Short Notes

We will look at representations of $GL_2(F)$ eventually
↳ 1st talk about Hecke Algebras and $GL_2(K)$.

⇒ if $|G| < \infty$, a rep of G ⟺ module over group algebra $\mathbb{C}[G]$
↳ this extends to smooth reps of locally profinite groups if we replace $\mathbb{C}[G]$ by $\mathcal{H}(G)$, the Hecke alg.

Note: assume G is unimodular here
Recall this means $\delta_G = \mathrm{Id}$

Fix Haar measure μ on G. For $f_1, f_2 \in C_c^\infty(G)$ def:

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x) \qquad \text{(convolution)}$$

with $f_1 * (f_2 * f_3)(g) = (f_1 * f_2) * f_3\ (g)$ (associative) (proof: direct)

Then $\mathcal{H}(G) = (C_c^\infty(G), *)$ is an associative $\mathbb{C}$-alg. (in general with no unit, commutative if G is)

Remark: the algebra structure depends on the choice of Haar measure, however $\exists\, c > 0$ s.t. $c\mu = \nu$ and then $f \longmapsto c^{-1} f$ is an alg. iso $\mathcal{H}_\mu(G) \to \mathcal{H}_\nu(G)$

Remark: if G is discrete $\mathcal{H}(G) \cong \mathbb{C}[G]$
$\mathcal{H}(G)$ has a unit iff this is the case

Lots of idempotent elements. Let K be a compact open subgroup.

$$e_K(x) := \begin{cases} \mu(K)^{-1} & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

$\mathcal{P}$: (1) $e_K * e_K = e_K$

(2) $e_K * f = f \iff f(Kg) = f(g)\ \forall k \in K, g \in G, f \in H(G)$

(3) the space $e_K * H(G) * e_K$ is a sub-alg. of $H(G)$ with unit element e_K

(proof: direct)

Note: $H(G,K) := e_K * H(G) * e_K$ i.e. $f \in H(G)$ satisfying $f(k_1 g k_2) = f(g)\ \forall k_1, k_2 \in K, g \in G$

1: left $H(G)$-mod w action $(f,m) \longmapsto$ "$f * m$"

M is smooth if $H(G) * M = M$

$H(G) = \bigcup_K H(G,K) \Rightarrow$ M is smooth iff

$\forall m \in M\ \exists\ K$ st $e_K * m = m$

Category: $H(G)$-Mod

(π, V) smooth rep of G. For $f \in H(G)$, $v \in V$:

Set $\pi(f)v := \int_G f(g)\pi(g)v\, d\mu(g)$

integrand $\in C_c^\infty(G;V) = \{ f: G \to V$ loc const. compactly supp$\}$

$\cong C_c^\infty(G) \otimes V$

integral $\in V$

It, choose K that fixes v and f (under right translation)

then int $= \pi(f)v = \mu(K) \sum_{g \in G/K} f(g)\pi(g)v$

$\therefore \pi(e_K)v = v$, $v \in V^K \leftarrow$ space of $\pi(K)$-fixed vectors in V

op Let (π, V) be a smooth rep of G.
The operation $(f, v) \longmapsto \pi(f)v$ gives V the structure of a {smooth} $H(G)$ module. If (π', V') is another smooth rep and $\phi : V \to V'$ is a G-homom, then ϕ is also an $H(G)$-homom i.e.

$$\phi \circ \pi(f) = \pi'(f) \circ \phi$$

---

lternatively let M be a smooth $H(G)$-module and denote the action by $H(G)$ on M by $f * m$ as before.

op: $\exists!$ G-homomorphism $\pi : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(M)$ st
(π, M) is a smooth rep. of G and
$\pi(f)m = f * m$. Moreover, if M' is another smooth module $\overline{\omega}$ (π', M') then any $H(G)$-homom. $M \to M'$ is a G-homom $\pi \to \pi'$

---

3) Look at $H(G, K)$ with unit element e_K

emma: (π, V) smooth rep of G.
The operator $\pi(e_K)$ is the K-projection $V \longrightarrow V^K$ with kernel $V(K)$. The space V^K is an $H(G,K)$ module on which e_K acts as the identity

Recall that $V = \bigcup_K V^K$ where V^K is the space of $\pi(K)$-fixed vectors

Recall that $V(K) = \bigoplus_{\substack{\rho \in \hat{K} \\ \rho \neq 1}} V^\rho$ where $\hat{K}$ denotes the equivalence classes of irreducible smooth representations of K.

hen, if $\rho \in \hat{K}$, def V^ρ to be the sum of all irreducible K-subspaces of V of class ρ.

$\divideontimes$ In particular, $V = V^K \oplus V(K)$ and $V(K)$ is the unique K-complement of V^K in V

The $\mathcal{H}(G,K)$-module V^K has info about the rep (π, V):

<u>Prop</u>: (1) let (π, V) be irr. smooth. $V^K = 0$ or a simple $\mathcal{H}(G,K)$-module.

(2) the process $(\pi, V) \longmapsto V^K$ induces a bijection between:

(a) equivalence classes of irreducible smooth reps of G s.t. $V^K \neq 0$

(b) isomorphism classes of simple $\mathcal{H}(G,K)$-modules.

<u>Cor</u>: (π, V) smooth, $V \neq 0$. (π, V) irreducible <u>iff</u> $\forall K$, $V^K = 0$ or $\mathcal{H}(G,K)$-simple

---

~~Notes~~.

Σ if G is finite, Key: $\mathbb{C}[G]$ is semisimple

$\iff$ Jacobson radical of $\mathbb{C}[G]$ trivial

$\iff$ if $x \in \mathbb{C}[G]$, $x \neq 0$ $\exists$ simple $\mathbb{C}[G]$-module M st $xM \neq 0$

ypothesis: $\forall K$, G/K is countable

$\Rightarrow$ if (π, V) is irr. $\Rightarrow$ $\dim_{\mathbb{C}} V$ is countable

<u>paration property</u>: let $f \in \mathcal{H}(G)$, $f \neq 0$ $\exists$ irr. smooth (π, V) st $\pi(f) \neq 0$

For proof, use <u>lemma</u>

inite Fields and $GL_2(K)$

irst, facts about $GL_2(F)$ for F arbitrary field:

algebraic subgroups: $B = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right\}$ Borel subgroup

$N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$ unipotent radical of B

$B = T \ltimes N$

$T = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\}$ split maximal torus

$F^\times \cong Z = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\}$ centre of G

Now $G = B \cup BwB$ where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
(Bruhat Decomposition) ie $\{1, w\}$ is a set of representatives for the coset space $B \backslash G / B$

$\left(B = NT = TN \text{ and } w \in N_G(T) \text{ ie } Tw = wT\right)$

$\therefore$ $BwB = NwB = BwN$ & the map $B \times N \longrightarrow BwN$, $(b,n) \longmapsto bwn$ is bijective

---

jugacy classes

et $ch_g(t) := \det(tI - g)$ the characteristic poly of g.

$ch_g(t)$ is a monic quadratic polynomial $\bar{w}$ coeff in F and $ch_g(0) = \det g \neq 0$

: let $g \in G = GL_2(F)$ and $f(t) = ch_g(t)$

(i) suppose $f(t)$ is irreducible over F. The subalg $F[g]$ of $A = M_2(F)$ is a field & the G-centralizer of g is $F[g]^\times$.
If $f(t) = t^2 + at + b$, then g is G-conjugate to:

$$\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$$

[illegible] $h \in G$ is G-conjugate to g iff $ch_h(t) = f(t)$

2) if $f(t)$ has distinct roots $a, b \in F^\times$
Then g is conjugate to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

) $f(t)$ has repeated root $a \in F^\times$
g conj to exactly one of:
$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

---

low let K be the field with q elements.
lassify the irreducible reps of $GL_2(K)$

) $|G| = (q^2-1)(q^2-q)$

<u>Lemma</u>: G has exactly q^2-1 conjugacy classes
$\therefore$ q^2-1 irreducible reps (up to iso)

2) $N \cong (K, +)$ via $x \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

fix nontrivial character ψ of K
the function $a\psi : x \longmapsto \psi(ax)$, $x \in K$
ranges over the characters of K as a ranges over K

By the action of T (or B) on N, the characters of N fall into 2 conjugacy classes:
- trivial character
- all non-trivial ones

3) Let χ_1, χ_2 be characters of $K^\times$

Form the character of T $\chi = \chi_1 \otimes \chi_2 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto \chi_1(a)\,\chi_2(b)$

View this as a character of B trivial on N via $B \twoheadrightarrow B/N \cong T$

Form induces representations $\mathrm{Ind}_B^G \chi$ of G and consider the irreducible components.

Lemma: Let π be an irreducible rep. of G. TFAE:

(1) π is equivalent to a G-subspace of $\mathrm{Ind}_B^G \chi$ for some character χ of T

(2) π contains the trivial character of N

Use $\chi^\omega : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto \chi_2(a)\,\chi_1(b)$

Prop: Let χ, ξ be characters of T, viewed as characters of B which are trivial on N.

(1) the space $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\xi))$ is trivial unless $\chi = \xi$ or $\chi = \xi^\omega$

(2) the spaces $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\chi))$, $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi), \mathrm{Ind}_B^G(\chi^\omega))$ have the same dimension. This dim is 2 if $\chi = \chi^\omega$ and otherwise is 1.

Cor: Let χ be a character of T, viewed as a character of B that is trivial on N.

(1) $\mathrm{Ind}_B^G \chi$ is irreducible iff $\chi \neq \chi^\omega$

(2) if $\chi = \chi^\omega$, the rep $\mathrm{Ind}_B^G \chi$ has length 2 with distinct composition factors.

↳ these are called the principal series

Cor: up to isomorphism there are $\frac{1}{2}(q^2+q)-1$ of these

An irreducible rep of G <u>not</u> containing the trivial character of N is called ~~a~~ <u>cuspidal</u>.

↳ ~~[illegible]~~ Any two non-trivial characters of N are B-conjugate, so such a rep σ contains all non trivial characters of N.

Let l/K be a quadratic field extension.

The non-trivial K-automorphism of l is $x \longmapsto x^q$

A character θ of $l^\times$ is <u>regular</u> if $\theta^q \neq \theta$

Choosing a K-basis of l identifies it to $K \oplus K$ and $G = GL_2(K) \cong \mathrm{Aut}_K(l)$. The action of $l^\times$ on l embeds $l^\times$ in G, with conjugacy class of Type (1).

$\subset$ $E := l^\times$ subgroup of G.

<u>Note</u>: if $g \in G$ w $ch_g(t)$ irreducible, it is conjugate to an element of E.

Let θ be a regular character of E, ψ a non-trivial character of N. Define a character θ_ψ of ZN:

$$\theta_\psi : \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} u \longmapsto \theta(a)\psi(u) \quad \text{for } a \in K^\times,\ u \in N$$

<u>Thm</u>: (1) the virtual representation

$$\pi_\theta = \mathrm{Ind}_{ZN}^{G} \theta_\psi - \mathrm{Ind}_E^G \theta$$

is an irreducible cuspidal rep of G of dim $q-1$

(2) θ_1, θ_2 regular characters of E; $\pi_{\theta_1} \cong \pi_{\theta_2}$ <u>iff</u> $\theta_1 = \theta_2$ or $\theta_2 = \theta_1^q$

(3) <u>Every</u> irreducible cuspidal rep of G is of the form π_θ.