$p$-divisible groups

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January 19, 2016

1 Notation. — In these notes, instead of $\lim_{\to}$ we write $\text{colim}$, and instead of $\lim_{\leftarrow}$ we write $\lim$.

2 — Let $K$ be a number field. Let $A$ be an abelian variety over $K$. Let $\ell$ be a prime number. Let $A_{\ell^\infty}(\bar{K}) = \bigcup_n A[\ell^n](\bar{K})$ denote the $\ell$-divisible subgroup of $A(\bar{K})$. Let $W$ be a Gal($\bar{K}/K$)-stable subgroup of $A_{\ell^\infty}(\bar{K})$. For every $n$, let $B(n)$ denote the quotient $A/W_{\ell^n}$.

We want to bound the height of $B(n)$. More precisely, we want the following result.

3 Theorem. — For $n \gg 0$, the height $h(B(n))$ does not depend on $n$.

In this talk I will not prove this theorem. Rather, I will present an overview of some facts about $p$-divisible groups, and in the end prove a proposition that will be very useful for proving theorem 3.

4 — We will use [1] as main reference.

Let $R$ be a ring (or scheme). Let $p$ be a prime number. Let $h$ be an integer $\geq 0$. By definition, a group scheme over $R$ has rank $h$ if it is locally free of rank $h$ over $R$ (in other words, it is defined by a Hopf algebra that is locally free of rank $h$ over $R$).

5 Definition. — A $p$-divisible group of height $h$ is an inductive system

$$G = (G_\nu, i_\nu)_{\nu \geq 0}$$

where $G_\nu$ is a finite group scheme over $R$ of order $p^{\nu h}$, and such that for each $\nu \geq 0$, the sequence

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1}[p^\nu] \longrightarrow G_{\nu+1}$$

is exact. (So $G_\nu$ is the set of $p^\nu$-torsion points in $G_{\nu+1}$.)

A homomorphism of $p$-divisible groups is what you think it is. Probably the best known example of a $p$-divisible group is given by

$$G_\nu = (\mathbb{Z}/p^\nu \mathbb{Z})^h \quad \text{and} \quad G = \text{colim} G_\nu = (\mathbb{Q}_p/\mathbb{Z}_p)^h.$$ 

The next best known example is $A_{\ell^\infty}(\bar{K}) = \text{colim} A[\ell^n](\bar{K})$, where $A$ is an abelian variety over a field $K$, as in §2.
6 Consequences of the definition. — Let $G$ be a $p$-divisible group. By iteration, we obtain closed immersions $i_{\nu,\mu} : G_{\nu} \to G_{\nu+\mu}$, for all $\nu, \mu \geq 0$. (Note that $i_{\nu,1} = i_\nu$.) These maps $i_{\nu,\mu}$ identify $G_{\nu}$ with the kernel of $[p^\nu]$ in $G_{\nu+\mu}$.

Consider the following diagram, with exact row and column.

\[
\begin{array}{ccc}
0 & \longrightarrow & G_{\mu} \\
\downarrow & \downarrow & \downarrow \quad i_{\nu,\mu} \\
G_{\nu+\mu} & \longrightarrow & G_{\nu+\mu} \\
\downarrow & \downarrow & \downarrow \\
G_{\nu} & \longrightarrow & 0 \\
\end{array}
\]

Since the composition $[p^\nu] \circ [p^\mu] = [p^{\nu+\mu}]$ is identically 0 on $G_{\nu+\mu}$, we see that $[p^\mu]$ factors via a map $j_{\nu,\mu} : G_{\nu+\mu} \to G_{\nu}$.

\[
\begin{array}{ccc}
0 & \longrightarrow & G_{\mu} \\
\downarrow & \downarrow & \downarrow \quad i_{\nu,\mu} \\
G_{\nu+\mu} & \longrightarrow & G_{\nu+\mu} \\
\downarrow & \downarrow & \downarrow \\
G_{\nu} & \longrightarrow & 0 \\
\end{array}
\]

Observe that $i_{\nu,\mu} \circ j_{\mu,\nu} = [p^\mu]$. Because $i_{\nu,\mu}$ is an immersion, the sequence

\[
0 \longrightarrow G_{\mu} \xrightarrow{i_{\nu,\mu}} G_{\nu+\mu} \xrightarrow{j_{\mu,\nu}} G_{\nu}
\]

is exact. In fact, since the order of $G_{\mu}$ and $G_{\nu}$ add up to the order of $G_{\nu+\mu}$, we find that the last map is in fact a quotient map, and we obtain the short exact sequence

\[
0 \longrightarrow G_{\mu} \xrightarrow{i_{\nu,\mu}} G_{\nu+\mu} \xrightarrow{j_{\mu,\nu}} G_{\nu} \longrightarrow 0.
\]

We will write $j_\nu$ for $j_{1,\nu}$.

7 Tate modules. — Let $R$ be an integral domain, with field of fractions $K$. Assume char $K = 0$ and let $\bar{K}$ be an algebraic closure of $K$. Let $G$ be a $p$-divisible group over $R$ of height $h$. The Tate module of $G$ is denoted $T(G)$, and is by definition $\lim_G G_{\nu}(\bar{K})$, where limit is taken over the morphisms $j_\nu$. Dually, one defines $\Phi(G)$ as $\text{colim}_G G_{\nu}(\bar{K})$, where the colimit is over the maps $i_\nu$.

N.b.: There is a notion of “points of $G$” which we do not need for the main result of this talk. It coincides with $\Phi(G)$ when $G$ is étale, but contains $\Phi(G)$ as torsion subgroup in the general situation.
Since \( K \) has characteristic 0, the groups \( G_\nu \otimes K \) are étale, and hence \( T(G) \) is isomorphic as \( \mathbb{Z}_p \)-module to \( \mathbb{Z}_p^b \), while \( \Phi(G) \) is isomorphic to \( \left( \mathbb{Q}_p / \mathbb{Z}_p \right)^b \). Furthermore, there is a continuous action of \( \text{Gal}(\bar{K} / K) \) on \( T(G) \) and \( \Phi(G) \). There are canonical isomorphisms (of Galois modules)

\[
\Phi(G) \cong T(G) \otimes_{\mathbb{Z}_p} \left( \mathbb{Q}_p / \mathbb{Z}_p \right) \quad \text{and} \quad T(G) \cong \text{Hom}(\mathbb{Q}_p / \mathbb{Z}_p, \Phi(G)).
\]

Observe that one can recover the Galois module \( G_\nu(\bar{K}) \) from \( \Phi(G) \) by taking the kernel of \( [p^\nu] \).

Using the well-known fact that a finite étale group scheme over a field is determined by its Galois module of \( \bar{K} \)-points, we may thus recover the generic fibre \( G \otimes_R K \) from \( \Phi(G) \) or \( T(G) \).

8 Corollary. — The assignment \( G \mapsto T(G) \) establishes an equivalence of categories between the category of \( p \)-divisible groups over \( K \) and free \( \mathbb{Z}_p \)-modules of finite rank with a continuous action of \( \text{Gal}(\bar{K} / K) \).

9 Proposition (Prp. 12 of [4]). — Let \( R \) be an integrally closed, Noetherian, integral domain, with field of fractions \( K \). Fix a prime number \( p \). Let \( G \) be a \( p \)-divisible group over \( R \). Let \( T(G) \) be the Tate module of \( G \). Let \( W \) be a direct summand of \( T(G) \) over \( \mathbb{Z}_p \) that is stable under the action of \( \text{Gal}(\bar{K} / K) \). Then there exists a \( p \)-divisible group \( \Gamma \) over \( R \), and a morphism \( \phi : \Gamma \to G \) such that \( \phi \) induces an isomorphism \( T(\Gamma) \cong W \).

Proof. By corollary 8 we immediately obtain a \( p \)-divisible subgroup \( H_s \subset G \otimes K \). We want to take the closure \( H \) of \( H_s \) in \( G \). To make this precise, let \( B_\nu \) be the \( R \)-algebra corresponding to \( G_\nu \). Let \( A_\nu \) be the \( K \)-algebra corresponding to \( H_\nu \), and consider \( u_\nu : B_\nu \otimes_R K \to A_\nu \) corresponding to \( H_\nu \subset G_\nu \otimes K \). Let \( A_\nu \) be the image \( u_\nu(B_\nu) \) and put \( H_\nu = \text{Spec}(A_\nu) \). Observe that \( A_\nu \) is a cocommutative Hopf algebra, and therefore \( H_\nu \) is a commutative group scheme.

\[
\begin{array}{ccc}
B_\nu \otimes_R K & \longrightarrow & A_\nu \\
\uparrow & & \downarrow \iota \\
B_{\nu+1} \otimes_R K & \longrightarrow & A_{\nu+1} \\
\sigma & & \\
A_{\nu+1} & \longrightarrow & A_{\nu+1}
\end{array}
\]

By construction \( \iota \) is injective, while \( \sigma \) is surjective. Hence we obtain a map

\[
\begin{array}{ccc}
B_{\nu} \otimes_R K & \longrightarrow & A_{\nu} \\
\uparrow & & \downarrow \iota \\
B_{\nu+1} \otimes_R K & \longrightarrow & A_{\nu+1} \\
\sigma & & \\
A_{\nu+1} & \longrightarrow & A_{\nu+1}
\end{array}
\]

and thus maps \( H_\nu \to H_{\nu+1} \). Nevertheless, \( H \) is not necessarily a \( p \)-divisible group. (The last lines of [4] provide an example by Serre, that illustrates this problem.) However, \( H \otimes K \cong H_s \) is a \( p \)-divisible group. As we will see, the failure of \( H \) being a \( p \)-divisible group is somehow only at a finite level. What I mean is this: for \( \nu \gg 0 \) we will see that \( H_{\nu} \to H_{\nu+1} \) satisfies the axioms for a \( p \)-divisible group. We will exploit this to define \( \Gamma \) in terms of \( H \).

Because all groups involved are finite, quotients such as \( H_{\mu+1} / H_\mu \) exist. By looking at the generic fibre, we see that \( H_{\mu+1} / H_\mu \) is killed by \( p \). In particular the map \([p]\) induces maps

\[
H_{\mu+\nu+1} / H_{\mu+1} \to H_{\mu+\nu} / H_\mu
\]
that are isomorphisms on the generic fibre. (After all, on the generic fibre both the source and the
target are isomorphic to $H_\nu$, and the kernel of the map is 0.) Let $D_\mu$ be the algebra corresponding
to $H_{\mu+1}/H_\mu$. By the above observation, the algebra $D_\mu \otimes_R K$ does not depend on $\mu$; and the
$D_\mu$ form an increasing sequence of orders inside a finite separable $K$-algebra.

From some point onwards, say $\mu_0$, this sequence stabilises: $D_\mu = D_{\mu_0}$ for $\mu \geq \mu_0$. Now we
may put $\Gamma_\nu = H_{\nu+\mu_0}/H_{\mu_0}$. Note that $[p^{\mu_0}]$ induces maps $\Gamma_\nu \to H_\nu$ that are isomorphisms on
the generic fibre. Hence (if we assume for a moment that $\Gamma$ is $p$-divisible), it is immediate that
$\Gamma \mapsto G$ induces an isomorphism $T(\Gamma) \to W$.

We are done if we show that $\Gamma$ is $p$-divisible. To see this, consider the following diagram.

\[
\begin{array}{ccc}
H_{\nu+\mu_0+1}/H_{\mu_0} & \xrightarrow{[p^{\nu}]} & H_{\nu+\mu_0+1}/H_{\mu_0} \\
\downarrow \alpha & & \downarrow \gamma \\
H_{\nu+\mu_0+1}/H_{\nu+\mu_0} & \xrightarrow{\beta} & H_{\mu_0+1}/H_{\mu_0}
\end{array}
\]

Here
- $\alpha$ is the canonical surjection.
- $ \beta $ is the map induced by $[p^{\nu}]$, and is an isomorphism by the choice of $\mu_0$.
- $ \gamma $ is the canonical inclusion.

Observe that both objects in the top row are isomorphic to $\Gamma_{\nu+1}$. We conclude that the kernel
of $[p^{\nu}]: \Gamma_{\nu+1} \to \Gamma_{\nu+1}$ is isomorphic to the kernel of $\alpha$, which is $H_{\nu+\mu_0}/H_{\mu_0}$. By definition this
is $\Gamma_\nu$. We conclude that $\Gamma$ is indeed $p$-divisible.

\[\square\]

References