Smooth Representations

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1 Locally profinite groups

One of the two sets in the Weil correspondence for GL_2 is the set of equivalence classes of smooth representations of $\mathrm{GL}_2(F)$ for a nonarchimedean local field F. This talk will be about defining smooth representations and giving some properties. I start by defining a type of group of which $\mathrm{GL}_2(F)$ is an example.

Definition 1.1. A *locally profinite group* is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G.

A locally profinite group is locally compact, and any compact open subgroup is profinite; this explains the terminology.

Example 1.2. Let F be a non-Archimedean local field with ring of integers $\mathfrak{o}$ and prime ideal $\mathfrak{p}$; furthermore, assume $\mathfrak{o}/\mathfrak{p}$ is finite with cardinality q. Let ω be a prime element, i.e. $\mathfrak{p} = (\omega)$. Then any open neighbourhood of 0 contains an 'open sphere'

$$\{x \in F : ||x|| < q^{-n-1}\} = \mathfrak{p}^n = \omega^n \mathfrak{o},$$

which is a compact open subgroup of the additive group F. This shows that F is a locally profinite group.

Example 1.3. Similarly, the unit group $F^\times$ is locally profinite, as the congruence unit groups $U_F^j = 1 + \mathfrak{p}^j$ are compact open. In general $\mathrm{GL}_n(F)$ has compact open subgroups $1 + \mathfrak{p}^j \, \mathrm{M}_n(\mathfrak{o})$ that form a fundamental system of open neighbourhoods of 1. This makes $\mathrm{GL}_n(F)$ a locally profinite group.

Definition 1.4. Let G be a locally profinite group. A *character* of G is a continuous homomorphism $G \to \mathbf{C}^\times$, where $\mathbf{C}^\times$ is given the archimedean topology.

2 Smooth representations of locally profinite groups

Definition 2.1. Let G be a locally profinite group, and let (π, V) be a representation of G, i.e. a pair of a complex vector space V and a group homomorphism $\pi : G \to \mathrm{Aut}_{\mathbf{C}}(V)$. The representation (π, V) is called *smooth* if for every $v \in V$ there is a compact open subgroup K of G such that $\pi(K)v = v$.

A smooth representation (π, V) is called *admissible* if the space $V^K = \{v \in V : \pi(K)v = v\}$ is finite-dimensional for every compact open subgroup $K \subset G$. It is called irreducible if it has no G-stable subspaces other than 0 and V.

The category of smooth representations of G will abusively be denoted $\mathrm{Rep}(G)$; the category of 'abstract' representations of G will be denoted $\mathrm{ARep}(G)$ (and will occur much less in our seminar). Note that $\mathrm{Rep}(G)$ is abelian. There is a left exact functor $\cdot^\infty : \mathrm{ARep}(G) \to \mathrm{Rep}(G)$ defined as follows. Given $(\pi, V) \in \mathrm{ARep}(G)$ one may define

$$V^\infty = \bigcup_{K \subset G \text{ compact open}} V^K.$$

This is a G-stable subspace of V, which allows us to define $\pi^\infty(g) = \pi(g)|_{V^\infty}$. Then by definition (π^∞, V^∞) is a smooth representation of G. If (ρ, W) is another smooth representation, then any morphism of representations $W \to V$ factors through V^∞.

Proposition 2.2. *Let G be a locally profinite group and let (π, V) be a smooth representation of G. Then the following are equivalent:*

1. *V is the sum of its irreducible G-subspaces;*
2. *V is the direct sum of a family of irreducible G-subspaces;*
3. *any G-subspace of V has a G-complement in V.*

A representation satisfying these properties is called semisimple.

Proof. See [1, 2.2]. □

Example 2.3. Let G be a profinite group, and let V be an irreducible presentation of G. Let $v \in V$, and let $K \subset G$ be compact open such that $v \in V^K$. Since G is compact, K is of finite index in G. Let X be a set of representatives of G over K; then V is spanned by $\{gv : g \in X\}$, so V is pointwise stabilised by $\bigcap_{g \in X} gKg^{-1} = K'$, which is normal and open in G, hence of finite index. But then V is an irreducible representation of the finite discrete group G/K', so it has finite dimension. in general, let G be locally profinite, let V be a representation of G, and let K be a compact open subgroup of G. Let $v \in V$. Then K is profinite, and as before there is a normal open subgroup $K' \subset K$ of finite index such that K' acts trivially on the K-subspace W of V generated by v. Then W is a finite-dimensional representation of K/K', and as such is a direct sum of K-irreducible subspaces. The same then holds for V, so we have shown that V is *K-semisimple.*

Proposition 2.4. *Let V be a smooth representation of a locally profinite group G and let K be a compact open subgroup of G. Let $\hat{K}$ be the set of equivalence classes of irreducible smooth representations of K, and for $\rho \in \hat{K}$, let V^ρ be the sum of all irreducible K-subspaces of V of class ρ.*

1. *$V = \bigoplus_{\rho \in \hat{K}} V^\rho$.*
2. *Let W be another smooth representation of G. For any G-homomorphism $f : V \to W$ and for any $\rho \in \hat{K}$ we have $f(V^\rho) = W^\rho \cap f(V)$.*

Proof.

1. By the above example we can compose V into a direct sum of irreducible K-subspaces; choose such a sum, and let $V(\rho)$ be the sum of all of the summands of type ρ. Then $V = \bigoplus_{\rho \in \hat{K}} V(\rho)$, and we need to show that $V(\rho) = V^\rho$. Suppose $W \subset V$ is an irreducible K-subspace of type ρ. If $W \not\subset V(\rho)$, there would be a an irreducible $U \subset V$ of type not equal to ρ in the summation such that the projection $W \to U$ is nontrivial, which is a contradiction.
2. This follows from the direct sum $V = \bigoplus_{\rho \in \hat{K}} V^\rho$ and the fact that any map $V^\rho \to W^\tau$ is trivial.

□

This proposition has two important corrolaries.

Corollary 2.5. *A short exact sequence of smooth representations $U \to V \to W$ is exact if and only if the induced sequence $U^K \to V^K \to W^K$ is exact for every compact open subgroup K of G.*

Corollary 2.6. *Let G be a locally profinite group, let K be a compact open subgroup of G, and let V be a smooth representation of G. Define $V(K)$ to be the linear subspace of V spanned by $\{v - kv : v \in V, k \in K\}$. Then*

$$V(K) = \bigoplus_{\rho \in \hat{K} \setminus \{1\}} V^{\rho} and V = V^K \oplus V(K).$$

$V(K)$ is the unique K-complement of V^K in V.

Proof. $\subset$ follows from the decomposition into V^ρ. For $\supset$ note that for any irreducible U of class $\neq 1$ one has $U(K) = U$. In general, $V(K)$ is contained in the kernel of any G-morphism $V \to V^K$. □

3 Induced representations

Let G be a locally profinite group, and let H be a closed subgroup of G. Then H itself is locally profinite, and restriction of scalars gives us a functor $R : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$. The aim of this section is to find both a left adjoint and a right adjoint to this functor.

The right adjoint of R is constructed as follows. Let (σ, W) be a smooth representation of H. Let $I(\sigma)$ be the space of functions $f : G \to W$ such that $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and $g \in G$. There is an action of G on $I(\sigma)$ by defining $(g \cdot f)(g') = f(g'g)$. This makes $I(\sigma)$ an abstract representation of G. We now define $\mathrm{Ind}_H^G : H \to G$ as $\cdot^\infty \circ I$; we call $\mathrm{Ind}_H^G(\sigma)$ the representation of G *smoothly induced by σ*.

Proposition 3.1. *Let G be a locally profinite group, and let H be a closed subgroup of G.*

1. *$R \dashv \mathrm{Ind}_H^G$.*

2. *Ind_H^G is additive and exact.*

Proof. 1. For every $(\sigma, W) \in \mathrm{Rep}(H)$, the map $a_\sigma : \mathrm{Ind}_H^G(\sigma) \to W : f \mapsto f(1)$ is an H-morphism. To show that Ind_H^G is right adjoint to R it is enough to show that for every $(\pi, V) \in \mathrm{Rep}(G)$ and for every H-morphism $F : V \to W$ there is a unique G-morphism F' such that $F' = a_\sigma F$. This is done by defining $F'(v) = (g \mapsto f(\pi(g)v))$.

2. Additivity follows from the additivity of I and $\cdot^\infty$. It is clear that I is exact and $\cdot^\infty$ is left exact, so we need to show that Ind_H^G is right exact. So let $f : (\sigma, W) \to (\tau, U)$ be an H-surjection, and let $\varphi \in \mathrm{Ind}_H^G(\tau)$. Let $K \subset G$ be a compact open subgroup fixing φ. Then the support of φ is a union of cosets of the form HgK; let X be a set of representatives for such g. For $g \in X$, $\varphi(g)$ is fixed by $H \cap gKg^{-1}$. Applying corollary 2.5 we find that there is an $w_g \in W$, fixed by $H \cap gKg^{-1}$, such that $f(w_g) = \varphi(g)$. Now define $\Phi : G \to W$ as having the same support as φ and $\Phi(hgk) = \sigma(h)w_g$, for $h \in H$, $g \in X$, $k \in K$. Then Φ is fixed by K, so $\Phi \in \mathrm{Ind}_H^G(\sigma)$, and $f_*(\Phi) = \varphi$.

□

If H is open we can construct a left adjoint to R. For $(\sigma, W) \in \mathrm{Rep}(H)$, let $c\text{-}\mathrm{Ind}_H^G(\sigma)$ be the subspace of $\mathrm{Ind}_H^G(\sigma)$ consisting out of all functions $f : G \to W$ such that $\mathrm{supp}\, f$ is compact in $H \backslash G$. This gives rise to a functor $c\text{-}\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. Furthermore, define the H-homomorphism

$$\begin{aligned} \alpha_\sigma^c : W &\to c\text{-}\mathrm{Ind}_H^G \sigma \\ w &\mapsto f_w \end{aligned}$$

Where f_w is supported in H and $f_w(h) = \sigma(h)w$. Notice that the space $c\text{-}\mathrm{Ind}_H^G\sigma$ can be constructed for closed subgroups in general, but $\alpha_\sigma^c(w)$ will not be an element of $c\text{-}\mathrm{Ind}_H^G\sigma$ unless H is open.

Proposition 3.2. *Let G be a locally profinite group and let H be an open subgroup.*

1. *c-*Ind_H^G *is a left adjoint to R, i.e. for any, $(W, \sigma) \in \text{Rep}(H)$, $(V, \pi) \in \text{Rep}(G)$ and any H-morphism $F : W \to V$ there is a G-morphism $F' : V \to$ c-*$\text{Ind}_H^G(\sigma)$ *such that $F = F' \circ \alpha_\sigma^c$.*

2. *c-*Ind_H^G *is additive and exact.*

Proof. See [1, 2.5]. □

4 Duality

Let G be a locally profinite group, and let (π, V) be a smooth representation of G. Let V^* be the dual vector space, and $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbf{C}$ the canonical pairing. We can define a representation π^* of G on V^* by stating

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle,$$

for all $g \in G$, $v^* \in V^*$, $v \in V$. Such a representation, however, is not always smooth. Therefore we define the *contragredient* or *smooth dual* of V to be $V^\vee = (V^*)^\infty$. Note that $(V^\vee)^K = (V^K)^*$ for all compact open $K \subset G$. We also get a canonical map $\delta : V \to (V^\vee)^\vee$, which is an isomorphism if and only if V is admissible, which can be seen by looking at the level of V^K's. On this level one sees, by corollary 2.5, that the contravariant functor $V \to V^\vee$ is exact.

5 Haar Measures

Let G be a locally profinite group. We consider the complex vector space $C_c^\infty(G)$ of functions $f : G \to \mathbf{C}$ that are locally constant and of compact support. Let f be such a function. For every $x \in \mathbf{C}$ the preimage $f^{-1}(x)$ is open in G; hence there are only finitely many x for which this is nonempty. Take such an x, then for every g in its preimage there is a compact open subgroup K_g such that $f(gK_g) = f(K_g g) = x$. Then $\{gK_g : g \in f^{-1}(x)\}$ is a cover of the compact set $f^{-1}(x)$. Taking a finite subcover and then taking the intersection of the corresponding compact open subgroups, we find a compact open subgroup K_x of G such that $f^{-1}(x)$ is closed under left and right multiplication of K_x. Doing this for all x and taking intersections, we find a compact open subgroup K of G such that f is a finite linear combination of characteristic functions of double cosets KgK.

G acts on $C_c^\infty(G)$ be left translation $((\lambda_g f)(x) = f(g^{-1}x)$ and right translation $((\rho_g f)(x)) = f(xg))$, and both of these turn $C_c^\infty(G)$ into a smooth G-representation.

Definition 5.1. A *right Haar integral* on G is a non-zero right invariant linear functional $I : C_c^\infty(G) \to \mathbf{C}$ such that $I(f) \geq 0$ for any $f \geq 0$. A *left Haar integral* is defined analogously.

Proposition 5.2. *There exists a right Haar integral on G, which is unique up to multiplication by a positive real scalar.*

Proof. For any compact open subgroup K of G, let ${}^K C_c^\infty(G)$ be the subspace of $C_c^\infty(G)$ of $\lambda(K)$-invariant functions; this is a G-subspace when $C_c^\infty(G)$, which we may regard as c-$\text{Ind}_K^G 1_K$. Then $\text{Hom}_G({}^K C_c^\infty(G), \mathbf{C}) = \text{Hom}_H(\mathbf{C}, \mathbf{C}) \cong \mathbf{C}$, which shows the uniqueness of a right Haar integral on ${}^K C_c^\infty(G)$. A $\mathbf{C}$-basis of ${}^K C_c^\infty(G)$ is given by the characteristic functions f_g of the cosets Kg. The functional $I_K : f_g \mapsto 1$ has the desired properties.

Now let $(K_n)_{n\geq 1}$ be a descending sequence of compact open subgroups of G such that $\bigcap_n K_n = 1$. Take $I_1 = I_{K_1}$ as above, and let I_n be the unique multiple of I_{K_n} such that K_n is mapped to $(K_1 : K_n)^{-1}$. Then $I_{n+1}|_{{}^{K_n} C_c^\infty} = I_n$, so taking the union over the I_n gives us the required functional I; its uniqueness follows from the uniqueness of the I_n. □

Remark 5.3. The existence and uniqueness of left Haar integrals on G can be proven similarly. A left Haar integral is not necessarily a right Haar integral; if it is we call G *unimodular*. If Γ_S is the characteristic function of a compact open set S, then one may write $\mu_G(S) = I(\Gamma_S)$ to introduce the notion of measure, and this allows for the notations

$$I(f) = \int_G f(x)d\mu_G(x) \text{ and } \int_G f(x)\Gamma_S d\mu_G(x) = \int_S f(x)d\mu_G(x).$$

We can also integrate more general functions. Let f be a function on G invariant under left translation by a compact open subgroup K of G. Then for any $g \in K \setminus G$ the integral

$$\int_{K_g} f(x)d\mu_G(x)$$

is well-defined. If

$$\sum_{g\in K\setminus G} \int_{K_g} |f(x)|d\mu_G(x)$$

converges, then so does the series without absolute values, and we can set

$$\int_G f(x)d\mu_G(x) = \sum_{g\in K\setminus G} \int_{K_g} f(x)d\mu_G(x).$$

Now suppose μ_G is a left Haar measure on G (corresponding to a functional I). For $g \in G$, the functional

$$\begin{aligned} I_g : C_c^\infty(G) &\to \mathbf{C} \\ f &\mapsto \int_G f(xg)d\mu_G(x) \end{aligned}$$

is again a left Haar integral on G; let $\delta_G(g) \in \mathbf{R}^+$ be such that $\delta_G(g)I_g = I$. This defines a homomorphism $\delta_G : G \to \mathbf{R}_{>0}$, which is trivial if G is abelian or, more generally, unimodular. δ_G is called the *module* of G. Note that δ_G is trivial on any open compact subgroup of G, and the functional

$$f \mapsto \int_G \delta_G(x)^{-1}f(x)d\mu_G(x)$$

is a right Haar integral on G.

Now let H be a closed subgroup with module δ_H, and let $\delta : H \to \mathbf{C}^\times$ be a character of G. Let $C_c^\infty(H \setminus G, \vartheta)$ be the space of functions $f : G \to \mathbf{C}$ which are compactly supported modulo H and which satisfy $f(hg) = \vartheta(h)f(g)$ for all $h \in H$, $g \in G$. Let G act on this space by right translation.

Proposition 5.4. *The following are equivalent:*

1. *There exists a non-zero right invariant linear functional* $I_\vartheta : C_c^\infty(H \setminus G, \vartheta) \to \mathbf{C}$*;*
2. $\vartheta\delta_H = \delta_G|_H$.

If these conditions hold, I_ϑ *is determined up to a constant factor.*

Proof. **1.** $\Rightarrow$ **2.** Let μ_H and μ_G be left Haar measures. Define a G-morphism

$$\begin{aligned} Q : C_c^\infty(G) &\to C_c^\infty(H \setminus G, \vartheta) \\ f &\mapsto \int_H \vartheta(h)\delta_H(h)^{-1}f(hg)d\mu_H(h). \end{aligned}$$

Note that $Q(\lambda_k f) = \delta_H(k)\vartheta(k)^{-1}Q(f)$. Since 1 holds the induced map $I_\vartheta \circ Q : C_c^\infty(G) \to \mathbf{C}$ is a nontrivial G-homomorphism, hence it must be a scalar multiple of a right Haar integral. This also shows that I_ϑ is determined up to a constant factor.

Any function of the form $\lambda_h f - \delta_H(h)\vartheta(h)^{-1}f$, for $h \in H$, lies in the kernel of Q. Applying the right Haar integral to these functions we get

$$\int_G (\lambda_h f(g) - \delta_H(h^{-1})\vartheta(h^{-1})f(g))\delta_G(g)^{-1}d\mu_G(g)$$

$$= (\delta_G(h^{-1}) - \delta_H(h^{-1})\vartheta(h^{-1}))\int_G f(g)\delta_G(g)^{-1}d\mu_G(g),$$

which vanishes for all f and h if and only if 2. holds.

2. $\Rightarrow$ 1. Let K be a compact open subgroup of G. Then a coset of the form HgK supports at most a onedimensional space of functions in $C_c^\infty(H \setminus G, \vartheta)^K$, and these subspaces span $C_c^\infty(H \setminus G, \vartheta)^K$. The such a space lies in the image of Q, since it is the image of the charactersitic function of a coset gK. This shows that Q is surjective on K-fixed functions, hence it is surjective in general. To show 1. it is now sufficient to show that the right Haar integral factors through Q. So let $f \in C_c^\infty(G)$ be such that $Q(f) = 0$. We may assume that $\mathrm{supp}(f) \subset HgK$ for some $g \in G$ and some compact open subgroup K of G. Then f is a finite linear combination of the characteristic function of cosets h_igK, with h_i in H. For such a characteristic function we find

$$\begin{aligned} Q(\Gamma_{h_igK})(g') &= \int_H \vartheta(h)\delta_H(h)^{-1}\Gamma_{h_igK}(hg')d\mu_H(h) \\ &= \int_{H\cap h_igKg^{-1}} \vartheta(h)\delta_H(h)^{-1}d\mu_H(h). \end{aligned}$$

However, $\vartheta\delta_H^{-1}$ is trivial on the compact subgroup $H \cap gKg^{-1}$ of H, so this is equal to $\mu_H(H \cap h_igKg^{-1})\vartheta(h_i)\delta_H(h_i)^{-1} = \mu_H(H \cap gKg^{-1})\vartheta(h_i)\delta_H(h_i)^{-1}$. For f this means that for all $g \in G$ one has

$$\begin{aligned} 0 &= \mu_H(H \cap gKg^{-1})\sigma_i\vartheta(h_i)\delta_H(h_i)^{-1}f(h_ig) \\ &= \mu_H(H \cap gKg^{-1})\delta_G(h_i)^{-1}f(h_ig) \\ &= \frac{\mu_H(H \cap gKg^{-1})}{\mu_G(K)}\int_G f(x)\delta_G(x)^{-1}d\mu_G(x), \end{aligned}$$

which shows that $I(f) = 0$, as was to be shown.

□

Remark 5.5. The character ϑ as found above is now uniquely determined and is denoted $\delta_{H\setminus G}$. The measure corresponding to ϑ is denoted $\mu_{H\setminus G}$ and is called *semi-invariant.*

Theorem 5.6. *Let μ be a positive semi-invariant measure on $H \setminus G$. Let (σ, W) be a smooth representation of H. There is a natural isomorphism*

$$(c\text{-}Ind_H^G\sigma)^\vee \cong Ind_H^G(\delta_{H\setminus G} \otimes \sigma^\vee)$$

induced by the G-invariant pairing

$$\begin{aligned} Ind_H^G(\delta_{H\setminus G} \otimes \sigma^\vee) \times c\text{-}Ind_H^G\sigma &\to \mathbf{C} \\ (\Phi, \varphi) &\mapsto \int_{H\setminus G}\langle\Phi(g), \varphi(g)\rangle d\mu(g). \end{aligned}$$

Proof. See [1, 3.5]. □

References

[1] Colin J Bushnell and Guy Henniart. *The local Langlands conjecture for GL (2)*, volume 335. Springer, 2006.