Local factors of zeta functions of algebraic varieties (definitions and conjectures)

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This is a translation by Johan Commelin of the article

*Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*

by J. P. Serre.
Note by the translator

This is a translation of the article Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures) by J. P. Serre. The translation is quite concordant, in the sense that sentences are translated literally and word by word as much as possible (while producing correct English sentences). Now and then this results in long statements, that might have been cut in two or three separate sentences.

All references are taken over literally, so the bibliography is unchanged. (I have not taken the effort to look up translations of certain books and articles and change citations accordingly.)

Concerning typography and layout. Most of the phrases that were underlined in the original article are set in italic in this translation. Further the equation numbering has changed.

Concerning mathematical notation a few subtle though innocent changes have been made. Most of these have to do with the better typesetting facilities anno 2012. Sometimes a dot has been replaced by parentheses (as in dim. $V$ becomes dim($V$)). The canonical rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, and $\mathbb{F}_q$ are set in blackboard face, instead of placing a twiddle below them. Further, in the original text the cardinality of the residue field $k(v)$ is denoted $N_0$, which is replaced by $N_v$ to avoid confusion. The subscript $nr$ to indicate a maximal unramified (non ramifiée) extension is replaced by the English $ur$.

Other than these, all formulas are copied as is, with the exception of eq. 2.3. In this equation $V/V_{n+1}$ is replaced with $V_{n+1}/V$ since I guess the former was a typo.

May 24, 2012, Leiden, The Netherlands — Johan Commelin
1 Statement of the problem

1.1 Standard conjectures over finite fields

We briefly review what they consist of (see [6, 7, 19], for more details).

Let an integer $m \geq 0$, and a non-singular projective variety $Y$ over a finite field $k$ of $q = p^f$ elements be given. We write $\pi : y \mapsto y^q$ for the Frobenius endomorphism of $Y$.

Let $k$ be an algebraic closure of $k$, and let $Y = Y \times_k k$ be the variety deduced from $Y$ by extending scalars from $k$ to $k$. If $\ell$ is a prime number $\neq p$, M. Artin and A. Grothendieck have defined the cohomology group $H^m(Y, \mathbb{Q}_\ell)$ and showed that this is a finite-dimensional vector space over the field $\mathbb{Q}_\ell$ of $\ell$-adic numbers (cf. [2]). By functoriality, $\pi$ induces an endomorphism $\pi_m^{\ell, \ell}$ of $H^m(Y, \mathbb{Q}_\ell)$; which allows us to define the polynomial

$$P_\ell^m(T) = \det(1 - T \pi_m^{\ell, \ell}),$$

which has coefficients in $\mathbb{Z}_\ell$. Moreover, the product

$$Z(T) = \prod_{m=0}^{m=2r} P_\ell^m(T)^{(-1)^{m+1}}, \quad r = \dim(Y),$$

coincides (cf. [2]) with the zeta function of $Y$ in the sense of Weil ([13, 19]); in particular, it is a rational function of $T$, with coefficients in $\mathbb{Q}$, and it does not depend on the choice of $\ell$.

The "standard conjectures" for a given dimension $m$ consist of the following two statements:

$C_1$ The polynomial $P_\ell^m(T)$ has coefficients in $\mathbb{Z}$ and does not depend on the choice of the prime number $\ell$. (Therefore we can write $P_m(T)$, without specifying $\ell$.)

$C_2$ (Weil conjecture) If we decompose $P_m(T)$ over $\mathbb{C}$ in linear factors $(1 - \lambda_i T)$, we have $|\lambda_i| = q^{m/2}$ for all $i$.

For $m \leq 1$ these statements are true, since they are equivalent to classical results of Weil ([18, §8ix, x]). For $m \geq 2$, on the contrary, we do not know much, because we cannot transfer the results of Kähler theory to characteristic $p$ (cf. [6, 7, 11]).

1.2 The “good factors” of zeta functions over global fields

1.2.1 Notation

The letter $K$ denotes a global field, i.e., an algebraic number field or a function field in one variable over a finite field. We write $\Sigma_k$ (resp. $\Sigma_k^\infty$) for the set of ultrametric (resp. archimedean) places of $K$. If $v \in \Sigma_k \cup \Sigma_k^\infty$, we write $K_v$ for the completion of $K$ at $v$. If $v$ is ultrametric, we write $O_v, k(v)$ and $p_v$ for the valuation ring of $K_v$, its residue field, and its residue characteristic; we put $N_v = \text{Card}(k(v))$. We have

$$N_v = p_v^\deg(v), \quad \text{with } \deg(v) = [k(v) : \mathbb{F}_{p_v}].$$

(1.3)
The letter $X$ denotes a non-singular projective variety over $K$. The letter $m$ denotes an integer $\geq 0$.

1.2.2 Good reduction

We choose a finite subset $S$ of $\Sigma_K$ such that $K$ has “good reduction outside $S$”. This implies that for all $v \in \Sigma_K - S$, there exists projective smooth $O_v$-scheme $X_v$ such that $X_v \times_{O_v} K_v$ is identified with $X \times_K K_v$. We choose such an $X_v$, and we write $X(v)$ for its reduction at $v$, in other words the scheme $X_v \times_{O_v} k(v)$. It is an non-singular projective variety over the finite field $k(v)$.

We suppose in all that follows that $X(v)$ satisfies the standard conjectures (C1) and (C2) in dimension $m$. So we can talk about the corresponding polynomial $P_m(T)$, which has integral coefficients; as this polynomial depends on $v$, we write $P_{m,v}(T)$.

The function $\zeta_S$

It depends on the choice of $S$ (and obviously also on $X$, $K$, and $m$). We define (cf. Tate [17]) it via the Euler product

$$\zeta_S(s) = \prod_{v \in \Sigma_K - S} \frac{1}{1 - \lambda_{X,v}N_v^{-s}},$$

which converges absolutely for $\Re(s) > 1 + m/2$. We deduce that $\zeta_S(s)$ is holomorphic and non-zero on the half-plane $\Re(s) > 1 + m/2$ and that it is a Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$, with integral coefficients, converging absolutely on the half-plane in question.

1.3 Analytic continuation and functional equation of $\zeta_S$

In general, we do not know whether the function $\zeta_S$ as defined above might be continued analytically to the left of the line $\Re(s) = 1 + m/2$. Anyhow, it has been verified in several special cases (notably, abelian varieties with complex multiplication, modular curves). In each of these cases, we found that $\zeta_S$ could be continued to a meromorphic function on the entire complex plane, and satisfied a functional equation of the form $s \leftrightarrow m + 1 - s$. More precisely, in each case, one is able to define the following objects:

i) A rational number $A > 0$;
ii) For every \( v \in S \), a polynomial \( P_{m,v}(T) = \prod (1 - \lambda_{a,v}T) \) with integral coefficients, of degree \( B_{m,v} \leq B_m \);

iii) For every \( v \in \Sigma^\infty_K \) a "gamma factor" \( \Gamma_v(s) \) (see section 3).

Moreover, if we put
\[
\zeta(s) = \zeta_S \cdot \prod_{v \in S} \frac{1}{P_{m,v}(N_v^{-s})} = \prod_{v,a} \frac{1}{(1 - \lambda_{a,v}N_v^{-s})}
\]  

and
\[
\xi(s) = A^{s/2} \zeta(s) \prod_{v \in \Sigma^\infty_K} \Gamma_v(s),
\]  

then the functional equation is:
\[
\xi(s) = w \xi(m + 1 - s), \quad \text{with } w = \pm 1.
\]

When we examine how \( A \), the \( P_{m,v} \), and the \( \Gamma_v \) are defined in each of these special cases, then one observes that it is possible to give a general definition of them: \( A \) and the \( P_{m,v} \) (\( v \in S \)) only depend on the \( \ell \)-adic cohomology of the varieties \( X \times_K K_v \), and the \( \Gamma_v \) (\( v \in \Sigma^\infty_K \)) only depend on the Hodge decomposition of the complex cohomology of \( X \times_K K_v \). It is this general definition that I propose to give in this article; it can be found in §4 (the §§ 2 and 3 contain several preliminaries of local nature).

I stress that I do nothing more than giving a definition, accompanied by a certain number of conjectures denoted with the symbols \( C_3, \ldots, C_9 \). The problem of proving these conjectures (and in particular the analytic continuation and the functional equation) remain entirely. Moreover, such a proof will probably not be possible without a convenient combination of methods from algebraic geometry and the theory of modular functions, following the path taken by Weil ([20]) and followed, among others, by Langlands. We are still far from our goal...

# 2 Ultrametric local invariants

In this section, we denote with \( K_v \) a field that is complete for a normalised discrete valuation \( v \). As in section 1.2, we write \( O_v, k(v) \), and \( p_v \) for the valuation ring of \( K_v \), its residue field, and its residue characteristic. We assume \( k(v) \) is perfect; from section 2.2 on, we assume that \( k(v) \) is even finite, and we write \( N_v \) for the number of its elements.

We choose a separable closure \( K_{v,s} \) of \( K_v \); we write \( G \) for the Galois group of \( K_{v,s} \) over \( K_v \), and \( I \) for its inertia group.

## 2.1 \( \ell \)-adic representations and conductors

Let \( \ell \) be a prime number \( \neq p_v \), let \( V \) be a vector space over \( \mathbb{Q}_\ell \) of finite dimension \( d \), and let
\[
\rho: G \to \text{Aut}(V) \cong \text{GL}(d, \mathbb{Q}_\ell)
\]
be a continuous homomorphism from $G$ to $\text{Aut}(V)$, in other words an $\ell$-adic representation of $G$ in $V$ in the sense of ([15]). We are going to attach two positive integers $\varepsilon$ and $\delta$ to $\rho$ that measure in some sense the “ramification” of $\rho$.

The invariant $\varepsilon$ is the most easy to define: If $V^I$ denotes the subspace of $V$ consisting of the elements that are invariant under the inertia group $I$, we put

$$\varepsilon = \text{codim} V^I = d - \text{dim} V^I. \tag{2.1}$$

We have $\varepsilon = 0$ if and only if $\rho$ is unramified, i.e., if $\rho(I) = \{1\}$.

To define $\delta$, we state the following condition:

$H_\rho$ There exists an open subgroup $I'$ of $I$ such that $\rho(g)$ is unipotent for all $g \in I'$.

(Recall that an endomorphism is called unipotent if all its eigenvalues are equal to 1.)

2.2 Remark. Grothendieck has shown ([16, p. 515]) that the condition $(H_\rho)$ is satisfied if $k(v)$ is finite. It is also satisfied, according to Deligne (not published), if $\rho$ is the representation defined by the $\ell$-adic cohomology of a non-singular projective variety over $K_\nu$, see section 2.3.

We assume that $(H_\rho)$ is satisfied, and we take a fixed $I'$ in $I$. For every integer $n \geq 0$, let $V_n$ denote the set of $x \in V$ such that $(\rho(g) - 1)^n x = 0$ for all $g \in I'$. The $V_n$ form a increasing filtration of $V$, stable under $I$, and we have $V_n = V$ for $n$ big enough, according to a theorem by Lie-Kolchin applied to $I'$. If we put

$$\text{gr}V = \bigoplus_{n=0}^{\infty} V_{n+1}/V_n, \tag{2.3}$$

the group $I$ acts on $\text{gr}V$ through the finite group $\Phi = I/I'$. In particular we deduce that, if $g \in I$, the trace of $\rho(g)$ in $V$ only depends on the image of $g$ in $\Phi$, in that way we obtain a function $\text{Tr}_\rho : \Phi \to \mathbb{Q}_\ell$ which we call the character of $\rho$ on $I$.

Now let $K_{v,\text{ur}}$ be the maximal unramified extension of $K_v$ in $K_{v,\text{ur}}$; we have $I = \text{Gal}(K_{v,\text{ur}}/K_v)$ and the group $\Phi$ is thus the Galois group of a finite extension $K_\Phi/K_{v,\text{ur}}$, which is totally ramified. Let $\nu_\Phi$ be the normalized valuation of $K_\Phi$, and let $t$ be a uniformizer of $K_\Phi$; we have $\nu_\Phi(t) = 1$. We define a function $b_\Phi$ on $\Phi$ by the formulas

$$b_\Phi(g) = 1 - \nu_\Phi(g(t) - t), \quad g \in \Phi - \{1\},$$

$$b_\Phi(1) = - \sum_{g \neq 1} b_\Phi(g). \tag{2.4}$$

The function $b_\Phi$ is a character of the group $\Phi$ (it is the character that Grothendieck calls “de Swan”, cf. [9] as well as [14, p. III–20]). The invariant $\delta$ is then defined as the scalar product $\langle \text{Tr}_\rho, b_\Phi \rangle$ of the characters $\text{Tr}_\rho$ and $b_\Phi$:

$$\delta = \langle \text{Tr}_\rho, b_\Phi \rangle = \frac{1}{\text{Card}(\Phi)} \sum_{g \in \Phi} \text{Tr}_\rho(g)b_\Phi(g). \tag{2.5}$$
It is an integer $\geq 0$. It does not depend on the choice of $I'$. We have $\delta = 0$ if and only if the action of $I$ on $\text{gr}V$ is tame, i.e., if $p_v = 0$ or (in the case $p_v \neq 0$) if the Sylow-$p_v$-subgroup of $I$ acts trivially on $\text{gr}V$ (which also amounts to saying that this subgroup acts in a unipotent way on $V$).

Now that $\epsilon$ and $\delta$ are defined, we put
\[ f = \epsilon + \delta; \]
which is the exponent of the conductor of $\rho$.

2.7 Remark.

1) The integers $\epsilon$, $\delta$ and $f$ only depend on the restriction of $\rho$ to $I$; they do not change if we make an unramified field extension of the base (they are “geometric” invariants).

2) We can give another interpretation to $\delta$: Let $\Lambda$ be a $\mathbb{Z}_\ell$-lattice of $V$ stable under $G$, and let $V(\ell) = \Lambda/\ell\Lambda$ be the reduction modulo $\ell$; it is a Galois module of finite order prime to $p_v$, and its “measure of wild ramification” $\delta(K_v, V(\ell))$ is defined (cf. Ogg [9, §1]); it is not difficult to prove (with the same arguments as in [16, §3]) that we have
\[ \delta = \delta(K_v, V(\ell)). \]
[We could have taken this formula as definition of $\delta$; that would have had the advantage of avoiding the condition $(H_\rho)$.]

3) If the image of $\rho$ is finite, the Artin conductor of the character of $\rho$ is defined (cf. for example [12, chap. vi, §2]) and we verify without problems that its exponent is equal to $f$, which justifies the used terminology.

2.2 The case of a finite residue field

We add to the hypotheses of the preceding section that $k(v)$ is finite. The group $G/I$ is then equipped with a canonical generator, the Frobenius generator $F: \lambda \mapsto \lambda^{N_v}$. Since $I$ acts trivially on $V^I$, $F$ defines an automorphism of the vector space $V^I$; the inverse of this automorphism is called the “geometric” Frobenius automorphism; we denote it with $\pi_\rho$, or simply $\pi$. We put
\[ P_\rho(T) = \det(1 - \pi_\rho T). \] (2.8)
(We stress that $\pi_\rho$ is an automorphism of $V^I$ and not of $V$ as a whole.)

The polynomial $P_\rho$ has coefficients in $\mathbb{Z}_\ell$; its degree is equal to $\dim V^I = d - \epsilon$.

2.3 Application to the cohomology

We now return to the situation of section 1; let $Y$ be a non-singular projective variety over $K_v$, and let $m$ be an integer $\geq 0$. If $\ell$ is a prime number $\neq p_v$, let $V_\ell$ be the $m$-th cohomology group of $Y \times_{K_v} K_{v, \ell}$ with coefficients in $\mathbb{Q}_\ell$. The group $G$ acts on $V_\ell$ by transport of the structure; this defines an $\ell$-adic representation $\rho_\ell$, to which we can apply the above definitions.
C₃ The integers ε, δ and f associated to ρᵣ are independent of the choice of ℓ.

The part of this conjecture concerning δ can be made precise in the following way (cf. [16, p. 514, problem 1]):

C₄ The restriction to ℓ of the function Trₗ, has values in ℤ and is independent of ℓ.

If we assume C₃ and C₄ (which we do), the integers ε, δ and f are defined without ambiguity.

Now suppose that the residue field k(ν) of K, is finite; for all ℓ ≠ pₙ, the polynomial Pₗ associated to the representation ρₗ is defined (see section 2.2).

C₅ The coefficients of Pₙ lie in ℤ and are independent of the choice of ℓ.

We assume this conjecture and we write Pₙ(T) for the polynomial with coefficients in ℤ defined by any one of the Pₙ. We decompose Pₙ over ℂ in a product of linear factors:

\[ Pₙ(T) = \prod (1 - \alpha_n T). \] (2.9)

C₆ For all a, there exists an integer m(a) between 0 and m such that |λₙ| = Nₚⁿ(a)/2.

C₇ If ε = 0, all the m(a) are equal to m.

2.10 Remark.

1) The conjectures C₅, C₆, C₇ only concern the action of G on Vₙ; there are more general conjectures, associated to the action of G on the whole of Vₙ, for example the following (cf. [16, p. 514, problem 2]):

C₈ Let g be an element of G whose image in G/I is an integral power of the Frobenius generator F. The characteristic polynomial of ρₙ(g) has coefficients in Q, and does not depend on ℓ.

2) We assume the theorems of LEFSCHETZ (of the form A(X)(a) in [6]) can be applied to Y, this is for example the case if Kᵣ is of characteristic zero. Therefore we can, for all ℓ ≠ pₙ, construct a non-degenerate bilinear form on Vₙ, having values in that which Grothendieck calls Qₚₙ[a], and that is invariant under G (taking into account the action of G on Vₙ as well as on Qₚₙ[a]). If we suppose that ρₙ is unramified (i.e., ε = 0), this implies that, for every index a, there exists an index β such that

\[ \lambdaₐ \lambdaₜ = Nₚⁿ. \] (2.11)

From this follows m(α) + m(ρ) = 2m, and since m(α) and m(ρ) are between 0 and m according to C₆, we have m(α) = m(ρ) = m. Therefore, C₆ implies C₇, at least in characteristic zero.

3) We suppose that Y has good reduction at ν, and let Y(ν) be such a reduction. According to [2], the ℓ-adic cohomology of Y coincides with that of Y(ν); in particular, we have ε = 0. Moreover, the polynomial Pₙ associated to Y (in the above sense) coincides with the polynomial Pₙ associated to Y(ν) in the sense of section 1.1; see on this subject TATE [17, §3]. The conjectures C₅ and C₇ hence reduce to the standard conjectures C₁ and C₂.

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2.4 Example: elliptic curves (cf. Ogg [9])

We suppose that \( Y \) is an elliptic curve, and that \( m = 1 \). The cohomology space \( V \) is the dual of the Tate module \( T \) of \( Y \). There are three cases to distinguish, on the basis of the structure of the connected component \( Y(v) \) of the fibre of the Néron model of \( Y \):

a) Good reduction (i.e., \( Y(v) \) is an elliptic curve).
   
   We have \( \epsilon = \delta = f = 0 \). The polynomial \( P_m \) is of degree 2; it is given by
   
   \[
   P_m(T) = 1 - a_v T + N_v T^2, \tag{2.12}
   \]
   
   where \( a_v \) is the trace of the Frobenius endomorphism of the reduced curve \( Y(v) \). The number of points of \( Y(v) \) over \( k(v) \) is equal to \( P_m(1) = 1 - a_v + N_v \), which gives a way to calculate \( a_v \).

b) Bad reduction of multiplicative type (i.e., \( Y(v) \) is a torus of dimension 1).

   We have \( \epsilon = 1, \delta = 0, f = 1 \). The polynomial \( P_m \) is of degree 1; it is given by
   
   \[
   P_m(T) = 1 - c_v T, \quad c_v = \pm 1, \tag{2.13}
   \]
   
   with \( c_v = +1 \) (resp. \( c_v = -1 \)) if the torus \( Y(v) \) is split (resp. not split).

b) Bad reduction of multiplicative type (i.e., \( Y(v) \) is isomorphic to the additive group).

   We have \( \epsilon = 2 \) and \( P_m = 1 \). The invariant \( \delta \) is equal to 0 if \( p_v \) is different from 2, 3. If \( p_v = 2 \) or 3, the calculation of \( \delta \) is indicated (at least partially) in Ogg [9].

   More general, the conjectures \( C_3, \ldots, C_8 \) are true for all abelian varieties. This is shown (by the arguments of [16, §3]) starting from the theorem of Grothendieck-Mumford which says that such a variety admits a Néron model without additive component after a finite extension of the base field (cf. Grothendieck [4, exposé iv]).

3 Archimedian local invariants

In all the known functional equations, the gamma factors are accompanied by a power of \( \pi \). To be able to treat the two simultaneously, we put

\[
\Gamma_R(s) = \pi^{-s/2}\Gamma(s/2), \tag{3.1}
\]

\[
\Gamma_C(s) = (2\pi)^{-s}\Gamma(s). \tag{3.2}
\]

We have

\[
2\Gamma_C(s) = \Gamma_R(s)\Gamma_R(s + 1). \tag{3.3}
\]

[This last formula suggests to replace \( \Gamma_C(s) \) by \( 2\Gamma_C(s) \) in all that follows; this type of change is inoffensive since it multiplies both sides of the functional equation with the same power of \( 2 \).]

3.1 Hodge decomposition over \( \mathbb{C} \)

Let \( V \) be a complex vector space of finite dimension. A \( \mathbb{C} \)-Hodge decomposition of \( V \) is a decomposition of \( V \) into a direct sum \( V = \bigoplus V^{p,q} \), indexed by \( \mathbb{Z} \times \mathbb{Z} \).
We put
\[ h(p, q) = \dim V^{p,q}. \tag{3.4} \]
The gamma factor attached to \( V \) is defined as the product:
\[ \Gamma_V(s) = \prod_{p,q} \Gamma_C(s - \ln(p,q))^{h(p,q)}. \tag{3.5} \]

### 3.2 Hodge decomposition over \( \mathcal{R} \)

Let \( V \) be as above. An \( \mathcal{R} \)-Hodge decomposition of \( V \) is a pair formed of a \( \mathcal{C} \)-Hodge decomposition \( (V^{p,q}) \) and an automorphism \( \sigma \) of \( V \) such that \( \sigma^2 = 1 \) and \( \sigma(V^{p,q}) = V^{q,p} \) for all pairs \( (p,q) \).

We suppose that \( V \) is equipped with such a structure. We put \( h(p, q) = \dim V^{p,q} \) as above.

If \( n \) is an integer, the automorphism \( \sigma \) stabilizes \( V_{n,n} \); which allows us to decompose \( V_{n,n} \) into a direct sum of two subspaces
\[ V_{n,n} = V_{n,+} \oplus V_{n,-}, \tag{3.6} \]
with
\[ V_{n,+} = \{ x | x \in V_{n,n}, \sigma(x) = (-1)^nx \}, \]
\[ V_{n,-} = \{ x | x \in V_{n,n}, \sigma(x) = (-1)^{n+1}x \}. \]

We put
\[ h(n, +) = \dim V_{n,+} \quad \text{and} \quad h(n, -) = \dim V_{n,-}, \tag{3.7} \]
so that \( h(n,n) = h(n,+) + h(n,-) \).

The gamma factor associated to \( V \) is then defined as the product
\[ \Gamma_V(s) = \prod_n \Gamma_{\mathcal{R}}(s - n)^{h(n,+)} \Gamma_{\mathcal{R}}(s - n + 1)^{h(n,-)} \prod_{p<q} \Gamma_C(s - p)^{h(p,q)}. \tag{3.8} \]

### 3.3 Applications to cohomology

We return to the situation of section 1. Let \( K_0 \) be a field that is complete for an archimedian absolute value; we know that \( K_0 \) is isomorphic to \( \mathcal{R} \) or to \( \mathcal{C} \) (cf. Bourbaki, Alg., Comm., Chap. vi, §6). Let \( Y \) be a non-singular projective variety over \( K_0 \), and let \( m \) be an integer \( \geq 0 \). We can attach a \( K_0 \)-Hodge structure to these, and therefore also a gamma factor, due to the preceding. We distinguish two cases:

a) \( K_0 \) is isomorphic to \( \mathcal{C} \). Let \( Z \) be the topological space \( Y(K_0) \) of points of \( Y \) in \( K_0 \), and we take for \( V \) the cohomology group \( H^m(Z, \mathcal{C}) \). Once we have chosen an isomorphism of \( K_0 \) to \( \mathcal{C} \), the space \( Z \) is equipped with the structure of a complex analytic variety and the Hodge theory provides a decomposition \( V = \oplus V^{p,q} \) of \( V \) into a sum of subspaces of type \( (p,q) \). This gives a \( \mathcal{C} \)-Hodge structure on \( V \), in the sense of section 3.1. If we replace the isomorphism \( K_0 \cong \mathcal{C} \) by its conjugate, we get the symmetric Hodge structure of the preceding \( (V^{p,q} \) instead of \( V^{q,p} \), and the corresponding gamma factor is the same. So we have associated to \( (K_0, Y, m) \) a well-defined gamma factor, which we denote \( \Gamma_v(s) \).
b) $K_v = \mathbb{R}$. Because $C$ is a quadratic extension of $K_v$, the space $Z = Y(C)$ of points of $Y$ in $C$ is defined. We equip as above $V = H^m(Z, C)$ of the Hodge structure defined by the canonical complex analytic structure of $Z$. The complex conjugation $z \mapsto \bar{z}$ is an anti-holomorphic automorphism of $Z$; it induces an automorphism $\sigma$ on $V$ that transforms $V^{\rho, \rho}$ into $V^{\rho, \rho}$, and we have $\sigma^2 = 1$. So we obtain an $\mathbb{R}$-Hodge structure on $V$ in the sense of section 3.2, which gives a gamma factor, that we denote $\Gamma_v(s)$.

3.9 Example. We take $m = 1$, and let $Y$ be an abelian variety of dimension $r$. We have $h(1, 0) = h(0, 1) = r$, the other $h(p, q)$ are zero. We can deduce from this, if $K_v = C$, the gamma factor is equal to $\Gamma_C(s)^{2r}$. If $K_v = \mathbb{R}$, it is equal to $\Gamma_C(s)^r$. «

4 The functional equation

4.1 Exposition

We return to the notation and hypotheses of section 1.3. In particular, $X$ denotes a non-singular projective variety over a global field $K$, and $m$ denotes an integer $\geq 0$. We are going to define the invariants $A$, $P_{m,v}$ and $\Gamma_v$ as promised in section 1.3:

i) The conductor

It is the positive divisor $f$ of $K$ defined (in additive notation) by the formula

$$f = \sum_{v \in \Sigma_K} f(v) \cdot v,$$

(4.1)

where $f(v)$ is “the invariant $f$” associated to the cohomology of dimension $m$ of the variety $X \times_K K_v$ over the local field $K_v$. (Observe that $f(v)$ is zero if $X$ has good reduction at $v$, which ensures that the sum in eq. 4.1 is finite.)

If $K$ is a number field, $f$ is identified with an integer ideal of the field $K$.

ii) Definition of $A$

If $K$ is a number field, we put

$$D = |d_{K/Q}|,$$

(4.2)

where $d_{K/Q}$ is the discriminant of $K$ over $Q$.

If $K$ is a function field of genus $g$ over a finite field with $q$ elements, we put

$$D = q^{2g-2}.$$

(4.3)

(Observe that $D$ is the factor that occurs in the definition of the Tamagawa measure.)

Let the other part $N(f) = \prod_{v \in \Sigma_K} N_v^{f(v)}$ be the norm of the conductor $f$ as defined above. The invariant $A$ is defined by

$$A = N(f) \cdot D^{B_m},$$

(4.4)

where $B_m$ is the $m$-th Betti number of $X$. 11
iii) **Definition of the** $P_{m,v}, v \in \Sigma_K$

They are the polynomials $P_m$ associated to the variety $X \times_K K_v$, see section 2.3. If $v \not \in S$, they coincide with the $P_{m,v}$ that where defined in section 1.2.

iv) **Definition of the gamma factors** $\Gamma_v(s), v \in \Sigma_K^\infty$

They are the gamma factors associated to the varieties $X \times_K K_v$, see section 3.3.

We can therefore define $\zeta(s)$ and $\xi(s)$ by the formulas

$$\zeta(s) = \prod_{v \in \Sigma_K} \frac{1}{P_{m,v}(N_v^{-s})}$$  \hspace{1cm} (1.6)

and

$$\xi(s) = A^{d/2} \zeta(s) \prod_{v \in \Sigma_K^\infty} \Gamma_v(s),$$  \hspace{1cm} (1.7)

and formulate the conjecture:

$C_9$ The functions $\zeta(s)$ and $\xi(s)$ can be extended to meromorphic functions on the entire complex plane. We have

$$\zeta(s) = w \zeta(m+1-s),$$ \hspace{1cm} with $w = \pm 1$.  \hspace{1cm} (1.8)

In the exposition by Tate [17] we find supplementary conjectures concerning the zeros and poles of $\zeta(s)$.

**4.5 Example.**

a) We suppose that $K$ is a function field of genus $g$ over a finite field with $q$ elements, and that the theorems of Lefschetz can be applied to $X$ (which is for example the case if we can lift $X$ to characteristic zero). Then conjecture $C_9$ is true (Grothendieck, non-published). In a more precise way, although the preceding conjectures $C_1, \ldots, C_8$ are not yet proven, Grothendieck works with a fixed $\ell$. He considers $\zeta(s)$ as a formal series $Z(T)$, with coefficients in $Q_\ell$, in the variable $T = q^{-s}$, and he shows that it is a rational function in $T$, and that there is a functional equation

$$Z(T) = w(q^{(m+1)/2})^a Z(1/q^{m+1}T),$$  \hspace{1cm} (4.6)

with

$$a = \deg(f) + 2g - 2, \hspace{1cm} i.e., q^a = A.$$  \hspace{1cm} (4.7)

The formula 4.6 is equivalent with eq. 1.8.

The proof of Grothendieck can in fact be applied to every $\ell$-adic representation over $K$; the functional equation then relates the zeta function of the representation with that of its dual (the theorem of Lefschetz simply helps to determine this dual in the particular case considered here).

Obviously, it is this result of Grothendieck that is mainly at the basis of the definitions that we have adopted for the invariants $f(v), P_{m,v}$ and $A$.

b) Let $K$ be a number field, let $X = \text{Spec}(K)$ be a variety consisting of a point, and we take $m = 0$. We have $N(f) = 1, A = D, P_{m,v}(T) = 1 - T$ for all $v \in \Sigma_K$, and the function $\zeta$ is the zeta function of the field $K$, in the usual sense; the equation 1.8 is its functional equation, as shown by Hecke; the constant $w$ is equal to 1 here.
c) We take $K = \mathbb{Q}$, $m = 1$, and we choose for $X$ an elliptic curve. We have $D = 1$, $A$ is the conductor of $X$, $\Gamma_v(s) = \Gamma_C(s)$ and eq. 1.8 is the functional equation conjectured by Weil [20]. If $X$ has complex multiplication, its zeta function is essentially a Hecke-$L$-function of “Grössencharacter”, which permits to calculate its functional equation (Deuring); we observe that this is identical to eq. 1.8. Besides this case, eq. 1.8 has been verified for certain curves associated to modular curves, but unfortunately not for all (due to lack of good knowledge of the properties of ramification of $\ell$-adic representations attached to modular forms in the sense of Deligne [3]). It is even more unfortunate because, according to another conjecture by Weil, every elliptic curve over $\mathbb{Q}$ should be associated (in a sense that is easily made precise) to a modular curve.

4.2 Appendix

There are analogous expositions for the functions $L(s, \chi)$ of Artin associated to the action of a finite group on $X$ and the datum of a character $\chi$ of this group. (The case dealt by Artin [1] corresponds to $m = 0$.) Obviously, the functional equation then relates $L(s, \chi)$ to $L(m + 1 - s, \overline{\chi})$, where $\overline{\chi}$ denotes the conjugate of $\chi$. Moreover, the constant $w(\chi)$ of the functional equation is no longer necessarily equal to $\pm 1$. The explicit expression for $w(\chi)$ is of particular interest; one such expression is given for Artin-$L$-functions by Dwork [5] (up to a sign) and Langlands [8]. Using these results, Deligne has formulated a general conjecture giving, in all these cases, the value of $w(\chi)$ in terms of local invariants; we will find this in the exposition following this one; the case of elliptic curves has moreover been remarked by Langlands (non-published).

We remark also that the above definitions and conjectures can be given in the framework of the “motives” of Grothendieck, in other words, grosso modo, the direct factors of the $H^n$ given by algebraic idempotents. This type of generalization is useful if you want, for example, to discuss the properties of tensor products of cohomology groups, or, which amounts to the same, product varieties. The case of the product of two elliptic curves is particularly interesting (cf. Ogg [10]).

References


Local factors of zeta functions of algebraic varieties


