin},0}$ is given by the bivector:

$$\pi_{\text{lin},0} = \frac{1}{2}\sum_{i,j,k} C^k_{i,j} x_k \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j},$$

where the numbers $C^k_{i,j}$ are

$$C^k_{i,j} = \frac{\partial \pi_{i,j}}{\partial x_k}(0).$$

We see that $\pi_{\text{lin},0}$ is a linear Poisson structure, and therefore (see the discussion in item 2.23) it corresponds to a Lie algebra $(\mathfrak{g}_p, [\cdot,\cdot])$, called the **isotropy Lie algebra** of π at p. Coordinate free, this Lie algebra can be described as follows:

$$\mathfrak{g}_p := T^*_pM, \quad [a,b] := [\pi^\sharp(\alpha)+\alpha, \pi^\sharp(\beta)+\beta]|_p, \quad a,b\in T^*_pM,$$

where $\alpha,\beta\in\Omega^1(M)$ are any 1-forms so that $\alpha_p = a$ and $\beta_p = b$, and the bracket on the right is the Dorfman bracket. Because $\pi_p = 0$, this operation is independent on the chosen extensions. Thus, can identify $\pi_{\text{lin},p} = \pi_{\mathfrak{g}_p}$. In this setting, we have:

Theorem (Conn [3]*)* If the isotropy Lie algebra $\}_p$ at p is semisimple of compact type, then π is linearizable around p.

The "compactness assumption" in this case can be restated as any of the following equivalent conditions:

(a) $\mathfrak{g}_p$ is semisimple of compact type;
(b) the Killing form of $\mathfrak{g}_p$ is negative definite;
(c) the connected and simply connected Lie group integrating $\mathfrak{g}_p$ is compact.

Finally, let us mention that there is a generalization of Conn's theorem to compact symplectic leaves of a Poisson manifold; the reader is referred to [6, 10].

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