Quantum Groups and Special Functions

Bizerte Lectures

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Chapter 1

Preface

Quantum groups are deformations of groups, and there are many relations to special functions, especially to those of basic hypergeometric type. There are many books that cover parts of this relationship in some depth, see e.g. [4], [22], [23], [61, Vol. 3]. For lecture notes on this subject see e.g. [26], [37]. The emphasis in these books on q-analogues of the classical relation between group theory and special functions. So one calculates the matrix elements of irreducible representations explicitly in terms of special functions and next uses this interpretation to find all kinds of properties of these special functions.

The purpose of these lecture notes is to give an introduction to locally compact quantum groups on the level of von Neumann algebras, which is defined initially by Kustermans and Vaes [42], [43] in 2000, see also [41], [58], [60] for overviews. Masuda, Nakagami and Woronowicz [47] have developed a similar theory around the same time, but we stick to the Kustermans-Vaes approach. We emphasise the special case, corresponding to the quantum analogue of $SU(1,1)$ or better the normaliser of $SU(1,1)$ in $SL(2,\mathbb{C})$. The quantum group analogue has been studied for quite some time, starting with the work of [39], [46], [3], and slightly later [21], [20]. By a negative result of Woronowicz [62], recalled in Theorem 7.1.1, the Hopf $*$-algebra approach could not be lifted to a operator algebra approach. The work by Korogodsky [38] showed a way out, which was followed up in [28] as well as in the unfinished work of Woronowicz [64]. The approach in [28] heavily relies on special functions, and we indicate how this can be done. We also discuss the dual locally compact quantum group for this particular case, which is taken from [16].

The contents of these lecture notes are as follows. In Chapter 2 we recall briefly the notion of Hopf algebras and Hopf $*$-algebras. Chapter 3 recalls some basics of basic hypergeometric series and orthogonal polynomials. Chapter 4 discusses the main example, the algebraic construction for the Hopf algebra associated to the quantum group analogue of $SL(2,\mathbb{C})$. In Chapter 5 there is a crash course on operator theory, so that we can develop the notion of locally compact quantum groups à la Kustermans and Vaes in Chapter 6. Chapter 7 is then devoted to the main example of such a locally compact quantum group, namely for the normaliser of $SU(1,1)$ in $SL(2,\mathbb{C})$. This construction is based on special functions, and we recall in Chapter 4 some ideas that have led to this construction. Finally, Chapter 8 gives a short outlook to other subjects and related research questions.

We emphasise that we intend to discuss the ideas behind several construction without going into all the details of the proofs. There are plenty of exercises throughout the text as well as pointers to the literature.

Finally, we thank professor Fethi Bouzeffour for the invitation to lecture on this subject at the Faculté des Sciences de Bizerte, Tunisia, in February 2010.

Chapter 2

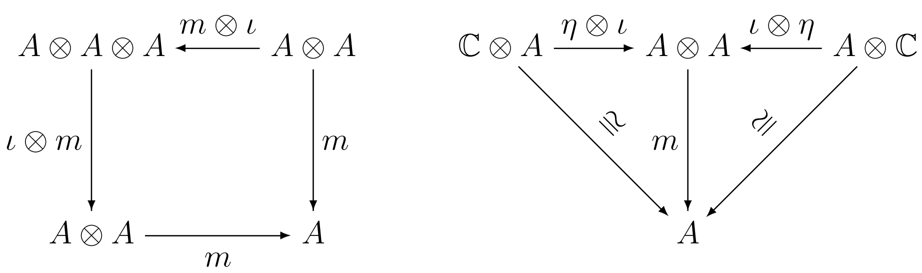
Hopf algebras

The concept of a *Hopf algebra* is fundamental for the theory of quantum groups on the algebraic level. We study this concept in some detail and we treat some important examples.

The ground field is $\mathbb{C}$, although most goes through when working over a commutative ring with unit. The tensor product $V \otimes W$ of two linear spaces is the algebraic tensor product, which means that elements of $V \otimes W$ consist of finite linear combinations of the form $v \otimes w$, $v \in V$, $w \in W$.

2.1 Algebras, bi-algebras and Hopf algebras

Recall that an *algebra*, or better, an associative algebra with unit, is a linear space A (over $\mathbb{C}$) with a bilinear mapping $A \times A \to A$, $(a, b) \mapsto ab$, called *multiplication*, and a distinguished non-zero element $1 \in A$, called the *unit*, such that $a(bc) = (ab)c$ and $1a = a = a1$ for all $a, b, c \in A$. This leads to two mappings, $m\colon A \otimes A \to A$, also called multiplication, and $\eta\colon \mathbb{C} \to A$, also called unit, defined by $m(a \otimes b) = ab$ and $\eta(z) = z1$. Then we can rephrase the associativity and unit in terms of the following commuting diagrams:



(2.1.1)

where we use $\mathbb{C} \otimes A \cong A \cong A \otimes \mathbb{C}$ by identifying $z \otimes a$ and $a \otimes z$ with za for $z \in \mathbb{C}$ and $a \in A$. An algebra homomorphism always means a unital algebra homomorphism, i.e. mapping unit onto unit.

Here and elsewhere ι denotes the identity map on the appropriate space.

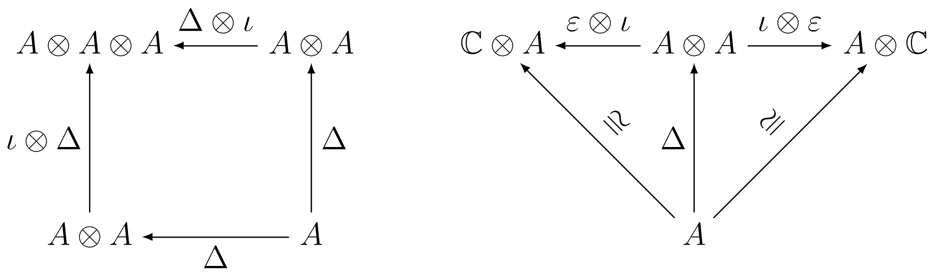
For an algebra A the tensor product $A \otimes A$ is again an algebra with multiplication $(a \otimes$

$b)(c \otimes d) = ac \otimes bd$ and unit $1 \otimes 1$. Note that

$$m_{A\otimes A}\colon A^{\otimes 4} \to A^{\otimes 2}, \qquad m_{A\otimes A} = (m_A \otimes m_A) \circ (\iota \otimes \Sigma \otimes \iota) \tag{2.1.2}$$

where $\Sigma\colon A \otimes A \to A \otimes A$ is the flip automorphism, $\Sigma(a \otimes b) = b \otimes a$. Finally, note that commutativity of A is equivalent to the condition $m \circ \Sigma = m$.

Definition 2.1.1. *A* ***coalgebra****, or better, a coassociative coalgebra with counit, is a linear space A (over $\mathbb{C}$) with a linear mapping $\Delta\colon A \to A \otimes A$, called the comultiplication, and a non-zero linear mapping $\varepsilon\colon A \to \mathbb{C}$, called the counit, such that the following diagram is commutative*



The commutative diagrams of Definition 2.1.1 are obtained from (2.1.1) by reversing the arrows.

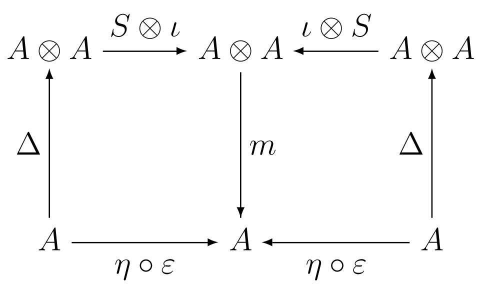
We say that the coalgebra is *cocommutative* if $\Sigma \circ \Delta = \Delta$.

Definition 2.1.2. *A* ***bialgebra*** *is an algebra A, such that A is also a coalgebra and the comultiplication Δ and counit ε are algebra homomorphisms.*

Remark 2.1.3. An equivalent definition of a bialgebra can be obtained replacing the condition that Δ and ε are algebra homomorphism by the condition that m and η are coalgebra homomorphisms.

Exercise 2.1.4. Give a definition of a coalgebra homomorphism, and prove the equivalence of the definitions in Definition 2.1.2 and Remark 2.1.3.

Definition 2.1.5. *A* ***Hopf algebra*** *is a bialgebra A with a linear mapping $S\colon A \to A$, the* ***antipode****, such that the following diagram is commutative;*



A ***Hopf algebra morphism*** *$\phi\colon A \to B$ of two Hopf algebras A and B is an algebra morphism $\phi\colon A \to B$ such that $\varepsilon_B \circ \phi = \varepsilon_A$, $\Delta_B \circ \phi = \phi \otimes \phi \circ \Delta_A$ and $S_B \circ \phi = \phi \circ S_A$.*

Proposition 2.1.6. *(i) If A is a bialgebra and S an antipode making A into a Hopf algebra, then S is unique.*

(ii) Let A be a Hopf algebra, then the antipode S is unital, counital, antimultiplicative and anticomultiplicative. Or, with Σ the flip automorphism,

$$S(1) = 1, \quad \varepsilon \circ S = \varepsilon, \quad S \circ m = m \circ \Sigma \circ (S \otimes S), \quad \Delta \circ S = \Sigma \circ (S \otimes S) \circ \Delta.$$

Proof. (i) Let F and G be linear mappings of A into itself, then we define the convolution product $F * G$ by $F * G = m \circ (F \otimes G) \circ \Delta$. This convolution product is associative, which follows from the associativity of m and the coassociativity of Δ. Moreover, $\eta \circ \varepsilon \colon A \to A$ is the unit for the convolution product. So the endomorphism algebra of A, $\mathrm{End}(A)$, becomes an algebra. Now assume that A is a Hopf algebra with antipode S, then $S * \iota = \eta \circ \varepsilon = \iota * S$. Or, S is a two-sided inverse of the identity mapping in $\mathrm{End}(A)$ with respect to the convolution product and thus unique.

(ii) Apply the diagram of Definition 2.1.5 to $1 \in A$ to find $S(1) = 1$. To ease notation we introduce

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \qquad (\iota \otimes \Delta)\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \tag{2.1.3}$$

which is well-defined by the coassociativity of Δ. The notation (2.1.3) is known as the Sweedler notation. Then by Definition 2.1.2 we have $\sum_{(a)} \varepsilon(a_{(1)})a_{(2)} = a$. Apply S and next ε to see that

$$\varepsilon\big(S(a)\big) = \varepsilon \otimes \varepsilon\big((\iota \otimes S)\Delta(a)\big) = \varepsilon \circ m \circ (\iota \otimes S) \circ \Delta(a) = \varepsilon \circ \eta \circ \varepsilon(a) = \varepsilon(a).$$

Next, using $\varepsilon(a) = \sum_{(a)} a_{(1)}S(a_{(2)}) = \sum_{(a)} S(a_{(1)})a_{(2)}$ repeatedly, as well as the antipode axiom for the second equality, and the fact that Δ, ε are homomomorphisms,

$$\begin{aligned} S(b)S(a) &= \sum_{(a),(b)} S(b_{(1)})S(a_{(1)})\varepsilon(a_{(2)}b_{(2)}) = \sum_{(a),(b)} S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)}) \\ &= \sum_{(a),(b)} \varepsilon(a_{(1)})\varepsilon(b_{(1)})S(a_{(2)}b_{(2)}) = S\big(\sum_{(a),(b)} \varepsilon(a_{(1)}b_{(1)})\, a_{(2)}b_{(2)}\big) = S(ab). \end{aligned}$$

The last statement is left as an exercise. □

Exercise 2.1.7. Prove the last statement of Proposition 2.1.6 (ii). In order to do this, define the convolution product for $F, G \in \mathrm{Hom}_k(A, A \otimes A)$ as $F * G = m_{A*A} \circ (F \otimes G) \circ \Delta$. So that $\mathrm{Hom}_k(A, A \otimes A)$ becomes an algebra. First show that $\Delta \circ \eta \circ \varepsilon$ is a unit in this algebra. Then show that both $\Delta \circ S$ and $\Sigma \circ (S \otimes S) \circ \Delta$ are inverses of Δ.

Exercise 2.1.8. Assume that S is bijective. Show that the opposite bialgebra A^{op}, i.e. with reversed multiplication, is a Hopf algebra with antipode S^{-1}. Prove the same statement for the co-opposite bialgebra A^{cop}, i.e. with comultiplication $\Delta^{\mathrm{cop}} = \Sigma \circ \Delta$. Conclude that for a commutative or a cocommutative Hopf algebra S^2 is the identity.

Example 2.1.9. (i) In case G is a finite group we set $A = F(G) = \{f\colon G \to \mathbb{C}\}$. Upon identifying $A \otimes A$ with $F(G \times G)$ we can define the Hopf algebra mappings as $\big(\Delta f\big)(x,y) = f(xy)$, $\varepsilon(f) = f(e)$ ($e \in G$ being the identity element), $\big(Sf\big)(x) = f(x^{-1})$. It is then straightforward to check that A is a Hopf algebra. This Hopf algebra is commutative, and in general not cocommutative (unless G is commutative).

(ii) Let $\mathfrak{g}$ be a Lie algebra with corresponding universal enveloping algebra $U(\mathfrak{g})$, then $U(\mathfrak{g})$ is a Hopf algebra by defining $\varepsilon(X) = 0$, $S(X) = -X$, $\Delta(X) = 1 \otimes X + X \otimes 1$ for $X \in \mathfrak{g}$ and extending Δ and ε as homomorphisms and S as an anti-homomorphism. We need to check that this is well-defined, e.g. to see that $\Delta([X,Y]) = \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X)$ we calculate

$$\begin{aligned}
\Delta(X)\Delta(Y) \;&=\; (1\otimes X + X\otimes 1)(1\otimes Y + Y \otimes 1) \;=\; 1\otimes XY + Y\otimes X + X\otimes Y + XY\otimes 1 \\
\Longrightarrow \qquad & \Delta(X)\Delta(Y) - \Delta(Y)\Delta(X) \;= \\
& 1\otimes (XY - YX) + (XY - YX)\otimes 1 = \Delta(XY - YX) = \Delta([X,Y])
\end{aligned}$$

and similarly for the other cases. This Hopf algebra is cocommutative, but in general not commutative (unless $\mathfrak{g}$ is abelian).

2.2 A characterisation of a Hopf algebra

One of the main problems in the developments of quantum groups has been the antipode S. In the analytic development, as well as for the examples on the level of operator algebras, the antipode turned out to be an unbounded operator. The step from the algebraic theory to the analytic theory has been facilitated by the following result, due to Van Daele, see [58, §1.3.4]

Theorem 2.2.1. *A unital bialgebra A is a Hopf algebra if and only if the maps T_1 and T_2 are bijections. Here*

$$\begin{aligned}
T_1 \;&=\; (\iota\otimes m)\circ(\Delta\otimes\iota)\colon A\otimes A \to A\otimes A \quad & T_1(a\otimes b) \;&=\; \Delta(a)(1\otimes b) \\
T_2 \;&=\; (m\otimes\iota)\circ(\iota\otimes\Delta)\colon A\otimes A \to A\otimes A \quad & T_2(a\otimes b) \;&=\; (a\otimes 1)\Delta(b)
\end{aligned}$$

Theorem 2.2.1 is related to the observation that a semigroup with cancellation property is actually a group.

Proof. We show that T_1, T_2 being bijective implies the existence of an antipode S. We define

$$S\colon A \to A, \qquad a \mapsto (\varepsilon\otimes\iota)\big(T_1^{-1}(a\otimes 1)\big)$$

and we need to check, see Definition 2.1.5, $\sum_{(a)} S(a_{(1)})a_{(2)} = \varepsilon(a)1$ and $\sum_{(a)} a_{(1)}S(a_{(2)}) = \varepsilon(a)1$.

In order to prove $\sum_{(a)} S(a_{(1)})a_{(2)} = \varepsilon(a)1$ we observe that $T_1 \circ (\iota\otimes m) = (\iota\otimes m)\circ(T_1\otimes\iota)$ as map from $A\otimes A\otimes A \to A\otimes A$ by the associativity of the multiplication. So

$$\begin{aligned}
\sum_{(a)} S(a_{(1)})a_{(2)} \;&=\; \sum_{(a)} (\varepsilon\otimes\iota)\big(T_1^{-1}(a_{(1)}\otimes 1)\big)\,a_{(2)} \;=\; (\varepsilon\otimes\iota)\big(T_1^{-1}(\sum_{(a)} a_{(1)}\otimes a_{(2)})\big) \\
&= (\varepsilon\otimes\iota)(a\otimes 1) \;=\; \varepsilon(a)1
\end{aligned}$$

since $T_1(a\otimes 1)=\Delta(a)$.

In order to prove that $\sum_{(a)} a_{(1)}S(a_{(2)})=\varepsilon(a)1$, we need that the counit can be expressed in terms of the map T_1;

$$\varepsilon\colon A\to\mathbb{C},\qquad \varepsilon(a)1\;=\;m\big(T_1^{-1}(a\otimes 1)\big). \tag{2.2.1}$$

Assuming this for the moment, we deduce from $(\iota\otimes T_1)\circ(\Delta\otimes\iota)=(\Delta\otimes\iota)\circ T_1$ as a map from $A\otimes A\to A\otimes A\otimes A$, as follows from the coassociativity of the comultiplication. So

$$\begin{aligned}
\sum_{(a)} a_{(1)}S(a_{(2)}) \;&=\; \sum_{(a)} a_{(1)}(\varepsilon\otimes\iota)\big(T_1^{-1}(a_{(2)}\otimes 1)\big)\\
&=\sum_{(a)}\big(m\circ(\iota\otimes\varepsilon\otimes\iota)\circ(\iota\otimes T_1^{-1})\big)(a_{(1)}\otimes a_{(2)}\otimes 1)\\
&=\big(m\circ(\iota\otimes\varepsilon\otimes\iota)\circ(\iota\otimes T_1^{-1})\big)(\Delta(a)\otimes 1)\\
&=\big(m\circ(\iota\otimes\varepsilon\otimes\iota)\circ(\Delta\otimes\iota)\circ T_1^{-1}\big)(a\otimes 1)\;=\;\big(m\circ T_1^{-1}\big)(a\otimes 1)\;=\;\varepsilon(a)1
\end{aligned}$$

where we used the counit axiom $(\iota\otimes\varepsilon)\circ\Delta=\iota$ and the expression (2.2.1).

It remains to prove that (2.2.1) is valid. Define $E\colon A\to A$ by $m\big(T_1^{-1}(a\otimes 1)\big)$, the right hand side of (2.2.1). We calculate

$$\begin{aligned}
(\iota\otimes E)\big(a\otimes 1)\Delta(b)\big)\;&=\;(a\otimes 1)\,(\iota\otimes m)\big(\iota\otimes T_1^{-1})(\Delta(b)\otimes 1)\big)\\
&=(a\otimes 1)\,(\iota\otimes m)\big(\Delta\otimes\iota)(T_1^{-1})(b\otimes 1))\big)\;=\;(a\otimes 1)\,T_1(T_1^{-1})(b\otimes 1))\big)\;=\;ab\otimes 1,
\end{aligned}$$

which shows that $\iota\otimes E$ maps elements of the form $\big(a\otimes 1)\Delta(b)\big)$ to $A\otimes 1$. Since, by surjectivity of T_2, these elements span A we see that E actually maps to $\mathbb{C}1\subset A$. So we find $E(a)=\varepsilon(a)1$, and it remains to show that ε defined in this way coincides with the counit. By taking $a=1$ in the above calculation we get $(\iota\otimes\varepsilon)\Delta(b)=b$. It remains to check that $(\varepsilon\otimes\iota)\Delta(b)=b$, and that ε is an algebra homomorphism, which we refer to Exercise 2.2.3.

To prove the converse of the statement, we note that, given the antipode S, the following maps yield inverses to T_1 and T_2. Indeed, the inverse of T_1 is given by

$$R_1\;=\;(\iota\otimes m)\circ(\iota\otimes S\otimes\iota)\circ(\Delta\otimes\iota)\colon A\otimes A\to A\otimes A$$

and the inverse to T_2 is given by

$$R_2\;=\;(m\otimes\iota)\circ(\iota\otimes S\otimes\iota)\circ(\iota\otimes\Delta)\colon A\otimes A\to A\otimes A.$$

The proof is left to Exercise 2.2.4. □

Remark 2.2.2. So it is possible to prove that the counit $\varepsilon\colon A\to\mathbb{C}$ can also completely be described in terms of the (inverse of the) maps T_1 and T_2 by (2.2.1), see also Exercise 2.2.3, so that we only need the comultiplication Δ and the bijectivity of the maps T_1 and T_2 in order to obtain a Hopf algebra.

Exercise 2.2.3. Prove that ε defined in the proof of Theorem 2.2.1 does indeed satisfy $(\varepsilon\otimes\iota)\Delta(b)=b$. (Hint: use again $T_1\circ(\iota\otimes m)=(\iota\otimes m)\circ(T_1\otimes\iota)$.) Prove also that ε defined in this way is a homomorphism. (Hint: use the first part to prove $\sum_{(a),(b)}\varepsilon(a_{(1)})\varepsilon(b_{(1)})a_{(2)}b_{(2)}c = abc = \sum_{(a),(b)}\varepsilon(a_{(1)}b_{(1)})a_{(2)}b_{(2)}c$ for all $a,b,c\in A$, and next the surjectivity of T_1.)

Exercise 2.2.4. Prove that indeed R_1 is the inverse of T_1 by checking that $R_1(T_1(a\otimes b))=a\otimes b$ and $T_1(R_1(a\otimes b))=a\otimes b$ using the properties of the antipode and counit. Similarly, prove that the inverse of T_2 is R_2.

2.3 Hopf $*$-algebras and duality

A $*$-operator on an algebra A is an antilinear, antimultiplicative involution, i.e. $(\lambda a+\mu b)^*=\bar\lambda a^*+\bar\mu b^*$, $(ab)^*=b^*a^*$, $(a^*)^*=a$ for $\lambda,\mu\in\mathbb{C}$, $a,b\in A$. An algebra A equipped with a $*$-operator is a $*$-**algebra**. We make $A\otimes A$ into a $*$-algebra by $(a\otimes b)^*=a^*\otimes b^*$.

Definition 2.3.1. *A Hopf algebra A which is also a $*$-algebra is a* ***Hopf $*$-algebra*** *if the comultiplication Δ and counit ε are $*$-homomorphims, i.e. $(*\otimes *)\circ\Delta=\Delta\circ *$, $\varepsilon(a^*)=\overline{\varepsilon(a)}$ for all $a\in A$*

Proposition 2.3.2. *Let A be a Hopf $*$-algebra, then the antipode satisfies $S\circ *\circ S\circ * = 1$.*

Proof. Check that $*\circ S\circ *$ satisfies the properties of an antipode for the opposite Hopf algebra A^{op}. By Exercise 2.1.8 and the unicity of S we find $S^{-1}=*\circ S\circ *$. □

A pairing between two vector spaces A and U is a bilinear mapping $U\times A\to\mathbb{C}$, $(a,u)\mapsto\langle u,a\rangle$. We say that the pairing is non-degenerate, or better, doubly non-degenerate, if $\langle u,a\rangle=0$ for all $u\in U$ implies $a=0$ and $\langle u,a\rangle=0$ for all $a\in A$ implies $u=0$. Such a pairing can be extended to a pairing of $U\otimes U$ and $A\otimes A$ by $\langle u\otimes v,a\otimes b\rangle=\langle u,a\rangle\langle v,b\rangle$.

Definition 2.3.3. *The Hopf $*$-algebras A and U are in duality as Hopf $*$-algebras if there exists a doubly non-degenerate pairing $\langle\cdot,\cdot\rangle\colon U\times A\to\mathbb{C}$ so that*

$$\langle XY,\xi\rangle=\langle X\otimes Y,\Delta(\xi)\rangle,\qquad \langle X,\xi\eta\rangle=\langle\Delta(X),\xi\otimes\eta\rangle, \tag{2.3.1}$$
$$\langle 1,\xi\rangle=\varepsilon(\xi),\qquad \langle X,1\rangle=\varepsilon(X), \tag{2.3.2}$$
$$\langle S(X),\xi\rangle=\langle X,S(\xi)\rangle,\qquad \langle X^*,\xi\rangle=\overline{\langle X,S(\xi)^*\rangle}. \tag{2.3.3}$$

The duality can be used to define a left and right action of U on A. For $X\in U$ and $\xi\in A$ we define elements $X\cdot\xi$ and $\xi\cdot X$ of A by

$$X\cdot\xi=(\iota\otimes X)\circ\Delta(\xi),\qquad \xi\cdot X=(X\otimes\iota)\circ\Delta(\xi), \tag{2.3.4}$$

where the pairing between A and U is used in the second, respectively the first, part of the tensor product. Using the duality we can rewrite (2.3.4) as

$$\langle Y,X\cdot\xi\rangle=\langle YX,\xi\rangle,\qquad \langle Y,\xi\cdot X\rangle=\langle XY,\xi\rangle. \tag{2.3.5}$$

Writing $\Delta(X)=\sum_{(X)} X_{(1)}\otimes X_{(2)}$, then (2.3.1) implies

$$X\cdot(\xi\eta)=\sum_{(X)}\big(X_{(1)}\cdot\xi\big)\big(X_{(2)}\cdot\eta\big),\qquad (\xi\eta)\cdot X=\sum_{(X)}\big(\xi\cdot X_{(1)}\big)\big(\eta\cdot X_{(2)}\big). \tag{2.3.6}$$

Using (2.3.4) and (2.3.3) we prove

$$X\cdot\xi^*=\sum_{(\xi)}\xi^*_{(1)}\langle X,\xi^*_{(2)}\rangle=\Big(\sum_{(\xi)}\xi_{(1)}\langle S(X)^*,\xi_{(2)}\rangle\Big)^*=\big(S(X)^*\cdot\xi\big)^*. \tag{2.3.7}$$

2.4 (Co-)representations of Hopf algebras

Definition 2.4.1. *A representation of an algebra A is an algebra homomorphism $\pi\colon A\to End(V)$ for some vector space V. In case A is a $*$-algebra, we require V to be a Hilbert space and that $\pi\colon A\to B(V)$ is a $*$-homomorphism.*

Note that $B(V)$ is a $*$-algebra with respect to the adjoint of an operator. However, it occurs regularly that it is not possible to represent the $*$-algebra by bounded operators. Then one has to take unbounded operators, and care has to be taken when considering domains, extensions, etc.

Definition 2.4.2. *A representation $\pi\colon A\to B(V)$ has an invariant subspace $W\subset V$ if W is a closed linear subspace so that $\pi(a)W\subset W$ for all $a\in A$. The representation π is irreducible if there are non non-trivial (i.e. not equal to $\{0\}$ and V) invariant subspaces. An intertwiner between two representations $\pi\colon A\to B(V)$ and $\tau\colon A\to B(W)$ is a linear map $T\colon V\to W$ so that $T\pi(a)=\tau(a)T$ for all $a\in A$. T is the intertwiner. Two representations $\pi\colon A\to B(V)$ and $\tau\colon A\to B(W)$ are (unitarily) equivalent in case there exists intertwiner which is an isometric isomorphism.*

The notion of a representation has a natural analogue as a corepresentation.

Definition 2.4.3. *A corepresentation of a co-algebra A is a linear map $\pi\colon V\to A\otimes V$ for some vector space V, so that*

$$(\iota\otimes\pi)\circ\pi\;=\;(\Delta\otimes\pi)\circ\pi,\qquad(\varepsilon\otimes\iota)\circ\pi=\iota.$$

In case A is a Hopf $*$-algebra, we require V to be a Hilbert space and $\langle\pi(v),\pi(w)\rangle=\langle v,w\rangle\,1$.

Here the inner product is extended to $A\otimes V$ by $\langle a\otimes v,b\otimes w\rangle=\langle v,w\rangle b^*a$.

Definition 2.4.4. *A unitary corepresentation of a Hopf $*$-algebra A in a Hilbert space V is a unitary element $U\in A\otimes B(V)$ satisfying $(\Delta\otimes\iota)(U)=U_{13}U_{23}$.*

Definition 2.4.5. *A square matrix $t=(t_{n,m})$ of elements of the Hopf $*$-algebra A is called a unitary matrix corepresentation if*

$$\Delta(t_{n,m})=\sum_k t_{n,k}\otimes t_{k,m},\quad\varepsilon(t_{n,m})=\delta_{n,m},\quad S(t_{n,m})=t^*_{m,n}.$$

Exercise 2.4.6. Write out the Definition 2.4.3 in terms of a commutative diagram (as in Definition 2.1.1) and reverse arrows. Relate the resulting diagram to Definition 2.4.1.

The interdependence between Definitions 2.4.3, 2.4.4 and 2.4.5 is the following. Take a corepresentation π as in Definition 2.4.3, and take a basis $v_{i}{}_{i=1}^{n}$ of the finite-dimensional space V. In case V is an inner proct space, i.e. a finite-dimensional Hilbert space, we assume moreover that this basis is orthonormal. Since it is a basis we define $\pi_{ij} \in A$ for $1 \leq i, j \leq n$ by $\pi(v_i) = \sum_{i=1}^{n} \pi_{ij} \otimes v_j$. The conditions in Definition 2.4.3 then translate into

$$\sum_{j,k=1}^{n} \pi_{ij} \otimes \pi_{jk} \otimes v_k = \sum_{i=1}^{n} \Delta(\pi_{ij}) \otimes v_j, \qquad \sum_{j=1}^{n} \varepsilon(\pi_{ij}) v_j = v_i$$

and using the fact that $\{v_k\}_k$ is a basis we obtain $\sum_{j=1}^{n} \pi_{ij} \otimes \pi_{jk} = \Delta(\pi_{ik})$ and $\varepsilon(\pi_{ij}) = \delta_{ij}$, so we obtain the first two statements of Definition 2.4.4. Note that $\varepsilon(\pi_{ij}) = \delta_{ij}$ follows from the other condition and the counit axiom if the matrix elemenst π_{ij} are linearly independent in A.

Note that antipode axioma leads to

$$\sum_{j=1}^{n} \pi_{ij} S(\pi_{jk}) = \delta_{ik}, \qquad \sum_{j=1}^{n} S(\pi_{ij}) \pi_{jk} = \delta_{ik},$$

so that the matrix $\big(S(\pi_{ij})\big)_{i,j=1}^{n}$ is the inverse of the matrix $\big(\pi_{ij}\big)_{i,j=1}^{n}$ in $M_n(A) = M_n(\mathbb{C}) \otimes A$. In case of unitarity we want $\big(\pi_{ji}^{*}\big)_{i,j=1}^{n}$ to be this inverse, leading to the last requirement in Definition 2.4.5.

For V a Hilbert space define $\theta_{ab} \colon V \to V$, $a, b \in V$, by $\theta_{ab} v = \langle v, b \rangle a$, then $\theta_{ab}^{*} = \theta_{ba}$ and $\theta_{ab}\theta_{cd} = \langle c, b \rangle \theta_{ad}$. Then put $U = \sum_{i,j=1}^{n} \pi_{ij} \otimes \theta_{v_i v_j}$, so that

$$U_{13} U_{23} = \sum_{i,j,k,l=1}^{n} \pi_{ij} \otimes \pi_{kl} \otimes \theta_{v_i v_j} \theta_{v_l v_k} = \sum_{i,j,l=1}^{n} \pi_{ij} \otimes \pi_{jl} \otimes \theta_{v_i v_l} = \sum_{r,s=1}^{n} \Delta(\pi_{rs}) \otimes \theta_{v_r v_s} = (\Delta \otimes \iota)(U)$$

which leads to the required expression. The unitarity $U^{*}U = 1$ leads to $\sum_{i=1}^{n} \pi_{il}^{*} \pi_{ik} = \delta_{kl}$ and similarly for $UU^{*} = 1$.

Chapter 3

Special functions of basic hypergeometric type

3.1 Basic hypergeometric series

A very short introduction to basic hypergeometric series can be found here, but (almost) everything you might want to know about this subject can be obtained from Gasper and Rahman [15].

Recall that the series $\sum_{n=0}^{\infty} u_n$ is a hypergeometric series if the quotient u_{n+1}/u_n is a rational function of n. Similarly, a series $\sum_{n=0}^{\infty} v_n$ a basic hypergeometric series (with base q) if the quotient v_{n+1}/v_n is a rational function of q^n for a fixed base q. The most general form of the quotient is

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\ (-q^n)^{1+s-r}\, z. \tag{3.1.1}$$

assuming $v_0 = 1$. We define an ${}_r\phi_s$ *basic hypergeometric series* by

$$\begin{aligned} {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) &\equiv {}_r\varphi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\ ; q, z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{3.1.2}$$

with $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s+1$. Here

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \end{cases} \tag{3.1.3}$$

is the q-*shifted factorial.* In case $0 < |q| < 1$ we can take the limit $k \to \infty$ to get

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{3.1.4}$$

for $|q| < 1$.

If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 \cdots b_s q|/|a_1 a_2 \cdots a_r|$. It diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| > |b_1 b_2 \cdots b_s q|/|a_1 a_2 \cdots a_r|$, unless it terminates. As is customary, the ${}_r\phi_s$ notation is also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *basic hypergeometric functions*) outside the circle of convergence. To switch from base q to base q^{-1} we note

$$ {}_r\varphi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1^{-1}, \ldots, a_r^{-1}; q^{-1})_n}{(q^{-1}, b_1^{-1}, \ldots, b_s^{-1}; q^{-1})_n} \left(\frac{a_1 \cdots a_r z}{b_1 \cdots b_s q}\right)^n \tag{3.1.5} $$

assuming the upper and lower parameters are non-zero. In case one of the upper parameters is of the form q^{-n} for $n \in \mathbf{N}$ the series in (3.1.2) terminates. From now on, unless stated otherwise, whenever $q^{-j}, q^{-k}, q^{-m}, q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers. For terminating series it is sometimes useful to switch the order of summation, which is given by

$$ \begin{aligned} {}_{r+1}\varphi_s\left(\begin{matrix} a_1, \ldots, a_r, q^{-n} \\ b_1, \ldots, b_s \end{matrix} ; q, z\right) = & \frac{(a_1, \ldots, a_r; q)_n}{(b_1 \ldots, b_s; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1} \\ & \times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1, \ldots, q^{1-n}/b_s, q^{-n}; q)_k}{(q, q^{1-n}/a_1, \ldots, q^{1-n}/a_r; q)_k} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r} \frac{q^{n+1}}{z}\right)^k \end{aligned} \tag{3.1.6} $$

for non-zero parameters.

Observe that the series (3.1.2) has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is again of the form (3.1.2) with r replaced by $r-1$.

The basic hypergeometric series

$$ {}_{r+1}\varphi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z\right) $$

is called *k-balanced* if $b_1 b_2 \cdots b_r = q^k a_1 a_2 \cdots a_{r+1}$ and $z = q$, and a 1-balanced series is called *balanced* (or *Saalschützian*). The basic hypergeometric series ${}_{r+1}\phi_r$ is *well-poised* if the parameters satisfy the relations

$$ q a_1 = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r; $$

very-well-poised if, in addition, $a_2 = q a_1^{\frac{1}{2}}, a_3 = -q a_1^{\frac{1}{2}}$.

For very-well-poised series the following notation is in use:

$$ {}_{r+1}W_r\left(a_1; a_4, a_5, \ldots, a_{r+1}; q, z\right) = {}_{r+1}\varphi_r\left(\begin{matrix} a_1, q a_1^{\frac{1}{2}}, -q a_1^{\frac{1}{2}}, a_4, \ldots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, q a_1/a_4, \ldots, q a_1/a_{r+1} \end{matrix} ; q, z\right). \tag{3.1.7} $$

The series in (3.1.7) is *very-well-poised balanced* if $(a_1 \cdots a_{r+1})z = (\pm(a_1 q)^{\frac{1}{2}})^{r-3}$ (with either sign).

The q-binomial theorem states that

$$ {}_1\varphi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1,\ |q| < 1, \tag{3.1.8} $$

which is the q-analogue of the binomial theorem $\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}$, $|z| < 1$.

The q-binomial theorem leads to various transformation and summation formulas. Heine showed

$$ {}_2\phi_1(a, b; c; q, z) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1(c/b, z; az; q, b), \tag{3.1.9} $$

where $|z| < 1$ and $|b| < 1$. By iterating the result

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(c/b, bz;q)_\infty}{(c, z;q)_\infty}\, {}_2\phi_1(abz/c, b; bz; q, c/b) \\ &= \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a, c/b; c; q, abz/c). \end{aligned} \tag{3.1.10} $$

with appropriate conditions on the parameters for the last two series to be convergent. To prove (3.1.9) Heine writes

$$ \begin{aligned} {}_2\phi_1(a, b; c; q, z) &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n (cq^n;q)_\infty}{(q;q)_n (bq^n;q)_\infty} z^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} (bq^n)^m \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (zq^m)^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \\ &= \frac{(b, az;q)_\infty}{(c, z;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right) \end{aligned} \tag{3.1.11} $$

by (3.1.8), which gives (3.1.9). The implied convergence of the series above is assumed to hold.

The q-Saalschütz summation formula

$$ {}_3\varphi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q\right) = \frac{(c/a, c/b;q)_n}{(c, c/ab;q)_n}, \qquad n = 0, 1, \cdots. \tag{3.1.12} $$

can also be derived from thet q-binomial theorem. The summation has the important property that the ${}_3\phi_2$-series is balanced and terminating.

The q-integral is defined as follows

$$\int_a^b f(t)\; d_qt = \int_0^b f(t)\; d_qt - \int_0^a f(t)\; d_qt, \qquad \int_0^a f(t)\; d_qt = a(1-q)\sum_{n=0}^{\infty} f(aq^n)\, q^n. \quad (3.1.13)$$

For fixed $q \neq 1$, the q-derivative operator D_q is defined by

$$D_q f\,(x) \;=\; \frac{f(x)-f(qx)}{(1-q)x}, \qquad (3.1.14)$$

and its iterates by $D_q^n f \;=\; D_q(D_q^{n-1}f)$, $n = 1, 2, \cdots$. Since formally

$$D_q \int_0^x f(t)\, d_qt \;=\; f(x) \qquad (3.1.15)$$

the q-integral (3.1.13) is an inverse of D_q.

Exercise 3.1.1. Prove the q-binomial formula by showing that the left hand side of 3.1.8 satisfies the difference equations $h_a(z) - h_{aq}(z) = -azh_{aq}(z)$ and $h_a(z) - h_a(qz) = (1-a)zh_{aq}(z)$. Eliminating $h_{aq}(z)$ from these two equations gives a recursion relation. Finish the proof of the q-binomial theorem.

Exercise 3.1.2. Prove the q-Saalschütz summation formula (3.1.12). Proceed by multipliying $(abz/c;q)_\infty/(z;q)_\infty$ by ${}_2\phi_1(c/a, c/b; c; q, abz/c)$ and collecting the coefficients of the powers of z using the q-binomial theorem, then equate coefficients of the same powers after applying Heine's transformation (3.1.9).

3.2 Orthogonal polynomials

Consider the Hilbert space $L^2(\mu)$, where μ is a non-negative Borel measure on the real line. Assume that all moments exist, $\int_{\mathbb{R}} |x|^k\, d\mu(x) < \infty$, so that all polynomials are integrable. In applying the Gram-Schmidt orthogonalisation process to the sequence $\{1, x, x^2, x^3, \ldots\}$ we may end up in one of the following situations: (a) the polynomials are linearly dependent in $L^2(\mu)$, or (b) the polynomials are linearly independent in $L^2(\mu)$. In case (a) it follows that there is a non-zero polynomial p such that $\int_{\mathbb{R}} |p(x)|^2\, d\mu(x) = 0$. This implies that μ is a finite sum of Dirac measures at the zeros of p. From now on we exclude this case, but the reader may consider this case him/herself. In case (b) we end up with a set of orthonormal polynomials as in the following definition.

Definition 3.2.1. *A sequence of polynomials* $\{p_n\}_{n=0}^\infty$ *with* $\deg(p_n) = n$ *is a set of orthonormal polynomials with respect to* μ *if* $\int_{\mathbb{R}} p_n(x)p_m(x)\, d\mu(x) = \delta_{n,m}$.

Note that the polynomials p_n are real-valued for $x \in \mathbb{R}$, so that its coefficients are real. Moreover, from the Gram-Schmidt process it follows that the leading coefficient is positive. Only the moments $m_k = \int_{\mathbb{R}} x^k \, d\mu(x)$ of μ play a role in the orthogonalisation process.

The following theorem describes the fundamental property of orthogonal polynomials in these notes.

Theorem 3.2.2 (Three term recurrence relation.)**.** *Let $\{p_k\}_{k=0}^{\infty}$ be a set of orthonormal polynomials in $L^2(\mu)$, then there exist sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$, with $a_k > 0$ and $b_k \in \mathbb{R}$, such that*

$$\begin{aligned} x\,p_k(x) &= a_k p_{k+1}(x) + b_k p_k(x) + a_{k-1} p_{k-1}(x), \qquad k \geq 1, \\ x\,p_0(x) &= a_0 p_1(x) + b_0 p_0(x). \end{aligned}$$

Moreover, if μ is compactly supported, then the coefficients a_k and b_k are bounded.

Note that the three term recurrence, together with the initial condition $p_0(x) = 1$, completely determine the polynomials $p_k(x)$ for all $k \in \mathbb{N}$.

Proof. The degree of $xp_k(x)$ is $k+1$, so there exist constants c_i so that $x\,p_k(x) = \sum_{i=0}^{k+1} c_i\, p_i(x)$. By the orthonormality properties of p_k it follows that

$$c_i = \int_{\mathbb{R}} p_i(x) x p_k(x)\, d\mu(x).$$

Since the degree of $xp_i(x)$ is $i+1$, we see that $c_i = 0$ for $i + 1 < k$. Then

$$b_k = c_k = \int_{\mathbb{R}} x\big(p_k(x)\big)^2\, d\mu(x) \in \mathbb{R}.$$

Moreover, $c_{k+1} = \int_{\mathbb{R}} p_{k+1}(x) x p_k(x)\, d\mu(x)$ and $c_{k-1} = \int_{\mathbb{R}} p_{k-1}(x) x p_k(x)\, d\mu(x)$ display the required structure for the other coefficients. The positivity of a_k follows by considering the leading coefficient.

For the last statement we observe that

$$\begin{aligned} |a_k| = \Big| \int_{\mathbb{R}} x p_{k+1}(x) p_k(x)\, d\mu(x) \Big| &\leq \int_{\mathbb{R}} |p_{k+1}(x)||p_k(x)|\, d\mu(x) \sup_{x \in \mathrm{supp}(\mu)} |x| \\ &\leq \|p_{k+1}\|_{L^2(\mu)} \|p_k\|_{L^2(\mu)} \sup_{x \in \mathrm{supp}(\mu)} |x| = \sup_{x \in \mathrm{supp}(\mu)} |x| < \infty, \end{aligned}$$

since $\|p_k\|_{L^2(\mu)} = 1$ and $\mathrm{supp}(\mu)$ is compact. In the second inequality we have used the Cauchy-Schwarz inequality. Similarly,

$$|b_k| \leq \|p_k\|^2_{L^2(\mu)} \sup_{x \in \mathrm{supp}(\mu)} |x| = \sup_{x \in \mathrm{supp}(\mu)} |x| < \infty$$

gives the estimate on the coefficients b_k. $\square$

The converse to Theorem 3.2.2 is called Favard's theorem. It states that any set $\{p_k\}_{k=0}^\infty$ of polynomials generated by a recurrence as in Theorem 3.2.2 with the initial condition $p_0(x) = 1$ with $a_k > 0$, $b_k \in \mathbb{R}$ are in fact orthonormal polynomials with respect to some positive probability measure μ on the real line. However, the measure need not to be determined completely, and it may be hard to find in explicit cases.

In case the sequences $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ are bounded, the polynomials generated by the recurrence in Theorem 3.2.2 are orthogonal with respect to a uniquely determined measure μ and the $\mathrm{supp}(\mu)$ is compact.

3.3 Some explicit orthogonal polynomials of basic hypergeometric type

We collect some results on specific families of orthogonal polynomials that we use. You can find more information in [15], or the references in the compendium [24].

3.3.1 Little q-Jacobi polynomials

The little q-Jacobi polynomials

$$p_n(x; a, b; q) \;=\; {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) \tag{3.3.1}$$

are q-analogues of the Jacobi polynomials. The orthogonality relations are

$$\sum_{x=0}^\infty \frac{(bq;q)_x}{(q;q)_x}(aq)^x p_n(q^x; a, b; q) p_m(q^x; a, b; q) = 0, \qquad n \neq m. \tag{3.3.2}$$

Note that the orthogonality measure is essentially the q-binomial formula (3.1.8).

Exercise 3.3.1. Prove the orthogonality relations (3.3.2) using the q-binomial theorem and the q-Saalschütz formula. You might want to consider the sum

$$\sum_{x=0}^\infty \frac{(bq;q)_x}{(q;q)_x}(aq)^x\, q^{xk} p_m(q^x; a, b; q), \qquad 0 \leq k \leq m,$$

first.

We need the case $b = 0$, which is known as the Wall polynomials. The three-term recurrence is then

$$x\, p_n(x) \;=\; -q^n(1-aq^{n+1})\, p_{n+1}(x) + q^n(1+a-aq^n-aq^{n+1})\, p_n(x) - q^n(a-aq^n)\, p_{n-1}(x) \tag{3.3.3}$$

where $p_n(x) = p_n(x; a, 0; q)$. The orthogonality can be rewritten as

$$\sum_{k=0}^{\infty} P_n(q^k; a; q)\, P_m(q^k; a; q) \;=\; \delta_{n,m}$$

where

$$P_n(q^k; a; q) \;=\; \sqrt{\frac{(aq;q)_\infty (q^{k+1};q)_\infty (aq;q)_n}{(q;q)_\infty (q;q)_n}}(-1)^n (aq)^{(k-n)/2}\, p_n(q^k; a, 0; q).$$

Exercise 3.3.2. Prove that also

$$\sum_{n=0}^{\infty} P_n(q^k; a; q)\, P_n(q^l; a; q) \;=\; \delta_{k,l}.$$

Hint: Use that for an orthonormal basis $\{v_i\}_{i\in I}$ (I some index set) of the Hilbert space $\ell^2(\mathbb{N})$ with standard orthonormal basis $\{e_n\}_{n=0}^{\infty}$ both the orthogonality relations

$$\sum_{n=0}^{\infty} \langle v_i, e_n\rangle\langle e_n, v_j\rangle = \langle v_i, v_j\rangle = \delta_{ij}$$

and the dual orthogonality relations

$$\sum_{i\in I} \langle e_m, v_i\rangle\langle v_i, e_n\rangle = \langle e_m, e_n\rangle = \delta_{nm}.$$

3.3.2 Al-Salam and Carlitz polynomials

The Al-Salam–Carlitz polynomials

$$U_n^{(a)}(x; q) \;=\; (-a)^n q^{\frac{1}{2}n(n-1)}\; {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}\; ; q, \frac{qx}{a}\right) \tag{3.3.4}$$

satisfy the orthogonality relations

$$\int_a^1 (qx, qxa; q)_\infty\, U_n^{(a)}(x;q)\, U_m^{(a)}(x;q)\, d_qx \;=\; \delta_{nm}(-a)^n(1-q)(q;q)_n(q, a, q/a; q)_\infty q^{\frac{1}{2}n(n-1)} \tag{3.3.5}$$

for $a < 0$. Here the q-integral is defined as in (3.1.13). The Al-Salam–Carlitz polynomials satisfy the following second-order q-difference relation

$$(1-q^n)x^2\, y(x) = aq^{n-1}\, y(qx) - \big(aq^{n-1} + q^n(1-x)(a-x)\big)\, y(x) + q^n(1-x)(a-x)\, y(q^{-1}x), \tag{3.3.6}$$

for $y(x) = U_n^{(a)}(x; q)$.

3.3.3 q-Laguerre polynomials

The q-Laguerre polynomials are defined by

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix};q,-xq^{n+\alpha+1}\right). \tag{3.3.7}$$

They satisfy the orthogonality relations

$$\sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-cq^k;q)_\infty} L_n^{(\alpha)}(cq^k;q)\, L_m^{(\alpha)}(cq^k;q) \;=\; \delta_{n,m}\, q^{-n} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \frac{(q,-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}{(q^{\alpha+1},-c,-q/c;q)_\infty}, \tag{3.3.8}$$

for $\alpha > -1$ and $c > 0$. The summation that corresponds to the case $n = m = 0$ is a special case of Ramanujan's ${}_1\psi_1$-summation formula, see [15, (5.2.1)]. In particular, we see that q-Laguerre polynomials are orthogonal with respect to various inequivalent measures on the real line (just vary over c). This means that the corresponding moment problem is indeterminate.

By $L^2(\mu^{(\alpha;c)})$ we denote the space of square integrable functions on the set $\{cq^k \mid k \in \mathbb{Z}\}$, $c > 0$, with positive weight $q^{k(\alpha+1)}/(-cq^k;q)_\infty$ at cq^k, $k \in \mathbb{Z}$, i.e. $f \in L^2(\mu^{(\alpha;c)})$ if

$$\mathcal{L}(|f|^2) := \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-cq^k;q)_\infty} |f(cq^k)|^2 < \infty.$$

Here we take $\alpha > -1$. In particular, each of the discrete measures $\mu^{(\alpha;c)}$ is an orthogonality measure for the q-Laguerre polynomials. From the general theory of indeterminate moment problems, see e.g. [1], we know that the polynomials are not dense in $L^2(\mu^{(\alpha;c)})$ (i.e. these measures are not Nevanlinna-extremal). So one can wonder if one can find a suitable complementary orthogonal basis, and indeed the result is the following.

Define the functions, which we can think of big q-Bessel functions,

$$\begin{aligned} M_p^{(\alpha;c)}(x;q) &= \frac{(q^{\alpha+1};q)_\infty}{(q,-cq^{\alpha+1};q)_\infty}\, {}_1\varphi_1\left(\begin{matrix} -cq^{\alpha-p} \\ q^{\alpha+1} \end{matrix};q,\frac{xq^{p+1}}{c}\right) \\ &= \frac{(xq^{p+1}/c;q)_\infty}{(q,-cq^{\alpha+1};q)_\infty}\, {}_1\varphi_1\left(\begin{matrix} -x \\ xq^{p+1}/c \end{matrix};q,q^{\alpha+1}\right) \end{aligned} \tag{3.3.9}$$

for $p \in \mathbb{Z}$. The second equality is by a limiting case of Heine's transformation (3.1.10). Then the functions $M_p^{(\alpha;c)}(\cdot;q)$, $p \in \mathbb{Z}$, together with the q-Laguerre polynomials $L_n^{(\alpha)}(\cdot;q)$, $n \in \mathbb{N}_0$, form an orthogonal basis for $L^2(\mu^{(\alpha;c)})$, $c > 0$, $\alpha > -1$. Explicitly,

$$\begin{aligned} \mathcal{L}\big(L_n^{(\alpha)}(\cdot;q)L_p^{(\alpha)}(\cdot;q)\big) &= \delta_{n,p} q^{-p} \frac{(q^{\alpha+1};q)_p}{(q;q)_p} \frac{(q,-cq^{\alpha+1},-q^{-\alpha}/c;q)_\infty}{(q^{\alpha+1},-c,-q/c;q)_\infty}, \\ \mathcal{L}\big(M_p^{(\alpha;c)}(\cdot;q)M_r^{(\alpha;c)}(\cdot;q)\big) &= \delta_{p,r} c q^\alpha q^{-p} \frac{(-q^{p+1}/c,-q^{-\alpha}/c;q)_\infty}{(-q^{p+1-\alpha}/c,-cq^{\alpha+1};q)_\infty} \frac{1}{(-c,-q/c;q)_\infty}, \\ \mathcal{L}\big(M_p^{(\alpha;c)}(\cdot;q)L_n^{(\alpha)}(\cdot;q)\big) &= 0. \end{aligned} \tag{3.3.10}$$

There are at least two ways of proving (3.3.10), one by direct series manipulation using transformation and summation formulas, and one by using spectral analysis of a second order difference operator, see [5]. For more general information on the spectral decomposition of a second order difference operator, see [34, App. A], [27], [45].

Exercise 3.3.3. Work out the dual orthogonality relations corresponding to 3.3.10. Why do the dual orthogonality relations hold?

3.3.4 Askey-Wilson polynomials

The Askey-Wilson polynomials and the q-Racah polynomials form the top layer in the q-Askey scheme, see [24]. All the orthogonal polynomials discussed here are limiting cases of these orthogonal polynomials.

The Askey-Wilson polynomials are defined by

$$p_n(\cos\theta; a, b, c, d \mid q) = a^{-n}(ab, ac, ad; q)_n \; {}_4\varphi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, a^{-i\theta} \\ ab, \; ac, \; ad \end{matrix}; q, q\right) \qquad (3.3.11)$$

The Askey-Wilson polynomial is symmetric in the parameters a, b, c and d. To stress the fact that Askey-Wilson polynomials generalise Jacobi polynomials

$$p_n^{(\alpha,\beta)}(x; s, t \mid q) = p_n(x; q^{1/2}t/s, q^{1/2+\alpha}s/t, -q^{1/2}/(st), -stq^{1/2+\beta} \mid q). \qquad (3.3.12)$$

The orthogonality relations for the Askey-Wilson polynomials depend on the values of the parameters a, b, c and d. First we introduce some notation;

$$\begin{aligned} w\big(\frac{1}{2}(z+z^{-1})\big) &= \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}, \\ h_n &= \frac{(1-q^{n-1}abcd)(q, ab, ac, ad, bc, bd, cd; q)_n}{(1-q^{2n-1}abcd)(abcd; q)_n} h_0, \\ h_0 &= \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}, \end{aligned}$$

where we suppressed the dependence on a, b, c and d in the notation for w and h. For $z = e^{i\theta}$ we use $w(\cos\theta)$.

Proposition 3.3.4. *Let a, b, c and d be real and let all the pairwise products of a, b, c and d be less than 1. Then the Askey-Wilson polynomials $p_n(x) = p_n(x; a, b, c, d \mid q)$ satisfy the orthogonality relations*

$$\frac{1}{2\pi}\int_0^\pi p_n(\cos\theta)p_m(\cos\theta)w(\cos\theta)d\theta + \sum_k p_n(x_k)p_m(x_k)w_k = \delta_{n,m}h_n.$$

The points x_k are of the form $\frac{1}{2}(eq^k + e^{-1}q^{-k})$ for e any of the parameters a, b, c or d with absolute value greater than 1; the sum is over $k \in \mathbb{N}$ such that $|eq^k| > 1$ and w_k is the residue of $z \mapsto w\big(\frac{1}{2}(z+z^{-1})\big)$ at $z = eq^k$ minus the residue at $z = e^{-1}q^{-k}$.

The orthogonality relations remain valid for complex parameters a, b, c and d, if they occur in conjugate pairs. If all parameters have absolute value less than 1, the Askey-Wilson orthogonality measure is absolutely continuous.

We use the notation $dm(x) = dm(x; a, b, c, d \mid q)$ for the normalised orthogonality measure. So for any polynomial p

$$\int_{\mathbb{R}} p(x)\, dm(x) \;=\; \frac{1}{h_0}\Big(\frac{1}{2\pi}\int_{-1}^{1} p(x)w(x)\frac{dx}{\sqrt{1-x^2}} + \sum_k p(x_k)w_k\Big). \tag{3.3.13}$$

Chapter 4

Example: quantum $SL(2, \mathbb{C})$ group at the Hopf $*$-algebra level

4.1 The quantised universal enveloping algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$

Let $U_q(\mathfrak{sl}(2, \mathbb{C}))$ be the complex unital associative algebra generated by A, B, C, D subject to the relations

$$AD = 1 = DA, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - D^2}{q - q^{-1}}. \qquad (4.1.1)$$

On the level of generators we define the comultiplication, counit and antipode by

$$\begin{gathered} \Delta(A) = A \otimes A, \quad \Delta(B) = A \otimes B + B \otimes D, \\ \Delta(C) = A \otimes C + C \otimes D, \quad \Delta(D) = D \otimes D, \\ \varepsilon(A) = \varepsilon(D) = 1, \qquad \varepsilon(C) = \varepsilon(B) = 0, \\ S(A) = D, \quad S(B) = -q^{-1}B, \quad S(C) = -qC, \quad S(D) = A. \end{gathered} \qquad (4.1.2)$$

Here q is thought of as a deformation parameter, and at first we take $q \in \mathbb{C}\backslash\{-1, 0, 1\}$. (It is also possible to view $U_q(\mathfrak{sl}(2, \mathbb{C}))$ as an algebra over $\mathbb{C}(q)$.) The case $q \to 1$ is considered in a moment. However, we will always assume that q is not a root of unity, i.e. $q^m \neq 1$ for all $m \in \mathbb{N}$. Observe that S is invertible, but $S^2 \neq 1$ since $q^2 \neq 1$.

Proposition 4.1.1. *Define Δ and ε on $U_q(\mathfrak{sl}(2, \mathbb{C}))$ by* (4.1.2) *as (unital) algebra homomorphisms and S by* (4.1.2) *as (unital) anti-algebra homomorphisms, then $U_q(\mathfrak{sl}(2, \mathbb{C}))$ is a Hopf algebra.*

Proof. We have to check that Δ, ε and S are well-defined and that they satisfy the axioms of a Hopf algebra. These are straighforward computations. E.g.

$$\Delta(AB) = A^2 \otimes AB + AB \otimes 1 = q(A^2 \otimes BA + BA \otimes 1) = q\Delta(BA)$$

and

$$m \circ (id \otimes S) \circ \Delta(B) = AS(B) + BS(D) = -q^{-1}AB + BA = 0 = \varepsilon(B).$$

Continuing in this way proves the proposition. □

Exercise 4.1.2. Show that $U_q(\mathfrak{sl}(2,\mathbb{C}))^{\mathrm{op}}$ is isomorphic to $U_{q^{-1}}(\mathfrak{sl}(2,\mathbb{C}))$. What are the possible Hopf algebra isomorphisms $\Phi\colon U_q(\mathfrak{sl}(2,\mathbb{C})) \to U_p(\mathfrak{sl}(2,\mathbb{C}))$?

The element

$$\Omega = \frac{q^{-1}A^2 + qD^2 - 2}{(q^{-1}-q)^2} + BC = \frac{qA^2 + q^{-1}D^2 - 2}{(q^{-1}-q)^2} + CB \tag{4.1.3}$$

is the Casimir element of the quantised universal enveloping algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$. Ω belongs to the centre of $U_q(\mathfrak{sl}(2,\mathbb{C}))$, as can be checked by direct verification.

Exercise 4.1.3. Show that the name for this Hopf algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is jusitified. Replace A by $\exp((q-1)H/2)$, and hence D by $\exp((1-q)H/2)$ and let $q \uparrow 1$. Deduce from (4.1.1) that in the limit we formally get

$$[H,B] = 2B, \qquad [H,C] = -2C, \qquad [B,C] = H.$$

Hence we obtain the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

As before we fix $0 < q < 1$, then there are two possible $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ (up to Hopf $*$-algebra isomorphism).

Proposition 4.1.4. *For $0 < q < 1$ there are two $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$.*

- $A^* = A$, $B^* = C$, $C^* = B$, $D^* = D$ *(the compact real form $U_q(\mathfrak{su}(2))$);*
- $A^* = A$, $B^* = -C$, $C^* = -B$, $D^* = D$ *(the non-compact real form $U_q(\mathfrak{su}(1,1))$).*

Observe that for both $*$-structures the Casimir operator is self-adjoint.

Exercise 4.1.5. Prove Proposition 4.1.4. Is it possible to obtain more $*$-structures on $U_q(\mathfrak{sl}(2,\mathbb{C}))$ if you drop the condition $0 < q < 1$?

Theorem 4.1.6. *For each spin $l \in \frac{1}{2}\mathbb{N}$ there exists a unique $(2l+1)$-dimensional representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ such that the spectrum of A is contained in $q^{\frac{1}{2}\mathbb{Z}}$. Equip $\mathbb{C}^{2l+1}$ with orthonormal basis $\{e^l_n\}$, $n = -l, -l+1, \dots, l$ and denote the representation by t^l. The action of the generators is given by*

$$t^l(A)\, e^l_n = q^{-n} e^l_n, \qquad t^l(D)\, e^l_n = q^n\, e^l_n,$$

$$t^l(B)\, e^l_n = \frac{\sqrt{(q^{-l+n-1} - q^{l-n+1})(q^{-l-n} - q^{l+n})}}{q^{-1}-q}\, e^l_{n-1}$$

$$t^l(C)\, e^l_n = \frac{\sqrt{(q^{-l+n} - q^{l-n})(q^{-l-n-1} - q^{l+n+1})}}{q^{-1}-q}\, e^l_{n+1},$$

where $e^l_{l+1} = 0 = e^l_{-l-1}$.

Exercise 4.1.7. (i) Prove Theorem 4.1.6 by considering eigenvectors and eigenvalues for the operator corresponding to A, and check the action of B and C on these eigenvectors.
(ii) Show that the representations of Theorem 4.1.6 are unitary for the $*$-structure corresponding to the compact real form.
(iii) Calculate the action of the Casimir $t^l(\Omega)$.

Exercise 4.1.8. Show that a linear basis for $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is given by $D^l C^k B^m$ for $k, m \in \mathbb{N}$, $l \in \mathbb{Z}$ with the convention $D^{-l} = A^l$ for $l \in \mathbb{N}$. (Use Theorem 4.1.6.)

4.2 The Hopf algebra $A_q(SL(2,\mathbb{C}))$

Fix $q \in (0,1)$. $A_q(SL(2,\mathbb{C}))$ is the complex unital associative algebra generated by α, β, γ, δ subject to the relations

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \\ &\beta\gamma = \gamma\beta, \quad \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma = 1. \end{aligned} \tag{4.2.1}$$

Lemma 4.2.1. *A linear basis for $A_q(SL(2,\mathbb{C}))$ is given by $\{\alpha^n\beta^m\gamma^l \mid m, l \in \mathbb{N}, n \in \mathbb{Z}\}$, with the convention $\alpha^{-n} = \delta^n$ for $n \in \mathbb{N}$.*

Exercise 4.2.2. How would you prove Lemma 4.2.1?

The algebra $A_q(SL(2,\mathbb{C}))$ is an example of a Hopf $*$-algebra. The comultiplication Δ, the counit ε, the antipode S and the $*$-operator are given on the generators by

$$\begin{aligned} &\Delta(\alpha) = \alpha\otimes\alpha + \beta\otimes\gamma, \quad \Delta(\beta) = \alpha\otimes\beta + \beta\otimes\delta, \\ &\Delta(\gamma) = \gamma\otimes\alpha + \delta\otimes\gamma, \quad \Delta(\delta) = \gamma\otimes\beta + \delta\otimes\delta, \end{aligned} \tag{4.2.2}$$

and

$$\varepsilon\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}, \tag{4.2.3}$$

Exercise 4.2.3. Check that this defines a Hopf algebra structure on $A_q(SL(2,\mathbb{C}))$.

Proposition 4.2.4. *Define the pairing $\langle\cdot,\cdot\rangle\colon U_q(\mathfrak{sl}(2,\mathbb{C})) \times A_q(SL(2,\mathbb{C})) \to \mathbb{C}$ on the generators by*

$$\begin{aligned} &\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\rangle = \begin{pmatrix} q^{\frac12} & 0 \\ 0 & q^{-\frac12} \end{pmatrix}, \quad \langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\rangle = \begin{pmatrix} q^{-\frac12} & 0 \\ 0 & q^{\frac12} \end{pmatrix} \\ &\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and extend by Definition 2.3.3, then the Hopf algebras U and $A_q(SL(2,\mathbb{C}))$ are in duality.

Exercise 4.2.5. Prove Proposition 4.2.4 by showing that the pairing does preserves the defining relations of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $A_q(SL(2,\mathbb{C}))$. What is the relation to the representations of Theorem 4.1.6? Check the case $l=\frac{1}{2}$ and compare with Proposition 4.2.4.

Now that we have the pairing as Hopf algebras between $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and $A_q(SL(2,\mathbb{C}))$ we can transport the two $*$-structures from $U_q(\mathfrak{sl}(2,\mathbb{C}))$ to $A_q(SL(2,\mathbb{C}))$. This leads to two $*$-structures on $A_q(SL(2,\mathbb{C}))$.

Proposition 4.2.6. *Up to Hopf $*$-algebra isomorphisms the Hopf algebra $A_q(SL(2,\mathbb{C}))$ has two inequivalent $*$-structures for $0<q<1$:*

- $\alpha^*=\delta$, $\beta^*=-q\gamma$, $\gamma^*=-q^{-1}\beta$, $\delta^*=\alpha$ *(the compact real form $A_q(SU(2))$);*
- $\alpha^*=\delta$, $\beta^*=q\gamma$, $\gamma^*=q^{-1}\beta$, $\delta^*=\alpha$ *(the non-compact real form $A_q(SU(1,1))$).*

In the case of $A_q(SU(2))$ we know that there exists a unique functional $\varphi\colon A_q(SU(2))\to\mathbb{C}$ satisfying

$$\begin{aligned}\varphi(a^*a)\geq 0,\quad \varphi(a^*a)=0 \iff a=0 \\ (\varphi\otimes\iota)\Delta(a) = \varphi(a)1 = (\iota\otimes\varphi)\Delta(a).\end{aligned}\tag{4.2.4}$$

There are several ways of proving the existence of the Haar functiona φ, see e.g. [8], [58]. An explicit expression in terms of the linear basis is given by

$$\varphi(\alpha^n(\gamma^*)^m\gamma^l)=\delta_{n,0}\delta_{ml}\frac{1-q^2}{1-q^{2m+2}}.\tag{4.2.5}$$

4.3 Corepresentations of $A_q(SU(2))$

Since the Hopf $*$-algebras $U_q(\mathfrak{su}(2))$ and $A_q(SU(2))$ are in duality as Hopf $*$-algebras, we see that we can obtain the corepresentations of $A_q(SU(2))$ in each dimension $2l+1$, $l\in\frac{1}{2}\mathbb{N}$. It has already been proved that the corresponding matrix elements can be expressed explicitly in terms of the little q-Jacobi polynomials, see Section 3.3.1. Work by Koornwinder [36] introduced new degrees of freedom making it possible to interpret the full four parameters Askey-Wilson polynomials as matrix elements, see [25] and references given there. This idea has lead to very general situation, see Letzter [44] for more references and developments.

4.4 Representations of $A_q(SU(2))$

Now $A_q(SU(2))$ is a $*$-algebra, given by the following relations

$$\alpha\gamma=q\gamma\alpha,\quad \alpha\gamma^*=q\gamma^*\alpha,\quad \gamma\gamma^*=\gamma^*\gamma,\quad \alpha\alpha^*+q^2\gamma^*\gamma=\alpha^*\alpha+\gamma\gamma^*=1.$$

Note that when representing this $*$-algebra by operators on a suitable Hilbert space H, the last relation shows that the operator corresponding to α and γ are bounded (by 1).

Theorem 4.4.1. *The following is a list of mutually inequivalent irreducible $*$-representations of $A_q(SU(2))$:*

- *one-dimensional representations $\alpha \mapsto e^{i\theta}$, $\gamma \mapsto 0$ for $\theta \in [0, 2\pi)$;*
- *representation π_ϕ ($\phi \in [0, 2\pi)$) in the Hilbert space $\ell^2(\mathbb{N})$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and the action given by*

$$\pi_\phi(\alpha)\, e_n \;=\; \sqrt{1-q^{2n}}\, e_{n-1}, \qquad \pi_\phi(\gamma)\, e_n \;=\; e^{i\phi}\, q^n\, e_n.$$

Exercise 4.4.2. Prove Theorem 4.4.1 under the additional assumption that the normal operator $\gamma^*\gamma$ has an eigenvector. Consider the case of eigenvalue zero separately.

Exercise 4.4.3. Consider the representation π in the Hilbert space $\ell^2(\mathbb{N}) \otimes L^2(\mathbb{T})$ with orthonormal basis $e_n \otimes \zeta^m$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$ and with actions

$$\pi(\alpha)\, e_n \otimes \zeta^m \;=\; \sqrt{1-q^{2n}}\, e_{n-1} \otimes \zeta^m, \qquad \pi(\gamma)\, e_n \otimes \zeta^m \;=\; q^n\, e_n \otimes \zeta^{m+1}.$$

Show that this defines a $*$-representation, and show that this representation is faithful. Moreover, show that $\pi \cong \int^{\oplus} \pi_\phi\, d\phi$.

Exercise 4.4.4. Show that the Haar functional (4.2.4) can be written as

$$\varphi(a) \;=\; \frac{(1-q^2)}{2\pi} \int_0^{2\pi} \mathrm{Tr}\big(D\pi_\phi(a)\big)\, d\phi, \qquad a \in A_q(SU(2)),$$

where $D\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $De_n = q^{2n} e_n$, by checking that (4.2.5) is fulfilled.

Using the comultiplication we can define tensor product representations by

$$\pi_1 \otimes \pi_2\colon A_q(SU(2)) \to B(H_1 \otimes H_2), \qquad \pi_1 \otimes \pi_2(a) = \pi_1 \otimes \pi_2\big(\Delta(a)\big)$$

Put $\pi = \pi_0$ (so $\phi = 0$). As an example we consider $\pi \otimes \pi$. In order to find the suitable decomposition we consider the normal operator $\pi \otimes \pi(\gamma^*\gamma)$. Explicitly, we find

$$\begin{aligned} \pi \otimes \pi(\gamma^*\gamma)\, e_n \otimes e_m \;&=\; (\pi\otimes\pi)\Big(\big(\gamma\otimes\alpha + \alpha^*\otimes\gamma\big)^*\big(\gamma\otimes\alpha+\alpha^*\otimes\gamma\big)\Big)\, e_n \otimes e_m \\ &= \Big(\pi(\alpha\gamma)\otimes\pi(\gamma^*\alpha) + \pi(\gamma^*\gamma)\otimes\pi(\alpha^*\alpha) + \pi(\alpha\alpha^*)\otimes\pi(\gamma^*\gamma) + \pi(\gamma^*\alpha^*)\otimes\pi(\alpha^*\gamma)\Big)\, e_n \otimes e_m \\ &= q^{n+m-1}\sqrt{(1-q^{2n})(1-q^{2m})}\, e_{n-1}\otimes e_{m-1} \;+\; \Big(q^{2n}(1-q^{2m}) + q^{2m}(1-q^{2n+2})\Big)\, e_n \otimes e_m \\ &\quad +\; q^{n+m+1}\sqrt{(1-q^{2m+2})(1-q^{2n+2})}\, e_{n+1}\otimes e_{m+1}. \end{aligned} \tag{4.4.1}$$

So we see that the subspace $H_p = \{e_n \otimes e_m \mid n - m = p\} \subset \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, $p \in \mathbb{Z}$, is invariant for $\pi \otimes \pi(\gamma^*\gamma)$. Assuming $p \geq 0$, we replace $n = m + p$ and put $f_m = e_{m+p} \otimes e_m$, $m \in \mathbb{N}$, so that we find

$$\begin{aligned} \pi \otimes \pi(\gamma^*\gamma)\, f_m \;=\; & q^{2m+p-1}\sqrt{(1-q^{2m+2p})(1-q^{2m})}\, f_{m-1} + \\ & \Big(q^{2m+2p}(1-q^{2m}) + q^{2m}(1-q^{2m+2p+2})\Big)\, f_m \;+\; q^{2m+p+1}\sqrt{(1-q^{2m+2})(1-q^{2m+2p+2})}\, f_{m+1}. \end{aligned}$$

So we can look for eigenvectors of the form $\sum_{m=0}^{\infty} c_m f_m$ for $\pi\otimes\pi(\gamma^*\gamma)$, and for this we have to consider the three term recurrence relation. Comparing with the three-term recurrence (3.3.3) for the Wall polynomials we see that we can find eigenvectors of $\pi\otimes\pi(\gamma^*\gamma)$.

Proposition 4.4.5 (Koornwinder [35]). *Set* $v_k^p = \sum_{m=0}^{\infty} P_m(q^{2k}; q^{2p}; q^2)\, e_{m+p}\otimes e_p$, *then* $\{v_k^p \mid k\in\mathbb{N}\}$ *is an orthonormal basis of* $\bigoplus_{m\in\mathbb{N}} e_{m+p}\otimes e_m$ *and*

$$\pi\otimes\pi(\gamma^*\gamma)\, v_k^p \;=\; q^{2k}\, v_k^p.$$

Exercise 4.4.6. Check the remainder of the proof of Proposition 4.4.5.

Exercise 4.4.7. Show that $\pi\otimes\pi \cong \frac{1}{2\pi}\int^{\oplus} \pi_\phi\, d\phi$. For this you need to calculate the action of $\pi\otimes\pi(\alpha)$ and $\pi\otimes\pi(\gamma)$ on v_k^p. This requires the following contiguous relations for the Wall polynomials;

$$\begin{aligned} p_n(q^x; a/q; q) \;&= \frac{1-aq^n}{1-a}\, p_n(q^x; a; q) \;-\; \frac{1-q^n}{1-a}\, p_{n-1}(q^x; a; q), \\ (1-q^x)\, p_n(q^{x-1}; a; q) \;&= (1-aq^{n+1})\, p_{n+1}(q^x; a; q) \;-\; aq^n\, p_{n-1}(q^x; a; q). \end{aligned}$$

Prove the contiguous relations as well.

The result of Exercise 4.4.7 can be extended to Clebsch-Gordan coefficients and Racah in different bases, and this is unpublished work (2003) by Wolter Groenevelt.

Chapter 5

Operator theory

This chapter contains all the preperatory work for an operator algebraic approach to quantum groups. Basically, this will be an introduction to von Neumann algebras, Tomita-Takesaki theory and weight theory on von Neumann algebras. The chapter requires some basic knowledge of functional analysis, see for instance [6] or [50]. For the ones who are interested in the proofs, backgrounds and related theory, we give some references: [10], [19], [49], [50], [56], [57].

5.1 Hilbert spaces and $B(\mathcal{H})$

A Hilbert space $\mathcal{H}$ is linear space over $\mathbb{C}$ equipped with an inner product, or a sesquilinear form, $\langle \cdot, \cdot \rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, which is linear in the first variable and antilinear in the second, so that the space is complete as a normed space with respect to the norm $\|v\| = \sqrt{\langle v, v \rangle}$.

The Hilbert space $\mathcal{H}$ is taken to be separable. We denote $B(\mathcal{H})$ for the space of bounded operators on $\mathcal{H}$.

5.1.1 Operator topology

For $T \in B(\mathcal{H})$, the operator norm is defined as the norm as a linear map between Hilbert spaces:

$$\|T\| = \sup_{v \in \mathcal{H}} \frac{\|Tv\|}{\|v\|}. \tag{5.1.1}$$

The topology it induces is called the **operator topology** or **uniform topology**. $B(\mathcal{H})$ is then a Banach space, i.e. a complete normed space, in this topology. Note that the following (in)equalities hold. For $T, S \in B(\mathcal{H})$:

$$\|TS\| \leq \|T\| \|S\|, \qquad \|T^*T\| = \|T\|^2. \tag{5.1.2}$$

Therefore $B(\mathcal{H})$ is a C*-algebra, i.e. an involutive, normed algebra that is complete and satisfies the relations (5.1.2) above. We will come back to C*-algebras later. It follows from these equations that the involution and the product are continuous.

5.2 Compact operators, the Trace and Trace class operators

The aim of this section is twofold. Here we introduce the trace and the trace class operators in order to define new topologies on $B(\mathcal{H})$ in the next section. We could have given the definitions of these topologies straightaway, but it is much nicer to see how they (at least some of them) arise naturally. Also, this section provides one of the main examples for the theory in Section 5.8.

We start with recalling the main results on compact operators. Let $D_v(r) \subseteq \mathcal{H}$ be the norm-closed ball with center $v \in \mathcal{H}$ and radius $r > 0$. We recall the following definition.

Definition 5.2.1. *An operator $T \in B(\mathcal{H})$ is called* ***compact*** *if $TD_0(1)$ is compact.*

We denote the set of compact operators by $K(\mathcal{H})$. Not every operator is compact. For example the indentity fails to be compact unless $\dim(\mathcal{H})$ is finite. Finite rank operators are examples of compact operators. In fact, we have the following theorem for compact operators.

Theorem 5.2.2. *An operator $T \in B(\mathcal{H})$ is compact if and only if it is the (operator) norm limit of finite rank operators.*

In particular, $K(\mathcal{H})$ is a norm-closed subspace of $B(\mathcal{H})$, since it is the closure of the finite rank operators. One can now prove that:

Exercise 5.2.3. If $T \in K(\mathcal{H})$ and $S \in B(\mathcal{H})$. Show that: (1) ST is compact; (2) T^* is compact; (3) TS is compact.

It follows that $K(\mathcal{H})$ is a closed two-sided ideal of $B(\mathcal{H})$. The following theorem is known as the spectral theorem for compact operators.

Theorem 5.2.4. *Let $T \in K(\mathcal{H})$. Then there exists a family of positive constants $\alpha_n \in \mathbb{R}^+$, and two orthonormal sequences $e_n \in \mathcal{H}$ and $f_n \in \mathcal{H}$, such that for every $x \in \mathcal{H}$:*

$$Tx = \sum_n \alpha_n \langle x, e_n \rangle f_n. \tag{5.2.1}$$

Furthermore, $\alpha_n \to 0$ as $n \to \infty$.

If T is self-adjoint and compact, one has $e_n = f_n$ and we see that the expression in the previous theorem gives an eigenvalue decomposition of T. (Actually the theorem is proved by first assuming that T is self-adjoint. Then, one shows that T has an eigenvalue of value $\|T\|$ or $-\|T\|$. The eigenspace as well as its complement are invariant since T is self-adjoint, so that the decomposition follows by an induction argument. Then for non-self-adjoint T one applies a trick to obtain the theorem). The theorem allows us to introduce the trace class operators. If T is an arbitrary compact operator, one sees that α_n^2 are the eigenvalues of T^*T. Therefore, the sum in the next definition is independent of the decompostion (5.2.1).

Definition 5.2.5. *A compact operator $T \in K(\mathcal{H})$ is called **trace class** if $\|T\|_1 := \sum \alpha_n < \infty$, where the α_n are the constants as they appear in Theorem 5.2.4. Then $\| \, \|_1$ defines a norm on the set of trace class operators.*

We denote the trace class operators by $L^1(\mathcal{H})$. The following map will allow us to introduce the σ-weak topology in the next section. Also, it will prepare us for the theory of weights on von Neumann algebras. Let $T \in L^1(\mathcal{H})$, we define the **trace** of T to be:

$$\mathrm{Tr}(T) = \sum_n \langle Te_n, e_n \rangle,$$

where $(e_n)_n$ is an orthonormal base of $\mathcal{H}$. One easily checks that the sum is independent of the orthonormal base and that the sum converges (and is finite). As an exercise, one can show that:

Exercise 5.2.6. Show that if $T \in L^1(\mathcal{H})$ and $S \in B(\mathcal{H})$, then ST, T^* and TS are trace class. In particular, $L^1(\mathcal{H})$ is a two-sided ideal of $B(\mathcal{H})$.

Furthermore, quite tautological, the trace satisfies the trace property:

$$\mathrm{Tr}(ST) = \mathrm{Tr}(TS), \qquad \forall S \in B(\mathcal{H}), T \in L^1(\mathcal{H}),$$

moreover one can prove that $|\mathrm{Tr}(ST)| \leq \|S\| \|T\|_1$.

The next theorem shows some important relations between the spaces $K(\mathcal{H}), L^1(\mathcal{H})$ and $B(\mathcal{H})$. Note that if $T \in L^1(\mathcal{H})$ and $S \in B(\mathcal{H})$, $K(\mathcal{H}) \ni A \mapsto \mathrm{Tr}(AT)$ is a continuous functional on $K(\mathcal{H})$ and $L^1(\mathcal{H}) \ni A \mapsto \mathrm{Tr}(AS)$ is a continuous functional on $L^1(H)$. These maps are actually isometric isomorphisms.

Theorem 5.2.7. *We have an isometric isomorphism $K(\mathcal{H})^* \simeq L^1(\mathcal{H})$. The isomorphism being given by sending $T \in L^1(\mathcal{H})$ to the functional $\mathrm{Tr}(\cdot T)$. We have an isometric isomorphism $L^1(\mathcal{H})^* \simeq B(\mathcal{H})$. The isomorphism being given by sending $T \in B(\mathcal{H})$ to the functional $\mathrm{Tr}(\cdot T)$.*

It immediately follows that $L^1(\mathcal{H})$ is a Banach space, since it is the dual of $K(\mathcal{H})$. Having these relations at hand we are capable of introducing new topologies on $B(\mathcal{H})$.

5.3 Locally convex topologies on $B(\mathcal{H})$

The space $B(\mathcal{H})$ contains a lot of interesting and many used topologies. The topologies are necessary for a good understanding of operator algebras and in particular l.c. qunatum groups. All the topologies are locally convex, of which we recall the definition.

Let E be any Banach space. A topology on E is called a **locally convex topology** if it is defined by a family of seminorms $p_i, i \in I$ on E, such that $\cap_{i \in I} \mathrm{Ker}(p_i) = 0$. That is, the topology on E is induced by the semi-norms; it is the smallest topology for which these semi-norms are continuous. The assumption that the intersections of the kernels must be trivial is made in order to let this topology be Hausdorff. In particular the norm topology on $B(\mathcal{H})$ is a locally convex topology, since it is induced by the single (semi-)norm $\| \, \|$. The name 'locally convex' may appear a bit strange at first sight; the name is justified by [6, Proposition IV.1.15].

5.3.1 Strong topology

For $v \in \mathcal{H}$ define the seminorm $p_v \colon B(\mathcal{H}) \to \mathbb{R}$ by $p_v(T) = \|Tv\|$. The strong topology (or strong operator topology) is the topology induced by the seminorms p_v for $v \in \mathcal{H}$. It gives $B(\mathcal{H})$ the structure of a locally convex vector space. A basis of neighbourhoods of an operator $T_0 \in B(\mathcal{H})$ is given by

$$V(T_0; x_1, \ldots, x_k; \varepsilon) = \{T \in B(\mathcal{H}) \mid \|(T - T_0)x_i\| < \varepsilon \ \forall i = 1, \ldots, k\}.$$

Remark 5.3.1. Note that if we have two families $\mathcal{F}_1$, $\mathcal{F}_2$ of seminorms on $B(\mathcal{H})$ such that for each $p \in \mathcal{F}_1$ there exists a constant C and a $q \in \mathcal{F}_2$ such that $p(T) \leq Cq(T)$ for all $T \in B(\mathcal{H})$, then the topology induced by $\mathcal{F}_1$ is weaker than the topology induced by $\mathcal{F}_2$. Indeed, by the assumed estimate it follows that each functional p is continuous with respect to the topology induced by $\mathcal{F}_2$. Since the topology induced by $\mathcal{F}_1$ is the weakest topology such that all $p \in \mathcal{F}_1$ are continuous the observation follows. In particular, if $\mathcal{F}_1 \subset \mathcal{F}_2$ the topology induced by the seminorms of $\mathcal{F}_1$ is weaker than the topology induced by $\mathcal{F}_2$,

Since $p_v(T) \leq \|v\| \, \|T\|$ it follows that the strong topology is weaker than the operator topology.

Exercise 5.3.2. We examine properties of the strong topology.

1. Let $\mathcal{H}$ be infinite dimensional and let e_n be an orthonormal basis. Let P_N be the projection onto span $\{e_1, \ldots, e_N\}$. Let $T \in B(\mathcal{H})$, show that $P_N T$ converges to T in the strong operator topology. Use Theorem 5.2.2 to infer that the strong operator topology is strictly weaker than the norm topology.

2. Let $S : l^2(\mathbb{N}) \to l^2(\mathbb{N}) : (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots)$ be the shift operator. Show that $(S^*)^n \to 0$ strongly as $n \to \infty$, but S^n does not converge stongly. This shows that the adjoint operation is not strongly continuous!

5.3.2 Strong-$*$ topology

The adjoint operation $T \mapsto T^*$ is not continuous for the strong topology (see Exercise 5.3.2), and we define the strong-$*$ topology as the locally convex topology induced by the seminorms p_v, q_v for $v \in \mathcal{H}$ defined by $p_v(T) = \|Tv\|$, $q_v(T) = \|T^*v\|$. Note that the strong topology is weaker than the strong-$*$ topology.

The strong-$*$ topology can be defined as well by the seminorms $T \mapsto \left(\|Tv\|^2 + \|T^*v\|^2\right)^{1/2}$.

Exercise 5.3.3. Check that the adjoint operation is strong-$*$ continuous.

Exercise 5.3.4. Show that $T_n \to T$ in the strong-$*$ topology if and only if both $T_n \to T$ and $T_n^* \to T^*$ in the strong topology.

5.3.3 σ-weak topology

Recall from Theorem 5.2.7, that $B(\mathcal{H}) = L^1(\mathcal{H})^*$, so that $B(\mathcal{H})$ carries the weak-$*$ topology, or $\sigma(B(\mathcal{H}), L^1(\mathcal{H}))$ topology. This topology is by definition the σ-weak topology.

There is an alternative decription of the σ-weak toplogy giving a direct description of the set of semi-norms. We call an operator $T \in K(\mathcal{H})$ **Hilbert-Schmidt** if $T^*T \in L^1(\mathcal{H})$. We denote the Hilbert-Schmidt operators with $L^2(\mathcal{H})$. One can show that $L^2(\mathcal{H})$ forms a two-sided ideal in $B(H)$ and furhtermore $L^2(\mathcal{H})L^2(\mathcal{H}) = L^1(\mathcal{H})$. Moreover, for $T, S \in L^2(\mathcal{H})$, Tr satisfies the property $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$. Let $R \in L^1(\mathcal{H})$ and let $S, T \in L^2(\mathcal{H})$ be such that $ST^* = R$. Let e_n be an orthonormal basis of $\mathcal{H}$ and let $Se_n = v_n$, $Te_n = w_n$. Note that the Hilbert-Schmidt condition is equivalent to the fact that $\sum_n \|v_n\|^2, \sum_n \|w_n\|^2 < \infty$. Then, for $A \in B(\mathcal{H})$:

$$\mathrm{Tr}(AR) = \mathrm{Tr}(T^*AS) = \sum_n \langle ASe_n, Te_n\rangle = \sum_n \langle Av_n, w_n\rangle,$$

so that the σ-weak topology is defined by all the semi-norms of the form $A \mapsto \sum_n \langle Av_n, w_n\rangle$, where $\sum_n \|v_n\|^2, \sum_n \|w_n\|^2 \leq \infty$.

The following Theorem is just a special case of Alaoglu's theorem.

Theorem 5.3.5. *The (norm)-closed unit ball in $B(\mathcal{H})$ is σ-weakly compact.*

5.3.4 Weak topology

For $v, w \in \mathcal{H}$ define $\omega_{v,w} \colon B(\mathcal{H}) \to \mathbb{C}$ by $\omega_{v,w}(T) = \langle Tv, w\rangle$. The weak topology (or weak operator topology) on $B(\mathcal{H})$ is the topology induced by the linear functionals $\omega_{x,y}$ for all $x, y \in \mathcal{H}$. It gives a locally convex vector space according to the seminorms $|\omega_{x,y}|$, $x, y \in \mathcal{H}$. A basis of neighbourhoods of an operator $T_0 \in B(\mathcal{H})$ is given by

$$V(T_0; x_1, \ldots, x_k, y_1, \ldots, y_k; \varepsilon) = \{T \in B(\mathcal{H}) \mid |\langle (T - T_0)x_i, y_i\rangle| < \varepsilon \ \forall i = 1, \ldots, k\}.$$

Remark 5.3.6. We warn the reader that the weak topology is not the $\sigma(B(\mathcal{H}), B(\mathcal{H})^*)$-topology, i.e. the topology induced by the continuous linear functionals on $B(\mathcal{H})$. The definition of 'weak' is different then what you might expect from general Banach space theory.

Since $|\omega_{x,y}(T)| \leq \|Tx\| \, \|y\|$ it follows that the weak topology is weaker than the strong topology. For $\mathcal{H}$ infinite dimensional, the weak topology is strictly weaker as the following example shows.

Exercise 5.3.7. Let S be the shift operator of Exercise 5.3.2. Show that $S^n \to 0$ as $n \to \infty$ in the weak topology, but not in the strong topology.

Exercise 5.3.8. Let P_N be as in Exercise 5.3.2. Show that $1-P_N \to 0$ as $N \to \infty$ in the weak topology, but not in the $\sigma(B(\mathcal{H}), B(\mathcal{H})^*)$-topology. HINT: use the Hahn-Banach theorem.

We have the following theorems.

Theorem 5.3.9. *The relative weak and $\sigma-$weak topology coincides on the (norm-)closed unit ball in $B(\mathcal{H})$. In particular the (norm-)closed unit ball in $B(\mathcal{H})$ is weakly compact.*

Proof. The idenity map from the closed unit ball with the σ-weak topology to the closed unit ball with the weak topology is continuous. The map goes from a compact space to a Hausdorff space and therefore is a homeomorphism. □

Theorem 5.3.10. *A convex set in $B(\mathcal{H})$ which is strongly closed is also weakly closed. In particular, strongly closed subspaces of $B(\mathcal{H})$ are also weakly closed.*

5.3.5 σ-strong topology

For any sequence v_n of elements from $\mathcal{H}$ with $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$ define the seminorm

$$B(\mathcal{H}) \ni T \mapsto \left(\sum_{n=1}^{\infty} \|Tv_n\|^2 \right)^{1/2} .$$

The σ-strong topology (or σ-strong operator topology) on $B(\mathcal{H})$ is the topology induced by the seminorms given above. Note that all functionals used to define the strong topology are included as special cases of the seminorms defining the σ-strong topology, so that the strong topology is weaker than the σ-strong topology.

Theorem 5.3.11. *The strong topology and σ-strong topology coincide on bounded subsets of $B(\mathcal{H})$.*

The σ-strong topology is weaker than the norm topology. The adjoint operation is not continuous for the σ-strong topology.

Exercise 5.3.12. Let S be the shift operator as in Exercise 5.3.2. Show that $(S^*)^n \to 0$ as $n \to 0$ σ-strongly, but not in the norm topology.

5.3.6 σ-strong-$*$ topology

For any sequence $\{v_n\}_{n=1}^{\infty}$ of elements in $\mathcal{H}$ with $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$ define the seminorms

$$B(\mathcal{H}) \ni T \mapsto \left(\sum_{n=1}^{\infty} \|Tv_n\|^2 \right)^{1/2} , \quad B(\mathcal{H}) \ni T \mapsto \left(\sum_{n=1}^{\infty} \|T^*v_n\|^2 \right)^{1/2} .$$

The σ-strong-$*$ topology (or σ-strong-$*$ operator topology) on $B(\mathcal{H})$ is the topology induced by the seminorms given above. Note that all functionals used to define the strong-$*$ topology are included as special cases of the seminorms defining the σ-strong-$*$ topology, so that the strong-$*$ topology is weaker than the σ-strong-$*$ topology.

Theorem 5.3.13. *The strong-$*$ topology and σ-strong-$*$ topology coincide on bounded subsets of $B(\mathcal{H})$.*

The σ-strong-$*$ topology is weaker than the norm topology.

5.4 Interrelations of topologies

We collect the results on the various topologies in one scheme. With $\mathcal{T} \succ \mathcal{S}$ meaning that the topology $\mathcal{T}$ is weaker than $\mathcal{S}$ we have:

$$\begin{array}{ccccccc} \sigma\text{-weak} & \succ & \sigma\text{-strong} & \succ & \sigma\text{-strong-}* & \succ & \text{uniform} \\ \curlywedge & & \curlywedge & & \curlywedge & & \\ \text{weak} & \succ & \text{strong} & \succ & \text{strong-}*. & & \end{array}$$

Indeed, the vertical inclusions and σ-strong $\succ$ σ-strong-$*$ are due to Remark 5.3.1. We proved the bottom row of inclusions in the previous Sections. The inclusions σ-weak $\succ$ σ-strong and σ-strong-$*$ $\succ$ uniform follow by the following estimates. For x_n, y_n, such that $\sum_n \|v_n\|^2 < \infty$, $\sum_n \|w_n\|^2 < \infty$:

$$|\sum_n \langle Tv_n, w_n\rangle| \leq \sum_n \|Tv_n\|\|w_n\| \leq \left(\sum_n \|Tv_n\|^2\right)^{\frac{1}{2}} \left(\sum_n \|w_n\|^2\right)^{\frac{1}{2}},$$

and similarly:

$$\sum_n \|Tv_n\|^2 \leq \|T\|^2 \sum_n \|v_n\|^2.$$

All of the inclusions are strict. Furhtermore, the following theorem reveals some interesting relations between the topologies.

Theorem 5.4.1. *Let M be a σ-weakly closed subspace of $B(\mathcal{H})$ and let ω be a continuous linear functional on M. $D_v(r)$ denotes the closed ball with center $v \in \mathcal{H}$ and radius $r > 0$. For $v, w \in \mathcal{H}$, we denote $\omega_{v,w}$ for the functional on M defined by $\omega_{v,w}(T) = \langle Tv, w\rangle$.*

1. *The following are equivalent:*

   (a) *ω is weakly continuous;*

   (b) *ω is strongly continuous;*

   (c) *ω is strongly-$*$ continuous;*

   (d) *$\omega = \sum_{i=1}^n \omega_{v_i,w_i}$, where $v_i, w_i \in \mathcal{H}$.*

2. *The following are equivalent:*

   (a) *ω is σ-weakly continuous;*

(b) ω *is* σ*-strongly continuous;*

(c) ω *is* σ*-strongly-$*$ continuous;*

(d) $\omega = \sum_{i=1}^{\infty} \omega_{vi,w_i}$*, where* $v_i, w_i \in \mathcal{H}$ *and* $\sum_{i=1}^{\infty} \|v_i\|^2 < \infty$ *and* $\sum_{i=1}^{\infty} \|w_i\|^2 < \infty$*;*

(e) ω *is weakly continuous on* $M \cap D_0(1)$*;*

(f) ω *is strongly continuous on* $M \cap D_0(1)$*;*

(g) ω *is strongly-$*$ continuous on* $M \cap D_0(1)$.

3. *Let* $K \subseteq M$ *be convex. The following are equivalent:*

(a) K *is* σ*-weakly closed;*

(b) K *is* σ*-strongly closed;*

(c) K *is* σ*-strongly-$*$ closed;*

(d) $K \cap D_0(r)$ *is weakly (therefor* σ*-weakly) closed for every* $r > 0$*;*

(e) $K \cap D_0(r)$ *is strongly (therefor* σ*-strongly) closed for every* $r > 0$*;*

(f) $K \cap D_0(r)$ *is strongly-$*$ (therefor* σ*-strongly-$*$) closed for every* $r > 0$.

5.5 C*-algebras

This section gives a very concise overview of results on C*-algebras. We will not need much of the general theory of C*-algebras. Even though it is good to be aware of the basic results of C*-algebra theory, since it plays an essential rôle in abstract harmonic analysis. For example, the Plancherel theorem relies on the understanding of C*-algebras. Also, this section enables us to introduce some concepts that are of great importance for von Neumann algebras as will be introduced later. We treat the Gelfand-Naimark theorem, positivity and give the two most relevant examples of C*-algebras, the universal and reduced group C*-algebra.

Let A be an algebra over $\mathbb{C}$. We call a map $^* : A \to A$ an **involution** if it satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$, $(\lambda x)^* = \overline{\lambda} x^*$, $(x+y)^* = x^* + y^*$, for all $x, y \in A$ and $\lambda \in \mathbb{C}$. If A carries an involution, we say that A is an involutive algebra.

Definition 5.5.1. *A* ***C*-algebra*** *A is an algebra over $\mathbb{C}$ together with an involution $*$ and a norm $\| \|$, such that A is complete. Moreover, A satisfies the axioms $\|x^*x\| = \|x\|^2$ and $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A is called* ***unital*** *if A carries a unit. A is called* ***commutative*** *if $xy = yx$ for all $x, y \in A$.*
Let A and B be C-algebras. A moprhism of C*-algebras (or $*$-homomorphism) is an algebra homomorphism $\Phi : A \to B$, such that $\Phi(x^*) = \Phi(x)^*$. If Φ is bijective, it is an isomorphism, and its inverse is automatically a morphism of C*-algebras. If A and B are unital, Φ in addition has to preserve the unit of A and B.*

One can prove that a $*$-homomorphism Φ between C*-algebras is automatically continuous, with $\|\Phi\| \leq 1$. If Φ is injective, it is automatically isometric, i.e. $\|\Phi(x)\| = \|x\|$.

Examples are $B(\mathcal{H})$, $K(\mathcal{H})$ with the operator norm and the adjoint as involution. Every involutive norm-closed subalgebra of $B(\mathcal{H})$ is a C*-algebra. In fact, one can prove that these are all possible examples!

Theorem 5.5.2 (Gelfand, Naimark)**.** *Every C*-algebra A is isomorphic to a norm-closed involutive subalgebra of $B(\mathcal{H})$.*

Let us briefly comment on how to prove this theorem for the case that A is unital. A **state** on a unital C*-algebra A is a continuous linear functional $\omega : A \to \mathbb{C}$, such that $\omega(x^*x) \geq 0$ and $\omega(1) = 1$. One can construct a semi-innerproduct on A by defining

$$\langle x, y\rangle = \omega(y^*x).$$

One sets $\mathcal{H}$ to be the completion of A with respect to this inner-product after dividing out by the left ideal $N = \{x \mid \omega(x^*x) = 0\}$. There is a representation π_ω of A on $\mathcal{H}$ by defining $\pi_\omega(x)y = xy \in \mathcal{H}$ (the map is well-defined by the estimate $\omega(y^*x^*xy) \leq \|x\|^2\omega(y^*y)$). In general, π_ω is not faithful and a priori it is not clear that there exist any states at all. The next step in the proof is to show that there are lots of states, that is, the direct sum over all states ω of representations π_ω is faithful.

The construction of the representations π_ω is called the **GNS-construction** and we will see a similar construction later. We remark that GNS stands for Gelfand, Naimark and Segal. The GNS-construction is not only a key concept for the theory of operator algebras, it is also very useful for the study of (abstract) harmonic analysis and quantum groups.

If X is a locally compact, Hausdorff space, then $C_b(X)$ and $C_0(X)$, i.e. repsectively the continuous bounded functions and the continuous functions vanishing at infinity are commutative C*-algebras. $C_b(X)$ is unital, $C_0(X)$ is unital if and only if X is compact. The assumptions on X being locally compact and Hausdorff are not strictly necessary. However, we have the following important theorem.

Theorem 5.5.3 (Gelfand)**.** *Suppose that A is a commutative C*-algebra, then there is a locally compact, Hausdorff space X such that A is isomorphic to $C_0(X)$. X is compact if and only if A is unital.*

The combination of Theorems 5.5.3 and 5.5.2 is called the **Gelfand-Naimark theorem**. C*-algebras are important for the theory of representations of locally compact groups and essential for operator algebraic approaches to quantum groups. The following two examples of C*-algebras are typical.

Example 5.5.4. Let G be a locally compact group. Let $L^1(G)$ be the space of measurable functions whose absolute value is integrable with respect to the left Haar measure $\int_G d_lx$ on G. For $f \in L^1(G)$, define the norm $\|f\|_1 = \int_G |f(x)|d_lx$ and the involution $f^*(x) = \overline{f(x^{-1})}\delta_G(x^{-1})$, where $\delta_G(\cdot)$ denotes the modular function on G. Recall that the modular function is the Radon-Nikodym derivative $\frac{d_lx}{d_rx} = \delta_G(x)$, see e.g. [13]. The modular function $\delta_G\colon G \to \mathbb{R}_{>0}$ is a homeomorphism. The group G is unimodular if $\delta_G = 1$, i.e. the left

Haar measure is also right invariant. All compact groups and all connected semisimple Lie groups are unimodular. Define an algebra structure on $L^1(G)$ by convolution: $(f * g)(y) = \int f(x)g(x^{-1}y)d_l x$. Then, one can show that $L^1(G)$ satisfies all the properties of a C*-algebra, except for the equality $\|f^* * f\|_1 = \|f\|_1^2$. Now define the universal norm for $f \in L^1(G)$ as:

$$\|f\|_u = \sup_\pi \|\pi(f)\|,$$

where the supremum runs over all unitary representations of G. One can prove that $\| \, \|_u$ is a norm on $L^1(G)$ and that the completion of $L^1(G)$ in this norm is a C*-algebra. We call it the **(universal) group C*-algebra** of the group G and denote it by $C^*(G)$.

Note that in particular $L^1(G)$ acts on the Hilbert space $L^2(G)$ by convolution. That is, $L^1(G) \ni f \mapsto L_f \in B(L^2(G))$, where $L_f g = f * g \in L^2(G)$, for $g \in L^2(G)$. One defines a norm:

$$\|f\|_r = \|L_f\|.$$

The completion of $L^1(G)$ with respect to this norm is again a C*-algebra and it is called the reduced group C*-algebra of G. We denote it by $C_r^*(G)$.

Let A be a C*-algebra. We call an element $a \in A$ **positive** if $b = a^*a$ for some $a \in A$. We denote the positive elements in A by A^+. We define a relation on A by defining $a \leq b$ if $b - a$ is positive. $\leq$ defines partial order on A^+ (the transitivity and anti-symmetry require some technicalities including the Gelfand-Naimark theorem to be proved).

Exercise 5.5.5. Let A be a C* algebra. Prove that if $a \in A$ is such that $a^*a = 0$, then $a = 0$.

5.6 Von Neumann algebras

For a subset $M \subset B(\mathcal{H})$ we define its commutant:

$$M' = \{T \in B(\mathcal{H}) \mid TS = ST \, \forall S \in M\}.$$

Iteratively $M'' = (M')'$, $M''' = (M'')'$. Obviously, $M \subset M''$ and so $M''' \subset M' \subset M'''$, hence $M' = M'''$. Note that M' is an algebra, and since:

$$\omega_{x,y}(ST) = \langle STx, y\rangle = \langle Tx, S^*y\rangle = \omega_{x,S^*y}(T),$$

and similarly $\omega_{x,y}(TS) = \omega_{Sx,y}(T)$ it follows that M' is closed in the weak (operator) topology. The following characterisation of weakly closed $*$-subalgebras is due to John von Neumann.

Theorem 5.6.1 (von Neumann's bicommutant theorem)**.** *Assume $M \subset B(\mathcal{H})$ is a unital subalgebra and self-adjoint (i.e. $T \in M \;\Rightarrow\; T^* \in M$), then the following are equivalent:*

1. $M = M''$;

2. *M is weakly closed;*
3. *M is strongly closed.*

Definition 5.6.2. *We define:*

1. *A unital self-adjoint subalgebra M of $B(H)$ is a **von Neumann algebra** if $M = M''$.*
2. *A von Neumann algebra M is a **factor** if $M \cap M' = \mathbb{C}1$ in $B(\mathcal{H})$. In a factor, a non-trivial two-sided ideal is σ-weakly dense in M.*
3. *A von Neumann algebra is called **abelian** if $M = M'$.*

Let us give some examples. $B(\mathcal{H})$ is a von Neumann algebra, since it is weakly-closed. If (X, Ω) is some measure space, then one can prove that $L^\infty(X)$, the space of essentially bounded, measurable functions modulo the relation $f \sim g$ if $f(x) - g(x) = 0$ almost everywhere, is an abelian von Neumann algebra. The compact operators do not form a von Neumann algebra, since their weak closure equals $B(\mathcal{H})$. $B(\mathcal{H})$ is a factor, $L^\infty(X)$ is (generally) not. Remark that in particular, every von Neumann algebra is a C*-algebra. It follows directly from von Neumann's double commutant theorem, or from the defining relation $M'' = M$.

Definition 5.6.3. *The space of bounded linear functionals on a von Neumann algebra M which are σ-weakly continuous is denoted M_* (as a subset of the dual M^*), and is called the **predual**. The functionals in M_* are called **normal** functionals.*

M_* is a norm-closed subspace of M^* and therefore, it is a Banach space. Furthermore, one has $M = (M_*)^*$. So the predual of M is a space of which the dual is M. The σ-weak topology on M is the topology induced by M_*. It is actually quite remarkable that every von Neumann algebra is the dual of a Banach algebra. In fact, one can prove that a C*-subalgebra of $B(\mathcal{H})$ is a von Neumann algebra if and only if it is the dual of some Banach space. It is natural to ask what a morphism of von Neumann algebras is.

Definition 5.6.4. *Let M and N be von Neumann algebras. A $*$-homomorphism $\Phi : M \to N$ is called normal, if it is continuous with respect to the σ-weak topology.*

Using the notation of the previous definition, note that if $\omega \in N_*$, then $\omega \circ \Phi$ is a functional on M. This induces a map $\Phi^* : N_* \to M^*$. Φ is normal precisely if the image of Φ^* is contained in M_*. We stretch the importance of normal homomorphism by means of the following theorem.

Theorem 5.6.5. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. Let $M \subseteq B(\mathcal{H}_1)$ be a von Neumann algebra and suppose that $\Phi : M \to B(\mathcal{H}_2)$ is a normal $*$-homomorphism. Then $\Phi(M)$ is a von Neumann algebra.*

One of the nice features of von Neumann algebra is that it has a notion of a supremum. The following theorem is false for C*-algebras. We formulate it with the notion of a net, which is a generalisation of a sequence.

Theorem 5.6.6. *Let M be a von Neumann algebra. Suppose that $a_\lambda \in M^+$ is a bounded increasing net, that is a_λ is a net in M^+ such that for $\lambda_1 \leq \lambda_2$ we have $a_{\lambda_1} \leq a_{\lambda_2}$ and furthermore, there exists some $b \in M^+$ such that for all λ, $a_\lambda \leq b$. Then there exist a (necessarily unique) element $a \in M^+$ such that for all λ, $a_\lambda \leq a$ and if $a' \in M^+$ is such that for all λ, $a_\lambda \leq a'$, then $a' = a$. a is called the supremum, or least upper bound of a_λ. Notation: $a = \sup_\lambda a_\lambda$. Moreover, a_λ converges strongly to a.*

Exercise 5.6.7. Check that for the C*-algebra $C_0(\mathbb{R})$ we have a bounded increasing sequence $(\chi_{[-n,n]})_{n\in\mathbb{N}}$, $\chi_[- n, n]$ the indicator function of the interval $[-n, n]$, which does not have a supremum in the C*-algebra $C_0(\mathbb{R})$. Check that it has a supremum in the enveloping von Neumann algebra $L^\infty(\mathbb{R})$.

In terms of suprema there is another important characterisation of normal functionals. A functional $\omega \in M^*$ is called positive if $\omega(x^*x) \geq 0$ for all $x \in M$.

Theorem 5.6.8. *A positive functional $\omega \in M^*$ is normal if and only if $\sup \omega(a_\lambda) = \omega(\sup a_\lambda)$ for every bounded increasing net $a_\lambda \in M$.*

5.6.1 Tensor products

For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we put an inner product on the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ by defining

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle, \qquad x_1, y_1 \in \mathcal{H}_1, \quad x_2, y_2 \in \mathcal{H}_2$$

and extending bilinearly. Then this defines a positive definite sesquilinear form, and the closure gives the Hilbert space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_2$.

If M_1 (M_2) is a von Neumann algebra acting on $\mathcal{H}_1$ ($\mathcal{H}_2$), then we define the von Neumann algebraic tensor product:

$$M_1 \otimes M_2 = M_1 \bar{\otimes} M_2 = \{T_1 \otimes T_2 \in B(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2) \mid T_1 \in M_1, \ T_2 \in M_2\}''$$

i.e. the von Neumann algebra generated by the algebraic tensor product operators.

Theorem 5.6.9. *For M and N von Neumann algebras we have $(M \bar{\otimes} N)' = M' \bar{\otimes} N'$.*

5.7 Unbounded operators and closed linear maps

We briefly recall some aspects of unbounded operators. In this section $\mathcal{H}$ denotes a Hilbert space, and M denotes a von Neumann algebra acting on $\mathcal{H}$. We start with the definition of unbounded operators on $\mathcal{H}$.

Definition 5.7.1. *Let $Dom(T) \subseteq \mathcal{H}$ be a linear subspace. A linear map $T : \mathcal{D} \to \mathcal{H}$ is called an **unbounded operator** on $\mathcal{H}$ with **domain** $\mathcal{D}$ if $\sup_{v \in \mathcal{D}} \frac{\|Tv\|}{\|v\|} = \infty$. T is called **densely defined** if $\mathcal{D}$ is dense in $\mathcal{H}$.*

The operators that appear in these notes will always be densely defined. We will use the notation $\mathrm{Dom}(T)$ to refer to the domain of an unbounded operator T. We will simply speak about an unbounded operator T if it is clear what the domain of T is.

Example 5.7.2. Let $K(\mathbb{R})$ denote the continuous functions on $\mathbb{R}$ with compact support. The following operator m_x is unbounded, with $\mathrm{Dom}(A) = K(\mathbb{R})$:

$$m_x : K(\mathbb{R}) \to L^2(\mathbb{R}) : f(x) \mapsto xf(x).$$

For unbounded operators, the norm as defined in (5.1.1) does not make sense. However, one can still consider a interesting norm. This norm is defined as:

$$\|v\|_{\mathcal{G}} = \sqrt{\|v\|^2 + \|Tv\|^2}, \qquad v \in \mathrm{Dom}(T).$$

This norm is known as the **graph norm**. Indeed, it corresponds to the norm on $\mathcal{H} \oplus \mathcal{H}$ for an element (v, Tv) on the graph $\mathcal{G}(T) = \{(v, Tv) \mid v \in \mathrm{Dom}(T)\} \subset \mathcal{H} \oplus \mathcal{H}$ of an unbounded operator T.

Definition 5.7.3. *An unbounded operator T is called **closed** if its graph $\mathcal{G}(T)$ is closed with respect $\mathcal{H} \oplus \mathcal{H}$. An operator T is called **preclosed** or **closable** if the closure of $\mathcal{G}(T)$ with respect to the graph norm is the graph of an operator $\overline{T}$. The operator $\overline{T}$ is in this case called the **closure** of T.*

Note that the definition also makes sense for bounded operators. By the closed graph theorem we find that a closed operator whose domain equals $\mathcal{H}$ is necessarily bounded.

Exercise 5.7.4. Show that an operator T is preclosed if and only if for every $v_n \in \mathrm{Dom}(T)$ such that v_n converges in norm to 0 and such that Tv_n converges, Tv_n converges to 0.

Exercise 5.7.5. Show that an operator T is closed if and only if for every $v_n \in \mathrm{Dom}(T)$ such that v_n converges in norm to v and such that Tv_n converges, v is in the domain of T and Tv_n converges to Tv.

Let us give an example.

Example 5.7.6. The operator m_x of Example 5.7.2 is preclosed (check the property given in Exercise 5.7.4), but not closed since one can prove that for example e^{-x^2} is in the closure of m_x.

Note that if T is a preclosed operator, then $\overline{T}$ is an extension of T. That is $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(\overline{T})$ and T and $\overline{T}$ coincide on $\mathrm{Dom}(T)$. We write this as $T \subseteq \overline{T}$. More general, if S and T are two unbounded, not necessarily closable operators, then we write $S \subseteq T$ for $\mathrm{Dom}(S) \subseteq \mathrm{Dom}(T)$ and $\forall v \in \mathrm{Dom}(S) : Sv = Tv$.

Note that if T is a preclosed operator, then the graph of T is dense in the graph of $\overline{T}$, by defintion. The operator $\overline{T}$ is therefore fully determined by its values on $\mathrm{Dom}(T)$. This motivates the following definition for general unbounded operators.

Definition 5.7.7. *Let T be an unbounded closed operator. A subset $\mathcal{D} \subseteq Dom(T)$ is called a **core** for T if $\{(v, Tv) \mid v \in \mathcal{D}\}$ is dense in the graph $\mathcal{G}(T)$ with respect to the norm on $\mathcal{H} \oplus \mathcal{H}$.*

Exercise 5.7.8. In particular, we see that $\mathcal{D} \subseteq \mathrm{Dom}(T)$ is dense with respect to the graph norm $\| \cdot \|_T$. Show that the converse is also valid. That is, if $\mathcal{D} \subseteq \mathrm{Dom}(T)$ is dense with respect to the graph norm $\| \cdot \|_T$ for a closed operator T, that $\mathcal{D}$ is a core for T.

There are analogue notions for maps between locally convex vector spaces.

Definition 5.7.9. *Let E_1, E_2 be vector spaces and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two locally convex topologies on E_1 and E_2 respectively. Let $\mathcal{D} \subseteq E_1$ be a linear subspace and let $\Phi : \mathcal{D} \to E_2$ be a linear map. Then Φ is called $\mathcal{T}_1/\mathcal{T}_2 -$closed if the graph of Φ, denoted by $\mathcal{G}(\Phi)$, is closed with respect to the $\mathcal{T}_1$-topology on E_1 and the $\mathcal{T}_2$-topology on E_2.*

Suppose that Φ is $\mathcal{T}_1/\mathcal{T}_2 -$ closed. A linear subspace $\mathcal{D}_\prime \subseteq \mathcal{D}$ is called a core for Φ with respect to the $\mathcal{T}_1/\mathcal{T}_2$-topology if $\{(x, \Phi(x)) \mid x \in \mathcal{D}_\prime\}$ is dense $\mathcal{G}(\Phi)$ with respect to the $\mathcal{T}_1$-topology on E_1 and the $\mathcal{T}_2$-topology on E_2.

Using the notation of the definition above, note that a sequence $(x_n, \Phi(x_n)) \in \mathcal{G}(\Phi)$ converges to a point $(x, y) \in M_1 \times M_2$ in the $\mathcal{T}_1/\mathcal{T}_2$-topology if and only if $x_n \to x$ in the $\mathcal{T}_1$ topology and $\Phi(x_n) \to y$ in the $\mathcal{T}_2$-topology. Typically, the $\mathcal{T}_1$ and $\mathcal{T}_2$ are one of the locally convex topologies of Section 5.3. We will encounter closed operators in the generalized GNS-constructions that are defined in the next sections. Also, if we want to define an antipode for a von Neumann algebraic quantum group, then the closedness plays an important rôle.

Definition 5.7.10. *Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let T be an unbounded operator on $\mathcal{H}$. T is said to be **affiliated** with M (notation $T \eta M$) if for every unitary operator $U \in M'$, $UT = TU$.*

If T is a bounded operator that is affiliated with a von Neumann algebra M, then $T \in M$. Loosely speaking, the operators that are affiliated with M are the operators that should be elements of M, but cannot be for the fact that they are not bounded.

5.8 Weights on von Neumann algebras

Definition 5.8.1. *A weight ϕ on a von Neumann algebra M is a map $\phi\colon M_+ \to [0, \infty]$ such that $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in M_+$ (with the convention that $a + \infty = \infty$ for $a \in [0, \infty]$) and $\phi(\lambda x) = \lambda \phi(x)$ for $\lambda \geq 0$ and $x \in M_+$ (with the convention that $0 \cdot \infty = 0$).*

If a weight also satisfies the condition $\phi(x^* x) = \phi(x x^*)$, then the weight is called a trace. On an abelian von Neumann algebra, every weight is a trace.

We give some typical examples. If X is a measure space and μ is a positive measure on X, then integrating by means of the measure μ is a weight on $L^\infty(X)$. For $B(\mathcal{H})$, the trace Tr is a weight. Both these weights are tracial. If $T \in B(\mathcal{H})$, then we can define a weight

$\phi(x) = \mathrm{Tr}(T^* x T)$. In general ϕ is not a trace anymore. Actually, ϕ is a trace if and only if T is a multiple of the identity. Also every positive functional on M is a weight.

As a trivial example, the zero-functional is always a trace. This example is of course not very interesting.

Definition 5.8.2. *The weight ϕ is semi-finite if the set $\mathfrak{p}_\phi = \{x \in M^+ \mid \phi(x) < \infty\}$ generates M (i.e. $\mathfrak{p}_\phi'' = M$), faithful if $\phi(x) = 0$, $x \in M_+$ implies $x = 0$, and normal if $\phi(\sup_\lambda x_\lambda) = \sup_\lambda \phi(x_\lambda)$ for any bounded increasing net $\{x_\lambda\}_{\lambda\in\Lambda}$ in M^+. A normal, semi-finite, faithful weight is a n.s.f. weight.*

The normality can be interpreted as a continuity condition. Note that the terminology is consistent with the normality of functionals. That is, a normal positive functional is in particular a normal weight. For our purpose, all the interesting examples of weights are normal, semi-finite and faithful.

Theorem 5.8.3. *Every von Neumann algebra admits a n.s.f. weight. M is a semifinite von Neumann algebra if and only if M admits a n.s.f. trace.*

For the definition of semi-finite von Neumann algebras we refer to the literature, see for example [57] or [19]. The reader who is not familiar with semifiniteness might take the above theorem as a definition for it. Here we merely remark that not every von Neumann algebra is semifinite. The existence of a trace on a von Neumann algebra M makes the study of its structure quite a lot easier, see for example Exercise 6.1.16, which proves that all possible weights on M are in some sense a deformation of a n.s.f. trace.

Note that $\mathfrak{p}_\phi$ is hereditary convex subcone of M_+, (i.e. $\mathfrak{p}_\phi + \mathfrak{p}_\phi \subset \mathfrak{p}_\phi$, $\mathbb{R}_{\geq 0}\mathfrak{p}_\phi \subset \mathfrak{p}_\phi$, and $0 \leq y \leq x$ and $x \in \mathfrak{p}_\phi$ implies $y \in \mathfrak{p}_\phi$). For a hereditray subcone $\mathfrak{p}_\phi$ we have that:

- $\mathfrak{n}_\phi = \{x \in M \mid x^*x \in \mathfrak{p}_\phi\}$ is a left ideal in M
- $\mathfrak{m}_\phi = \{x \in M \mid x = \sum_{i=1}^n z_i^* y_i, y_i, z_i \in \mathfrak{n}_\phi\}$ is a hereditary $*$-subalgebra such that $\mathfrak{n}_\phi \cap M_+ = \mathfrak{p}_\phi$ and any element of $\mathfrak{m}_\phi$ can be written as a finite linear combination of 4 elements of $\mathfrak{p}_\phi$.

We denote the extension of $\phi : \mathfrak{p}_\phi \to [0, \infty)$ to $\phi : \mathfrak{m}_\phi \to \mathbb{C}$ by the same symbol. If φ is a trace, then $\mathfrak{n}_\phi$ is also a right ideal. In that case $\mathfrak{m}_\phi$ is a two-sided ideal. Let $x \in \mathfrak{m}_\phi$ and assume that ϕ is tracial. Then we write $x \cdot \phi$ for the linear map $M \ni y \mapsto \phi(xy) \in \mathbb{C}$.

5.8.1 GNS-construction for weights

This section contains a generalization of the GNS-construction for states on a C*-algebra. Let M be a von Neumann algebra and suppose that it comes with a normal, semi-finite weight ϕ. It is convenient to be aware of the following estimate.

$$\phi(y^*x^*xy) \leq \|x\|^2 \phi(y^*y), \qquad \forall x, y \in M.$$

Put $N_\phi = \{x \in M \mid \phi(x^*x) = 0\}$, then N_ϕ is a left ideal. Consider the space $\mathfrak{n}_\phi/N_\phi$ and the map:

$$\eta_\phi \colon \mathfrak{n}_\phi \to \mathfrak{n}_\phi/N_\phi, \qquad \eta_\phi(x) = x + N_\phi,$$

and put $\langle \eta_\phi(x), \eta_\phi(y)\rangle = \phi(y^*x)$ for $x, y \in \mathfrak{n}_\phi$. This makes $\mathfrak{n}_\phi/N_\phi$ a pre-Hilbert space and denote the completion $\mathcal{H}_\phi$. Note that $\mathfrak{n}_\phi/N_\phi$ is a left M-module (since $\mathfrak{n}_\phi$ and N_ϕ are left ideals), so $a\colon x + N_\phi \mapsto ax + N_\phi$ is a well-defined map $\mathfrak{n}_\phi/N_\phi \to \mathfrak{n}_\phi/N_\phi$. Since $\phi((ax)^*ax) \leq \|a\|^2\phi(x^*x)$ this map extends to a bounded map $\pi_\phi(a)\colon \mathcal{H}_\phi \to \mathcal{H}_\phi$, and $\pi_\phi(a)\eta_\phi(x) = \eta_\phi(ax)$ for $a \in M$, $x \in \mathfrak{n}_\phi$.

Proposition 5.8.4. *For a semi-finite normal weight ϕ the representation $\pi_\phi\colon M \to B(\mathcal{H}_\phi)$ is a normal non-degenerate $*$-representation. If ϕ is a n.s.f. weight, then the representation is also faithful.*

This construction is known as the **cyclic GNS-construction**, or many authors simply use the terminology **GNS-construction** so that the construction for states can be viewed as a special case. Also, one refers to the triple $(\mathcal{H}_\phi, \pi_\phi, \eta_\phi)$ as the GNS-construction with respect to ϕ.

In particular, if ϕ is n.s.f., then M is isomorphic to $\pi_\phi(M)$, which itself is a von Neumann algebra by Theorem 5.6.5. It is not true (in general) that the commutants M' and $\pi_\phi(M)'$ are isomorphic. We end this section with an exercise that shows this. It also prepares the reader for the Tomita-Takesaki theorem.

Exercise 5.8.5. Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ called cyclic for M if $M\Omega$ is dense in $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is called separating for M if $A\Omega = 0$ implies that $A = 0$ for all $A \in M$.

Suppose that M admits a vector Ω that is both cyclic and separating. Suppose that there exists a state ω on M, such that the inner product on $\mathcal{H}$ is given by $\langle A\Omega, B\Omega\rangle = \omega(B^*A)$ (note that this is well-defined!). Let $J : M\Omega \mapsto M\Omega$ be the anti-linear map defined by $J : A\Omega \to A^*\Omega$.

1. Show that J is well-defined and its domain and range are dense in $\mathcal{H}$.

2. Prove that J is an isometry. That is, J satisfies the property $\langle v, w\rangle = \langle Jw, Jv\rangle$ for every $v, w \in \mathcal{H}$. Conclude that J extends to an anti-linear, surjective isometry $J : \mathcal{H} \to \mathcal{H}$ that satisfies $J^2 = 1$.

3. Show that for every $A \in M$, we have $JAJ \in M'$.

4. Prove that $M' = JMJ$.

Consider the von Neumann algebra $M_n(\mathbb{C})$ acting in the usual way on $\mathbb{C}^n$ and let Tr be the trace. Let $(\mathcal{H}, \pi, \eta)$ be its GNS-construction.

5. Show that $M_n(\mathbb{C})'$ is not isomorphic to $\pi(M_n(\mathbb{C}))'$ and there cannot exist an anti-linear isometry J, such that $JM_n(\mathbb{C})J = M_n(\mathbb{C})'$.

6. Show that $M_n(\mathbb{C})$ does not admit a separating vector. Are there cyclic vectors?

7. Show that $\pi(M_n(\mathbb{C}))$ admits a vector Ω that is both cyclic and separating.

8. Let $A \in M_n(\mathbb{C})$. Give an explicit description of the operator $J\pi(A)J$. Is this depending on the choice of Ω?

5.9 Tomita-Takesaki theory

Let M be a von Neumann algebra. The exercise in the last section shows that under certain circumstances, there exist an anti-linear operator J such that $JMJ = M'$. The exercise has a far reaching generalization, which is known as te Tomita-Takesaki theorem. The Tomita-Takesaki theorem is formulated in terms of Hilbert algebras.

5.9.1 Hilbert algebras

Definition 5.9.1. *A left Hilbert algebra $\mathfrak{U}$ is an algebra over $\mathbb{C}$ with involution $\mathfrak{U} \ni \xi \mapsto \xi^\sharp \in \mathfrak{U}$ equipped with an inner product structure $\langle \cdot, \cdot \rangle : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ such that*

- *for $\xi \in \mathfrak{U}$ the left multiplication $\pi_l(\xi) : \mathfrak{U} \to \mathfrak{U}$, $\pi_l(\xi)\eta = \xi\eta$ is bounded (i.e. there exists $M > 0$ such that $\langle \pi_l(\xi)\eta, \pi_l(\xi)\eta \rangle \leq M \langle \eta, \eta \rangle$ for all $\eta \in \mathfrak{U}$)*
- *the map $\mathfrak{U} \ni \xi \mapsto \xi^\sharp \in \mathfrak{U}$ is preclosed (i.e. if $\xi_n \to 0$ and $\xi_n^\sharp \to \eta$ then $\eta = 0$)*
- *$\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\sharp \eta \rangle$ for all $\xi, \eta, \zeta \in \mathfrak{U}$*
- *$\mathfrak{U}^2 = \{\xi\eta \mid \xi, \eta \in \mathfrak{U}\}$ is dense in $\mathfrak{U}$ (i.e. with respect to the topology generated by the inner product).*

Similarly a right Hilbert algebra is defined.

Definition 5.9.2. *A right Hilbert algebra $\mathfrak{U}$ is an algebra over $\mathbb{C}$ with involution $\mathfrak{U} \ni \xi \mapsto \xi^\flat \in \mathfrak{U}$ equipped with an inner product structure $\langle \cdot, \cdot \rangle : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ such that*

- *for $\xi \in U$ the left multiplication $\pi_r(\xi) : \mathfrak{U} \to \mathfrak{U}$, $\pi_r(\xi)\eta = \eta\xi$ is bounded (i.e. there exists $M > 0$ such that $\langle \pi_r(\xi)\eta, \pi_r(\xi)\eta \rangle \leq M \langle \eta, \eta \rangle$ for all $\eta \in \mathfrak{U}$)*
- *the map $\mathfrak{U} \ni \xi \mapsto \xi^\flat \in \mathfrak{U}$ is preclosed (i.e. if $\xi_n \to 0$ and $\xi_n^\sharp \to \eta$ then $\eta = 0$)*
- *$\langle \eta\xi, \zeta \rangle = \langle \eta, \eta\xi^\flat \rangle$ for all $\xi, \eta, \zeta \in \mathfrak{U}$*
- *$\mathfrak{U}^2 = \{\xi\eta \mid \xi, \eta \in \mathfrak{U}\}$ is dense in $\mathfrak{U}$ (i.e. with respect to the topology generated by the inner product).*

For a left Hilbert algebra $\mathfrak{U}$ we denote by $\mathcal{H}$ the completion of $\mathfrak{U}$, and then $\pi_l\colon \mathfrak{U} \to B(\mathcal{H})$ extends to a non-degenerate $*$-representation of $\mathfrak{U}$ in $\mathcal{H}$. The left von Neumann algebra is $R_l(\mathfrak{U}) = \pi_l(\mathfrak{U})''$. Similarly, in case of a right Hilbert space $\pi_r\colon \mathfrak{U} \to B(\mathcal{H})$ extends to a non-degenerate antimultiplicative $*$-representation, and the right von Neumann algebra $R_r(\mathfrak{U}) = \pi_r(\mathfrak{U})''$.

Now fix a left Hilbert algebra $\mathfrak{U}$, its extension and put $S_0\colon \mathfrak{U} \to \mathfrak{U}$, $\xi \mapsto \xi^\sharp$, and let $(S, D^\sharp)$ be its closed extension. Equip the domain $D^\sharp$ with the inner product $\langle \xi, \eta\rangle_\sharp = \langle \xi, \eta\rangle + \langle S\eta, S\xi\rangle$ which makes $D^\sharp$ a Hilbert space. Note the interchange in the second term because of the antilinearity of S.

Lemma 5.9.3. *1. $S = S^{-1}$*

2. There exists a densely defined closed linear operator F with domain $D^\flat = \{\eta \in H \mid D^\sharp \ni \xi \mapsto \langle \eta, S\xi\rangle$ is continuous$\}$ such that $\langle S\xi, \eta\rangle = \langle F\eta, \xi\rangle$ for all $\xi \in D^\sharp$, $\eta \in D^\flat$ (so F is the "adjoint of S") and $F = F^{-1}$

3. $\Delta = FS$ is linear, positive (in particular self-adjoint), non-singular operator on $\mathcal{H}$ and $D(\Delta^{\frac{1}{2}}) = D^\sharp$

4. There exists a antilinear isometry $J\colon \mathcal{H} \to \mathcal{H}$ such that $J^2 = I$ and

(a) $J\Delta J = \Delta^{-1}$

(b) $S = J\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}J$ (in particular $J(D^\sharp) = D(\Delta^{-\frac{1}{2}})$). Moreover, this property with $D(\Delta^{\frac{1}{2}}) = D^\sharp$ determines J and Δ.

(c) $F = J\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}J$

Definition 5.9.4. *Δ is the modular operator and J the modular conjugation.*

Starting with a left Hilbert algebra $\mathfrak{U}$, we say $\eta \in \mathcal{H}$ is right bounded if $\sup_{\xi\in\mathfrak{U}, \|\xi\|\le 1} \|\pi_l(\xi)\eta\| \le M < \infty$. In that case there is an operator T, depending on η such that $T\xi = \pi_l(\xi)\eta$ for all $\xi \in \mathcal{H}$. Put $\pi_r(\eta) = T$. Let B' be the space of right bounded vectors, and then $\mathfrak{n}_r = \pi_r(B')$ is a left ideal in $R_l(\mathfrak{U})'$. Define $\mathfrak{U}' = B' \cap D^\flat$, then $\mathfrak{U}'$ is a right Hilbert algebra and $R_l(\mathfrak{U})' = R_r(\mathfrak{U}')$ and $\pi_r(\mathfrak{U}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*$. Similarly, one can go from the right Hilbert algebra $\mathfrak{U}'$ to a left Hilbert algebra $\mathfrak{U}''$. The left Hilbert algebra is full if $\mathfrak{U} = \mathfrak{U}''$.

Theorem 5.9.5 (Tomita, Takesaki)**.** *Let $\mathfrak{U}$ be a left Hilbert algebra with modular operator Δ and modular conjugation J, then*

$$\Delta^{it} R_l(\mathfrak{U})\Delta^{-it} = R_l(\mathfrak{U}), \qquad \Delta^{it} R_l(\mathfrak{U})\Delta^{-it} = R_l(\mathfrak{U}), \quad t \in \mathbb{R},$$
$$J R_l(\mathfrak{U}) J = R_l(\mathfrak{U})', \qquad J R_l(\mathfrak{U})' J = R_l(\mathfrak{U})$$

The 1-parameter unitary group Δ^{it}, $t \in \mathbb{R}$, acts on $\mathfrak{U}''(= \mathfrak{U}$ if $\mathfrak{U}$ is full) and on $\mathfrak{U}'$ as automorphisms. J maps $\mathfrak{U}''$ onto $\mathfrak{U}'$ anti-isomorphically (i.e. $J(\xi\eta) = J\eta J\xi$.) Moreover, for a central element $a \in R_l(\mathfrak{U})$ we have $JaJ = a^*$ and $\Delta^{it} a \Delta^{-it} = a$, $t \in \mathbb{R}$.

The main application and motivation for this theorems is given in the next paragraph.

5.9.2 Weights and full left Hilbert algebras

Let $\mathfrak{U}$ be a full left Hilbert algebra and $M = R_l(\mathfrak{U})$ the left von Neumann algebra acting on the Hilbert space $\mathcal{H}$. Define $\phi_l \colon M^+ \to [0, \infty]$ such that

$$\phi_l(x) = \begin{cases} \|\xi\|^2, & \text{if } x^{\frac{1}{2}} = \pi_l(\xi), \\ \infty & \text{otherwise} \end{cases}$$

Theorem 5.9.6. *In this setting ϕ_l is a n.s.f. weight on M. The action of M on $\mathcal{H}$ is unitarily equivalent with the action of M on the GNS-space $\mathcal{H}_{\phi_l}$ with the unitary given by $\xi \mapsto \eta_{\phi_l}(\pi_l(\xi))$.*

Theorem 5.9.7. *For a nsf weight ϕ define $U_\phi = \eta_\phi(\mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*)$ and define*

$$\begin{aligned} \eta_\phi(x)\eta_\phi(y) &= \eta_\phi(xy), \qquad x, y \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^* \\ \eta_\phi(x)^\sharp &= \eta_\phi(x^*), \qquad x \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^* \end{aligned}$$

then U_ϕ is a full left Hilbert algebra such that $\pi_\phi(M) = R_l(U)$ and ϕ coincides with the ϕ_l of Theorem 5.9.6.

In particular, the polar decomposition of $\eta_\phi(x) \mapsto \eta_\phi(x)^\sharp$ in $\mathcal{H}_\phi \to \mathcal{H}_\phi$ defines a modular operator Δ, as an unbounded, strictly positive operator on $\mathcal{H}_\phi$ and a modular conjugation $J \colon \mathcal{H}_\phi \to \mathcal{H}_\phi$ (anti-unitary involution) for the n.s.f. weight ϕ on M. Then

$$J\pi_\phi(M)J = \pi_\phi(M)', \qquad \Delta^{it}\pi_\phi(M)\Delta^{-it} = \pi_\phi(M), \quad t \in \mathbb{R}.$$

The modular operator is closely related to the modular automorphism group of the n.s.f. weight ϕ. Let M be von Neumann algebra and let $\gamma : \mathbb{R} \to M$. We say that γ is **strongly continuous** if this map is continuous with respect to the strong topology on M.

Theorem 5.9.8. *Let ϕ be a n.s.f. weight on the von Neumann algebra M with GNS-construction $(\mathcal{H}_\phi, \pi_\phi, \eta_\phi)$. There exists a unique strongly continuous one-parameter group $\mathbb{R} \ni t \mapsto \sigma_t \in \mathrm{Aut}(M)$ of $*$-automorphisms of M, so that (i) $\phi \circ \sigma_t = \phi$ and (ii) for all $x, y \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ there exists a bounded continuous complex-valued function $f \colon \{z \in \mathbb{C} \mid 0 \leq \Im z \leq 1\} \to \mathbb{C}$ such that f is analytic on the interior $\{z \in \mathbb{C} \mid 0 < \Im z < 1\}$ of this strip and*

$$f(t) = \phi\big(\sigma_t(x)y\big) \quad \text{and} \quad f(t+i) = \phi\big(x\sigma_t(y)\big), \quad \forall\, t \in \mathbb{R}.$$

Moreover, in the GNS-construction we have for $a \in \mathfrak{n}_\phi$, $b \in \mathfrak{n}_\phi \cap D(\sigma_{i/2})$,

$$\Delta^{it}\eta_\phi(a) = \eta_\phi(\sigma_t(a)), \quad J\eta_\phi(b) = \eta_\phi(\sigma_{i/2}(b)^*)$$

The function f depends on $x, y \in \mathfrak{n}_\phi \cap \mathfrak{n}_\phi^*$ and is the two-point function for x, y. This condition is known as the KMS-property (for Kubo-Martin-Schwinger). The idea is to define σ_t using the faithful representation $\pi \colon M \to B(\mathcal{H}_\phi)$, so $\pi_\phi(\sigma_t(x)) = \Delta^{it}\pi_\phi(x)\Delta^{-it}$ and show that it satisfies the required properties.

The definition of σ_w for $w \in \mathbb{C}$ is the following; $D(\sigma_w)$ is the space of all $x \in M$ such that the function $\mathbb{R} \ni t \mapsto \sigma_t(x) \in M$ admits an analytic extension to a bounded continuous function $f\colon \{z \in \mathbb{C} \mid 0 \leq z \leq \Im w\} \to M$ (assuming $\Im w > 0$) such that f is analytic on the interior with analyticity and continuity meaning that $\mathbb{R} \ni t \mapsto \omega\big(\sigma_t(x)\big) \in \mathbb{C}$ for all $\omega \in M_*$, i.e. analyticity and continuity with respect to the σ-weak topology on M. Then for $x \in D(\sigma_w)$ we define $\sigma_w(x)$ as the value of this M-valued function at $z = w$. The domain $D(\sigma_w)$ always contains

$$M_a = \{x \in M \colon \mathbb{R} \ni t \mapsto \sigma_t(x) \in M \text{ admits an entire extension}\}$$

which is a σ-weakly dense $*$-subalgebra of M.

Chapter 6

Von Neumann algebraic quantum groups

The theory of locally compact quantum groups was developped by Kustermans and Vaes and dates from around 2000. Until then, the approaches to quantum groups had been either algebraically or C*-algebraically. The Kustermans-Vaes definition is in the context of von Neumann algebras and one of their main motivation was to generalize the Pontrjagin duality theorem for locally compact abelian groups. Also, many other results from abstract harmonic analysis have a quantum group analogue as we will see.

Here we state the definition of a locally compact quantum group and describe their main features, including duality and Plancherel measure. We will relate the definition to the theory of locally compact groups and refer to this example as the 'classical' situation. It is good to mention that the von Neumann algebraic approach heavily uses results of operator algebras and many proofs are rather long and technically demanding. We omit most of the technical proofs and refer to the original sources instead, see the references in the surveys [41], [58].

6.1 The Kustermans-Vaes definition

Let us start with the classical example of a locally compact group G. Then we set $M = L^\infty(G)$, the abelian von Neumann algebra of measurable, essentially bounded functions on G (with respect to the left Haar measure). Denote by $m : G \times G \to G : (g,h) \mapsto gh$ the product on G and we consequently write $\int_G d_l g$ and $\int_G d_r g$ for the left and right Haar integrals. We may identify $L^\infty(G) \otimes L^\infty(G)$ with $L^\infty(G \times G)$, the identification is made by $f \otimes g \mapsto ((x,y) \mapsto f(x)g(y))$. We will use this identification implicitly several times. We can define a *coproduct* Δ_G on M by setting:

$$\Delta_G : M \to M \otimes M : f \mapsto f \circ m. \tag{6.1.1}$$

So that a function $g \mapsto x(g)$ is sent to $(g,h) \mapsto x(gh)$. One can easily check that precomposing with m preserves suprema and therefore Δ_G is a normal homomorphism. Furthermore, we define two weights φ and ψ on M by integrating against the left Haar measure, respectively the right Haar measure. That is,

$$\varphi(f) = \int_G x(g) d_l g, \qquad \psi(f) = \int_G x(g) d_r g.$$

These weights are normal, semi-finite and faithful; this follows from well-known results from the theory of locally compact groups see for instance [13]. Kustermans and Vaes turned the pair (M, Δ_G) together with the weights φ and ψ into a defintion.

Definition 6.1.1. *Consider a von Neumann algebra M together with a unital normal $*$-homomorphism $\Delta\colon M \to M \otimes M$ (the comultiplication) such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (coassociativity). Moreover, if there exist two normal semi-finite faithful weights φ, ψ on M such that*

$$\begin{aligned} \varphi\big((\omega \otimes \iota)\Delta(x)\big) &= \varphi(x)\omega(1), \qquad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\varphi^+ \qquad \textit{(left invariance)}, \\ \psi\big((\iota \otimes \omega)\Delta(x)\big) &= \psi(x)\omega(1), \qquad \forall\, \omega \in M_*^+, \forall\, x \in \mathfrak{m}_\psi^+ \qquad \textit{(right invariance)}, \end{aligned}$$

*then (M, Δ) is a **von Neumann algebraic locally compact quantum group**, or simply locally compact quantum group.*

We omit the Haar weights in the notation and simply write (M, Δ) for a locally compact quantum group. Note that the definition entails that $(\omega \otimes \iota)\Delta(x) \in \mathfrak{m}_\varphi^+$ for all $x \in \mathfrak{m}_\varphi^+$ and for all positive normal functionals ω. We will abbreviate the term locally compact quantum group as l.c. quantum group.

Exercise 6.1.2. Verify the left invariance of φ for $(L^\infty(G), \Delta_G)$. Verify also the right invariance of ψ in this example.

Example 6.1.3. The previous exercise gives us our first example $(L^\infty(G), \Delta_G)$ of a locally compact quantum group.

Definition 6.1.4. *A l.c. quantum group (M, Δ) is called compact if the Haar measures φ and ψ are states, that is, if $\varphi(1) = \psi(1) = 1$. In that case we simply speak about a **compact quantum group**.*
*A l.c. quantum group (M, Δ) is called **unimodular** if $\varphi = \psi$.*

One immediately sees that $(L^\infty(G), \Delta_G)$ is compact if and only if G is a compact group, and that $(L^\infty(G), \Delta_G)$ is unimodular if and only if G is unimodular.

Since φ is a n.s.f. weight, we can use its GNS-construction, denoted by $(\mathcal{H}, \iota, \Lambda)$. In particular, we may assume that M is acting on $B(\mathcal{H})$, so $\iota\colon M \to B(\mathcal{H})$ is assumed to be the identity. From general results we can assume that the GNS representation for the right Haar weight ψ takes place in the same Hilbert space [57, Theorem VII.3.2], so $(\mathcal{H}, \iota, \Gamma)$ is the GNS-construction for the right invariant weight ψ.

The following theorem is fundamental for the duality theorems as they appear in the next section.

Theorem 6.1.5. *There exists a unique unitary element $W \in B(\mathcal{H} \otimes \mathcal{H})$ (known as the* ***multiplicative unitary****) such that:*

$$W^*\big(\Lambda(x) \otimes \Lambda(y)\big) = (\Lambda \otimes \Lambda)\big(\Delta(y)(x \otimes 1)\big), \qquad \forall\, x, y \in \mathfrak{n}_\varphi.$$

Then $\Delta(x) = W^*(1 \otimes x)W$, $x \in M$, $W \in M \otimes B(\mathcal{H})$ and W satisfies the ***pentagon equation*** *$W_{12}W_{13}W_{23} = W_{23}W_{12}$.*

Here we used the *leg-numbering notation*, i.e. $W_{12} = W \otimes 1$, $W_{13} = (1 \otimes \Sigma)(W \otimes 1)(1 \otimes \Sigma)$, $W_{23} = 1 \otimes W$, where $\Sigma \colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} : \xi \otimes \eta \mapsto \eta \otimes \xi$ is the *flip* operator. Both the left and right Haar weight are required for the construction of the multiplicative unitary W.

Remark 6.1.6. Pretty much of the structure of a l.c. quantum group (M, Δ) can be recovered from W. For example, the von Neumann algebra M can be recovered from W as we shall see later. Also the above theorem shows that the coproduct is fully determined by W. It is therefore no surprise that earlier approaches to quantum groups have been made by studying unitary operators W acting on a Hilbert space $\mathcal{H} \otimes \mathcal{H}$ that satisfies the pentagon equation and some additional axioms, see for instance [2] and [63].

Remark 6.1.7. To see how groups are included in this definition, set $(M, \Delta) = (L^\infty(G), \Delta_G)$. Then the von Neumann algebra M is acting by multiplication operators on the Hilbert space $\mathcal{H} = L^2(G)$, where the L^2-norm is defined with respect to the left Haar measure $d_l g$. So we consider M as a subalgebra of $B(L^2(G))$. Then $\varphi(x) = \int_G x(g)\, d_l g$ for $x \in L^\infty(G) \cap L^1(G) = \mathfrak{m}_\varphi$, and the corresponding GNS-construction of φ is $(L^2(G), \iota, \Lambda)$ where $\Lambda \colon \mathfrak{n}_\varphi = L^2(G) \cap L^\infty(G) \to L^2(G)$, $x \mapsto x$. In this case the predual is $M_* = L^1(G) \subset M^*$ by considering $L^1 \ni f \mapsto (L^\infty(G) \ni x \mapsto \int_G f(g)x(g)\, d_l g)$. For $f, h \in L^2(G)$ a normal functional $\omega_{f,h}$ is defined as the matrix element $\omega_{f,h}(x) = \langle xf, h\rangle = \int_G x(g)f(g)\overline{h(g)} d_l g$. In this case the multiplicative unitary W is

$$\begin{aligned} &W \colon L^2(G) \otimes L^2(G) \cong L^2(G \times G) \to L^2(G) \otimes L^2(G) \cong L^2(G \times G) \\ &\big(W^* x\big)(g, h) = x(g, gh), \qquad \big(Wx\big)(g, h) = x(g, g^{-1}h), \end{aligned}$$

and it is a straightforward check that the formula for the coproduct in Theorem 6.1.5 holds. In this case W is known as the structure operator, see [57, Section VII.3]. In Figure 6.1.7 we see that the pentagon equation for W^* is related to the associativity

Particular to the group case is that the *antipode* defined by $S \colon M \to M$, $(Sx)(g) = x(g^{-1})$ is bounded, but in the general case it is not. To indicate how the antipode can be obtained from the invariant weight in the general case note that for $x, y \in L^\infty(G) \cap L^2(G) = \mathfrak{n}_\varphi$:

$$\int_G x(h^{-1}g)\, y(g)\, d_l g = \int_G x(g)\, y(hg)\, d_l g$$

so that

$$S \colon (\iota \otimes \varphi)\big(\Delta(x)(1 \otimes y)\big) \mapsto (\iota \otimes \varphi)\big((1 \otimes x)\Delta(y)\big).$$

Compare to the maps in Theorem 2.2.1.

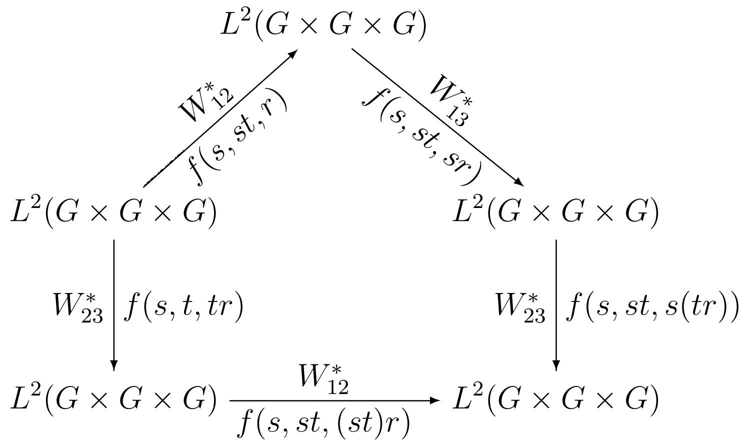


Figure 6.1: Pentagon equation for the group case: start with a function $f(s,t,r)$ in the left upper corner.

This motivates the following definition of the antipode.

Theorem 6.1.8. *Define*

$$S\Big((\iota\otimes\varphi)\big(\Delta(b^*)(1\otimes a)\big)\Big) = (\iota\otimes\varphi)\big((1\otimes b^*)\Delta(a)\big)$$

for $a, b \in \mathfrak{n}_\varphi$. Then S extends uniquely to a σ-strongly-$*$ closed linear operator with the linear span

$$\mathrm{Span}\Big\{(\iota\otimes\varphi)\big(\Delta(b^*)(1\otimes a)\big) \mid a, b \in \mathfrak{n}_\phi\Big\}$$

as a core (for the σ-strong-$*$ topology). The closure $S\colon D(S) \subset M \to M$ is the antipode for (M, Δ). Domain and range of S are dense in M for the σ-strong-$*$ topology.

Moreover, the antipode S satisfies the following properties:

- *$D(S)$ is a subalgebra of M and $S(xy) = S(y)S(x)$, for all $x, y \in D(S)$;*
- *$x \in D(S)$ implies $S(x)^* \in D(S)$ and $S(S(x)^*)^* = x$ for all $x \in D(S)$, so in particular $D(S^{-1}) = D(S)^*$ and $S^{-1}(x^*) = S(x)^*$ for $x \in D(S)$.*

Remark 6.1.9. In general a l.c. quantum group with its antipode S is not a Hopf-algebra, since S can be unbounded. Also, a counit is not included in its definition.

The left and right Haar weight and the antipode are closely related. The essential idea is the following decomposition theorem for the antipode.

Theorem 6.1.10. *There exists a unique $*$-anti-automorphism $R\colon M \to M$ and a unique strongly continuous one-parameter group of $*$-automorphisms $\tau\colon \mathbb{R} \to \mathrm{Aut}(M)$ such that*

$$S = R\circ\tau_{-i/2}, \quad R^2 = \mathrm{Id}, \quad \tau_t\circ R = R\circ\tau_t,\ \forall\, t\in\mathbb{R}.$$

R is called the unitary antipode and τ the scaling group. Moreover, $\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$, $\Delta \circ R = \chi \circ (R \otimes R) \circ \Delta$, where $\chi\colon M \otimes M \to M \otimes M$ is the flip $*$-automorphism.

An explicit formula for the unitary antipode R is given by:

$$R\big((\iota \otimes \omega_{v,w})(W)\big) = (\iota \otimes \omega_{Jw,Jv})(W).$$

For $(L^\infty(G), \Delta_G)$, the antipode is a $*$-anti-automorphism so that $R = S$ and the scaling group is trivial, $\tau_t = \iota$. In that case the algebraic properties of S, R and τ_t appearing in the previous theorem are easily checked.

Corollary 6.1.11. *We can assume $\psi = \varphi \circ R$, then the modular automorphism group σ'_t of ψ satisfies $\sigma'_t = R \circ \sigma_{-t} \circ R$ for all $t \in \mathbb{R}$. The three one-parameter groups mutually commute, and the relation with comultiplication is given by*

$$\Delta \circ \sigma_t = (\tau_t \otimes \sigma_t) \circ \Delta, \qquad \Delta \circ \sigma'_t = (\sigma'_t \otimes \tau_{-t}) \circ \Delta, \qquad \Delta \circ \tau_t = (\sigma_t \otimes \sigma'_{-t}) \circ \Delta.$$

As a result of this corollary, one might wonder if the existence of the right Haar weight has to be postulated in Defintion 6.1.1. However, the construction of the antipode as well as the other statements made in Theorem 6.1.5 rely on the fact that there exists a right Haar measure.

We end this section by showing that there is another important relation between φ and ψ.

Theorem 6.1.12. *There exists a unique number $\nu > 0$, the scaling constant, such that for all $t \in \mathbb{R}$ and all $x \in M^+$ we have*

$$\varphi\big(\tau_t(x)\big) = \nu^{-t}\,\varphi(x), \quad \psi\big(\tau_t(x)\big) = \nu^{-t}\,\psi(x), \quad \varphi\big(\sigma'_t(x)\big) = \nu^{t}\,\varphi(x), \quad \psi\big(\sigma_t(x)\big) = \nu^{-t}\,\psi(x).$$

Since for $(L^\infty(G), \Delta_G)$, we know that τ_t is trival, we find $\nu = 1$. In the group case, the left and right Haar measure are equivalent. That is, the left Haar measure is absolutely continuous with respect to the right Haar measure and vice versa. Therefore the Radon-Nikodym theorem says that there is a measurable function on G, say δ such that $d_r g = \delta(g) d_l g$. Note that δ corresponds to multiplication by δ_G^{-1}, where δ_G is the modular function for the group G. The Radon-Nikodym theorem has a generalization to weights on von Neumann algebras, due to Pedersen and Takesaki [51], which was later even further generalized by Vaes for applications to l.c. quantum groups [59]. We introduce the main ingredients for the next theorem.

Suppose that δ is a self-adjoint, strictly positive operator affiliated with the von Neumann algebra M. Define:

$$\mathfrak{n}_0 = \left\{x \in M \mid x\delta \text{ is bounded, and } \overline{x\delta} \in \mathfrak{n}_\varphi\right\},$$

where the $\overline{x\delta}$ is the closure of $x\delta$. Note that $\mathfrak{n}_0$ is a left ideal. Then we consider the map:

$$\Lambda_0 : \mathfrak{n}_0 \to H : x \mapsto \Lambda(\overline{x\delta}).$$

Vaes proves that $\Lambda_0(\mathfrak{n}_0^* \cap \mathfrak{n}_0^*)$ is a left Hilbert-algebra and the von Neumann algebra it generates is isomorphic to M. Now define φ_δ to be the weight associated to the Hilbert algebra $\Lambda_0(\mathfrak{n}_0 \cap \mathfrak{n}_0^*)''$ as is defined in Section 5.9.2.

Formally one has $\varphi_\delta(x) = \varphi(\delta^{\frac{1}{2}} x \delta^{\frac{1}{2}})$, however this expression has to be interpreted with much more care, see [59, Proposition 3.3]. Some authors call δ the Radon-Nikodym derivative (of φ_δ with respect to φ). We remark that this construction actually works for an arbitrary von Neumann algebra M together with a n.s.f. weight φ and operator δ affiliated to M. Now we apply the construction of φ_δ to a l.c. quantum group (M, Δ); by [59, Proposition 5.5], the previous theorem implies the following relation between the left and right Haar weight.

Theorem 6.1.13. *There exists a unique injective positive operator δ on $\mathcal{H}$, the modular element, such that δ is affiliated to M, and (i) $\sigma_t(\delta) = \nu^t \delta$ for all $t \in \mathbb{R}$, (ii) $\psi = \varphi_\delta$ (i.e. formally $\psi(x) = \varphi(\delta^{1/2} x \delta^{1/2})$).*

In the classical case, as the discussion above shows, we find that δ is given by the Radon-Nikodym derivative of the left Haar weight relative to the right Haar weight.

Example 6.1.14. For $SU_q(2)$, which as a von Neumann algebra equals $B(l^2(\mathbb{N})) \otimes L^\infty(\mathbb{T})$, we see that $\varphi = (\mathrm{Tr} \otimes \mathrm{Tr})_{\gamma^*\gamma}$, by the expression in Exercise 4.4.4.

Exercise 6.1.15. Prove that if (M, Δ) is a unimodular locally compact quantum group, then $\delta = 1$ and $\nu = 1$.

Exercise 6.1.16. Let M be a von Neumann algebra that admits a n.s.f. trace τ. Show that every n.s.f. weight φ on M is of the form τ_δ for some strictly positive operator δ affiliated with M.

Exercise 6.1.17. Consider the von Neumann algebra of $n \times n$-matrices $M_n(\mathbb{C})$. Let Tr be the usual trace. Let $A, B \in M_n(\mathbb{C})^+$ be invertible, and set $\varphi(x) = \mathrm{Tr}(Ax)$, $\psi(x) = \mathrm{Tr}(Bx)$.

1. Show that φ and ψ are n.s.f. weights (if one considers their domains to be restricted to $M_n(\mathbb{C})^+$).
2. Show that $\sigma_t^\varphi(x) = A^{it} x A^{-it}$ and similarly $\sigma_t^\psi(x) = B^{it} x B^{-it}$.
3. Remark that $\mathrm{Tr}(\sigma_t^\varphi(x)) = \mathrm{Tr}(x)$ and show that $\varphi = \mathrm{Tr}_A$, where Tr_A is given by the definition above.
4. Show that there is an invertible operator $C \in M_n(\mathbb{C})^+$, such that $\varphi_C = \psi$ if and only if A and B commute. Express C in terms of A and B.

6.2 Duality

This section deals with the main motivation for the Kustermans-Vaes definition, namely a generalized version of the Pontrjagin duality theorem. To every l.c. compact quantum group, one can associate a quantum group $(\hat{M}, \hat{\Delta})$ called the Pontrjagin dual of (M, Δ). We show how it is defined and we state the generalized duality theorem as was proved by Kustermans and Vaes. Finally, we show how the classical situation fits into their duality theorem.

Fix a l.c. quantum group (M, Δ). As a von Neumann algebra, the dual locally compact quantum group is defined as:

$$\hat{M} = \overline{\{(\omega \otimes \iota)(W) \mid \omega \in B(\mathcal{H})_*\}} \subset B(\mathcal{H}), \tag{6.2.1}$$

where the closure is with respect to the σ-strong-$*$ topology and $\mathcal{H}$ is the GNS-space for the left invariant weight φ. The point is that $\hat{M}$ can be equipped with the structure of a l.c. quantum group.

Theorem 6.2.1 ([43])**.** *$\hat{M}$ is a von Neumann algebra acting on $\mathcal{H}$, and there exists a unique normal injective $*$-homomorphism $\hat{\Delta}\colon \hat{M} \to \hat{M} \otimes \hat{M}$, $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma$ for $x \in \hat{M}$. Moreover, $(\hat{M}, \hat{\Delta})$ is a locally compact quantum group; the* ***Pontrjagin dual*** *of (M, Δ) or the dual locally compact quantum group.*

Here $\Sigma\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ denotes again the flip operator $\Sigma\colon a \otimes b \mapsto b \otimes a$. Also, it is convenient to introduce the following map:

$$\lambda : M_* \to \hat{M} : \omega \mapsto (\omega \otimes \iota)(W).$$

In order to let this map make sense, notice that we must have $M_* \subseteq B(\mathcal{H})_*$. This is indeed true, since the restriction of a normal functional $\omega \in B(\mathcal{H})_*$ to M defines a normal functional on M which again is normal since the inclusion $M \subseteq B(\mathcal{H})$ is normal. One can prove that every element in M_* is such a restriction.

Remark 6.2.2. Theorem 6.2.1 implicitly says that $(\hat{M}, \hat{\Delta})$ comes with a dual left and right Haar weight $\hat{\varphi}$ and $\hat{\psi}$. Let us show how these are constructed. First, define

$$\mathcal{I} = \{\omega \in M_* \mid \Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi \text{ is a continuous funtional on } \mathcal{H}\}.$$

The Riesz theorem for Hilbert spaces implies that for $\omega \in \mathcal{I}$, there is a vector $\xi(\omega) \in \mathcal{H}$, such that $\forall x \in \mathfrak{n}_\varphi : \langle \xi(\omega), \lambda(x) \rangle = \omega(x^*)$. A priori it is not clear that $\mathcal{I}$ contains any element at all, but one can prove that there are plenty of elements in $\mathcal{I}$, that is $\{\xi(\omega) \mid \omega \in \mathcal{I}\}$ is dense in $\mathcal{H}$. Now, the dual left Haar weight $\hat{\varphi}$ is defined as the unique n.s.f. weight on $\hat{M}$ with GNS-construction $(H, \iota, \hat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a σ-strong*-norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega), \omega \in \mathcal{I}$. Then $\hat{\varphi}$ indeed turns out to be left invariant:

$$\hat{\varphi}\big((\omega \otimes \iota)\hat{\Delta}(x)\big) \;=\; \hat{\varphi}(x)\omega(1), \qquad \forall\, \omega \in \hat{M}_*^+, \forall\, x \in \mathcal{M}_{\hat{\varphi}}^+.$$

The normal way to define the dual right Haar weight $\hat{\psi}$ is by constructing the dual multiplicative unitary $\hat{W}$ and the dual unitary antipode $\hat{R}$ first. We set $\hat{W} = \Sigma W^* \Sigma$, and then define $\hat{R} : (\iota \otimes \omega_{v,w})(\hat{W}) \mapsto (\iota \otimes \omega_{\hat{J}v, \hat{J}w})(\hat{W})$, where $\hat{J}$ is the modular operator of $\hat{\varphi}$. Then one can show that $\hat{R}$ is a $*$-anti-automorphism of $\hat{M}$ that satisfies $\hat{\Delta} \circ \hat{R} = \Sigma \circ (\hat{R} \otimes \hat{R}) \circ \hat{\Delta}$. One defines $\hat{\psi} = \hat{\varphi} \circ \hat{R}$, so that the right invariance of $\hat{\psi}$ follows from the left invariance of $\hat{\varphi}$.

We denote all dual objects by adding a hat to its notation, i.e. $\hat{W}$, $\hat{S}$, $\hat{R}$, ... $\hat{W}$ as introduced in Remark 6.2.2 is indeed the multiplicative unitary associated to $(\hat{M}, \hat{\Delta})$ as was defined in Definition 6.1.5. Similarly, $\hat{R}$ is indeed the unitary antipode of the dual quantum group. Furthermore, the following relations hold:

$$R(x) = \hat{J}\, x^*\, \hat{J}, \qquad \forall\, x \in M, \qquad \hat{R}(x) = J\, x^*\, J, \qquad \forall\, x \in \hat{M}, \tag{6.2.2}$$

where we consider M and $\hat{M}$ acting on the GNS-space $\mathcal{H}$. Moreover, we have $W \in M \otimes \hat{M}$ and hence $\hat{W} \in \hat{M} \otimes M$.

Example 6.2.3. Let us work out the classical example $(L^\infty(G), \Delta)$ and see what its dual quantum group is. By definition, $\hat{M}$ is generated in the σ-strong*-closure of

$$\left\{ (f \cdot \varphi \otimes \iota)(W) \mid f \in L^1(G) \right\},$$

where we used the notation $f \cdot \varphi(g) = \varphi(gf)$, so that $f \cdot \varphi$ is a normal functional for $f \in L^1(G)$. Recall that the GNS-space $\mathcal{H}$ is given by $L^2(G)$ and the multiplicative unitary is given by (6.1.5). Fix $f \in L^1(G)$ and let $g \in L^2(G)$ be such that $g^2 = f$. Let $h \in L^2(G)$. Then one computes that:

$$\begin{aligned}
\left(\left((f \cdot \varphi \otimes \iota) W\right) h\right)(y) &= (g \cdot \varphi \otimes \iota)\left(W(g \otimes h)\right)(y) \\
&= \int g(x)(g(x) h(x^{-1}y)) dx \\
&= (f * h)(y),
\end{aligned} \tag{6.2.3}$$

Thus, $\hat{M}$ is the σ-strong* closure of the convolution algebra $L^1(G)$, which we denote by $\mathcal{L}(G)$. $\mathcal{L}(G)$ is know as the *group von Neumann algebra* of G; it is also the σ-weak closure of the group C*-algebra $C^*(G)$ and the reduced group C*-algebra $C^*_r(G)$, see [9]. The algebra comes with a natural involution (see Exercise 6.2.4) being given by:

$$f^*(x) = \overline{f(x^{-1})}\delta(x^{-1}), \qquad f \in L^1(G). \tag{6.2.4}$$

We introduce the following notation to distinguish between the functions in M and the convolution operators in $\hat{M}$. For $f \in L^1(G) \cap L^\infty(G)$, we will write $\hat{f}$ for the convolution operator:

$$\hat{f} : L^2(G) \to L^2(G) : g \mapsto f * g.$$

We have shown that $\hat{f} \in \hat{M}$ and $\hat{f} = \lambda(f \cdot \varphi) = (f \cdot \varphi \otimes \iota)(W)$. We will prove that in the abelian case $\hat{M}$ can be identified with $L^\infty(\hat{G})$, where $\hat{G}$ is the dual group of irreducible, unitary representations of G. The map $f \mapsto \hat{f}$ will turn out to be the Fourier transform under this identification.

Next we determine the dual Haar measures. If $f \in L^1(G) \cap L^\infty(G)$, we have $f \cdot \varphi \in \mathcal{I}$. Indeed, by the Cauchy-Schwartz inequality, we see that for $g \in \mathfrak{n}_\varphi = L^2(G) \cap L^\infty(G)$ we have

$\varphi(\overline{g}f) \leq \varphi(|g|^2)^{\frac{1}{2}}\varphi(|f|^2)^{\frac{1}{2}} = \varphi(|g|^2)^{\frac{1}{2}}\|\Lambda(f)\|$, so that the definition of $\mathcal{I}$ (see (6.2.2)) yields $f \in \mathcal{I}$. Moreover, note that (6.2.3) and the definition of the dual GNS-map $\hat{\Lambda}$ give that for $f \in L^1(G) \cap L^\infty(G)$, we have:

$$\hat{\Lambda}(\hat{f}) = \xi(f \cdot \varphi) = \Lambda(f).$$

Let μ_e denote the point measure at $e \in G$. Let $K(G)$ denote the compactly supported continuous functions. $K(G)$ is a subalgebra of the convolution algebra $L^1(G)$, also eqquiped with the $\| \|_1$-norm. For $f, g \in K(G)$, we find:

$$\langle \hat{\Lambda}(\hat{f}), \hat{\Lambda}(\hat{g})\rangle = \langle \Lambda(f), \Lambda(g)\rangle = \int \overline{g(x)}f(x)d_lx = \int \overline{g(x^{-1})}f(x^{-1})\delta(x^{-1})d_lx = \int g^* * f d\mu_e. \tag{6.2.5}$$

Now, it can be shown that integrating against μ_e, as a (non-continuous) linear map on $K(G)$, has a unique extension to a n.s.f. weight on the Von Neumann algebra $\hat{M}$. This is basically done by the techniques of Section 5.9.2; we shortly sketch the construction for this specific situation. Let $\phi = \int d\mu_e : K(G) \to \mathbb{C}$. Then, ϕ induces a GNS-representations $(H_\phi, \Lambda_\phi, \pi_\phi)$. $\Lambda_\phi(K(G))$ is then a left Hilbert algebra (this is not obvious however; a proof can be found in [9, Proposition 17.2.2]). We find that the closure of $\Lambda_\phi(K(G))$ equals $H_\phi = L^2(G)$. The associated full left Hilbert algebra $\Lambda_\phi(K(G))''$ defines a n.s.f. weight $\hat{\varphi}$ on $R_l(\Lambda_\phi(K(G)))$ (see Section 5.9.2), which extends ϕ in the sense that $\phi = \hat{\varphi} \circ \pi_\phi$. The algebra $\pi_\phi(K(G))$ can be proved to be (norm-)dense in the convolution algebra $L^1(G)$ (see the proof of [9, Proposition 17.2.2]) and therefore the von Neumann algebra $R_l(\Lambda_\phi(K(G)))$, which is generated by $\pi_\phi(K(G))$ is also generated by $L^1(G)$. Then by definition of $\hat{M}$, we have $\hat{M} = R_l(\Lambda_\phi(K(G)))$. This shows that the n.s.f. weight $\hat{\varphi}$ must be the extension of ϕ. Actually, $\hat{\varphi}$ is tracial, since for $f, g \in K(G)$ one sees that:

$$\langle \hat{\Lambda}(\hat{f}), \hat{\Lambda}(\hat{g})\rangle = \int g^* * f d\mu_e = \int \overline{g(x^{-1})}f(x^{-1})\delta(x^{-1})d_lx = \overline{\int f^* * g d\mu_e} = \langle \hat{\Lambda}(\hat{g}^*), \hat{\Lambda}(\hat{f}^*)\rangle.$$

So that the operator $\hat{\Lambda}(\hat{f}) \mapsto \hat{\Lambda}(\hat{f}^*)$, $f \in K(G)$ extends to an antilinear, surjective isometry on $L^2(G)$. By their definitions, we find that $\hat{\nabla} = 1$ and $\hat{J} : L^2(G) \to L^2(G) : f \mapsto f^*$, where the involution is given by (6.2.4).

Now we compute the dual coproduct $\hat{\Delta}$. In order to do this, we define for $x \in G$ the operator $L_x : L^2(G) \to L^2(G) : f \mapsto L_x f$, where $(L_x f)(y) = f(x^{-1}y)$. So $x \mapsto L_x$ is actually the left regular representation of G. We claim that $L_x \in \hat{M}$. To prove this, the Tomita-Takesaki theorem is very useful. Since $\hat{M}$ is generated by convolution operators in $K(G)$, $\hat{M}'$ is generated by $\hat{J}K(G)\hat{J}$, where the operator $\hat{J}$ is given by $\hat{J} : L^2(G) \to L^2(G) : f \mapsto f^*$. For $f \in K(G)$ and $h \in L^2(G)$, it is straightforward to check that $\hat{J}(L_f)\hat{J}h = h * f^*$, so that $\hat{M}'$ is the σ-strong*-closure of the algebra of convolutions from the right side with functions in $K(G)$. Then, for $f \in K(G)$, $h \in L^1(G) \cap L^\infty(G)$:

$$((L_x h) * f)(y) = \int h(x^{-1}z)f(z^{-1}y)d_lz = \int h(z)f(z^{-1}x^{-1}y)d_lz = (L_x(h * f))(y),$$

so that indeed $L_x \in \hat{M}'' = \hat{M}$. Recall that a formula for the dual coproduct is given by Theorem 6.2.1. For $g, h \in L^2(G)$,

$$\begin{aligned}\left(\hat{\Delta}(L_z)(g \otimes h)\right)(x, y) &= (\Sigma W(L_x \otimes 1)W^*\Sigma(g \otimes h))(x, y) \\ &= (\Sigma W(L_z \otimes 1)(h \otimes g))(x, xy) \\ &= (\Sigma W(h \otimes g))(z^{-1}x, z^{-1}xy) \\ &= (g \otimes h)(z^{-1}x, z^{-1}y) \\ &= ((L_z \otimes L_z)(g \otimes h))(x, y).\end{aligned} \tag{6.2.6}$$

So we find the formula $\hat{\Delta}(L_z) = L_z \otimes L_z$. Since one can prove that the operators L_z are σ-weakly dense in $\mathcal{L}(G)$ and $\hat{\Delta}$ is normal, this fully determines the coproduct.

Exercise 6.2.4. In the previous example, prove that (6.2.4) indeed gives the adjoint of a function $f \in L^1(G)$ considered as a convolution operator acting on $L^2(G)$.

Exercise 6.2.5. For the previous example, determine $\hat{\Delta}(\hat{f})$, $f \in L^1(G) \cap L^\infty(G)$.

Exercise 6.2.6. For the previous example, verify the relation $R(f) = \hat{J}\overline{f}\hat{J}$, $\forall f \in L^\infty(G)$.

The nice feature of the von Neumann algebraic quantum groups is the following theorem, due to Kustermans and Vaes [42], [43]. It is a far-reaching generalization of the Pontrjagin duality theorem.

Theorem 6.2.7. $(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$.

We recall the classical Pontrjagin duality theorem and show how it relates to the quantum group version. Let G be a locally compact abelian group. Let $\hat{G}$ be the set of equivalence classes of irreducible, unitary representations of G. In particular, the representations in $\hat{G}$ are one-dimensional. The representations in $\hat{G}$ are in one-to-one correspondence with non-degenerate *-representations of $L^1(G)$; that is, $\pi \in \hat{G}$ corresponds to:

$$f \mapsto \pi(f) = \int_G f(x)\pi(x)d_l x,$$

see also [9, Paragraph 13.3] or [13]. Since G is abelian, $\hat{G}$ is by its definition the Gelfand spectrum of $L^1(G)$ and therefore it carries the weak-$*$ topology in which it is locally compact. Furthermore, $\hat{G}$ carries a group structure, simply by $(\pi \cdot \rho)(x) = \pi(x)\rho(x)$, $\pi, \rho \in \hat{G}, x \in G$. This defines the dual $\hat{G}$ as a l.c. group. Now the Pontrjagin theorem states that: $\hat{\hat{G}} = G$.

Exercise 6.2.8. Prove that if G is discrete, then $\hat{G}$ is compact.

Example 6.2.9. If we can prove that for an abelian group G, $(\mathcal{L}(G), \hat{\Delta}_G)$ is isomorphic to $(L^\infty(\hat{G}), \Delta_{\hat{G}})$, we see that the dual quantum group construction is a generalization of the classical dual group. In particular the Kustermans-Vaes duality Theorem 6.2.7 is a generalization of the Pontrjagin duality theorem. We will show that such an isomorphism indeed exists.

We recall that the Fourier transform $\mathcal{F} : L^2(G) \to L^2(\hat{G})$, is a unitary transformation that is determined by:

$$(\mathcal{F}(f))(\pi) = \pi(f) = \int_G \overline{\pi(x)} f(x) d_l x, \qquad \forall f \in L^1(G) \cap L^2(G).$$

For $f \in L^1(G) \cap L^2(G)$ and $g \in L^2(G)$, it is well known (see [13, Proposition 4.36]) that:

$$(\mathcal{F}(f * g)) = \mathcal{F}(f)\mathcal{F}(g).$$

Recall that for $f \in L^1(G)$ we used the notation $\hat{f}$ for the convolution operator on $L^2(G)$ defined by $h \mapsto f * h$. Then the previous equation is equivalent to:

$$\mathcal{F}\hat{f}\mathcal{F}^{-1} = \mathcal{F}(f) \in L^\infty(\hat{G}).$$

The operators $\hat{f}$, where $f \in L^1(G) \cap L^2(G)$ are σ-weakly dense in $\mathcal{L}(G)$ (since we saw that $K(G)$ is σ-weakly dense in $\mathcal{L}(G)$). Furthermore, the functions $\mathcal{F}(f), f \in L^1(G) \cap L^2(G)$ are σ-weakly dense in $L^\infty(\hat{G})$. This shows that $\mathcal{F}\mathcal{L}(G)\mathcal{F}^{-1} = L^\infty(\hat{G})$. Since $\mathcal{F}$ is a unitary transformation, the map $\mathcal{L}(G) \ni x \mapsto \mathcal{F}x\mathcal{F}^{-1}$ defines a normal isomorphism between the von Neumann algebras $\mathcal{L}(G)$ and $L^\infty(\hat{G})$. It remains to check wether the coproduct is preserved. We leave this as an exercise.

Exercise 6.2.10. In the previous example, show that:

$$\Delta_{\hat{G}} \circ (\mathcal{F}x\mathcal{F}^{-1}) = (\mathcal{F} \otimes \mathcal{F})\hat{\Delta}_G(x)(\mathcal{F}^{-1} \otimes \mathcal{F}^{-1}).$$

6.3 Corepresentations

This section covers corepresentations of l.c. quantum groups. These are the analogues of group representations. We give their main properties and definitions and work out the classical example.

6.3.1 Definition and examples

Definition 6.3.1. *A unitary corepresentation U of a l.c. quantum group (M, Δ) on a Hilbert space $\mathcal{K}$ is a unitary element $U \in M \otimes B(\mathcal{K})$ such that $(\Delta \otimes \iota)(U) = U_{13}U_{23} \in M \otimes M \otimes B(\mathcal{K})$, where the standard leg-numbering is used in the right hand side.*

Example 6.3.2. It follows from the pentagonal identity $W_{12}W_{13}W_{23} = W_{23}W_{12}$ and $\Delta(x) = W^*(1 \otimes x)W$ that the multiplicative unitary W defines a unitary corepresentation of M on the GNS-space $\mathcal{H}$.

Example 6.3.3. Consider $(L^\infty(G), \Delta_G)$. Suppose that $\pi : G \to B(\mathcal{K})$ is a unitary representation on a separable Hilbert space $\mathcal{K}$. Then π can be considered as an element of $L^\infty(G, B(\mathcal{K}))$ which in turn is isomorphic to $L^\infty(G) \otimes B(\mathcal{K})$ ([10, Proposition II.3.3]). We denote U_π for the corresponding element, and using these correspondences we see that

$$((\Delta \otimes \iota)(U_\pi))\,(g, h) \simeq U_\pi(gh) = U_\pi(g)U_\pi(h) = (U_\pi)_{13}(U_\pi)_{23},$$

so that U_π is a corepresentation. Similarly one shows that if $U_\pi \in M \otimes B(\mathcal{K})$ is a unitary corepresentation, then it corresponds to a representation $\pi : G \to B(\mathcal{K})$.

Under the correspondence in the previous example the multiplicative unitary of the quantum group $(L^\infty(G), \Delta_G)$ corresponds to the left regular representation of G. Therefore W is also called the **left regular corepresentation**. We leave the proof of this fact as an exercise.

Exercise 6.3.4. Consider the quantum group $(L^\infty(G), \Delta_G)$. As a Banach space, we identify $L^\infty(G)_*$ with $L^1(G)$. For $y \in L^1(G)$, we write ω_y for the normal functional on $L^\infty(G)$, defined by $x \mapsto \int_G x(g)y(g)d_lg$. For $\omega_y, \omega_z \in L^1(G)$, we define the functional:

$$(\omega_x * \omega_y) = (\omega_x \otimes \omega_y) \circ \Delta.$$

The functional is normal, since Δ is normal and the tensor product of normal functionals is again a normal functional on the tensor product of two von Neumann algebras.

1. Show that $(\omega_x * \omega_y) = \omega_{x*y}$, so that $*$ turns $L^\infty(G)_*$ into a Banach-$*$-algebra that is isomorphic to $L^1(G)$. We define $(\omega_y)^* = \omega_{y^*}$

2. Let U be a corepresentation of $(L^\infty(G), \Delta_G)$ on a Hilbert space H_U. Show that $\lambda_U : L^\infty(G)_* \to B(H_U) : \omega \mapsto (\omega \otimes \iota)(U)$ is a $*$-representation of M_* on H_U.

3. Recall that there is a bijective correspondence between unitary representations of G and non-degenerate $*$-representations of $L^1(G)$. Show that if π is a unitary representation of G, then λ_{U_π} is the representation of $L^1(G)$ that corresponds to π. See Example 6.3.3 for the definition of U_π.

4. Let $L : G \to B(L^2(G))$ be the left regular representation. Show that $U_L = W$.

We also introduce the **right regular corepresentation**. It is proved in [42] that there is a well-defined unitary operator V acting on $\mathcal{H} \otimes \mathcal{H}$ given by:

$$V\,(\Gamma(x) \otimes \Gamma(y)) = (\Gamma \otimes \Gamma)\,(\Delta(x)(1 \otimes y)) \quad \text{for all} \quad x, y \in \mathcal{N}_\psi.$$

The right regular corepresentation is defined as $\Sigma V \Sigma$, where Σ is the flip. We denote this operator by $\chi(V)$ and one can indeed check that the corepresentation relation $(\Delta \otimes \iota)\chi(V) = (\chi(V))_{13}(\chi(V))_{23}$ is satisfied. In the classical case, this corepresentation corresponds to the right regular representation.

6.3.2 Basic concepts and constructions

Typical concepts from representation theory have corepresentation theoretic analogues. Here, we mention some of them.

Invariant subspaces and equivalence

Let U be a unitary corepresentation on a Hilbert space $\mathcal{K}$. A closed subspace $\mathcal{L} \subseteq \mathcal{K}$ is an invariant subspace if $(\omega \otimes \iota)(U)$ preserves $\mathcal{L}$ for all $\omega \in M_*$. In particular, it follows from Definition 6.2.1 that an invariant subspace of W is precisely a closed subspace invariant for the action of the dual von Neumann algebra $\hat{M}$, since it is generated by $(\omega \otimes \iota)(W)$, $\omega \in M_*$. A unitary corepresentation U in the Hilbert space $\mathcal{K}$ is irreducible if there are only trivial (i.e. equal to $\{0\}$ or the whole Hilbert space $\mathcal{K}$) invariant subspaces. In particular, $\{(\omega \otimes \iota)(U) \mid \omega \in M_*\}'' = B(\mathcal{K})$ implies that U is an irreducible unitary corepresentation.

Let $U_0 \in M \otimes B(\mathcal{K}_0), U_1 \in M \otimes B(\mathcal{K}_1)$ be unitary corepresentations. We call a bounded linear map $T : \mathcal{K}_0 \to \mathcal{K}_1$ an intertwiner between U_0 and U_1 if $(1 \otimes T)U_0 = U_1(1 \otimes T)$. We write $\mathcal{C}(U_0, U_1)$ for the space intertwiners between U_0 and U_1.

Exercise 6.3.5. Let U_0, U_1, U_2 be unitary corepresentations and let $T \in \mathcal{C}(U_0, U_1)$ and $S \in \mathcal{C}(U_0, U_1)$. Show that $ST \in \mathcal{C}(U_0, U_2)$.

U and U' are called unitarily equivalent if there exists an intertwiner T between U and U' that is unitary. In that case T^* is an intertwiner between U' and U. One easily checks that unitary equivalence defines an equivalence relation. We denote $\mathrm{IC}(M)$ for the equivalence classes of irreducible unitary corepresentations of (M, Δ). We denote by $\mathrm{IR}(M)$ the equivalence classes of irreducible unitary representations (as a *-algebra) of M. Note that Example 6.3.3 set up a correspondence between the representations of a l.c. group G and $\mathrm{IC}(L^\infty(G))$.

The contragredient corepresentation

Let U be a corepresentation of (M, Δ) on a Hilbert space $\mathcal{K}$. Let $\mathcal{J} : \mathcal{K} \to \overline{\mathcal{K}} : v \mapsto \overline{v}$, where $\overline{\mathcal{K}}$ denotes the conjugate Hilbert space. Then we define:

$$\overline{U} = (\hat{J} \otimes \mathcal{J})U^*(\hat{J} \otimes \mathcal{J}^*).$$

$\overline{U}$ is called the **contragredient corepresentation** of U.

Exercise 6.3.6. Check that $\overline{U}$ is a corepresentation.

Tensor product

Let U_0 and U_1 be corepresentations on Hilbert spaces $\mathcal{K}_0$ and $\mathcal{K}_1$, respectively. Then we define the **tensor product** of U_0 and U_1 as $(U_0)_{12}(U_1)_{13}$.

Exercise 6.3.7. Show that $(U_0)_{12}(U_1)_{13}$ is indeed a corerepresentation on $\mathcal{K}_0 \otimes \mathcal{K}_1$.

Let (M, Δ) be a l.c. quantum group. It is rather straightforward to check that $(M\otimes M, \Delta_2)$ is a l.c. quantum group, with $\Delta_2 = \Sigma_{23}(\Delta \otimes \Delta)$ and Σ_{23} the flip in the second and right leg. Then we define a corepresentation $\tilde{U}$ of $(M \otimes M, \Delta_2)$ on $B_2(\mathcal{K}) \simeq \mathcal{K} \otimes \overline{\mathcal{K}}$ as $\tilde{U} = U_{13}\overline{U}_{24}$. Here $B_2(\mathcal{K})$ denotes the space of Hilbert-Schmidt operators on $\mathcal{K}$, see Section 5.2. It is the quantum group analogue of the representation $\pi \otimes \overline{\pi}$ of $G \times G$, where π is a representation of G. This construction is needed for the Plancherel theorem in the next section.

6.4 Plancherel theorems

This section covers a Plancherel theorem for l.c. quantum groups that satisfy certain technical assumptions. We first recall the classical Plancherel theorem and give a proof for the special case of abelian groups and compact groups. Then we turn back to quantum groups and give the Plancherel theorem as was proved by Desmedt [7]. We will not give a proof, since it is too technical. We rather give the classical example and refer to Chapter 7 for a non-classical example.

6.4.1 The classical Plancherel theorem

We start with recalling the Plancherel theorem for unimodular, locally compact quantum groups that satisfy certain additional technical conditions. The theorem gives simultaneously a decomposition of the left and right regular representation. It was first proved by Segal [54], [55] and Mautner [48] in a slightly different form as presented in this paragraph. A proof of the Plancherel statement as stated below may be found in Dixmier's book [9, Théorème 18.1.1]. The proof relies on the theory of C*-algebras, in particular traces on C*-algebras and representations of C*-algebras. Here we will only give the proof for abelian and compact groups, which is far less technical! Then we will provide the reader with the nessecary tools to understand the statements made in the more general Plancherel theorem.

Notational conventions. λ will denote the left regular representation. ρ denotes the right regular representation. For a l.c. G, $\hat{G}$ denotes the equivalence classes of irreducible unitary representations.

Plancherel for abelian groups

For abelian groups, the Plancherel theorem reduces to Fourier theory. Let us show how the Fourier transform gives a direct integral decomposition of the left and right regular representation. Let G be a l.c. abelian group. The Fourier transform $\mathcal{F}$ is a unitary transformation from $L^2(G)$ to $L^2(\hat{G}) = \int_{\hat{G}}^{\oplus} \mathbb{C} d_l g$. For $f \in L^1(G) \cap L^2(G)$, it is defined as:

$$\mathcal{F}(f)(\pi) = \int_G \overline{\pi(x)} f(x) d_l x. \tag{6.4.1}$$

Then

$$\left(\mathcal{F}\lambda(x)\mathcal{F}^{-1}\right)g(\pi)=\overline{\pi}(x)g(\pi)=\left(\left(\left(\int_{\hat{G}}^{\oplus}\overline{\pi}d_l\pi\right)(x)\right)\left(\int_{\hat{G}}^{\oplus}g(\pi)d_l\pi\right), \text{for} g\in L^2(\hat{G}), x\in G,$$

where $\overline{\pi}$ denotes the conjugate representation. The first equation follows by left invariance of the left Haar measure on G and the second equality is by definition. So the Fourier transform intertwines the left regular representation with the direct integral representation $\left(\int_{\hat{G}}^{\oplus}\overline{\pi}d_l\pi\right)$. Similarly, $\mathcal{F}$ intertwines the right regular representation with the direct integral $\left(\int_{\hat{G}}^{\oplus}\pi d_l\pi\right)$.

Exercise 6.4.1. Give a proof of the last claim.

Plancherel for compact groups

For compact groups, the Plancherel theorem reduces to the Peter-Weyl theorem. We need a bit of notation to make this precise. Let G be a compact group, let $\pi\in\hat{G}$ and let $\mathcal{H}_\pi$ be the representation space of π. Let d_π be the dimension of $\mathcal{H}_\pi$. We denote

$$\mathcal{E}_\pi=\operatorname{span}_{\mathbb{C}}\left\{(x\mapsto\langle\pi(x)v,w\rangle)\in C(G)\mid v,w\in\mathcal{H}_\pi\right\}.$$

The functions $\mathcal{E}_\pi$ are called **matrix coefficients** (of π).

Recall that $M_n(\mathbb{C})$ equipped with the inner product defined by $\langle S,T\rangle=\operatorname{Tr}(T^*S)$, $S,T\in M_n(\mathbb{C})$ is a Hilbert space and it can be identified with $\mathbb{C}^n\otimes(\mathbb{C}^n)^*$ (the * denotes the dual space). The identification is made by sending a matrix that has a 1 on the i-th row and j-th column and zeros elsewhere to $e_i\otimes e_j^*$.

By the Peter-Weyl theorem we have a direct sum decomposition $L^2(G)=\oplus_{\pi\in\hat{G}}\mathcal{E}_\pi$. Moreover, under the identification of $M_{d_\pi}(\mathbb{C})$ with $\mathbb{C}^{d_\pi}\otimes(\mathbb{C}^{d_\pi})^*$, the left regular representation acts on $\oplus_{\pi\in\hat{G}}\mathcal{E}_\pi$ as $\oplus_{\pi\in\hat{G}}\left(\operatorname{Id}_{d_\pi}\otimes\overline{\lambda}\right)$. Similarly, the right regular representation decomposes as $\oplus_{\pi\in\hat{G}}\left(\rho\otimes\operatorname{Id}_{d\pi}\right)$.

The intertwiner between λ and $\oplus_{\pi\in\hat{G}}\left(\operatorname{Id}_{d_\pi}\otimes\overline{\lambda}\right)$ is again given by the Fourier transform, which is given by the same equation as in the abelian case (6.4.1). Then:

$$\mathcal{F}(\lambda(y)f)(\pi)=\mathcal{F}(f)\overline{\pi}(y)^*=\left(\operatorname{Id}\otimes\overline{\pi}(y)\right)\mathcal{F}(f)(\pi).$$

Plancherel for unimodular groups

The following theorem can be found as a part of [9, Théorème 18.1.1].

Theorem 6.4.2. *Let G be a l.c. separable, post-liminal, unimodular group. Let λ and ρ denote the left and right regular representation. For $\zeta\in\hat{G}$, let $\mathcal{H}(\zeta)$ denote the representation space of ζ. There exists a positive measure μ on $\hat{G}$ and an isomorphism $\mathcal{R}$ from $L^2(G)$ to $\int_{\hat{G}}^{\oplus}(\mathcal{H}(\zeta)\otimes\overline{\mathcal{H}(\zeta)})d\mu(\zeta)$ with the property that $\mathcal{R}$ transforms λ into $\int_{\hat{G}}^{\oplus}(\zeta\otimes 1)d\mu(\zeta)$.*

The measure μ is called the **Plancherel measure**.

We briefly make remarks on the technical assumptions. The definition of separable is as in topology: G admits a countable dense subset. The definition of a post-liminal group is more technical. Under the separability condition it is equivalent to saying that the double commutant $\pi(G)''$ is a type I von Neumann algebra for every unitary (not necessarily irreducible) representation. In that case the group is called a type I group. The reader who is not familiar with type classification of von Neumann algebras may view this as a technical assumption. Examples of type I groups are abelian groups, compact groups and connected semisimple Lie groups [14]. A standard example of a non-type I group is the free group with 2 generators.

There are some variations of the Plancherel theorem. For example in [11] Duflo and Moore were able to prove a Plancherel theorem for non-unimodular groups. The result is not a decomposition of the left and right regular representation, but rather of their tensor product $\lambda \otimes \rho$.

In practice it not at all a straightforward exercise to determine the Plancherel measure of a l.c. group. A lot of work has been done by Harish-Chandra, see [14]. For examples of Plancherel measures we also refer to [13].

6.4.2 Plancherel for quantum groups.

The Plancherel theorem for groups has a quantum group analogue. It gives a decomposition of the biregular corepresentation, which is the analogue of the tensor product of the left and right regular representation. The result was obtained in the PhD-thesis of Desmedt [7]. For two unbounded operators A and B, we denote $A \cdot B$ for the closure of their product.

Theorem 6.4.3 (Desmedt, [7, Theorem 3.4.1]). *Let (M, Δ) be a locally compact quantum group such that $\hat{M}$ is a type I von Neumann algebra and such that $\hat{M}_u$ is separable. There exist a standard measure μ on $IC(M)$, a measurable field $(H_U)_U$ of Hilbert spaces, a measurable field $(D_U)_U$ of selfadjoint, strictly positive operators and an isomorphism $\mathcal{Q}_L$ of H_φ onto $\int^\oplus B_2(H_U)d\mu(U)$ with the following properties:*

1. *For all $\alpha \in \mathcal{I}$ and μ-almost all $U \in IC(M)$, the operator $(\alpha \otimes \iota)(U)D_U^{-1}$ is bounded and $(\alpha \otimes \iota)(U) \cdot D_U^{-1}$ is a Hilbert-Schmidt operator on H_U.*

2. *For all $\alpha, \beta \in \mathcal{I}$ one has the Parseval formula:*

$$\langle \xi(\alpha), \xi(\beta) \rangle = \int_{IC(M)} \mathrm{Tr}(((\beta \otimes \iota)(U)D_U^{-1})^*((\alpha \otimes \iota)(U)D_U^{-1}))d\mu(U),$$

   and $\mathcal{Q}_L$ is the isometric extension of:

$$\hat{\Lambda}(\lambda(\mathcal{I})) \to \int_{IC(M)}^\oplus B_2(H_U)d\mu(U) : \xi(\alpha) \mapsto \int_{IC(M)}^\oplus (\alpha \otimes \iota)(U)D_U^{-1}d\mu(U).$$

3. *$\mathcal{Q}_L$ satisfies the following intertwining property for all $\omega, \omega' \in M_*$:*

$$\mathcal{Q}_L(\omega \otimes \iota)(\chi(V))(\omega' \otimes \iota)(W) = \left(\int_{IC(M)}^{\oplus} (\omega \otimes \omega' \otimes \iota)(\tilde{U}) d\mu(U) \right) \mathcal{Q}_L.$$

4. *Moreover, if $\beta \in M_*$ such that $(\beta \otimes \iota)(W) \in \hat{M}^+$, then*

$$\hat{\varphi}((\beta \otimes \iota)(W)) = \int_{IC(M)} \mathrm{Tr}((\beta \otimes \iota)(U) \cdot D_U^{-2}) d\mu(U).$$

5. *The Haar weights on $(\hat{M}, \hat{\Delta})$ are tracial if and only if almost all D_U are multiples of the identity.*

6. *Suppose that the standard measure μ' and μ'-measurable fields $(H'_U)_U$ and $(D'_U)_U$ have the same properties. Then μ and μ' are equivalent, and we have for μ-almost all $U \in IC(M)$ that*

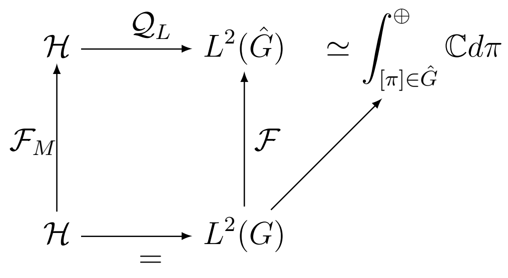
$$D'_U = \frac{d\mu}{d\mu'}(U) T_U D_U T_U^{-1}$$

where T_U is an isomorphism between H_U and H'_U for almost all U.

The proof of the theorem is quite technical and it requires the introduction of new aspects of quantum groups which are not presented in these notes. We refer to Desmedt's PhD-thesis [7] for a full proof. For the definition of $\hat{M}_u$ we refer to [40], mentioning only that in the classical case $\hat{M}_u$ equals $C^*(G)$, the universal group C*-algebra of a l.c. group G.

As an example we work out the classical case of an abelian l.c. group. Also, in Chapter 7 we give an example of a l.c. quantum group which has a decomposition of the left regular corepresentation W, see [16] for the precise result. Note that the Plancherel theorem gives a decomposition of the biregular representation $(\chi(V))_{13}(W)_{24}$ instead of the regular representation. The result in the next paragraph is somewhat similar and indeed the decomposition of the left regular representation allows one to determine the Plancherel measure in the theorem above.

Example 6.4.4. Consider the l.c. quantum group $(L^\infty(G), \Delta)$, where G is abelian. The Plancherel transformation $\mathcal{Q}_L$ can be derived from the following diagram.



We explain the notation. Here, $\mathcal{F}_M$ is defined by $\mathcal{F}_M : f \mapsto \xi(f \cdot \varphi)$, for $f \in L^1(G) \cap L^\infty(G)$, so that by the definition of the dual GNS-construction, $\mathcal{F}_M$ is just the identity map, see also Example 6.2.3. Since $\mathcal{H} = L^2(G)$ we can identify the objects in the bottom line of the diagram. $\mathcal{F}$ denotes the Fourier transform. Since there is a correspondence between irreducible, unitary representations of G and irreducible, unitary corepresentations of M, the Mackey Borel structure on $\hat{G}$ may be transfered to $IC(M)$ as well as the Haar measure on $\hat{G}$. The direct integral in the diagram then can be read as:

$$\int_{[\pi]\in\hat{G}}^{\oplus} \mathbb{C} d\pi = \int_{\mathrm{IC}(M)}^{\oplus} \mathbb{C} dU.$$

We claim that under these identifications the morphism $\mathcal{Q}$ is the Plancherel transformation, which then is simply the Fourier transform. Indeed, one computes that for $f \in L^1(G) \cap L^\infty(G)$:

$$\xi(f \cdot \varphi) \xrightarrow{\mathcal{F}_M^{-1}} f \xrightarrow{\quad\mathcal{F}\quad} \hat{f} \xrightarrow{\simeq} \int_{[\pi]\in\hat{G}}^{\oplus} \hat{f}(\pi) d\pi,$$

where

$$\hat{f}(\pi) = \int \overline{\pi(x)} f(x) dx = (f \cdot \varphi)(\overline{\pi}) = ((f \cdot \varphi) \otimes \iota)(U_\pi^*),$$

and $U_\pi \in \mathrm{IC}(M)$ corresponds to $\pi \in \hat{G}$. Then:

$$\mathcal{Q}_L : \xi(f \cdot \varphi) \mapsto \int_{\mathrm{IC}(M)}^{\oplus} ((f \cdot \varphi) \otimes \iota)(U^*) dU.$$

In particular we see that the operators D_U are all equal to 1. Actually, this is no surprise, since all corepresentations are one-dimensional, so that D_U is a complex number. Then one can incorporate the function $U \mapsto D_U$ in the Plancherel measure, defining $\mu' = D_U \mu$ if necessary.

Chapter 7

Example: quantum $SU(1,1)$ group as a von Neumann algebraic group

Warning. In this Chapter $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N}_0 = \{0, 1, 2, 3 \cdots\}$.

7.1 The Hopf $*$-algebra level

From Proposition 4.2.6 we know that there is another real form for the Hopf algebra corresponding to $SL(2,\mathbb{C})$ for real deformation parameter q. The real form corresponding to the compact group $SU(2)$, see the references in Chapter 1 The other real form corresponds to $SU(1,1)$ as real form, where

$$SU(1,1) = \left\{ g \in SL(2,\mathbb{C}) \mid g^* J g = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \mid a, c \in \mathbb{C},\ |a|^2 - |c|^2 = 1 \right\}$$

The Hopf $*$-algebra is $\tilde{A}$, the unital $*$-algebra generated by elements α_0, γ_0 and relations

$$\alpha_0^\dagger \alpha_0 - \gamma_0^\dagger \gamma_0 = 1, \qquad \alpha_0 \alpha_0^\dagger - q^2\, \gamma_0^\dagger \gamma_0 = 1,$$
$$\gamma_0^\dagger \gamma_0 = \gamma_0\, \gamma_0^\dagger, \qquad \alpha_0\, \gamma_0 = q\, \gamma_0\, \alpha_0, \qquad \alpha_0\, \gamma_0^\dagger = q\, \gamma_0^\dagger \alpha_0$$

where † denotes the $*$-operation on $\tilde{A}$ (in order to distinguish this kind of adjoint with the adjoints of possibly unbounded operators in Hilbert spaces). Then the comultiplication $\Delta_0 \colon \tilde{A} \to \tilde{A} \odot \tilde{A}$ such that

$$\Delta_0(\alpha_0) = \alpha_0 \otimes \alpha_0 + q\, \gamma_0^\dagger \otimes \gamma_0 \qquad \Delta_0(\gamma_0) = \gamma_0 \otimes \alpha_0 + \alpha_0^\dagger \otimes \gamma_0$$

and counit ε_0 and antipode S_0 determined by

$$S_0(\alpha_0) = \alpha_0^\dagger, \qquad S_0(\alpha_0^\dagger) = \alpha_0, \qquad S_0(\gamma_0) = -q\, \gamma_0 \qquad S_0(\gamma_0^\dagger) = -\frac{1}{q}\, \gamma_0^\dagger$$
$$\varepsilon_0(\alpha_0) = 1, \qquad \varepsilon_0(\gamma_0) = 0.$$

In order to perform the harmonic analysis we need to represent these relations, and it turns out that one needs unbounded operators on a Hilbert space. However, it turned out that it is impossible to realise the comultiplication on the level of operators.

Theorem 7.1.1 (Woronowicz (1991))**.** *For* (α^1,γ^1)*, resp.* (α^2,γ^2)*, closed operators on an infinite dimensional Hilbert space* H^1*, resp.* H^2*, representing the relations, there exist no closed operators* α*,* γ *acting on* $H^1\otimes H^2$ *representing the relations and extending* $\alpha^1\otimes\alpha^2 + q\,(\gamma^1)^*\otimes\gamma^2$*,* $\gamma^1\otimes\alpha^2+(\alpha^1)^*\otimes\gamma^2$*, such that* α^**,* γ^* *extend* $(\alpha^1)^*\otimes(\alpha^2)^*+q\,\gamma^1\otimes(\gamma^2)^*$*,* $(\gamma^1)^*\otimes(\alpha^2)^*+\alpha^1\otimes(\gamma^2)^*$*.*

So, Woronowicz's theorem [62] says that it is impossible to realise the comultiplication on the level of operators on a Hilbert space. However, it was pointed out by Korogodsky [38] in 1994 that we can adapt the Hopf $*$-algebra to represent the functions on the normaliser in $SL(2,\mathbb{C})$ of $SU(1,1)$.

So fix a number $0<q<1$. Define $\mathcal{A}_q$ to be the unital $*$-algebra generated by elements $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{e}$ and relations

$$\begin{gathered}\boldsymbol{\alpha}^\dagger\boldsymbol{\alpha}-\boldsymbol{\gamma}^\dagger\boldsymbol{\gamma}=\boldsymbol{e}\qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger-q^2\,\boldsymbol{\gamma}^\dagger\boldsymbol{\gamma}=\boldsymbol{e}\qquad \boldsymbol{\gamma}^\dagger\boldsymbol{\gamma}=\boldsymbol{\gamma}\,\boldsymbol{\gamma}^\dagger\\ \boldsymbol{\alpha}\,\boldsymbol{\gamma}=q\,\boldsymbol{\gamma}\,\boldsymbol{\alpha}\qquad \boldsymbol{\alpha}\,\boldsymbol{\gamma}^\dagger=q\,\boldsymbol{\gamma}^\dagger\boldsymbol{\alpha}\\ \boldsymbol{e}^\dagger=\boldsymbol{e}\qquad \boldsymbol{e}^2=1\qquad \boldsymbol{\alpha}\,\boldsymbol{e}=\boldsymbol{e}\,\boldsymbol{\alpha}\qquad \boldsymbol{\gamma}\,\boldsymbol{e}=\boldsymbol{e}\,\boldsymbol{\gamma}\end{gathered}\tag{7.1.1}$$

where $\dagger$ denotes the $*$-operation on $\mathcal{A}_q$ (again to distinguish this kind of adjoint with the adjoints of possibly unbounded operators in Hilbert spaces). In case we take $\boldsymbol{e}=1$ in (7.1.1) we obtain the $*$-algebra $\tilde{A}$ described above. The additional generator $\boldsymbol{e}$ has been introduced by Korogodsky.

For completeness we give the Hopf $*$-algebra structure on $\mathcal{A}_q$. By $\mathcal{A}_q\odot\mathcal{A}_q$ we denote the algebraic tensor product. There exists a unique unital $*$-homomorphism $\boldsymbol{\Delta}\colon\mathcal{A}_q\to\mathcal{A}_q\odot\mathcal{A}_q$ such that

$$\boldsymbol{\Delta}(\boldsymbol{\alpha})=\boldsymbol{\alpha}\otimes\boldsymbol{\alpha}+q\,(\boldsymbol{e}\,\boldsymbol{\gamma}^\dagger)\otimes\boldsymbol{\gamma}\qquad \boldsymbol{\Delta}(\boldsymbol{\gamma})=\boldsymbol{\gamma}\otimes\boldsymbol{\alpha}+(\boldsymbol{e}\boldsymbol{\alpha}^\dagger)\otimes\boldsymbol{\gamma}\qquad \boldsymbol{\Delta}(\boldsymbol{e})=\boldsymbol{e}\otimes\boldsymbol{e}\tag{7.1.2}$$

The counit $\boldsymbol{\varepsilon}\colon\mathcal{A}_q\to\mathcal{A}_q$ and antipode $\boldsymbol{S}\colon\mathcal{A}_q\to\mathcal{A}_q$ are given by

$$\begin{gathered}\boldsymbol{S}(\boldsymbol{\alpha})=\boldsymbol{e}\,\boldsymbol{\alpha}^\dagger\qquad \boldsymbol{S}(\boldsymbol{\alpha}^\dagger)=\boldsymbol{e}\,\boldsymbol{\alpha}\qquad \boldsymbol{S}(\boldsymbol{\gamma})=-q\,\boldsymbol{\gamma}\qquad \boldsymbol{S}(\boldsymbol{\gamma}^\dagger)=-\frac{1}{q}\,\boldsymbol{\gamma}^\dagger\qquad \boldsymbol{S}(\boldsymbol{e})=\boldsymbol{e}\\ \boldsymbol{\varepsilon}(\boldsymbol{\alpha})=1\qquad \boldsymbol{\varepsilon}(\boldsymbol{\gamma})=0\qquad \boldsymbol{\varepsilon}(\boldsymbol{e})=1\end{gathered}\tag{7.1.3}$$

This makes $\mathcal{A}_q$ into a Hopf $*$-algebra.

Exercise 7.1.2. Verify that $\mathcal{A}_q$ is a Hopf $*$-algebra.

To see that for $q=1$ we obtain the Hopf $*$-algebra of polynomials on the group $SU(1,1)$ (when restricting to the sub-Hopf $*$-algebra $\mathcal{A}_q^1$ given by $\boldsymbol{e}=1$) and on the normaliser $N_{SL(2,\mathbb{C})}(SU(1,1))$ of $SU(1,1)$ in $SL(2,\mathbb{C})$ we let $\boldsymbol{\alpha}(g)=a$, $\boldsymbol{\gamma}(g)=c$. Similarly,

$$\begin{aligned}N_{SL(2,\mathbb{C})}(SU(1,1)) &= \{g\in SL(2,\mathbb{C})\mid g^*Jg=\pm J\}\\ &=\left\{\begin{pmatrix}a & c\\ \varepsilon\bar{c} & \varepsilon\bar{a}\end{pmatrix}\mid a,c\in\mathbb{C},\ \varepsilon\in\{\pm1\}, |a|^2-|c|^2=\varepsilon\right\}=SU(1,1)\cup SU(1,1)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\end{aligned}$$

and we put $\boldsymbol{\alpha}(g) = a$, $\boldsymbol{\gamma}(g) = c$, $\boldsymbol{e}(g) = \varepsilon$.

We can represent the commutation relations (7.1.1) by unbounded operators acting on the Hilbert space $H = L^2(\mathbb{T}) \oplus L^2(I_q)$, where $I_q = -q^{\mathbb{N}} \cup q^{\mathbb{Z}}$ is equipped with the counting measure. Here $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the unit circle, $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. If $p \in I_q$, we define $\delta_p(x) = \delta_{x,p}$ for all $x \in I_q$, so the family $\{\delta_p \mid p \in I_q\}$ is the natural orthonormal basis of $L^2(I_q)$. For $L^2(\mathbb{T})$ we have the natural orthonormal basis $\{\zeta^m \mid m \in \mathbb{Z}\}$, with ζ the identity function on $\mathbb{T}$. Then $\{\zeta^m \otimes \delta_p \mid m \in \mathbb{Z}, p \in I_q\}$ is an orthonormal basis for H. Define linear operators α_0, γ_0, e_0 on the space E of finite linear combinations of $\zeta^m \otimes \delta_p$ by

$$\begin{aligned} &\alpha_0(\zeta^m \otimes \delta_p) = \sqrt{\operatorname{sgn}(p) + p^{-2}}\ \zeta^m \otimes \delta_{qp}, \\ &\gamma_0(\zeta^m \otimes \delta_p) = p^{-1}\ \zeta^{m+1} \otimes \delta_p, \qquad e_0(\zeta^m \otimes \delta_p) = \operatorname{sgn}(p)\ \zeta^m \otimes \delta_p. \end{aligned} \tag{7.1.4}$$

for all $p \in I_q$, $m \in \mathbb{Z}$. The actions of $\alpha_0^\dagger$ and $\gamma_0^\dagger$ on E can be given in a similar fashion by taking formal adjoints, and these satisfy the relations (7.1.1), and give a faithful representation of the algebra $\mathcal{A}_q$.

Exercise 7.1.3. (i) Write down the expressions for $\alpha_0^\dagger$ and $\gamma_0^\dagger$ on E, and show that α_0, γ_0, e_0, $\alpha_0^\dagger$, $\gamma_0^\dagger$ do satisfy the relations (7.1.1).
(ii) Prove that the representation on E obtained in this way is faithful.
(iii) Check that the operators α_0, γ_0, e_0, $\alpha_0^\dagger$, $\gamma_0^\dagger$ are closable as unbounded operators on H. Note that E is a dense subspace of H. Denote the closures of α_0, γ_0 by α, γ.
(iv) Check that $\alpha_0^\dagger \subset \alpha^*$, $\gamma_0^\dagger \subset \gamma^*$.
(v) Show that the adjoints α^* and γ^* are the closures of $\alpha_0^\dagger$, $\gamma_0^\dagger$.

So the operators α_0, γ_0 are closable with densely defined closed unbounded operators α, γ as their closure. Moreover, the adjoints α^* and γ^* are the closures of $\alpha_0^\dagger$, $\gamma_0^\dagger$. Let e be the closure of e_0, then e is a bounded linear self-adjoint operator on H. Consider the linear map $T\colon \zeta^m \otimes \delta_p \mapsto \zeta^m \otimes \delta_{-p}$, $T \in B(H)$, where we take $\delta_p = 0$ in case $p \notin I_q$, and let u be its partial isometry (in the polar decomposition)

Definition 7.1.4. *M is the von Neumann algebra in $B(H)$ generated by α, γ, e and u.*

To give meaning to a von Neumann algebra generated by unbounded operators we use the following definition. For $T_1, \ldots, T_n$ closed, densely defined (possibly unbounded) linear operators acting on a Hilbert space H we define the von Neumann algebra

$$N = \{x \in B(H) \mid xT_i \subseteq T_i x, \text{ and } xT_i^* \subseteq T_i^* x\ \forall\ i\}'.$$

Then N is the smallest von Neumann algebra so that $T_1, \ldots, T_n$ are affiliated to N, and we call N the von Neumann algebra generated by $T_1, \ldots, T_n$.

Note that in particular, α and γ are affiliated to M.

7.2 Special functions related to the multiplicative unitary

In order to understand how the definition of the multiplicative unitary W in this case can be obtained, we recall

$$\Delta(x) \;=\; W^*(1\otimes x)W, \qquad x\in M.$$

In particular, for two irreucible representations π_1, π_2 of the von Neumann algebra M we see that the tensor product $(\pi_1\otimes\pi_2)\circ\Delta$ can also be described in terms of the multiplicative unitary. In this interpretation the operator W can be viewed as a unitary intertwiner of the tensor product and some direct sum or integral of irreducible representations, so we expect to establish an explicit expression for the multiplicative unitary in the context of Clebsch-Gordan coefficients, cf. the Wall polynomials in Section 4.4.

Now checking the representation (7.1.4) of (7.1.1), we see that the element $\gamma_0^\dagger\gamma_0$ acts diagonally in these representations, so the idea is to study the operator corresponding to $\boldsymbol{\Delta}(\boldsymbol{\gamma}^\dagger\boldsymbol{\gamma})$ acting initially on the dense domain $E\odot E\subset H\otimes H$, and we denote this operator by $\Delta_0(\gamma_0^\dagger\gamma_0)$.

Exercise 7.2.1. Calculate the action of $\Delta_0(\gamma_0^\dagger\gamma_0)$ on $\zeta^m\otimes\delta_p$ explicitly using (7.1.1), (7.1.2), (7.1.4).

For use now, and later use as well, we introduce the following notation, which will be used throughout the chapter.

Definition 7.2.2. *(i)* $\chi\colon -q^{\mathbb{Z}}\cup q^{\mathbb{Z}}\to\mathbb{Z}$ *such that* $\chi(x)=\log_q(|x|)$ *for all* $x\in -q^{\mathbb{Z}}\cup q^{\mathbb{Z}}$*;*
(ii) $\kappa\colon\mathbb{R}\to\mathbb{R}$ *such that* $\kappa(x)=\operatorname{sgn}(x)\,x^2$ *for all* $x\in\mathbb{R}$*;*
(iii) $\nu\colon -q^{\mathbb{Z}}\cup q^{\mathbb{Z}}\to\mathbb{R}^+$ *such that* $\nu(t)=q^{\frac{1}{2}(\chi(t)-1)(\chi(t)-2)}$ *for all* $t\in -q^{\mathbb{Z}}\cup q^{\mathbb{Z}}$*;*
(iv) $s\colon\mathbb{R}_0\times\mathbb{R}_0\to\{-1,1\}$ *is defined such that*

$$s(x,y)=\begin{cases}-1 & \text{if } x>0 \text{ and } y<0\\ 1 & \text{if } x<0 \text{ or } y>0\end{cases}$$

for all $x,y\in\mathbb{R}_0=\mathbb{R}\backslash\{0\}$.

In order to study the eigenvalues and eigenvectors of $\Delta_0(\gamma_0^\dagger\gamma_0)$, we need a suitable self-adjoint operator extending $\Delta_0(\gamma_0^\dagger\gamma_0)$ with domain $E\odot E$. Define a linear map $L\colon\mathcal{F}(\mathbb{T}\times I_q\times\mathbb{T}\times I_q)\to\mathcal{F}(\mathbb{T}\times I_q\times\mathbb{T}\times I_q)$ by

$$\begin{aligned}(Lf)(\lambda,x,\mu,y) \;=\; &\left[\,x^{-2}(\operatorname{sgn}(y)+y^{-2})+(\operatorname{sgn}(x)+q^2\,x^{-2})\,y^{-2}\,\right]f(\lambda,x,\mu,y)\\ &+\operatorname{sgn}(x)\,q^{-1}\,\bar\lambda\mu\,x^{-1}y^{-1}\sqrt{(\operatorname{sgn}(x)+x^{-2})(\operatorname{sgn}(y)+y^{-2})}\,f(\lambda,qx,\mu,qy)\\ &+\operatorname{sgn}(x)\,q\,\lambda\bar\mu\,x^{-1}y^{-1}\sqrt{(\operatorname{sgn}(x)+q^2x^{-2})(\operatorname{sgn}(y)+q^2y^{-2})}\,f(\lambda,q^{-1}x,\mu,q^{-1}y)\end{aligned}$$

for all $\lambda,\mu\in\mathbb{T}$ and $x,y\in I_q$. Here $\mathcal{F}(\mathbb{T}\times I_q\times\mathbb{T}\times I_q)$ denotes functions on the space $\mathbb{T}\times I_q\times\mathbb{T}\times I_q$. A straightforward calculation reveals that if $f\in E\odot E$, then $\Delta_0(\gamma_0^\dagger\gamma_0)\,f=L(f)$.

From this, it is a standard exercise to check that $f \in D(\Delta_0(\gamma_0^\dagger \gamma_0)^*)$ and $\Delta_0(\gamma_0^\dagger \gamma_0)^* f = L(f)$ if $f \in L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$ and $L(f) \in L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$. Here we identify functions and elements of L^2-spaces.

Exercise 7.2.3. Prove the statements in this paragraph. Show also that $D(\Delta_0(\gamma_0^\dagger \gamma_0)^*)$ consists precisely of such elements f.

If $\theta \in q^{\mathbb{Z}}$, we define $\ell'_\theta = \{ (\lambda, x, \mu, y) \in \mathbb{T} \times I_q \times \mathbb{T} \times I_q \mid y = \theta x \}$. We consider $L^2(\ell'_\theta)$ naturally embedded in $L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$. It follows easily from the above discussion that $\Delta_0(\gamma_0^\dagger \gamma_0)^*$ leaves $L^2(\ell'_\theta)$ invariant. Thus, if T is a self-adjoint extension of $\Delta_0(\gamma_0^\dagger \gamma_0)$, the obvious inclusion $T \subseteq \Delta_0(\gamma_0^\dagger \gamma_0)^*$ implies that T also leaves $L^2(\ell'_\theta)$ invariant.

Therefore every self-adjoint extension T of $\Delta_0(\gamma_0^\dagger \gamma_0)$ is obtained by choosing a self-adjoint extension T_θ of the restriction of $\Delta_0(\gamma_0^\dagger \gamma_0)$ to $L^2(\ell'_\theta)$ for every $\theta \in q^{\mathbb{Z}}$ and setting $T = \oplus_{\theta \in q^{\mathbb{Z}}} T_\theta$. So fix $\theta \in q^{\mathbb{Z}}$. Define $J_\theta = \{ z \in I_{q^2} \mid \kappa(\theta)\, z \in I_{q^2} \}$ which is a q^2-interval around 0. On J_θ we define a measure ν_θ such that $\nu_\theta(\{x\}) = |x|$ for all $x \in J_\theta$.

Now define the unitary transformation $U_\theta \colon L^2(\mathbb{T} \times \mathbb{T} \times J_\theta) \to L^2(\ell'_\theta)$ such that $U_\theta(f) = g$ where $f \in L^2(\mathbb{T} \times \mathbb{T} \times J_\theta)$ and $g \in L^2(\ell'_\theta)$ are such that

$$g(\lambda, z, \mu, \theta z) = (\lambda \bar{\mu})^{\chi(z)}\, (-\mathrm{sgn}(\theta z))^{\chi(z)}\, |z|\, f(\lambda, \mu, \kappa(z))$$

for almost all $\lambda, \mu \in \mathbb{T}$ and all $z \in I_q$ such that $\theta z \in I_q$.

Define the linear operator $L_\theta \colon \mathcal{F}(J_\theta) \to \mathcal{F}(J_\theta)$ such that

$$\begin{aligned}(L_\theta f)(x) = \frac{1}{\theta^2 x^2} \big(& -\sqrt{(1+x)(1+\kappa(\theta)\, x)}\, f(q^2 x) - q^2 \sqrt{(1+q^{-2}x)(1+q^{-2}\kappa(\theta)\, x)}\, f(q^{-2}x) \\ & + [(1+\kappa(\theta)\, x) + q^2(1+q^{-2}x)]\, f(x) \big)\end{aligned} \tag{7.2.1}$$

for all $f \in \mathcal{F}(J_\theta)$ and $x \in J_\theta$.

Then an easy calculation shows that $U_\theta^* \, (\Delta_0(\gamma_0^\dagger \gamma_0)|_{E \odot E})\, U_\theta = 1 \odot (L_\theta|_{\mathcal{K}(J_\theta)})$, where $\mathcal{K}(J_\theta)$ is the space of compactly supported functions on J_θ. So our problem is reduced to finding self-adjoint extensions of $L_\theta|_{\mathcal{K}(I_q)}$. This operator $L_\theta|_{\mathcal{K}(I_q)}$ is a second order q-difference operator for which eigenfunctions in terms of q-hypergeometric functions are known. We can use a standard reasoning to get hold of the self-adjoint extensions of $L_\theta|_{\mathcal{K}(I_q)}$. Let $\beta \in \mathbb{T}$. Then we define a linear operator $L_\theta^\beta \colon D(L_\theta^\beta) \subseteq L^2(J_\theta, \nu_\theta) \to L^2(J_\theta, \nu_\theta)$ such that

$$\begin{aligned}D(L_\theta^\beta) = \{ f \in L^2(J_\theta, \nu_\theta) \,|\, & L_\theta(f) \in L^2(J_\theta, \nu_\theta), f(0+) = \beta\, f(0-) \\ & \text{and } (D_q f)(0+) = \beta\, (D_q f)(0-) \}\end{aligned}$$

and L_θ^β is the restriction of L_θ to $D(L_\theta^\beta)$. Here, D_q denotes the Jackson derivative, that is, $(D_q f)(x) = (f(qx) - f(x))/(q-1)x$ for $x \in J_\theta$. Also, $f(0+) = \beta\, f(0-)$ is an abbreviated form of saying that the limits $\lim_{x \uparrow 0} f(x)$ and $\lim_{x \downarrow 0} f(x)$ exist and $\lim_{x \downarrow 0} f(x) = \beta \lim_{x \uparrow 0} f(x)$.

Then L_θ^β is a self-adjoint extension of $L_\theta|_{\mathcal{K}(I_q)}$. If $\beta, \beta' \in \mathbb{T}$ and $\beta \neq \beta'$, then $L_\theta^\beta \neq L_\theta^{\beta'}$. At this point it is not clear which of these extensions has to be chosen.

Exercise 7.2.4. Check the statements of the above paragraphs.

So in order to find the special functions attached to the multiplicative unitary we need to study the solutions to (7.2.1), and this leads to special functions of basic hypergeometric type. The eigenfunctions can all be described in terms of the ${}_1\varphi_1$-series, see Section 3.1. We state the results as follows, and give the link to special functions later. For $a, b, z \in \mathbb{C}$, we define

$$\Psi\left(\begin{matrix} a \\ b \end{matrix}\ ; q,\ z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b\,q^n;q)_\infty}{(q\,;q)_n}\,(-1)^n\,q^{\frac{1}{2}n(n-1)}\,z^n \ = (b;q)_\infty\ {}_1\varphi_1\left(\begin{matrix} a \\ b \end{matrix}\ ; q, z\right). \tag{7.2.2}$$

This is an entire function in a, b and z. We use the normalisation constant

$$c_q = \frac{1}{\sqrt{2}\,q\,(q^2, -q^2; q^2)_\infty}$$

Definition 7.2.5. *If $p \in I_q$, we define the function $a_p\colon I_q \times I_q \to \mathbb{R}$ such that a_p is supported on the set $\{\,(x,y) \in I_q \times I_q \mid \mathrm{sgn}(xy) = \mathrm{sgn}(p)\,\}$ and is given by*

$$\begin{aligned} a_p(x,y) \ &= c_q\,s(x,y)\,(-1)^{\chi(p)}\,(-\mathrm{sgn}(y))^{\chi(x)}\,|y|\ \nu(py/x)\ \sqrt{\frac{(-\kappa(p), -\kappa(y); q^2)_\infty}{(-\kappa(x); q^2)_\infty}} \\ &\times \Psi\left(\begin{matrix} -q^2/\kappa(y) \\ q^2\kappa(x/y) \end{matrix}\ ; q^2,\ q^2\kappa(x/p)\right) \end{aligned}$$

for all $(x,y) \in I_q \times I_q$ satisfying $\mathrm{sgn}(xy) = \mathrm{sgn}(p)$.

Actually, the sign $s(x,y)$ in Definition 7.2.5 is chosen so that it corresponds to self-adjoint extensions $L_\theta^{\mathrm{sgn}(\theta)}$.

The extra vital information that we need is contained in the following proposition. For $\theta \in q^{\mathbb{Z}}$ we define $\ell_\theta = \{\,(x,y) \in I_q \times I_q \mid y = \theta x\,\}$.

Proposition 7.2.6. *Consider $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$. Then the family $\{\,a_p|_{\ell_\theta} \mid p \in I_q$ so that $\mathrm{sgn}(p) = \mathrm{sgn}(\theta)\,\}$ is an orthonormal basis for $l^2(\ell_\theta)$. In particular,*

$$\sum_{x \in I_q \ \text{so that}\ \theta x \in I_q} a_p(x,\theta x)\,a_r(x,\theta x) = \delta_{p,r}, \qquad p, r \in I_q.$$

This result implies also a dual result, stemming from the following simple duality principle. Consider a set I and suppose that $l^2(I)$ has an orthonormal basis $(e_j)_{j\in J}$. For every $i \in I$, we define a function f_i on J by $f_i(j) = e_j(i)$. Then $(f_i)_{i\in I}$ is an orthonormal basis for $l^2(J)$. If we apply this principle to the line ℓ_θ, Proposition 7.2.6 implies the next one.

Proposition 7.2.7. *Consider $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$ and define $J = q^{\mathbb{Z}} \subset I_q$ if $\theta > 0$ and $J = -q^{\mathbb{N}} \subset I_q$ if $\theta < 0$. For every $(x,y) \in \ell_\theta$ we define the function $e_{(x,y)} : J \to \mathbb{R}$ such that $e_{(x,y)}(p) = a_p(x,y)$ for all $p \in J$. Then the family $\{\,e_{(x,y)} \mid (x,y) \in \ell_\theta\,\}$ forms an orthonormal basis for $l^2(J)$. In particular,*

$$\sum_{p \in J} a_p(x,\theta x)\,a_p(y,\theta y) = \delta_{x,y}, \qquad x, y \in I_q.$$

Let us now discuss which special functions play a role. First of all, we let $\theta < 0$. Now Proposition 7.2.6 in case $\theta < 0$ corresponds to the orthogonality relations for the Al-Salam–Carlitz polynomials, see Section 3.3.2, with $a = \kappa(\theta)$ and base q^2. The fact that in this situation the functions $a_p(x,y)$ correspond to an eigenfunction for the operator L_θ follows from the second-order difference equation (3.3.6).

Exercise 7.2.8. Assume that $\theta < 0$, determine the orthogonal polynomials related to the dual orthogonality as in Proposition 7.2.7. You will need the q-Charlier polynomials, see e.g. [24].

In case $\theta > 0$ we are dealing with q-Laguerre polynomials, which correspond to an indeterminate moment problem, see Section 3.3.3. Now for $1 \geq \theta > 0$ Proposition 7.2.7 corresponds to (3.3.10) in base q^2 and with $c = 1$, $q^{2\alpha} = \kappa(\theta)$. Proposition 7.2.6 then follows from the dual orthogonality relations. For $1 \leq \theta$, we use base q^2 and with $c = 1$, $q^{2\alpha} = \kappa(\theta)^{-1}$.

Exercise 7.2.9. Work out the details of the dual orthogonality relations in case $\theta > 0$ and the corresponding second-order difference equation. You can find more results in [5] to solve this exercise.

Finally, these functions satisfy symmetry relations

$$\begin{aligned} a_p(x,y) &= (-1)^{\chi(yp)}\mathrm{sgn}(x)^{\chi(x)}\left|\frac{y}{p}\right| a_y(x,p); \\ a_p(x,y) &= \mathrm{sgn}(p)^{\chi(p)}\mathrm{sgn}(x)^{\chi(x)}\mathrm{sgn}(y)^{\chi(y)} a_p(y,x); \\ a_p(x,y) &= (-1)^{\chi(xp)}\mathrm{sgn}(y)^{\chi(y)}\left|\frac{x}{p}\right| a_x(p,y), \end{aligned} \tag{7.2.3}$$

which follow from transformation formulas for ${}_1\varphi_1$-series that can be obtained from limiting cases of Heine's transformation (3.1.9).

Exercise 7.2.10. Work out what the symmetry properties (7.2.3) mean for the ${}_1\varphi_1$-series, and prove (7.2.3) in this way.

Originally, the multiplicative unitary has been defined using the functions $a_p(x,y)$ of Definition 7.2.5. We now first introduce the Haar weight and its GNS-representation.

7.3 The GNS-construction for the Haar weight

We will not work in the Hilbert space H but in the GNS-space K, which we indicate how to construct this.

Proposition 7.3.1. *The von Neumann algebra M defined in Definition 7.1.4 equals* $L^\infty(\mathbb{T}) \otimes B\big(L^2(I_q)\big)$

The proof of Proposition 7.3.1 follows using the spectral theorem and showing that multiplication by $f \in L^\infty(\mathbb{T})$ and all rank-one operators on $L^2(I_q)$ are contained in M, see [28, Lemma 2.4].

We define the following operators in M, by Proposition 7.3.1,

$$\Phi(m,p,t)\colon \zeta^r \otimes \delta_x \mapsto \delta_{xt}\, \zeta^{m+r} \otimes \delta_p, \qquad m,r \in \mathbb{Z},\ p,t,x \in I_q.$$

A straightforward calculation gives

$$\Phi(m_1,p_1,t_1)\,\Phi(m_2,p_2,t_2) = \delta_{p_2,t_1}\,\Phi(m_1+m_2,p_1,t_2), \qquad \Phi(m,p,t)^* = \Phi(-m,t,p)$$

In particular the finite linear span of the operators $\Phi(m,p,t)$ form a σ-weakly dense $*$-subalgebra in M.

We now construct the nsf weight φ, for which we later show that it is actually the left Haar weight. We construct of the nsf weight by writing down its GNS-construction. Define $\mathrm{Tr} = \mathrm{Tr}_{L^\infty(\mathbb{T})} \otimes \mathrm{Tr}_{B(L^2(I_q))}$ on M, where $\mathrm{Tr}_{L^\infty(\mathbb{T})}$ and $\mathrm{Tr}_{B(L^2(I_q))}$ are the canonical traces on $L^\infty(\mathbb{T})$, i.e. $\mathrm{Tr}_{L^\infty(\mathbb{T})}(f) = \int_\mathbb{T} f(\zeta)\,d\zeta$ with normalization $\mathrm{Tr}_{L^\infty(\mathbb{T})}(1) = 1$, and on $B(L^2(I_q))$, normalized by $\mathrm{Tr}_{B(L^2(I_q))}(P) = 1$ for any rank one orthogonal projection. Note that Tr is a tracial weight on M so in particular its modular group is trivial. For Tr we have the following GNS-construction:

- a Hilbert space $\mathcal{K} = H \otimes L^2(I_q) = L^2(\mathbb{T}) \otimes L^2(I_q) \otimes L^2(I_q)$ equipped with the orthonormal basis $\{f_{mpt} \mid m \in \mathbb{Z},\ p,t \in I_q\}$;
- a unital $*$-homomorphism $\pi\colon M \to B(\mathcal{K})$, $\pi(a) = a \otimes \iota_{L^2(I_q)}$ for $a \in M$;
- $\Lambda_{\mathrm{Tr}}\colon \mathcal{N}_{\mathrm{Tr}} \to \mathcal{K}$, $a \mapsto \sum_{p \in I_q} (a \otimes \iota_{L^2(I_q)}))f_{0,p,p}$.

Exercise 7.3.2. Check that the above construction is the GNS-construction for Tr.

We define the left invariant nsf weight φ formally as $\varphi(x) = \mathrm{Tr}(|\gamma|\, x\, |\gamma|)$ with the operator $|\gamma|$ affiliated to M. We proceed by defining the set D as the set of elements of $x \in M$ such that $x|\gamma|$ extends to a bounded operator on H, denoted by $\overline{x|\gamma|}$, and such that $\overline{x|\gamma|} \in \mathcal{N}_{\mathrm{Tr}}$, and for $x \in D$ we put $\Lambda(x) = \Lambda_{\mathrm{Tr}}(\overline{x|\gamma|})$. The set D is then a core for the operator Λ which is closable for the σ-strong-$*$–norm topology.

Definition 7.3.3. *The nsf weight φ on M is defined by its GNS-construction $(\mathcal{K},\pi,\Lambda)$.*

So in particular, $\varphi(b^*a) = \langle \Lambda(a), \Lambda(b)\rangle_\mathcal{K}$ for all $a,b \in \mathcal{N}_\varphi$. Note the remarkable fact that for both the quantum group analogues of $SU(2)$ and $SU(1,1)$ the Haar weight are both of the form $\mathrm{Tr}_{\gamma^*\gamma}$, see Example 6.1.14.

Remark 7.3.4. Note that in particular we can use π to identify $M \subset B(H)$ with its image $\pi(M) \subset B(\mathcal{K})$. From now on we use this identification, and we work with M realized as von Neumann algebra in $B(\mathcal{K})$.

Lemma 7.3.5. *From the general theory of nsf weights, see Section 5.8, we know that φ comes with a modular automorphism group σ, a modular conjugation J and modular operator ∇. We have*

- $\sigma_t(x) = |\gamma|^{2it} x |\gamma|^{-2it}$ *for all* $x \in M$, $t \in \mathbb{R}$;
- $\Phi(m,p,t) \in \mathcal{N}_\varphi$ *and* $\Lambda(\Phi(m,p,t)) = |t|^{-1} f_{mpt}$;
- $\Phi(m,p,t) \in \mathcal{M}_\varphi$ *and* $\varphi(\Phi(m,p,t)) = |t|^{-2}\delta_{m,0}\delta_{p,t}$;
- $\Phi(m,p,t)$ *is analytic for* σ *and* $\sigma_z(\Phi(m,p,t)) = |p^{-1}t|^{2iz}\Phi(m,p,t)$ *for all* $z \in \mathbb{C}$;
- $J f_{mpt} = f_{-m,t,p}$;
- f_{mpt} *in the domain of* ∇ *and* $\nabla f_{mpt} = |p^{-1}t|^2 f_{mpt}$.

Proof. The first statement follows from the general discussion in Chapter 5. Observe that $\Phi(m,p,t)^*\Phi(m,p,t) = \Phi(0,t,t)$ is in $\mathcal{N}_{\mathrm{Tr}}$ and that $\Phi(m,p,t)|\gamma$ is a bounded operator, then the second statement follows from a calculation. The third statement follows by observing that $\Phi(m,p,t) = \Phi(0,p,p)\Phi(m,p,t)$, so that

$$\varphi\big(\Phi(m,p,t)\big) \;=\; \langle \Lambda(\Phi(m,p,t)), \Lambda(\Phi(0,p,p))\rangle \;=\; |pt|^{-1}\langle f_{mpt}, f_{0pp}\rangle$$

which gives the result.

A calculation shows that

$$|\gamma|^{2is}\Phi(m,p,t) \;=\; |p^{-1}t|^{2is}\Phi(m,p,t)|\gamma|^{2is}$$

and this implies $\sigma_z(\Phi(m,p,t)) = |p^{-1}t|^{2iz}\Phi(m,p,t)$ for all $z \in \mathbb{C}$ by the first result.

Consider the antilinear map S as in Chapter 5, then $Sf_{m,p,t} = |t| J\,\Lambda\big(\Phi(m,p,t)\big) = |t|\Lambda\big(\Phi(m,p,t)^*\big) = |t|\Lambda\big(\Phi(-m,t,p)\big) = \frac{|t|}{|p|} f_{-m,t,p}$. The last two statements follow from the polar decomposition of S, cf. Section 5.9.2. $\square$

From now on we work with $M \cong \pi(M) \subset B(\mathcal{K})$ in the GNS-representation.

7.4 The multiplicative unitary and the comultiplication

The multiplicative unitary $W \in B(\mathcal{K}\otimes\mathcal{K})$ has a useful description in terms of the functions $a_p(\cdot,\cdot)$, as defined in Definition 7.2.5,

$$\begin{aligned} W^*(f_{m_1,p_1,t_1}\otimes f_{m_2,p_2,t_2}) = \sum_{\substack{y,z\in I_q\\ \mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1\in I_q}} \left|\frac{t_2}{y}\right| a_{t_2}(p_1,y)a_{p_2}(z,\mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1) \\ \times f_{m_1+m_2-\chi(p_1p_2/t_2z),z,t_1}\otimes f_{\chi(p_1p_2/t_2z),\mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1,y}. \end{aligned} \tag{7.4.1}$$

For convenience we state the corresponding result for W as well, which follows directly from (7.4.1):

$$W(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) = \sum_{\substack{r,s\in I_q \\ \mathrm{sgn}(rp_2t_2)sp_1q^{m_2}\in I_q}} \left|\frac{s}{t_2}\right| a_s(\mathrm{sgn}(rp_2t_2)sp_1q^{m_2}, t_2)\, a_r(p_1,p_2) \times f_{m_1-\chi(sp_2/t_2),\mathrm{sgn}(rp_2t_2)sp_1q^{m_2},t_1} \otimes f_{m_2+\chi(sp_2/t_2),r,s}. \tag{7.4.2}$$

Proposition 7.4.1. *$W \in B(\mathcal{K}\otimes\mathcal{K})$ is unitary operator.*

Proof. This follows by a straightforward check using Propositions 7.2.6 and 7.2.7. □

Now we define the comultiplication $\Delta(x) = W^*(1\otimes x)W$ for $x \in M$.

Theorem 7.4.2. *Δ is coassociative, i.e. $(\Delta\otimes\iota)\circ\Delta = (\iota\otimes\Delta)\circ\Delta$.*

The proof of Theorem 7.4.2 is intense and laborious, and we will not repeat it here. The idea is to check it for generators of M, which has certain complications involving domains as α and γ are unbounded. We refer to [28] for details. Since the multiplicative unitary W is interpreted in terms of Clebsch-Gordan coefficients, one may wonder if Theorem 7.4.2 can be proved using ideas from Racah coefficients. This has not been proved. New developments by De Commer (unpublished) indicate that a different proof along the lines of Hopf-Galois modules might be feasible.

Theorem 7.4.3. *(M,Δ) is a locally compact quantum group with left and right Haar weight φ and multiplicative unitary W.*

Proof. According to Definition 6.1.1 and Theorem 7.4.2 it suffices to check that φ is a left Haar weight and that the right Haar weight can be taken equal to φ. This requires some von Neumann algebra techniques, for which we refer to [28]. □

In particular, W satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12} \in B(\mathcal{K}\otimes\mathcal{K}\otimes\mathcal{K}). \tag{7.4.3}$$

Here we use the standard leg notation.

7.5 The dual locally compact quantum group

According to the general definition the von Neumann we have to look at elements of the form $(\omega\otimes\iota)(W)$ for $\omega \in M_*$. However, we define

$$Q(p_1,p_2,n) = (\omega_{f,g}\otimes\iota)(W^*)\colon \mathcal{K}\to\mathcal{K}, \qquad f = f_{0,p_1,1},\ g = f_{n,p_2,1}, \tag{7.5.1}$$

where the normal functionals $\omega_{f,g} \in B(\mathcal{K})_*$ defined by $\omega_{f,g}(x) = \langle x\, f, g\rangle$, $f,g\in\mathcal{K}$.

Proposition 7.5.1. *The operators $Q(p_1,p_2,n) \in B(\mathcal{K})$, $p_1,p_2 \in I_q$, $n \in \mathbb{Z}$, are in $\hat{M}$ and the linear span is strong-$*$ dense in $\hat{M}$.*

Proof. We start by considering matrix elements of the more generally defined operator

$$(\omega_{f_{m_1,p_1,t_1},f_{m_2,p_2,t_2}} \otimes \iota)(W^*) \in B(\mathcal{K}),$$

with $m_1,m_2 \in \mathbb{Z}$ and $p_1,p_2,t_1,t_2 \in I_q$. For $n_1,n_2 \in \mathbb{Z}$ and $r_1,r_2,s_1,s_2 \in I_q$ we have

$$\begin{aligned}
&\big\langle (\omega_{f_{m_1,p_1,t_1},f_{m_2,p_2,t_2}} \otimes \iota)(W^*)\, f_{n_1,r_1,s_1}, f_{n_2,r_2,s_2}\big\rangle \\
&= \big\langle W^* f_{m_1,p_1,t_1} \otimes f_{n_1,r_1,s_1}, f_{m_2,p_2,t_2} \otimes f_{n_2,r_2,s_2}\big\rangle \\
&= \delta_{t_1,t_2}\delta_{n_1-n_2,m_2-m_1}\delta_{n_2,\chi(p_1r_1/s_1p_2)}\delta_{r_2,\mathrm{sgn}(r_1s_1)s_2p_2q^{n_1}/p_1} \\
&\quad \times \left|\frac{s_1}{s_2}\right| a_{s_1}(p_1,s_2)\, a_{r_1}(p_2,\mathrm{sgn}(r_1s_1)s_2p_2q^{n_1}/p_1),
\end{aligned}$$

where we used expression (7.4.1) for W^*. The dependence on $t_1,t_2 \in I_q$ and $m_1,m_2 \in \mathbb{Z}$ of the right hand side occurs only in the first two Kronecker deltas, so by (7.5.1) we have

$$(\omega_{f_{m_1,p_1,t_1},f_{m_2,p_2,t_2}} \otimes \iota)(W^*) = \delta_{t_1,t_2} Q(p_1,p_2,m_2-m_1). \tag{7.5.2}$$

We see that it suffices to restrict to the case $t_1=t_2=1$, $m_1=0$, $m_2=n$.

Recall

$$\big(\hat{J}\otimes J\big)\, W\, \big(\hat{J}\otimes J\big) \;=\; W^*, \tag{7.5.3}$$

where $\hat{J}\colon \mathcal{K}\to\mathcal{K}$ is the modular conjugation for the dual left invariant Haar weight $\hat{\varphi}$. The modular conjugation for the dual Haar weight can be evaluated explicitly;

$$\hat{J}\, f_{m,p,t} = \mathrm{sgn}(p)^{\chi(p)}\,\mathrm{sgn}(t)^{\chi(t)}\,(-1)^m\, f_{-m,p,t}, \qquad p,t \in I_q,\ m\in\mathbb{Z}. \tag{7.5.4}$$

(This can be proved from the results in [28] as follows. Since the right invariant weight equals the left invariant weight, we have $\hat{J}\Lambda(x) = \Lambda(R(x)^*)$ for $x \in \mathcal{N}$, see [43, Prop. 2.11]. Using [28, Prop. 4.14] for the explicit expression of the unitary antipode R we see that applying this expression with $x = \Phi(m,p,t)$ gives (7.5.4).)

We have to consider

$$\begin{aligned}
&(\omega_{f_{m_1,p_1,t_1},f_{m_2,p_2,t_2}} \otimes \iota)(W) = (\omega_{f_{m_1,p_1,t_1},f_{m_2,p_2,t_2}} \otimes \iota)\Big(\big(\hat{J}\otimes J\big)\, W^*\, \big(\hat{J}\otimes J\big)\Big) \\
&= J\,(\omega_{\hat{J} f_{m_1,p_1,t_1},\hat{J} f_{m_2,p_2,t_2}} \otimes \iota)\big(W^*\big)\, J \\
&= \mathrm{sgn}(p_1)^{\chi(p_1)}\mathrm{sgn}(p_2)^{\chi(p_2)}\mathrm{sgn}(t_1)^{\chi(t_1)}\mathrm{sgn}(t_2)^{\chi(t_2)}(-1)^{m_1+m_2} J\,(\omega_{f_{-m_1,p_1,t_1},f_{-m_2,p_2,t_2}} \otimes \iota)\big(W^*\big)\, J
\end{aligned}$$

using (7.5.3), $\hat{J}^2 = 1$, $\langle \hat{J}f, \hat{J}g\rangle = \langle g, f\rangle$, J being antilinear, and (7.5.4). It follows from the first paragraph, that we can restrict to the case $t_1=t_2=1$, $m_1=0$, $m_2=-n$. By (6.2.1) and $J^2=1$ we see that, up to a sign, $Q(p_1,p_2,n)$ equals $J\,(\omega\otimes\iota)(W)\,J$ for $\omega\in B(\mathcal{K})_*$. Recall from (6.2.2) that the unitary antipode $\hat{R}$ for the dual quantum group is given by $\hat{R}(x) = Jx^*J$, so that for $x\in\hat{M}$ we have $J\,x\,J = \hat{R}(x^*) \in \hat{M}$. Now we see that $Q(p_1,p_2,n)\in\hat{M}$.

In order to prove the density statement, we require more von Neumann algebra techniques, for which we refer to [16]. □

Using the definition and the matrix elements of W^* in (7.4.1) it is straightforward to check that

$$\langle Q(p_1,p_2,n)\, f_{uvw}, f_{lrs}\rangle \;=\; \delta_{u-l,n}\delta_{l,\chi(p_1v/p_2w)}\delta_{r,\mathrm{sgn}(vw)sp_2q^u/p_1}\left|\frac{w}{s}\right|\, a_w(p_1,s)\, a_v(p_2,r). \qquad (7.5.5)$$

Since the operators $Q(p_1,p_2,n)$ span $\hat{M}$ linearly, we calculate the structure constants, i.e. the multiplication table, and the comultiplication on the generators. Both results essentially depend on the pentagon equation for the multiplicative unitary, as will be clear from the proofs.

Proposition 7.5.2. *For $p_1,p_2,r_1,r_2\in I_q$, $n,m\in\mathbb{Z}$, we have $Q(p_1,p_2,n)\,Q(r_1,r_2,m)=0$ in case $|\frac{p_2}{p_1}|\neq q^m$ or $|\frac{r_1}{r_2}|\neq q^n$. In case $|\frac{p_2}{p_1}|=q^m$ and $|\frac{r_1}{r_2}|=q^n$ we have*

$$Q(p_1,p_2,n)\,Q(r_1,r_2,m) = \sum_{x_1,x_2\in I_q} a_{x_1}(r_1,p_1)\,a_{x_2}(r_2,p_2)\,Q(x_1,x_2,n+m)$$

where the coefficients $a_{x_i}(r_i,p_i)$, $i=1,2$, are defined in Definition 7.2.5.

Remark 7.5.3. In particular, the linear span of $Q(p_1,p_2,n)$, $p_1>0$, $p_2>0$, forms a sub von Neumann algebra of $\hat{M}$. To see this we need that for $p_1,p_2\in I_q$, $n\in\mathbb{Z}$, we have

$$Q(p_1,p_2,n)^* = (-q)^n\,\mathrm{sgn}(p_1)^{\chi(p_1)}\,\mathrm{sgn}(p_2)^{\chi(p_2)}\,Q(p_1,p_2,-n).$$

One way to see that this is true is to check that

$$\langle f_{mpt}, Q(p_1,p_2,n) f_{lrs}\rangle \qquad \text{and} \qquad \langle Q(p_1,p_2,-n)\, f_{mpt}, f_{lrs}\rangle$$

agree up to the factor $(-q)^n\mathrm{sgn}(p_1)^{\chi(p_1)}\,\mathrm{sgn}(p_2)^{\chi(p_2)}$ for all m,p,t,l,r,s using the matrix elements (7.5.5) of $Q(p_1,p_2,n)$

Since the von Neumann algebra $\hat{M}$ is equipped with a comultiplication, we can also calculate the action of $\hat{\Delta}$. The result is the following.

Proposition 7.5.4. *For $p_1,p_2\in I_q$, $n\in\mathbb{Z}$, we have*

$$\hat{\Delta}\big(Q(p_1,p_2,n)\big) = \sum_{m\in\mathbb{Z},\,p\in I_q} Q(p,p_2,n-m)\otimes Q(p_1,p,m),$$

where the sum converges in the σ-weak-topology of $\hat{M}\otimes\hat{M}$.

Proof of Proposition 7.5.2. First observe that as elements of $B(\mathcal{K})$

$$\big((\omega_{f,g}\otimes\iota)(W^*)\big)\,\big((\omega_{\xi,\eta}\otimes\iota)(W^*)\big) \;=\; \big(\omega_{\xi,\eta}\otimes\omega_{f,g}\otimes\iota\big)(W^*_{23}W^*_{13})$$

for arbitrary vectors $f,g,\xi,\eta\in\mathcal{K}$. Using the pentagonal equation $W_{12}W_{13}W_{23}=W_{23}W_{12}$ this can be rewritten in the compact form

$$\big((\omega_{f,g}\otimes\iota)(W^*)\big)\,\big((\omega_{\xi,\eta}\otimes\iota)(W^*)\big) \;=\; \big(\omega_{W(\xi\otimes f),W(\eta\otimes g)}\otimes\iota\big)(\iota\otimes W^*). \qquad (7.5.6)$$

We start with the choice $f = f_{0,p_1,1}$, $g = f_{n,p_2,1}$, $\xi = f_{0,r_1,1}$, $\eta = f_{m,r_2,1}$, so that the left hand side of (7.5.6) equals $Q(p_1,p_2,n)\,Q(r_1,r_2,m)$. In order to evaluate the right hand side of (7.5.6) we use (7.4.2), which leads to

$$\begin{aligned}\sum_{\substack{x_1,y_1\in I_q\\ \text{so that } \mathrm{sgn}(x_1p_1)y_1r_1\in I_q}} \ \sum_{\substack{x_2,y_2\in I_q\\ \text{so that } \mathrm{sgn}(x_2p_2)y_2r_2\in I_q}} & |y_1y_2|\, a_{x_1}(r_1,p_1)\, a_{x_2}(r_2,p_2) \\ \times\ & a_{y_1}(\mathrm{sgn}(x_1p_1)y_1r_1,1)\, a_{y_2}(\mathrm{sgn}(x_2p_2)y_2r_2q^n,1) \\ \times\ & \langle f_{-\chi(y_1p_1),\mathrm{sgn}(x_1p_1)y_1r_1,1}, f_{m-\chi(y_2p_2),\mathrm{sgn}(x_2p_2)y_2r_2q^n,1}\rangle \\ \times\ & \big(\omega_{f_{\chi(y_1p_1),x_1,y_1}, f_{n+\chi(y_2p_2),x_2,y_2}}\otimes\iota\big)(W^*).\end{aligned} \tag{7.5.7}$$

The inner product in the summand of (7.5.7) leads to

$$\delta_{-\chi(y_1p_1),m-\chi(y_2p_2)}\delta_{\mathrm{sgn}(x_1p_1)y_1r_1,\mathrm{sgn}(x_2p_2)y_2r_2q^n},$$

whereas, by (7.5.2), the last term in the summand is $\delta_{y_1,y_2}\,Q(x_1,x_2,n+\chi(p_2)-\chi(p_1))$. Combining this we see that the last two terms in the summand of (7.5.7) equal

$$\delta_{\chi(p_2),m+\chi(p_1)}\,\delta_{\mathrm{sgn}(x_1p_1)r_1,\mathrm{sgn}(x_2p_2)r_2q^n}\,\delta_{y_1,y_2}\,Q(x_1,x_2,n+m),$$

which is zero in case $|\frac{p_2}{p_1}|\neq q^m$ independent of x_1, y_1, x_2, y_2.

Assuming $|\frac{p_2}{p_1}| = q^m$ and inserting this into (7.5.7) leads to

$$\begin{aligned}&\sum_{x_1,x_2\in I_q} a_{x_1}(r_1,p_1)\, a_{x_2}(r_2,p_2)\, Q(x_1,x_2,n+m) \\ &\times\Big(\sum_{\substack{y_1\in I_q \text{ so that}\\ \mathrm{sgn}(x_1p_1)y_1r_1=\mathrm{sgn}(x_2p_2)y_1r_2q^n\in I_q}} y_1^2\, a_{y_1}(\mathrm{sgn}(x_1p_1)y_1r_1,1)\, a_{y_1}(\mathrm{sgn}(x_2p_2)y_1r_2q^n,1)\Big),\end{aligned} \tag{7.5.8}$$

where empty sums are zero. For the expression in (7.5.8) to be non-zero result we require $\mathrm{sgn}(x_1) = \mathrm{sgn}(r_1p_1)$ and $\mathrm{sgn}(x_2) = \mathrm{sgn}(r_2p_2)$, see Definition 7.2.5. Then we see that $\mathrm{sgn}(x_1p_1)y_1r_1 = y_1|r_1|$ and $\mathrm{sgn}(x_2p_2)y_1r_2q^n = y_1|r_2|q^n$, and so the inner sum is zero unless $|\frac{r_1}{r_2}| = q^n$. In this case the inner sum equals

$$\sum_{\substack{y_1\in I_q \text{ so that}\\ y_1|r_1|\in I_q}} y_1^2\,\big(a_{y_1}(y_1|r_1|,1)\big)^2 = \sum_{\substack{y_1\in I_q \text{ so that}\\ y_1|r_1|\in I_q}} \big(a_1(y_1,y_1|r_1|)\big)^2 = 1,$$

where the first equality follows from the symmetry relations (7.2.3), and the second equality is a special case of Proposition 7.2.6 (with $p=1$ and $\theta = |r_1|$).

Collecting the results finishes the proof of Proposition 7.5.2. □

Proof of Proposition 7.5.4. In order to calculate the action of the comultiplication of the dual quantum group on the elements $Q(p_1,p_2,n)$, we note that this can be done in greater generality. First observe

$$\begin{aligned} W\big((\omega_{f,g}\otimes\iota)(W^*)\otimes\iota\big)W^* &= (\omega_{f,g}\otimes\iota\otimes\iota)(W_{23}W_{12}^*W_{23}^*) \\ &= (\omega_{f,g}\otimes\iota\otimes\iota)(W_{13}^*W_{12}^*).\end{aligned} \tag{7.5.9}$$

The first equality is straightforward, and the second follows from the pentagonal equation for the multiplicative unitary, see Chapter 6. Using an orthonormal basis $\{e_k\}$ for the Hilbert space $\mathcal{K}$, so that we have $\langle x,y\rangle = \sum_k \langle x,e_k\rangle\langle e_k,y\rangle$ we get

$$\begin{aligned}\Sigma\,\hat{\Delta}\big((\omega_{f,g}\otimes\iota)(W^*)\big)\Sigma &= (\omega_{f,g}\otimes\iota\otimes\iota)(W^*_{13}W^*_{12})\\ &= \sum_k (\omega_{f,e_k}\otimes\iota)(W^*)\,\otimes\,(\omega_{e_k,g}\otimes\iota)(W^*).\end{aligned} \tag{7.5.10}$$

using the definition of $\hat{\Delta}$ and notation as in Theorem 6.2.1.

We use the general formula (7.5.10) with $f=f_{0,p_1,1}$, $g=f_{n,p_2,1}$, the orthonormal basis $f_{m,p,t}$ ($m,\in\mathbb{Z}$, $p,t\in I_q$) and next (7.5.2) to rewrite the right hand side in terms of the operators $Q(p_1,p_2,n)$. The series converges in the von Neumann algebra $\hat{M}\otimes\hat{M}$, so that we find convergence in the σ-weak topology. □

7.6 Results for special functions

In this section we state some results for the functions $a_p(x,y)$ that follow from Propositions 7.5.2, 7.5.4.

By taking the non-trivial structure constants of Proposition 7.5.2 and considering matrix coefficients at both sides we obtain the following theorem.

Theorem 7.6.1. *For $p_1,p_2,r_1,r_2\in I_q$, $l,n,m\in\mathbb{Z}$, $\varepsilon,\eta\in\{\pm\}$ and with $z\in I_q$ so that $\mathrm{sgn}(z)=\varepsilon$ and $\varepsilon\eta pq^l z\in I_q$ and with $w\in I_q$ so that $\mathrm{sgn}(w)=\varepsilon\mathrm{sgn}(r_1p_1)$ and $\varepsilon\eta\mathrm{sgn}(r_1p_1r_2p_2)pq^{l+m+n}w\in I_q$ we have*

$$\begin{aligned}&\sum_{\substack{x\in I_q \text{ so that } \mathrm{sgn}(x)=\mathrm{sgn}(r_1p_1)\\ \text{and } |x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}\in I_q}} a_z(x,w)\,a_x(r_1,p_1)\,a_{|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}}(r_2,p_2)\\ &\qquad\times\; a_{\varepsilon\eta pq^l z}(|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n},\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w) = \delta_{|\frac{r_1}{r_2}|,pq^{-2l-m}}\,\delta_{|\frac{p_1}{p_2}|,pq^{-2l-2m-n}}\\ &\times \sum_{\substack{u\in I_q \text{ so that } \mathrm{sgn}(u)=\mathrm{sgn}(r_1)\varepsilon\\ \text{and } \varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u\in I_q}} a_z(r_1,u)\,a_u(p_1,w)\,a_{\varepsilon\eta pq^l z}(r_2,\varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u)\\ &\qquad\times\; a_{\varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u}(p_2,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w),\end{aligned}$$

where the series on both sides converge absolutely.

Remark 7.6.2. The formula of Theorem 7.6.1 contains many special cases involving q-Laguerre polynomials, big q-Bessel functions, Al-Salam–Carlitz polynomials and q-Charlier polynomials as special cases by suitable specializing the signs in the formula. Note moreover that in all cases the sums are essentially sums over $q^{\mathbb{Z}}$ or $q^{\mathbb{N}}$. For each particular choice of the signs the square roots occurring in Definition 7.2.5 in Theorem 7.6.1 will cancel or can be taken together. It would be of interest to find a direct analytic proof.

Corollary 7.6.3. *For $r_1, r_2 \in I_q$, $l, m \in \mathbb{Z}$ and with $z \in I_q$ so that $\mathrm{sgn}(z) = \varepsilon$ and $\varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z \in I_q$ and we have*

$$\sum_{x\in q^{\mathbb{Z}}} \begin{array}{l}(-\eta)^{l+m}(\eta\mathrm{sgn}(r_1))^{\chi(r_1)}\,(\eta\mathrm{sgn}(r_2))^{\chi(r_2)}\,(\varepsilon\eta)^{\chi(z)} \\ x^2\, a_x(r_1,r_1)\, a_x(z,z)\, a_{xq^{-m}|\frac{r_2}{r_1}|}(r_2,r_2)\, a_{xq^{-m}|\frac{r_2}{r_1}|}(\varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z, \varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z) > 0\end{array}$$

and for $a \in \mathbb{Z}$ and $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$ we have

$$\sum_{k\in\mathbb{Z}} \frac{q^k}{(-q^k, -q^{k+a}; q)_\infty} L_{n_1}^{(0)}(q^k;q)\, L_{n_2}^{(0)}(q^k;q)\, L_{n_3}^{(0)}(q^{k+a};q)\, L_{n_4}^{(0)}(q^{k+a};q) \;> 0.$$

Note that the sum is closely related to one of the orthogonality measures for the q-Laguerre polynomials, which correspond to an indeterminate moment problem. A similar positivity result can be obtained for the q-Bessel functions involved.

Theorem 7.6.4. *For fixed $r \in q^{\mathbb{Z}}$, $m_1, m_2, M, n \in \mathbb{Z}$, $p_1, p_2 \in I_q$, $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \sigma \in \{\pm\}$ and for $z_1, z_2, w_1, w_2 \in I_q$ satisfying*

$$\begin{aligned}
&\mathrm{sgn}(z_i) = \varepsilon_i,\ (i=1,2), \quad \varepsilon_1\eta_1 q^{m_1} r z_1 \in I_q, \quad \varepsilon_2\eta_2 q^{-2m_1-m_2-n}\frac{z_2|p_2|}{r|p|} \in I_q,\\
&\mathrm{sgn}(w_1) = \mathrm{sgn}(p_1)\varepsilon_1, \quad \mathrm{sgn}(w_2) = \sigma\varepsilon_2, \qquad \sigma\mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1 \in I_q,\\
&\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{-2m_1-m_2-M}\frac{w_2|p_2|}{r|p|} \in I_q
\end{aligned}$$

and such that $a_{z_1}(p_1, w_1) \neq 0$ we have

$$\begin{aligned}
&\frac{1}{w_2^2} a_{ep_1\eta_1 q^{m_1} r z_1}(\sigma|p_1|rq^{2m_1+M}, \varepsilon_1\eta_1\sigma\mathrm{sgn}(p_1)w_1 r q^{m_1+M})\, a_{z_2}(\sigma|p_1|rq^{2m_1+M}, w_2)\\
&\qquad \times\, a_{\varepsilon_2\eta_2|\frac{p_2}{p_1}|\frac{z^2}{r}q^{-2m_1-m_2-n}}(p_2, \varepsilon_2\eta_2\sigma\frac{p_2 w_2}{|p_1|r}q^{-2m_1-m_2-M}) =\\
&\sum_{\substack{y,x\in I_q \text{ so that } \mathrm{sgn}(y)=\varepsilon_2\eta_1 \text{ and} \\ \mathrm{sgn}(p_1p_2)q^n x w_1/z_1 \in I_q,\ \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2} yx/rz_1 \in I_q}} \frac{1}{y^2}\, a_{z_2}(\varepsilon_1\eta_1 q^{m_1} r z_1, y)\, a_{w_2}(\mathrm{sgn}(p_1)\sigma\varepsilon_1\eta_1 q^{m_1+M} r w_1, y)\\
&\times\, a_{\varepsilon_2\eta_2\frac{z_2|p_2|}{r|p_1|}q^{-2m_1-m_2-n}}(x, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{yx}{rz_1})\, a_x(p_2, \mathrm{sgn}(p_1p_2)q^n\frac{xw}{z_1})\\
&\times\, a_{\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2\frac{w_2|p_2|}{r|p_1|}q^{-2m_1-m_2-M}}(\mathrm{sgn}(p_1p_2)q^n\frac{xw_1}{z_1}, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{yx}{rz_1})
\end{aligned}$$

where the left-hand-side is considered to be zero in case $\sigma|p_1|rq^{2m_1+M} \notin I_q$. The series converges absolutely.

Remark 7.6.5. (i) First note that the largest part of Remark 7.6.2 is also applicable to Theorem 7.6.4, except for the fact that the summation is more involved. Viewing the summation

as a sum over an area in $I_q \times I_q \subset \mathbb{R}^2$ (with x on the horizontal axis and y on the vertical axis), we see that the summation area is a subset of $I_q \times I_q$ bounded by a vertical line and a hyperbola. Depending on the sign choices there are eight possibilities for the location of the vertical line and the hyperbola.
(ii) Theorem 7.6.4 follows from the operator identity in Proposition 7.5.4, but the single term in the left hand side of Theorem 7.6.4 corresponds to summation on the left hand side of Proposition 7.5.4, whereas the double sum on the right hand side of Theorem 7.6.4 corresponds to the single term on the right hand side of Proposition 7.5.4.
(iii) Since the results in Theorems 7.6.1 and 7.6.4 both reflect the pentagonal equation for the multiplicative unitary, one might expect the resulting identities to be equivalent by using the orthogonality relations of Propositions 7.2.6 and 7.2.7. However, this is not the case as follows by considering the dependence of both results on the free parameters.

We start by introducing some notation. Let $p \in q^{\mathbb{Z}}$, $m \in \mathbb{Z}$ and $\varepsilon, \eta \in \{-,+\}$, and define

$$\begin{aligned} J(p,m,\varepsilon,\eta) &= \{\, z \in I_q \mid \varepsilon\eta\, q^m pz \in I_q \text{ and } \operatorname{sgn}(z) = \varepsilon\}, \\ \mathcal{K}_0(p,m,\varepsilon,\eta) &= \operatorname{span}\{\, f_{-m,\varepsilon\eta\, q^m pz,z} \mid z \in J(p,m,\varepsilon,\eta)\,\}. \end{aligned} \tag{7.6.1}$$

We denote by $\mathcal{K}(p,m,\varepsilon,\eta)$ the closure of $\mathcal{K}_0(p,m,\varepsilon,\eta)$ inside $\mathcal{K}$. Then $\mathcal{K}(p,m,\varepsilon,\eta) \cong \ell^2(J(p,m,\varepsilon,\eta))$, and we consider $v \in \mathcal{K}(p,m,\varepsilon,\eta)$ as a function $v\colon J(p,m,\varepsilon,\eta) \to \mathbb{C}$ by setting

$$v(z) = \langle v, f_{-m,\varepsilon\eta\, q^m pz,z}\rangle, \qquad z \in J(p,m,\varepsilon,\eta). \tag{7.6.2}$$

By convention, for $z \in \pm q^{\mathbb{Z}} \setminus J(p,m,\varepsilon,\eta)$ we set $v(z) = 0$. Note that $J(p,m,\varepsilon,\eta) = I_q^+ = q^{\mathbb{Z}}$ if $\varepsilon = \eta = +$. If $\varepsilon = -$ or $\eta = -$, then $J(p,m,\varepsilon,\eta)$ is a bounded q-halfline with 0 as only accumulation point.

Then the operators $Q(p_1,p_2,n)$ can be restricted to these subspaces as follows.

Lemma 7.6.6. *Let $p \in q^{\mathbb{Z}}$, $p_1, p_2 \in I_q$, $n, m \in \mathbb{Z}$, and $\varepsilon, \eta \in \{-,+\}$. If $q^{2m}p \neq q^{-n}|p_2/p_1|$, then*

$$Q(p_1,p_2,n)\big(\mathcal{K}(p,m,\varepsilon,\eta)\big) = \{0\}.$$

If $q^{2m}p = q^{-n}|p_2/p_1|$, then

$$Q(p_1,p_2,n)\colon \mathcal{K}(p,m,\varepsilon,\eta) \to \mathcal{K}(p,m+n,\operatorname{sgn}(p_1)\varepsilon,\operatorname{sgn}(p_2)\eta),$$

and $Q(p_1,p_2,n)$ is given explicitly by

$$\begin{aligned} Q(p_1,p_2,n)f &= (-1)^{m'}(\eta')^{\chi(p_1p_2)+m}\frac{|p_1p_2|}{q^m p} \\ \times &\sum_{w\in J(p,m',\varepsilon',\eta')}\Big(\frac{(\varepsilon'\eta')^{\chi(w)}}{|w|}\sum_{z\in J(p,m,\varepsilon,\eta)}\frac{f(z)}{|z|}\, a_{p_1}(z,w)\, a_{p_2}(\varepsilon\,\eta\, q^m p\, z, \varepsilon'\eta' q^{m'} p\, w)\Big) f_{-m',\varepsilon'\eta' q^{m'}pw,w}, \end{aligned}$$

where $f \in \mathcal{K}(p,m,\varepsilon,\eta)$, $\varepsilon' = \operatorname{sgn}(p_1)\varepsilon$, $\eta' = \operatorname{sgn}(p_2)\eta$, $m' = m+n$.

Exercise 7.6.7. Supply Lemma 7.6.6 with a proof.

Proof of Theorem 7.6.1. We start with the result of Proposition 7.5.2 and we next let the corresponding operator identity act on $f_{-l,\varepsilon\eta pq^l z,z} \in \mathcal{K}(p,l,\varepsilon,\eta)$. Lemma 7.6.6 shows that

$$Q(p_1,p_2,n)\,Q(r_1,r_2,m)\colon \mathcal{K}(p,l,\varepsilon,\eta) \to \mathcal{K}(p,l+m+n,\mathrm{sgn}(r_1p_1)\varepsilon,\mathrm{sgn}(r_2p_2)\eta)$$

is non-zero precisely if $q^{2l}p = q^{-m}|\frac{r_2}{r_1}|$ and $q^{2l+2m}p = q^{-n}|\frac{p_2}{p_1}|$. In particular, in case $q^{-n}|\frac{p_2}{p_1}| \neq q^m|\frac{r_2}{r_1}|$ we find $Q(p_1,p_2,n)\,Q(r_1,r_2,m) = 0$.

In order to calculate the appropriate matrix coefficient we proceed for $f_{-l-m-n,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w,w} \in \mathcal{K}(p,l+m+n,\mathrm{sgn}(r_1p_1)\varepsilon,\mathrm{sgn}(r_2p_2)\eta)$ as

$$\begin{aligned} &\langle Q(p_1,p_2,n)\,Q(r_1,r_2,m)\,f_{-l,\varepsilon\eta pq^l z,z}, f_{-l-m-n,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w,w}\rangle = \\ &\sum_{u\in J(p,l+m,\varepsilon\mathrm{sgn}(r_1),\eta\mathrm{sgn}(r_2))} \langle Q(r_1,r_2,m)\,f_{-l,\varepsilon\eta pq^l z,z}, f_{-l-m,\mathrm{sgn}(r_1r_2)\varepsilon\eta pq^{l+m}u,u}\rangle \\ &\qquad \times \langle Q(p_1,p_2,n)\,f_{-l-m,\mathrm{sgn}(r_1r_2)\varepsilon\eta pq^{l+m}u,u}, f_{-l-m-n,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w,w}\rangle \end{aligned}$$

using the orthogonal basis for the intermediate space $\mathcal{K}(p,l+m,\varepsilon\mathrm{sgn}(r_1),\eta\mathrm{sgn}(r_2))$. In this sum we can use (7.5.5) twice, and using (7.6.1) we find that this equals

$$\begin{aligned} &\delta_{|\frac{r_1}{r_2}|p,q^{-2l-m}}\,\delta_{|\frac{p_1}{p_2}|p,q^{-2l-2m-n}} \sum_{\substack{u\in I_q \text{ so that } \mathrm{sgn}(u)=\mathrm{sgn}(r_1)\varepsilon \\ \text{and } \varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u\in I_q}} \left|\frac{z}{w}\right| a_z(r_1,u)\,a_u(p_1,w) \\ &\quad\times\, a_{\varepsilon\eta pq^l z}(r_2,\varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u)\,a_{\varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u}(p_2,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w) \end{aligned} \tag{7.6.3}$$

Next observe

$$Q(x_1,x_2,m+n)\colon \mathcal{K}(p,l,\varepsilon,\eta) \to \mathcal{K}(p,l+m+n,\mathrm{sgn}(x_1)\varepsilon,\mathrm{sgn}(x_2)\eta)$$

is non-zero only if $q^{2l}p = q^{-m-n}|\frac{x_2}{x_1}|$, so that the double sum in Proposition 7.5.2 reduces to a single sum. Moreover, by Definition 7.2.5 shows that in the sum the functions $a_{x_i}(r_i,p_i)$ for $i=1,2$ are non-zero only if $\mathrm{sgn}(x_i) = \mathrm{sgn}(r_ip_i)$ for $i=1,2$. So the matrix element for the expression on the right hand side is

$$\sum_{x_1,x_2\in I_q} a_{x_1}(r_1,p_1)\,a_{x_2}(r_2,p_2)\,\langle Q(x_1,x_2,m+n)\,f_{-l,\varepsilon\eta pq^l z,z}, f_{-l-m-n,\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w,w}\rangle$$

and this reduces to a single sum and the summand is evaluated by (7.5.5). By eliminating x_2 and renaming x_1 by x we see that this equals

$$\begin{aligned} &\sum_{\substack{x\in I_q \text{ so that } \mathrm{sgn}(x)=\mathrm{sgn}(r_1p_1) \\ \text{and } |x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}\in I_q}} \left|\frac{z}{w}\right| a_x(r_1,p_1)\,a_z(x,w)\,a_{|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}}(r_2,p_2) \\ &\qquad\times\, a_{\varepsilon\eta pq^l z}(|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n},\mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w) \end{aligned} \tag{7.6.4}$$

Finally, equating (7.6.3) and (7.6.4) gives the result, where the conditions on the parameters in Theorem 7.6.1 follows from the fact that the matrix elements are taken with respect to vectors in the GNS Hilbert space. □

Proof of Corollary 7.6.3. In the proof of Theorem 7.6.1 we take $r_1 = p_1$, $r_2 = p_2$ and $m = -n$, and we multiply with the factor given in Remark 7.5.3. This shows that the sum considered on the left hand side in the proof of Theorem 7.6.1 is positive. Now rework the right hand side. The special case for the q-Laguerre polynomials corresponds to r_1, r_2 negative and z negative (so $\varepsilon = -1$) and $\eta = -1$. □

Proof of Theorem 7.6.4. For the proof it is easier to start by conjugating the result of Proposition 7.5.4 with the flip operator to obtain

$$\sum_{p \in I_q,\, m \in \mathbb{Z}} Q(p_1, p, m) \otimes Q(p, p_2, n-m) = W\big(Q(p_1, p_2, n) \otimes \iota\big) W^*, \tag{7.6.5}$$

which is a consequence of the proof of Proposition 7.5.4. We let both sides act on

$$f_{-m_1, \varepsilon_1\eta_1 q^{m_1} r_1 z_1, z_1} \otimes f_{-m_2, \varepsilon_2\eta_2 q^{m_2} r_2 z_2, z_2} \in \mathcal{K}(r_1, m_1, \varepsilon_1\eta_1) \otimes \mathcal{K}(r_2, m_2, \varepsilon_2\eta_2)$$

and we take inner products with

$$\begin{aligned} &f_{-m_1-M, \sigma \mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1, w_1} \otimes f_{-m_2-n+M, \sigma \mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{m_2+n-M} r_2 w_2, w_2} \\ &\qquad \in \mathcal{K}(r_1, m_1+M, \mathrm{sgn}(p_1)\varepsilon_1, \sigma\eta_1) \otimes \mathcal{K}(r_2, m_2+n-M, \sigma\varepsilon_2, \mathrm{sgn}(p_2)\eta_2). \end{aligned}$$

Then the sum over I_q and $\mathbb{Z}$ reduces to a single term by a double application of (7.5.5). Indeed, we find that we need $m = M$ and $\mathrm{sgn}(p) = \sigma$ for a non-zero contribution, but also both the conditions $q^{2m_1+M} = |\frac{p}{p_1 r_1}|$ and $q^{2m_2+n-M} = |\frac{p_2}{p r_2}|$ need to be satisfied. So for the matrix element of the left hand side of (7.6.5) to have a single non-zero term we require $r_1 r_2 q^{2m_1+2m_2} = q^{-n}|\frac{p_2}{p_1}|$, and in this case the left hand side equals

$$\begin{aligned} &\left|\frac{z_1 z_2}{w_1 w_2}\right| a_{z_1}(p_1, w_1)\, a_{\varepsilon_1\eta_1 q^{m_1} r_1 z_1}(\sigma|p_1|r_1 q^{2m_1+M}, \varepsilon_1\eta_1 w_1 \sigma \mathrm{sgn}(p_1) r_1 q^{m_1+M}) \\ &\times\, a_{z_2}(\sigma|p_1|r_1 q^{2m_1+M}, w_2)\, a_{\varepsilon_2\eta_2 q^{-2m_1-m_2-n}\frac{|p_2|z_2}{|p_1|r_1}}(p_2, \varepsilon_2\eta_2\sigma q^{-2m_1-m_2-M}\frac{w_2 p_2}{|p_1|r_1}) \end{aligned} \tag{7.6.6}$$

where we have chosen to eliminate r_2. Here all arguments of the function $a_p(x,y)$ are indeed elements of I_q, except possible $\sigma|p_1|r_1 q^{2m_1+M}$ and in case $\sigma|p_1|r_1 q^{2m_1+M} \notin I_q$ the expression has to be read as zero.

In order to calculate the same matrix element for the right hand side of (7.6.5) we rewrite this matrix element as

$$\begin{aligned} \Big\langle &\big(Q(p_1, p_2, n) \otimes \iota\big) W^*(f_{-m_1, \varepsilon_1\eta_1 q^{m_1} r_1 z_1, z_1} \otimes f_{-m_2, \varepsilon_2\eta_2 q^{m_2} r_2 z_2, z_2}), \\ &\qquad W^*(f_{-m_1-M, \sigma \mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1, w_1} \otimes f_{-m_2-n+M, \sigma \mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{m_2+n-M} r_2 w_2, w_2})\Big\rangle. \end{aligned} \tag{7.6.7}$$

In this expression we use (7.4.1) twice, with parameters y_1, x_1 (instead of y, z as in (7.4.1)) for the action of W^* in the left leg of the inner product and with parameters y_2, x_2 for the action of W^* in the left leg of the inner product. The resulting four-fold sum has the advantage that the inner product factorizes, and we obtain

$$\begin{aligned}
\sum \left|\frac{z_2 w_2}{y_1 y_2}\right| \, & a_{z_2}(\varepsilon_1\eta_1 q^{m_1} r_1 z_1, y_1)\, a_{\varepsilon_2\eta_2 q^{m_2} r_2 z_2}(x_1, \varepsilon_1\varepsilon_2\eta_1\eta_2 y_1 x_1 q^{-m_1-m_2}/r_1 z_1) \\
& \times\; a_{w_2}(\sigma \mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1, y_2) \\
& \times\; a_{\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{m_2+n-M} r_2 w_2}(x_2, \varepsilon_1\varepsilon_2\eta_1\eta_2 \mathrm{sgn}(p_1p_2) y_2 x_2 q^{-m_1-m_2-n}/r_1 w_1) \\
& \times\; \big\langle Q(p_1,p_2,n)\, f_{-2m_1-2m_2-\chi(r_1r_2z_1/x_1),x_1,z_1},\, f_{-2m_1-2m_2-2n-\chi(r_1r_2w_1/x_2),x_2,w_1}\big\rangle \\
& \times\; \big\langle f_{m_1+m_2+\chi(r_1r_2z_1/x_1),\varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2} y_1x_1/r_1z_1, y_1}, \\
& \qquad\qquad f_{m_1+m_2+n+\chi(r_1r_2w_1/x_2),\varepsilon_1\varepsilon_2\eta_1\eta_2\mathrm{sgn}(p_1p_2)q^{-m_1-m_2-n}y_2x_2/r_1w_1,y_2}\big\rangle
\end{aligned} \tag{7.6.8}$$

where the sum is four-fold; $y_1, x_1, y_2, x_2 \in I_q$ so that $\varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2} y_1 x_1/r_1 z_1 \in I_q$ and $\varepsilon_1\varepsilon_2\eta_1\eta_2 \mathrm{sgn}(p_1p_2) y_2 x_2 q^{-m_1-m_2-n}/r_1 w_1 \in I_q$.

The final term in the summand (7.6.8) gives three Kronecker delta's, which lead to the reduction of the four-fold sum to a double(!) sum since $y_2 = y_1$ and $x_2 = \mathrm{sgn}(p_1p_2) q^n x_1 w_1/z_1$ are required. Substituting this in the matrix element of $Q(p_1,p_2,n)$ in the summand in (7.6.8) gives

$$\big\langle Q(p_1,p_2,n)\, f_{-2m_1-2m_2-\chi(r_1r_2z_1/x_1),x_1,z_1},\, f_{-2m_1-2m_2-n-\chi(r_1r_2z_1/x_1),\mathrm{sgn}(p_1p_2)q^n x_1 w_1/z_1, w_1}\big\rangle$$

and by (7.5.5) this equals zero unless $r_1 r_2 = |\frac{p_2}{p_1}| q^{-2m_1-2m_2-n}$. In case this condition holds we see that the matrix coefficient of $Q(p_1,p_2,n)$ equals

$$\left|\frac{z_1}{w_1}\right| \, a_{z_1}(p_1, w_1)\, a_{x_1}(p_2, \mathrm{sgn}(p_1p_2) q^n \frac{x_1 w_1}{z_1}).$$

Eliminating again r_2 and using this we find that (7.6.8) equals

$$\begin{aligned}
\sum_{\substack{y_1,x_1\in I_q \text{ so that } \mathrm{sgn}(p_1p_2q^n x_1 w_1/z_1 \in I_q \\ \text{and } \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2} y_1x_1/r_1z_1 \in I_q}} & \left|\frac{z_2 w_2}{y_1^2}\right| \, a_{z_2}(\varepsilon_1\eta_1 q^{m_1} r_1 z_1, y_1) \\
\times\; a_{\varepsilon_2\eta_2 q^{-2m_1-m_2-n}\frac{z_2|p_2|}{r_1|p_1|}} & (x_1, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{y_1x_1}{r_1z_1})\, a_{w_2}(\sigma\mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1, y_1) \\
\times\; a_{\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{-2m_1-m_2-M}\frac{w_2|p_2|}{r_1|p_1|}} & (\mathrm{sgn}(p_1p_2) q^n x_1 w_1/z_1, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{y_1x_1}{r_1z_1}) \\
\times\; \left|\frac{z_1}{w_1}\right| \, & a_{z_1}(p_1, w_1)\, a_{x_1}(p_2, \mathrm{sgn}(p_1p_2) q^n \frac{x_1 w_1}{z_1}).
\end{aligned} \tag{7.6.9}$$

Equating (7.6.6) and (7.6.9) and canceling common factors and relabeling r_1, x_1, y_1 by r, x, y then proves Theorem 7.6.4 except for the sign constraint on y in the sum. This follows from Definition 7.2.5. $\square$

7.7 Relation between $\hat{M}$ and $U_q(\mathfrak{su}(1,1))$

It is clear from the construction that M is related to the Hopf $*$-algebra corresponding to $A_q(SU(1,1))$, so that we can expect that there is a relation between the dual $\hat{M}$ and the quantised universal enveloping algebra $U_q(\mathfrak{su}(1,1))$. This indeed the case, and we describe the results succintly without going into the proofs.

Recall that $U_q(\mathfrak{su}(1,1))$ is the complex unital $*$-algebra generated by $\mathbf{K}$, $\mathbf{K}^{-1}$, $\mathbf{E}$ and $\mathbf{F}$ subject to

$$\mathbf{K}\mathbf{K}^{-1} = 1 = \mathbf{K}^{-1}\mathbf{K}, \quad \mathbf{K}\mathbf{E} = q\,\mathbf{E}\mathbf{K}, \quad \mathbf{K}\mathbf{F} = q^{-1}\mathbf{F}\mathbf{K}, \quad \mathbf{F}\mathbf{E} - \mathbf{E}\mathbf{F} = \frac{\mathbf{K}^2 - \mathbf{K}^{-2}}{q - q^{-1}} \tag{7.7.1}$$

and where the $*$-structure is defined by $\mathbf{K}^* = \mathbf{K}$, $\mathbf{E}^* = \mathbf{F}$. Since we assume $0 < q < 1$, the $*$-structure is easily seen to be compatible with (7.7.1). (We identify (A, B, C, D) by $(\mathbf{K}, \mathbf{E}, -\mathbf{F}, \mathbf{K}^{-1})$.) The algebra $U_q(\mathfrak{su}(1,1))$ has more structure, since it can be made into a Hopf $*$-algebra. For completeness we recall the action of the antipode $\mathbf{S}$ and the comultiplication $\mathbf{\Delta}$ on the generators;

$$\mathbf{S}(\mathbf{K}) = \mathbf{K}^{-1}, \quad \mathbf{S}(\mathbf{E}) = -q^{-1}\mathbf{E}, \quad \mathbf{S}(\mathbf{F}) = -q\mathbf{F}, \quad \mathbf{S}(\mathbf{K}^{-1}) = \mathbf{K}. \tag{7.7.2}$$

and

$$\begin{aligned} &\mathbf{\Delta}(\mathbf{K}) = \mathbf{K} \otimes \mathbf{K}, \quad \mathbf{\Delta}(\mathbf{E}) = \mathbf{K} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{K}^{-1} \\ &\mathbf{\Delta}(\mathbf{F}) = \mathbf{K} \otimes \mathbf{F} + \mathbf{F} \otimes \mathbf{K}^{-1}, \quad \mathbf{\Delta}(\mathbf{K}^{-1}) = \mathbf{K}^{-1} \otimes \mathbf{K}^{-1}. \end{aligned} \tag{7.7.3}$$

The Casimir element

$$\mathbf{\Omega} = \frac{1}{2}\Big((q^{-1} - q)^2\mathbf{F}\mathbf{E} - q\mathbf{K}^2 - q^{-1}\mathbf{K}^{-2}\Big) = \frac{1}{2}\Big((q^{-1} - q)^2\mathbf{E}\mathbf{F} - q\mathbf{K}^{-2} - q^{-1}\mathbf{K}^2\Big) \tag{7.7.4}$$

is a central self-adjoint element in $U_q(\mathfrak{su}(1,1))$. The Casimir element $\mathbf{\Omega}$ generates the center of $U_q(\mathfrak{su}(1,1))$.

In order to represent the algebra $U_q(\mathfrak{su}(1,1))$ on the Hilbert space $\mathcal{K}$ of the GNS-representation some care has to be taken, since the operators are in general unbounded. We define the dense subspace $\mathcal{K}_0$ of $\mathcal{K}$ as the linear subspace consisting of finite linear combinations of the orthonormal basis elements f_{mpt}. Equivalently $\mathcal{K}_0$ can also be viewed as the linear span of elements of the form $\zeta^m \otimes f$ with $m \in \mathbb{Z}$, $f \in \mathcal{K}(I_q \times I_q)$, where $\mathcal{K}(I_q \times I_q)$ is the space of compactly supported function on $I_q \times I_q$. Note that $\mathcal{K}_0$ is dense in $\mathcal{K}$ and that $\mathcal{K}_0$ inherits the inner product of $\mathcal{K}$, so we can look at the space of adjointable operators $\mathcal{L}^+(\mathcal{K}_0)$ for $\mathcal{K}_0$. Recall that

$$\mathcal{L}^+(\mathcal{K}_0) = \{T\colon \mathcal{K}_0 \to \mathcal{K}_0 \text{ linear} \mid \exists S\colon \mathcal{K}_0 \to \mathcal{K}_0 \text{ linear so that } \langle Tx, y\rangle = \langle x, Sy\rangle \;\forall\, x, y \in \mathcal{K}_0\}$$

The $*$-operation in $\mathcal{L}^+(\mathcal{K}_0)$ will be denoted by $\dagger$.

Definition 7.7.1. *We define operators E_0, K_0 in $\mathcal{L}^+(\mathcal{K}_0)$ by*

$$\begin{aligned}(q-q^{-1})\, E_0\, f_{mpt} = \operatorname{sgn}(t)\; q^{-\frac{m-1}{2}}\, |p/t|^{\frac{1}{2}}\, \sqrt{1+\kappa(q^{-1}t)}\;\; f_{m-1,p,q^{-1}t} \\ -\; \operatorname{sgn}(p)\; q^{\frac{m-1}{2}}\, |t/p|^{\frac{1}{2}}\, \sqrt{1+\kappa(p)}\;\; f_{m-1,qp,t}\end{aligned} \tag{7.7.5}$$

and $K_0\, f_{mpt} = q^{-\frac{m}{2}}\, |p/t|^{\frac{1}{2}}\, f_{mpt}$ for all $m \in \mathbb{Z}$, $p, t \in I_q$.

Here sgn denotes the sign, and $\kappa(x) = \operatorname{sgn}(x)x^2$, see Definition 7.2.2.

One easily checks that $K_0^\dagger = K_0$ and that K_0 is invertible in $\mathcal{L}^+(\mathcal{K}_0)$. Also $E_0 \in \mathcal{L}^+(\mathcal{K}_0)$ and

$$\begin{aligned}(q-q^{-1})\, E_0^\dagger\, f_{mpt} = \operatorname{sgn}(t)\; q^{-\frac{m+1}{2}}\, |p/t|^{\frac{1}{2}}\, \sqrt{1+\kappa(t)}\;\; f_{m+1,p,qt} \\ -\; \operatorname{sgn}(p)\; q^{\frac{m+1}{2}}\, |t/p|^{\frac{1}{2}}\, \sqrt{1+\kappa(q^{-1}p)}\;\; f_{m+1,q^{-1}p,t}\end{aligned} \tag{7.7.6}$$

for all $m \in \mathbb{Z}$, $p, t \in I_q$.

The next proposition shows that E_0 and K_0 do satisfy the defining relations (7.7.1) for the $*$-algebra $U_q(\mathfrak{su}(1,1))$.

Proposition 7.7.2. *We have*

$$K_0\, E_0 = q\, E_0\, K_0 \qquad\qquad \textit{and} \qquad\qquad E_0^\dagger\, E_0 - E_0\, E_0^\dagger = \frac{K_0^2 - K_0^{-2}}{q-q^{-1}}\,.$$

and the elements from $\{\, K_0^m E_0^k (E_0^\dagger)^l \mid m \in \mathbb{Z}, k, l \in \mathbb{N}_0 \,\}$ are linearly independent.

Proposition 7.7.2 implies there exists a unique unital $*$-representation $\rho\colon U_q(\mathfrak{su}(1,1)) \to \mathcal{L}^+(\mathcal{K}_0)$ so that $\mathbf{E} \mapsto E_0$ and $\mathbf{K} \mapsto K_0$, hence $\mathcal{K}_0$ is turned into a $U_q(\mathfrak{su}(1,1))$-module. Define $\mathcal{U}$ to be the unital $*$-subalgebra of $\mathcal{L}^+(\mathcal{K}_0)$ generated by K_0, K_0^{-1} and E_0. This is a $*$-representation of $U_q(\mathfrak{su}(1,1))$ by unbounded operators, so that in particular each element of $\mathcal{U}$ is closable. The Poincaré-Birkhoff-Witt theorem implies that the $*$-representation $U_q(\mathfrak{su}(1,1)) \to \mathcal{L}^+(\mathcal{K}_0)$ is faithful and $\mathcal{U}$ is a concrete realization of $U_q(\mathfrak{su}(1,1))$.

An essential role in the representation theory of $U_q(\mathfrak{su}(1,1))$ is played by the Casimir operator (7.7.4). We define the Casimir element $\Omega_0 \in \mathcal{U} \subset \mathcal{L}^+(\mathcal{K}_0)$ as $\Omega_0 = \rho(\mathbf{\Omega})$, i.e.

$$\Omega_0 = \frac{1}{2}\left((q-q^{-1})^2\, E_0^\dagger\, E_0 - q\, K_0^2 - q^{-1}\, K_0^{-2} \right) = \frac{1}{2}\left((q-q^{-1})^2\, E_0 E_0^\dagger - q^{-1}\, K_0^2 - q\, K_0^{-2} \right).$$

By Definition 7.7.1 and (7.7.6) we have the explicit expression

$$\begin{aligned}&2\,\Omega_0\, f_{mpt} = -\operatorname{sgn}(pt)\, \sqrt{(1+\kappa(p))(1+\kappa(t))}\;\, f_{m,qp,qt} \\ &+ (q^{m-1}p|t| + q^{-m-1}t|p|)\, f_{mpt} - \operatorname{sgn}(pt)\, \sqrt{(1+\kappa(q^{-1}p))(1+\kappa(q^{-1}t))}\;\, f_{m,q^{-1}p,q^{-1}t}\end{aligned} \tag{7.7.7}$$

for all $m \in \mathbb{Z}$ and $p, t \in I_q$.

The Casimir operator is normalised in such a way that the continuous spectrum of the relevant self-adjoint extension of Ω_0 is given by $[-1,1]$ and the point spectrum of this extension has a maximal degree of symmetry with respect to the origin.

Not K_0, E_0 and Ω_0 are the operators relevant to the dual locally compact quantum group $(\hat{M},\hat{\Delta})$ introduced in Chapter 6, but rather the *right* closed extensions of these operators. Now K_0 is essentially self-adjoint, so it is clear what extension of K_0 to use. At this moment, it is not clear what kind of extension of E_0 we need, but Proposition 7.7.4 shows that the closure of E_0 is the natural extension in this setting. Next the Casimir operator is discussed.

Definition 7.7.3. *We define the densely defined, closed, linear operators E and K in $\mathcal{K}$ as the closures of E_0 and K_0 respectively.*

One expects at least that K and E are affiliated to the dual von Neumann algebra $\hat{M}$. This is indeed the case.

Proposition 7.7.4. *K is an injective positive self-adjoint operator in $\mathcal{K}$. The operators K and E are affiliated to the von Neumann algebra $\hat{M}$.*

Note that the spectrum $\sigma(K)$ consists of $q^{\frac{1}{2}\mathbb{Z}} \cup \{0\}$. Moreover, E^* is the closure of $E_0^\dagger$.

Next we want to define the Casimir operator on $\mathcal{K}$ as the *right* extension of Ω_0. Since $\Omega_0^\dagger = \Omega_0$, it is natural to look for a self-adjoint extension of Ω_0 to be this *right* extension. But Ω_0 is not essentially self-adjoint, thus, unlike the cases E and K, we can not merely use the closure of Ω_0.

Definition 7.7.5. *We define the Casimir operator Ω as the closure of the operator*

$$\frac{1}{2}\left((q-q^{-1})^2\, E^* E - q\, K^2 - q^{-1}\, K^{-2} \right).$$

At this point it is not clear that Definition 7.7.5 makes sense.

Theorem 7.7.6. *The Casimir operator Ω is a well-defined self-adjoint operator. Moreover, the Casimir operator Ω is the unique self-adjoint extension of Ω_0 that is affiliated to the von Neumann algebra $\hat{M}$.*

It turns out that Ω is not the closure of Ω_0.

Since (M,Δ) is a quantization of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$, and not of $SU(1,1)$, it is to be expected that the Casimir operator does not commute with all elements of $\hat{M}$. Indeed, the Casimir operator satisfies a graded commutation relation with the elements of $\hat{M}$, i.e. there exists a decomposition $\hat{M} = \hat{M}_+ \oplus \hat{M}_-$ such that the Casimir operator commutes with the elements of $\hat{M}_+$ and anti-commutes with elements of $\hat{M}_-$, see Proposition 7.7.8.

In order to formulate the graded commutation relation involving the Casimir operator we provide $\mathcal{K}$ and $\hat{M}$ with a natural $\mathbb{Z}_2$-grading.

Definition 7.7.7. *We define the closed subspaces $\mathcal{K}_+, \mathcal{K}_- \subseteq \mathcal{K}$ as*

$$\mathcal{K}_\pm = \overline{\text{Span}}\{\, f_{m,p,t} \mid m \in \mathbb{Z}, p,t \in I_q \textit{ so that } \operatorname{sgn}(pt) = \pm \,\},$$

So $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$. We define the σ-weakly closed subspaces $\hat{M}_+, \hat{M}_- \subseteq \hat{M}$ as

$$\hat{M}_+ = \{\, x \in \hat{M} \mid x\,\mathcal{K}_\pm \subseteq \mathcal{K}_\pm \,\} \qquad \textit{and} \qquad \hat{M}_- = \{\, x \in \hat{M} \mid x\,\mathcal{K}_\pm \subseteq \mathcal{K}_\mp \,\}\ .$$

Then $\hat{M}_+$ is a von Neumann algebra, and $\hat{M}_-$ is a self-adjoint subspace so that $\hat{M}_\pm \hat{M}_\mp \subseteq \hat{M}_-$ and $\hat{M}_- \hat{M}_- \subseteq \hat{M}_+$. In order to get a real $\mathbb{Z}_2$-grading on $\hat{M}$, we need the following result.

Proposition 7.7.8. *$\hat{M} = \hat{M}_+ \oplus \hat{M}_-$. Let $x \in \hat{M}_+$ and $y \in \hat{M}_-$, then $x\,\Omega \subseteq \Omega\,x$ and $y\,\Omega \subseteq -\Omega\,y$.*

Proposition 7.7.8 implies that E and K do not suffice to generate $\hat{M}$ because of Theorem 7.7.6. In order to determine $\hat{M}$, Proposition 7.7.8 also provides the key ingredient once we have determined the spectral decomposition of Ω explicitly. Indeed, Proposition 7.7.8 implies that elements of $\hat{M}$ can be described by mapping (generalized) eigenvectors for the eigenvalue λ of the Casimir operator to (generalized) eigenvectors for the eigenvalue $\pm\lambda$ of the Casimir operator. For this we have to study the Casimir operator restricted to suitable invariant subspaces on which the spectrum of Ω has simple spectrum. Now we can prove that $Q(p_1, p_2, n) \in \hat{M}_{\mathrm{sgn}(p_1 p_2)}$.

We have the following polar-type decomposition of these operators.

Lemma 7.7.9. *For fixed $p_1, p_2 \in I_q$, $n \in \mathbb{Z}$, there exists an orthogonal projection $P = P(p_1, p_2, n) \in B(\mathcal{K})$, a continuous function $H(\cdot) = H(\cdot; p_1, p_2, n)$ and a partial isometry $U = U_n^{\mathrm{sgn}(p_1),\mathrm{sgn}(p_2)}$ so that*

$$Q(p_1, p_2, n) = U\,H(\Omega)\,P.$$

In fact the function H can be determined in terms of ${}_2\varphi_1$-series explicitly, and having this realisation of $Q(p_1, p_2, n)$ gives the opportunity to find yet another special function identity from the structure constants in Proposition 7.5.2, see [16] for details.

Since the elements $H(\Omega)$ and P, as element of the spectral decomposition of K, are in the von Neumann algebra generated by E and K, we only need to incorporate the partial isometries. Now we can state the main theorem of this section, which gives an explicit description of the von Neumann algebra for the dual locally compact quantum group.

Theorem 7.7.10. *The von Neumann algebra $\hat{M}$ is generated by K, E, U_0^{+-}, U_0^{-+}.*

It is interesting to connect the comultiplication of the dual quantum group as in Theorem 6.2.1 with the comultiplication (7.7.3) of the quantized universal enveloping algebra.

Proposition 7.7.11. *We have $\hat{\Delta}(K) = K \otimes K$, and*

$$K_0 \odot E_0 + E_0 \odot K_0^{-1} \subset \hat{\Delta}(E) \qquad \textit{and} \qquad K_0 \odot E_0^\dagger + E_0^\dagger \odot K_0^{-1} \subset \hat{\Delta}(E^*).$$

In Proposition 7.7.11 the left hand side denotes the algebraic tensor product of the unbounded operators which are defined on the domain $\mathcal{K}_0 \odot \mathcal{K}_0 \subset \mathcal{K} \otimes \mathcal{K}$. So we see that the comultiplication of the dual quantum group corresponds to the comultiplication of the quantized universal enveloping algebra, see (7.1.2). Note that for an element x affiliated to $\hat{M}$ we can calculate $\hat{\Delta}(x)$ as an affiliated element of $\hat{M} \otimes \hat{M}$.

We conjecture that the von Neumann algebra generated by E and K, so the one generated by the quantised universal enveloping algebra, is the same as the von Neumann algebra generated by $Q(p_1, p_2, n)$ with $p_1 > 0$ and $p_2 > 0$. We already know that the latter does not have a suitable comultiplication as follows from Proposition 7.5.4.

7.8 Corepresentations in the left regular corepresentation

We know that the multiplicative unitary W corresponds to a unitary corepresentation, because of the pentagonal equation and its relation to the comultiplication, see Chapter 6. Decomposing W into irreducible corepresentations corresponds classically to decomposing the left regular representation of a group in the Hilbert space $L^2(G)$. In case of the Lie group $SU(1,1)$ we know that this decomposition involves discrete series representations as direct summands and principal unitary series representations as direct integrals. In fact, a similar situation is valid in this case. We only describe the result.

The Casimir operator Ω is a self-adjoint operator and the space $\mathcal{K}(p,m,\varepsilon,\eta) \subset \mathcal{K}$ is invariant for Ω. We can describe the spectral decomposition of $\Omega|_{\mathcal{K}(p,m,\varepsilon,\eta)}$ completely in terms of special functions. The spectrum is simple and consists of a continuous part $[-1,1]$ and a discrete part depending on $\mathcal{K}(p,m,\varepsilon,\eta)$ for $p \in q^{\mathbb{Z}}$, $m \in \mathbb{Z}$, $\varepsilon, \eta \in \{\pm 1\}$. We refer to (7.6.1) for the definition of these subspaces. Throughout this subsection we fix $p \in q^{\mathbb{Z}}$, $\lambda \in -q^{-\mathbb{N}} \cup q^{-\mathbb{N}}$ and set $x = \mu(\lambda) = \frac{1}{2}(\lambda + \lambda^{-1})$. Thus, x is an isolated point of the spectrum of the Casimir operator Ω if $x \in \sigma_d(\Omega)$. We denote $e_m^{\varepsilon,\eta}(p,x) \in D(\Omega) \cap \mathcal{K}(p,m,\varepsilon,\eta)$ to be the eigenvector of the Casimir operator Ω for the eigenvalue $\varepsilon\eta\, x$ in the subspace $\mathcal{K}(p,m,\varepsilon,\eta)$ of the GNS-space. We note that $e_m^{\varepsilon,\eta}(p,x) \neq 0$ if and only if Ω has an eigenvector with eigenvalue $\varepsilon\eta\, x$ inside $\mathcal{K}(p,m,\varepsilon,\eta)$. The eigenspace of Ω restricted to $\mathcal{K}(p,m,\varepsilon,\eta)$ is at most one-dimensional, so that $e_m^{\varepsilon,\eta}(p,x)$ is defined up to phase-factor after putting $\|e_m^{\varepsilon,\eta}(p,x)\| = 1$. The precise choice is not given.

Recall we have to find closed invariant subspaces for the action of $\hat{M}$, and we can define closed invariant subspaces in terms of the eigenvectors of the Casimir operator Ω.

Lemma 7.8.1. *We define the closed subspace $\mathcal{L}_{p,x}$ of $\mathcal{K}$ as*

$$\mathcal{L}_{p,x} = \overline{\mathrm{Span}}\{\, e_m^{\varepsilon,\eta}(p,x) \mid m \in \mathbb{Z}, \varepsilon, \eta \in \{-,+\}\,\}.$$

The space $\mathcal{L}_{p,x}$ is an invariant subspace of the corepresentation W of (M,Δ). If $\mathcal{L}_{p,x} \neq \{0\}$ we say that that (p,x) determines a discrete series corepresentation of (M,Δ). The element $W_{p,x} = W\big|_{\mathcal{K}\otimes\mathcal{L}_{p,x}}$ is a unitary corepresentation of (M,Δ) on $\mathcal{L}_{p,x}$.

Using the explicit actions of the generators of $\hat{M}$ as described in Theorem 7.7.10 on the eigenvectors of the Casimir operator we can classify the values of (p,x) such that $\mathcal{L}_{p,x}$ is a discrete series corepresentation of (M,Δ). The result is the following.

Proposition 7.8.2. *Consider $p \in q^{\mathbb{Z}}$ and $x = \mu(\lambda)$ where $\lambda \in -q^{2\mathbb{Z}+1}p \cup q^{2\mathbb{Z}+1}p$ and $|\lambda| > 1$. Let $j, l \in \mathbb{Z}$ be such that $|\lambda| = q^{1-2j}p^{-1} = q^{1+2l}p$, so $l < j$. Then (p,x) determines a discrete series corepresentation of (M,Δ) in the following 3 cases, and these are the only cases:*

(i) If $x > 0$, in which case

$$\{\, e_m^{++}(p,x) \mid m \in \mathbb{Z}\,\} \cup \{\, e_m^{-+}(p,x) \mid m \in \mathbb{Z}, m \leq l\,\} \cup \{\, e_m^{+-}(p,x) \mid m \in \mathbb{Z}, m \geq j\,\}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

(ii) If $x<0$, $l\geq 0$ and $j>0$, in which case

$$\{\, e_m^{-+}(p,x) \mid m\in\mathbb{Z}\,\} \cup \{\, e_m^{++}(p,x) \mid m\in\mathbb{Z}, m\leq l\,\} \cup \{\, e_m^{--}(p,x) \mid m\in\mathbb{Z}, m\geq j\,\}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

(iii) If $x<0$, $l<0$ and $j\leq 0$, in which case

$$\{\, e_m^{+-}(p,x) \mid m\in\mathbb{Z}\,\} \cup \{\, e_m^{--}(p,x) \mid m\in\mathbb{Z}, m\leq l\,\} \cup \{\, e_m^{++}(p,x) \mid m\in\mathbb{Z}, m\geq j\,\}$$

is an orthonormal basis for $\mathcal{L}_{p,x}$.

Proposition 7.8.2 gives a complete list of discrete corepresentations occurring in the left regular corepresentation. In each of the cases listed in Proposition 7.8.2 we can consider the representation of $\hat{M}$ as a representation of $U_q(\mathfrak{su}(1,1))$ (by unbounded operators in the sense of [52]), and then, by comparing the action of E and K, which requires a detailed calculation based on spectral analysis, with the listing above, we see that $\mathcal{L}_{p,x}$ in case (i), (ii) and (iii) of Proposition 7.8.2 corresponds to

$$\pi^S_{\frac{1}{2}(\chi(p)-1)+j,\epsilon(p)} \oplus D^-_{-\frac{1}{2}\chi(p)-l} \oplus D^+_{\frac{1}{2}\chi(p)+j} \tag{7.8.1}$$

as $U_q(\mathfrak{su}(1,1))$-module, where the decomposition corresponds to the order of the orthonormal basis. Here $\chi(p)\in\mathbb{Z}$ is defined in Definition 7.2.2 and $\epsilon(p)=\frac{1}{2}\chi(p) \mod 1$, so $\epsilon(p)=0$ for $p\in q^{2\mathbb{Z}}$ and $\epsilon(p)=\frac{1}{2}$ for $p\in q^{2\mathbb{Z}+1}$. So we see that a discrete series corepresentation in the left regular corepresentation decomposes in the same way as sum of three $U_q(\mathfrak{su}(1,1))$-representations involving a strange series representation in combination with a positive and negative discrete series representation. The definition in terms of the generators of $U_q(\mathfrak{su}(1,1))$ is given below.

Proposition 7.8.3. *Assume that (p,x) determines a discrete series corepresentation of the locally compact quantum group (M,Δ). Then $W_{p,x}$ is an irreducible corepresentation of (M,Δ).*

For completeness we recall the $U_q(\mathfrak{su}(1,1))$-representations involved.

Positive discrete series. The representation space is $\ell^2(\mathbb{N}_0)$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$. Let $k\in\frac{1}{2}\mathbb{N}$, define the action of the generators by

$$\begin{aligned} \mathbf{K}\cdot e_n &= q^{k+n}\, e_n, \qquad \mathbf{K}^{-1}\cdot e_n = q^{-k-n}\, e_n,\\ (q^{-1}-q)\ \mathbf{E}\cdot e_n &= q^{-\frac{1}{2}-k-n}\sqrt{(1-q^{2n+2})(1-q^{4k+2n})}\, e_{n+1},\\ (q^{-1}-q)\ \mathbf{F}\cdot e_n &= q^{\frac{1}{2}-k-n}\sqrt{(1-q^{2n})(1-q^{4k+2n-2})}\, e_{n-1}, \end{aligned} \tag{7.8.2}$$

with the convention $e_{-1}=0$. This representation is denoted by D_k^+ and $D_k^+(\mathbf{\Omega})=-\mu(q^{1-2k})$.

Negative discrete series. The representation space is $\ell^2(\mathbb{N}_0)$ with orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$. Let $k\in\frac{1}{2}\mathbb{N}$, and define the action of the generators by

$$\begin{aligned} \mathbf{K}\cdot e_n &= q^{-k-n}e_n, \qquad \mathbf{K}^{-1}\cdot e_n = q^{k+n}\, e_n,\\ (q^{-1}-q)\ \mathbf{E}\cdot e_n &= q^{\frac{1}{2}-k-n}\sqrt{(1-q^{2n})(1-q^{4k+2n-2})}\, e_{n-1},\\ (q^{-1}-q)\ \mathbf{F}\cdot e_n &= q^{-\frac{1}{2}-k-n}\sqrt{(1-q^{2n+2})(1-q^{4k+2n})}\, e_{n+1}, \end{aligned} \tag{7.8.3}$$

with the convention $e_{-1}=0$. This representation is denoted by D_k^- and $D_k^-(\mathbf{\Omega}) = -\mu(q^{1-2k})$.
Principal series. The representation space is $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. Let $0 \le b \le -\frac{\pi}{2\ln q}$ and $\varepsilon \in \{0, \frac{1}{2}\}$ and assume $(b,\varepsilon) \neq (0,\frac{1}{2})$. The action of the generators is defined by

$$\begin{aligned} \mathbf{K}\cdot e_n &= q^{n+\varepsilon}\, e_n, \qquad \mathbf{K}^{-1}\cdot e_n = q^{-n-\varepsilon}\, e_n, \\ (q^{-1}-q)\ \mathbf{E}\cdot e_n &= q^{-\frac{1}{2}-n-\varepsilon}\sqrt{(1-q^{2n+1+2\varepsilon+2ib})(1-q^{2n+1+2\varepsilon-2ib})}\, e_{n+1}, \\ (q^{-1}-q)\ \mathbf{F}\cdot e_n &= q^{\frac{1}{2}-n-\varepsilon}\sqrt{(1-q^{2n-1+2\varepsilon+2ib})(1-q^{2n-1+2\varepsilon-2ib})}\, e_{n-1}. \end{aligned} \tag{7.8.4}$$

We denote the representation by $\pi_{b,\varepsilon}$. In case $(b,\varepsilon)=(0,\frac{1}{2})$ this still defines an admissible unitary representation. It splits as the direct sum $\pi_{-\frac{\pi}{2\ln q},\frac{1}{2}} \cong D_{\frac{1}{2}}^+ \oplus D_{\frac{1}{2}}^-$ of a positive and negative discrete series representation by restricting to the invariant subspaces $\mathrm{span}\{e_n \mid n \geq 0\}$ and to $\mathrm{span}\{e_n \mid n<0\}$. We keep this convention for $\pi_{-\frac{\pi}{2\ln q},\frac{1}{2}}$. Note that $\pi_{b,\varepsilon}(\mathbf{\Omega}) = \mu(q^{2ib}) = \cos(-2b\ln q)$.
Strange series. The representation space is $\ell^2(\mathbb{Z})$ with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. Let $\varepsilon\in\{0,\frac{1}{2}\}$, and $a>0$. The action of the generators is defined by

$$\begin{aligned} \mathbf{K}\cdot e_n &= q^{n+\varepsilon}\, e_n, \qquad \mathbf{K}^{-1}\cdot e_n = q^{-n-\varepsilon}\, e_n, \\ (q^{-1}-q)\ \mathbf{E}\cdot e_n &= q^{-n-\varepsilon-\frac{1}{2}}\sqrt{(1+q^{2n+2\varepsilon+1+2a})(1+q^{2n+2\varepsilon-2a+1})}\, e_{n+1}, \\ (q^{-1}-q)\ \mathbf{F}\cdot e_n &= q^{-n-\varepsilon+\frac{1}{2}}\sqrt{(1+q^{2n+2\varepsilon-1+2a})(1+q^{2n+2\varepsilon-2a-1})}\, e_{n-1}. \end{aligned} \tag{7.8.5}$$

We denote this representation by $\pi^S_{a,\varepsilon}$. Note that $\pi^S_{a,\varepsilon}(\mathbf{\Omega}) = \mu(q^{2a})$.

Chapter 8

Outlook and research directions

8.1 General locally compact quantum groups

For the compact quantum group case there are still many examples to be studied. Focussing on general locally compact quantum groups, there are still many directions and open problems. One can think of appropriate quantum group generalisations of the C*-algebra approach to Lie groups (e.g. as given by Dixmier [9]), such as

- Plancherel decomposition, so the non-compact analogue of the Peter-Weyl decomposition, see [7] and Chapter 6;
- general Fourier transform, and related subjects such as the Paley-Wiener theorem, generalised translation, etc.;
- $L^p(G)$-spaces, and the interaction with the Fourier transform;
- homogeneous spaces and group actions, and related Fourier transforms;
- (generalised) Gelfand pairs;
- etc.

Naturally, one wants to develop these notions within the operator algebraic framework for quantum groups due to Kustermans and Vaes discussed in Chapter 6.

8.2 Specific locally compact quantum groups

It is natural to look for quantum groups that are analogues of non-compact Lie groups, especially those groups that play an important role in the relation between Lie groups and special functions, see e.g. Vilenkin and Klimyk [61]. Naturally, we want to link these to special functions, as q-analogues of the special functions featuring in [61]. Special cases have

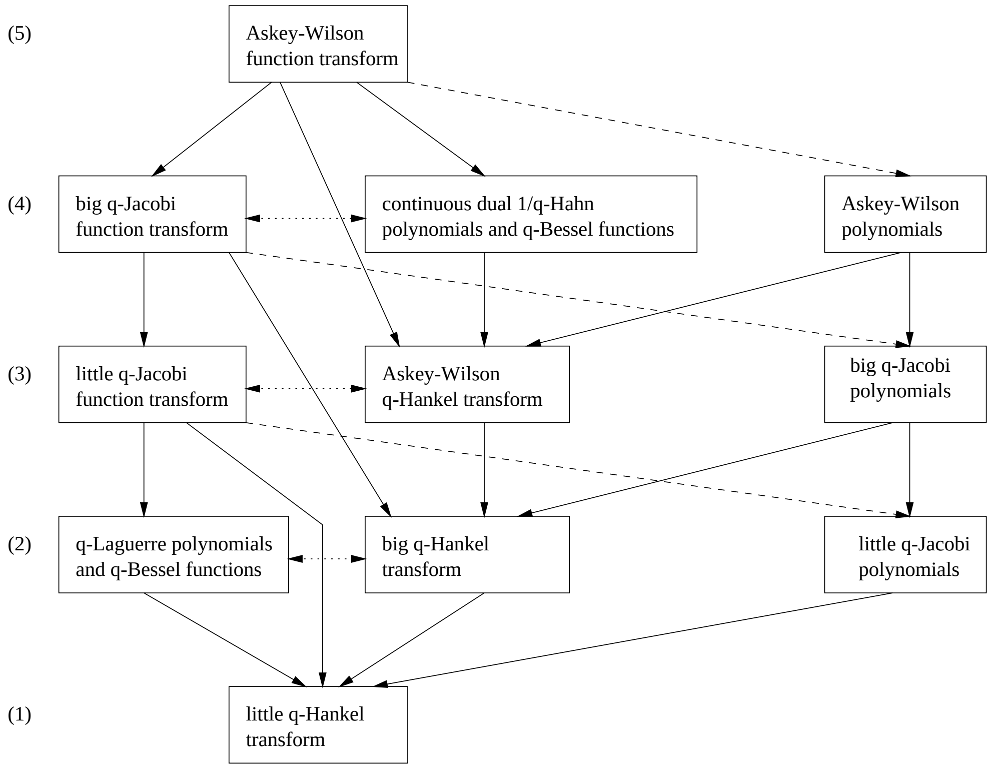


Figure 8.1: Non-polynomial q-Askey scheme

already been studied in depth, e.g. the quantum group analogue of the group of motions of the plane [18].

Important non-compact groups and related homogeneous spaces such as $SU(n,1)$, and more generally $SU(n,m)$, and related homogeneous spaces $SU(n,1)/U(n)$, and more generally $SU(n,m)/S(U(n)\times U(m))$, do not (yet) have an appropriate quantum group analogue in the sense of Chapter 6. The construction for the the quantum group analogue of $SU(1,1)$ suggests that special functions can play an important role in the construction.

8.3 Special functions

In these lecture notes the special functions are of basic hypergeometric type. The orthogonal polynomials fit in the q-analogue of the Askey scheme [24], and the non-polynomial special functions are extensions fit in a more general scheme.

Figure 8.3 is taken from [32]. The special functions are related to quantum groups in some suitable way, and all correspond to a suitable (q-)integral transformations. For all these cases one can study related questions:

- Paley-Wiener type theorems;

- convolution structures and generalised translation;
- transmutation properties;
- action on suitable L^p-spaces;
- etc.

For q-analogues of the Bessel function a large part of these questions have been resolved, especially for the third Jackson q-Bessel function (also known as ${}_1\varphi_1$-q-Bessel function or Hahn-Exton q-Bessel function) by Bouzeffour, Fitouhi and co-workers, but for many of the special functions in Figure 8.3 this is still open. Generalisations of this subject to the case of multivariable special functions is open, cf. Koornwinder's lectures.

8.4 Special functions related to quantum groups

There are many instances of relations between quantum groups and special functions, and there is an interesting cross fertilisation. Especially for locally compact quantum groups there are many open problems available, some related to the subjects mentioned in Section 8.1 for the cases mentioned in Section 8.2. For instance, is it possible to give a rigorous proof of the integral transform for (a subclass of) the Askey-Wilson functions (see the top of Figure 8.3) using the quantum group defined in Chapter 7? The analytic proof of the integral transform properties for the Askey-Wilson function is given in [33] and the quantum group theoretic approach, which is not complete, is given in [34].

In this setting many generalisations exist. We mention the generalisation to multivariable special functions (see Koornwinder's lectures in this course). Another generalisation is the extension of the notion of a Hopf algebra to a Hopf algebroid and related special functions, see e.g. [17], [29], [30], [31]

Also, in Section 4.4 we have calculated the Clebsch-Gordan coefficients in terms of Wall polynomials, and for the case of $A_q(SU(2))$ this has been generalised to Racah coefficients, both for this basis as different other types of basis by Wolter Groenevelt (2003). The Racah coefficients in the standard basis are given by certain ${}_2\varphi_1$-series, but are not (yet) related to orthogonal polynomials. In Section 7.2 this was used to write down explicitly the multiplicative unitary W for the quantum group analogue of the normaliser of $SU(1,1)$ in $SL(2,\mathbb{C})$. For the cases of compact quantum groups such interpretations can lead to explicit expressions for the multiplicative unitary for other quantum groups.It is known that the irreducible $*$-representations of such Hopf $*$-algebras are labeled by $\mathbb{T}^n \times W$, where W is the Weyl group associated to the group and n is the rank.

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