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Properties of graph C^*-algebras in the Cuntz–Krieger, Cuntz–Pimsner and groupoid models

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Introduction

In 1736 Euler wrote his famous paper on the Seven Bridges of Königsberg. This paper is now regarded as the first paper in the history of graph theory.

Around two centuries later, the theory of C^*-algebras started to develop. The study of C^*-algebras originated in both operator algebras on Hilbert spaces (von Neumann algebras) and commutative Banach algebras, and at that time had no connections to graph theory.

In 1977 this connection took form when Cuntz introduced a new class of C^*-algebras, that are now called the Cuntz algebras. Together with Krieger in 1980 he defined a class of algebras that generalize the Cuntz algebras, called the Cuntz–Krieger algebras, which are strongly related to topological Markov chains. These algebras $\mathcal{O}_A$ are defined to be the C^*-algebras generated by partial isometries satisfying certain relations determined by a given $n \times n$ matrix A with entries in $\{0, 1\}$.

In 1982 Watani realized that one could view the Cuntz–Krieger algebras as C^*-algebras associated to certain finite directed graphs in particular, by considering the $\{0, 1\}$-matrix A as the adjacency matrix of a directed graph. By the end of the century this class of C^*-algebras was expanded to accommodate even infinite directed graphs that are row-finite (i.e. all vertices emit a finite number of edges).

It was soon discovered that these graph C^*-algebras have a fascinating structure, in which various important C^*-algebraic properties of the algebra are related to the behaviour of paths in the directed graph. For example, the graph C^*-algebra is approximately finite if and only if the graph has no loops. This means that we can simply prove whether a graph C^*-algebra is approximately finite by merely looking at the drawing of the graph. Other than approximate finiteness, there are many other algebraic properties that can be verified this way.

What is also intriguing is that graph C^*-algebras can be realized using different models. In 1980 Renault introduced groupoid C^*-algebras. Not much later, together with Kumjian, Pask and Raeburn, he realized graph C^*-algebras as groupoid C^*-algebras, whenever the graph has no sinks [23]. In fact the ideal structure of graph C^*-algebras was first described for graphs without sinks using the groupoid model and this was later extended to all graphs.

Around the same time, Pimsner introduced a class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$ [29]. These algebras, called Cuntz–Pimsner algebras, were also found to generalize graph C^*-algebras. Pimsner's construction associates a universal C^*-algebra to a C^*-

correspondence, which is a special case of a Hilbert C^*-module. In fact, for every row-finite directed graph without sinks and sources, we can create two different correspondences, giving us two slightly different Cuntz–Pimsner models for the graph C^*-algebras. Having these different models gives us the chance to use results about groupoid C^*-algebras and Cuntz–Pimsner algebras and also helps us understand groupoid C^*-algebras and Cuntz–Pimsner algebras in general.

In this thesis we investigate structural properties of graph C^*-algebras, like simplicity and pure infiniteness, by translating them into corresponding properties of graphs, groupoids and correspondences. We verify these results using known results of the general case of groupoid C^*-algebras and Cuntz–Pimsner algebras.

In Chapter 1, the reader is introduced to the basic theory of C^*-algebras and Hilbert C^*-modules.

In Chapter 2, directed graphs are introduced first, after which paths and the path space are defined. In Section 2.2 the graph C^*-algebras $C^*(E)$ are defined and a method to evade sinks and sources is demonstrated. As we will see later in Chapter 3 and 4 the presence of sinks and sources is somewhat of an obstacle in our research.

In Chapter 3, groupoids are introduced and the path groupoid $\mathcal{G}_E$ coming from a directed graph E is defined. We subsequently discuss the general construction of groupoid C^*-algebras and apply this to the path groupoid, to come to the conclusion that $C^*(\mathcal{G}_E)$ is isomorphic to $C^*(E)$ in the case of row-finite directed graphs E without sinks.

Chapter 4 begins with the general construction of Cuntz–Pimsner algebras and some notable examples. Thereafter, two correspondences, the graph correspondence and the shift correspondence, are defined and the two corresponding Cuntz–Pimsner algebras are created. The chapter ends with the proof that these two Cuntz–Pimsner algebras are isomorphic to $C^*(E)$, whenever E has no sinks and no sources.

Finally, in Chapter 5 our main results about the structural properties of the graph C^*-algebras are discussed. In each of the three sections, this is done by first looking at the Cuntz–Krieger model, then at the groupoid model and finally at the two Cuntz–Pimsner models.

In the first section the ideal structure is discussed. We will see that the ideals correspond to saturated hereditary subsets of the vertex set of the graph, which corresponds to open invariant subsets of the path groupoid and invariant saturated ideals of the graph and shift correspondences. Furthermore, we will see that these results agree with the general results known about groupoid C^*-algebras and Cuntz–Pimsner algebras.

In the second section, simplicity is discussed. The notions of cofinality and condition (L) are shown to correspond to minimality of the path groupoid and essential principality. Furthermore, we will see that in the unital case these properties translate to minimality and non-periodicity of the graph and shift correspondences. Just as in the first section, we will show that these results agree with the general results known about groupoid C^*-algebras and Cuntz–Pimsner algebras as well.

In the last section, pure infiniteness is discussed. The structure of this section is slightly different from the first two sections, as the first subsection is dedicated to the notion of approximate finiteness in the Cuntz–Krieger case. (note that this notion is not translated to the other models.)

After that, the other subsections follow in the usual fashion, discussing pure infiniteness in the Cuntz–Krieger model, the groupoid model and the Cuntz–Pimsner models. In the Cuntz–Krieger model we will see that this notion relates to maximal tails in the graph.

In the groupoid model, we obtain a new result, as we were able to translate the necessary and sufficient condition of pure infiniteness to paradoxicality of the path groupoid.

Lastly, pure infiniteness in the Cuntz–Pimsner models is briefly discussed.

1. C^*-algebras and Hilbert C^*-modules

Before we can define what graph C^*-algebras actually are, we need to discuss the basics of operator theory. In this chapter, we give an overview of the theory that can be found for instance in [27, Chapter 2] and [33, Chapter 2].

1.1 C^*-algebras

First we define what a C^*-algebra is and describe some ways to construct a C^*-algebra. To understand what a C^*-algebra is and to formally define it, we need the following six definitions:

Definition 1.1.1.

1. An *algebra* is a vector space $\mathcal{A}$ together with a bilinear map
$$\mathcal{A}^2 \to \mathcal{A}, (a, b) \mapsto ab,$$
such that $a(bc) = (ab)c$ for all $a, b, c \in \mathcal{A}$.
2. A norm $\|.\|$ on $\mathcal{A}$ is said to be *submultiplicative* if $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. In this case the pair $(\mathcal{A}, \|.\|)$ is called a *normed algebra*.
3. A complete normed algebra is called a *Banach algebra*.
4. An *involution* on an algebra $\mathcal{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathcal{A}$, such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$.
5. The pair $(\mathcal{A}, *)$ is called an *involutive algebra*, or a *$*$-algebra*.
6. A *Banach $*$-algebra* is a $*$-algebra $\mathcal{A}$ together with a complete submultiplicative norm such that $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.

Definition 1.1.2. A *C^*-algebra* is a Banach $*$-algebra $\mathcal{A}$ such that
$$\|a^*a\| = \|a\|^2 \text{ for all } a \in \mathcal{A}.$$

Remark ([27, Corollary 2.1.2]). On any $*$-algebra there is at most one norm making it a C^*-algebra.

The following three examples are easy to understand, and the reader can check that they are indeed C^*-algebras.

Example 1.1.3. The scalar field $\mathbb{C}$ is a C^*-algebra with involution given by complex conjugation $\lambda \to \bar{\lambda}$.

Example 1.1.4. If Ω is a locally compact Hausdorff space, then $C_0(\Omega)$ is a C^*-algebra with involution $f \to \bar{f}$. Here $C_0(\Omega)$ is the set of functions $f : \Omega \to \mathbb{C}$ that vanishes at infinity, which means that for each positive number ϵ the set $\{\omega \in \Omega : |f(\omega)| > \epsilon\}$ is compact.

Every commutative C^*-algebra can be obtained this way.

Example 1.1.5. As as special case of the previous example, the set of continuous functions on the unit circle $C(\mathbb{T})$ is a C^*-algebra, with involution

$$f \to \bar{f} := x \mapsto \overline{f(x)},$$

where $\mathbb{T} = \{z \in \mathbb{C} \ : |z| = 1\}$.

One of the most important examples of a C^*-algebra is the set of bounded linear operators on a Hilbert space. Recall the definition of a Hilbert space.

Definition 1.1.6. A *Hilbert space* $\mathcal{H}$ is a vector space together with an inner product, such that $\mathcal{H}$ is closed in the norm induced by the inner product.

Definition 1.1.7. If X is a normed vector space, denote by $B(X)$ the set of all bounded linear maps from X to itself. These maps are called the *bounded operators* on X. We define a norm called the *operator norm* by

$$\|T\|_{\text{op}} = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|T(x)\| \qquad \text{for all } T \in B(X).$$

It is easy to see that $B(X)$ is a normed algebra with the pointwise-defined operations for addition and scalar multiplication, and with multiplication given by function composition $(T, S) \to T \circ S$.

Example 1.1.8. Let $\mathcal{H}$ be a Hilbert space, then $B(\mathcal{H})$ is a C^*-algebra with involution $T \to T^*$, where T^* is the operator defined by $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in \mathcal{H}$.

Example 1.1.9. As a special case of Example 1.1.8, let $\mathcal{H} = \mathbb{C}^n$. The bounded linear maps from $\mathbb{C}^n$ to itself are $n \times n$ matrices with matrix coefficients in $\mathbb{C}$. Therefore, in this case $B(\mathcal{H})$ is nothing but the matrix algebra $M_n(\mathbb{C})$. Thus, for every $n \in \mathbb{N}$, the *matrix algebra* $M_n(\mathbb{C})$ is a C^*-algebra.

Example 1.1.10. In general, for any algebra $\mathcal{A}$, we can create the matrix algebra $M_n(\mathcal{A})$, which denotes the algebra of all $n \times n$ matrices with entries in $\mathcal{A}$. (The operations are defined just as for scalar matrices.) If $\mathcal{A}$ is a $*$-algebra, so is $M_n(\mathcal{A})$, where the involution is given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$. Furthermore, if $\mathcal{A}$ is a C^*-algebra, then so is $M_n(\mathcal{A})$.

As we mentioned before, Example 1.1.8 is very important because every C^*-algebra can be thought of as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ by the Gelfand–Naimark Theorem stated below. The details of its proof can be found in [27].

Definition 1.1.11. A *representation* of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \phi)$ where $\mathcal{H}$ is a Hilbert space and $\phi : \mathcal{A} \to B(\mathcal{H})$ is a $*$-homomorphism, which is an algebra homomorphism such that $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. We say that $(\mathcal{H}, \phi)$ is *faithful* if ϕ is injective.

Theorem 1.1.12 (Gelfand-Naimark, [27, Theorem 3.4.1])**.** *Any C^*-algebra admits a faithful representation. Consequently, any C^*-algebra is isomorphic to a subalgebra of $B(\mathcal{H})$.*

Having discussed these elementary examples, we will now give some examples that are more difficult but will be important later on, as they are similar to the C^*-algebras we will construct in the following chapters.

Definition 1.1.13. We say that $T \in B(\mathcal{H})$ has *finite rank* if $T(\mathcal{H})$ is finite-dimensional. The operator $h \otimes \bar{k} : l \mapsto (l|k)h$ is a rank-one operator for all $h, k \in \mathcal{H}$ and $K(\mathcal{H}) = \overline{\operatorname{span}}\{h \otimes \bar{k} : h, k \in \mathcal{H}\}$.

Definition 1.1.14. An operator $T \in B(\mathcal{H})$ is said to be *compact* if $T(S)$ is relatively compact in $\mathcal{H}$, where S is the closed unit ball of $\mathcal{H}$. The set of compact operators on $\mathcal{H}$ is denoted by $K(\mathcal{H})$.

Example 1.1.15. The set $K(\mathcal{H})$ is a C^*-subalgebra of $B(\mathcal{H})$.

This is proven in [33, Chapter 1] by using the fact that a bounded operator is compact if and only if it is the norm-limit of a sequence of finite rank operators [33, Proposition 1.1].

Another way to think of C^*-algebras is through generators and relations on those generators, that is, universal C^*-algebras (see [6]).

Definition 1.1.16. Suppose we are given a set $G = \{x_i : i \in \Omega\}$ of generators and a set R of relations of the form $\|p(x_{i_1}, \ldots, x_{i_n}, x_{i_1}^*, \ldots, x_{i_n}^*)\| \leq \eta$, where p is a polynomial in $2n$ noncommuting variables with complex coefficients and $\eta \geq 0$. We could allow any kind of relations in R, but for specificity we will only consider these polynomials.

Define a *representation* of (G, R) to be a set $\{T_i : i \in \Omega\}$ of bounded operators on a Hilbert space $\mathcal{H}$ satisfying the relations in R. A representation of (G, R) defines a $*$-representation of the free $*$-algebra $\mathcal{A}$ on the set G.

Assume that there exists a representation of (G, R) and assume that whenever $\{y_i^\beta\}$ is a representation of (G, R) on $\mathcal{H}^\beta$ for all $\beta \in \Theta$, then $\bigoplus_\beta y_i^\beta \in \mathcal{L}\left(\bigoplus_\beta \mathcal{H}^\beta\right)$ for each i and $\{\bigoplus_\beta y_i^\beta\}$ is a representation of (G, R). Then we can define for $x \in \mathcal{A}$

$$\|x\| = \sup\{\|\pi(x)\| : \pi \text{ is a representation of } (G, R)\}.$$

Under the assumptions, this is a well-defined finite number and $\|\cdot\|$ is a C^*-seminorm on $\mathcal{A}$. The completion of $\mathcal{A} \setminus \{x : \|x\| = 0\}$ under $\|\cdot\|$ is called *the universal C^*-algebra* on (G, R), denoted $C^*(G, R)$.

To get a better understanding of universal C^*-algebras we will give three notable examples that will reappear in later chapters as well. Furthermore, we give another interesting example, for people who are familiar with crossed products.

Definition 1.1.17. An element u is called an isometry if $u^*u = 1$. If in addition $u^*u = uu^* = 1$, then u is called a unitary.

Example 1.1.18. Recall the C^*-algebra $C(\mathbb{T})$ from Example 1.1.5. Now let

$$G = \{u, 1\} \text{ and } R = \left\{1 = 1^* = 1^2, u1 = 1u = u, u^*u = uu^* = 1\right\},$$

and consider $C^*(G, R)$, the universal C^*-algebra generated by a single unitary. Then $C^*(G, R) \cong C(\mathbb{T})$.

Example 1.1.19. There is a universal C^*-algebra generated by a single isometry, called the *Toeplitz algebra* and denoted $\mathcal{T}$, that is, $\mathcal{T} = C^*(G, R)$, with $G = \{v, 1\}$ and

$$R = \left\{1^* = 1 = 1^2, \quad v1 = 1v = v, \quad v^*v = 1\right\}.$$

Take $\mathcal{H} = \ell^2(\mathbb{N})$ and let S be the *unilateral shift* defined by

$$S((a_0, a_1, \dots)) = (0, a_0, a_1, \dots).$$

Then $\mathcal{T} \cong C^*(S)$.

Example 1.1.20. Let $G = \{s_1, \dots, s_n, 1\}$ and

$$R = \left\{ s_i^* s_i = 1,\ 1 = 1^* = 1^2,\ s_i 1 = 1 s_i = s_i,\ \sum_{j=1}^{n} s_j s_j^* = 1 \ :\ 1 \leq i \leq n \right\}.$$

Then $C^*(G, R)$ is the *Cuntz algebra* $\mathcal{O}_n$, the universal (unital) C^*-algebra generated by n isometries whose range projections are mutually orthogonal and add up to the identity.

The following example involves crossed products and is only meant to serve as an example for readers who are already familiar with crossed products. The definition of the crossed product of a C^*-algebra with the integers $\mathbb{Z}$ can be found in [2, Definition 1.4.3].

Example 1.1.21. Let X be a compact metrizable space, $\alpha : X \to X$ a minimal homeomorphism, then $C(X) \rtimes_\alpha \mathbb{Z} = C^*\left(C(X), U \mid UfU^* = f \circ \alpha^{-1}\right)$.

We conclude this section by discussing multiplier algebras and full corners. These notions tell us more about the structure of C^*-algebras and will therefore be helpful in Chapter 5.

Before we can define full corners we need to define the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$, not to be confused with the matrix algebras $M_n(\mathcal{A})$. We follow the construction as in [27, Chapter 2].

Definition 1.1.22. A double centralizer for a C^*-algebra $\mathcal{A}$ is a pair (L, R) of bounded linear maps on $\mathcal{A}$, such that for all $a, b \in \mathcal{A}$, we have

$$L(ab) = L(a)b,\ R(ab) = aR(b) \text{ and } R(a)b = aL(b).$$

For all $c \in \mathcal{A}$, let $L_c(a) = ca$ and $R_c(a) = ac$. Then (L_c, R_c) is a double centralizer and $\|c\| = \|L_c\| = \|R_c\|$. In fact for all double centralizers (L, R) we have $\|L\| = \|R\|$. Indeed, as

$$\|aL(b)\| = \|R(a)b\| \leq \|R\|\|a\|\|b\|,$$

one has

$$\|L(b)\| = \sup_{\|a\| \leq 1} \|aL(b)\| \leq \|R\|\|b\|,$$

so $\|L\| \leq \|R\|$ and in a similar way we get $\|R\| \leq \|L\|$.

Next denote by $M(\mathcal{A})$ the set of double centralizers and define the norm of the double centralizer (L, R) to be $\|L\| = \|R\|$.

Then it is easy to see that $M(\mathcal{A})$ is a closed vector subspace of $B(\mathcal{A}) \oplus B(\mathcal{A})$, where $B(\mathcal{A})$ is the set of bounded linear maps on $\mathcal{A}$.

As explained in [27, Chapter 2], $M(\mathcal{A})$ becomes a C^*-algebra with multiplication and involution defined by

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 L_2, R_2 R_1), \quad L^*(a) = (L(a^*))^* \text{ and } (L, R)^* = (R^*, L^*).$$

We call this C^*-algebra $M(\mathcal{A})$ the multiplier algebra of $\mathcal{A}$.

Furthermore, it is easy to see that $\phi : \mathcal{A} \to M(\mathcal{A})$, $c \mapsto (L_c, R_c)$ is an isometric $*$-homomorphism, hence we can identify $\mathcal{A}$ as a C^*-subalgebra of $M(\mathcal{A})$ and $\mathcal{A} \cong \phi(\mathcal{A})$ is an ideal of $M(\mathcal{A})$, because

$$(L_c, R_c) \cdot (L_2, R_2) = (L_c L_2, R_2 R_c) \text{ and } (L_1, R_1) \cdot (L_c, R_c) = (L_1 L_c, R_c R_1),$$

where

$$\begin{aligned} L_c L_2(a) &= c L_2(a) = R_2(c) a = L_{R_2(c)}(a), \\ R_2 R_c(a) &= R_2(ac) = a R_2(c) = R_{R_2(c)}(a), \\ L_1 L_c(a) &= L_1(ca) = L_1(c) a = L_{L_1(c)}(a) \\ \text{and } R_c R_1(a) &= R_1(a) c = a L_1(c) = R_{L_1(c)}(a). \end{aligned}$$

Therefore, $M(\mathcal{A})$ is a C^*-algebra, containing $\mathcal{A}$ as an ideal. Furthermore, $M(\mathcal{A})$ is unital as $(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}}) \in M(\mathcal{A})$ is clearly the unit.

Definition 1.1.23. Let $\mathcal{A}$ be a C^*-algebra and let $M(\mathcal{A})$ denote the multiplier algebra of $\mathcal{A}$. For every projection $p \in M(\mathcal{A})$, $p\mathcal{A}p$ is a C^*-subalgebra of $\mathcal{A}$. A C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called a *corner* of $\mathcal{A}$ if there exists a projection $p \in M(\mathcal{A})$ such that $\mathcal{B} = p\mathcal{A}p$.

A corner is called *full* if it is not contained in any proper closed two-sided ideal of $\mathcal{A}$, that is, if $\mathrm{span}\{\mathcal{A}p\mathcal{A}\}$ is dense in $\mathcal{A}$. Two corners, $p\mathcal{A}p$ and $q\mathcal{A}q$, are called *complementary* if $p + q = \mathbb{1}_{M(\mathcal{A})} = (\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$.

Full corners $p\mathcal{A}p$ inherit a lot of properties from the ambient C^*-algebra $\mathcal{A}$. For example, $p\mathcal{A}p$ and $\mathcal{A}$ have the same ideal theory, as shown by the following lemma.

Lemma 1.1.24 ([32, Lemma 5.10])**.** *Suppose that $p\mathcal{A}p$ is a full corner in a C^*-algebra $\mathcal{A}$, then the map $I \mapsto pIp$ is a bijection between the set of ideals in $\mathcal{A}$ and the set of ideals in $p\mathcal{A}p$, with inverse given by*

$$J \mapsto \overline{\mathcal{A}J\mathcal{A}} = \overline{\mathrm{span}}\{abc \ : \ b \in J \text{ and } a, c \in \mathcal{A}\}.$$

1.2 Hilbert C^*-modules

In this section we discuss Hilbert modules, which are generalized forms of Hilbert spaces, as we will see in the examples below. As with Hilbert spaces, the sets of operators on these modules will form useful C^*-algebras.

Our main reference for this section is [33, Chapter 2].

Definition 1.2.1. Let $\mathcal{A}$ be a C^*-algebra, a *right $\mathcal{A}$-module* is a pair $(\mathsf{X}, (\cdot,\cdot))$, where X is a vector space over $\mathbb{C}$ and $(x,a) \mapsto x \cdot a$, $\mathsf{X} \times \mathcal{A} \to \mathsf{X}$ is a map, satisfying the following:

1. $(x+y)\cdot a = x \cdot a + y \cdot a$ for all $x, y \in \mathsf{X}$ and $a \in \mathcal{A}$,
2. $x \cdot (a+b) = x \cdot a + x \cdot b$ for all $x \in \mathsf{X}$ and $a, b \in \mathcal{A}$,
3. $x \cdot (ab) = (x \cdot a) \cdot b$ for all $x \in \mathsf{X}$ and $a, b \in \mathcal{A}$,
4. $(\lambda x)a = \lambda(x \cdot a) = x \cdot (\lambda a)$ for all $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$.

A *left $\mathcal{A}$-module* is defined analogously.

The following two examples help understand this notion, and will also be important later on in this chapter.

Example 1.2.2. The set of complex numbers $\mathbb{C}$ is a C^*-algebra, so every vector space V over $\mathbb{C}$ is a right $\mathbb{C}$-module with $v \cdot a = av$, where $a \in \mathbb{C}$ and $v \in V$.

Example 1.2.3. Every C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module, with $a \cdot b = ab$.

Now that we have defined what $\mathcal{A}$-modules are, we can look at a more specific type of $\mathcal{A}$-module.

Definition 1.2.4. A *right inner product $\mathcal{A}$-module* is a right $\mathcal{A}$-module X with a pairing $\langle \cdot,\cdot \rangle_{\mathcal{A}} : \mathsf{X} \times \mathsf{X} \to \mathcal{A}$ such that for all $x, y, z \in \mathsf{X}$, $\lambda, \mu \in \mathbb{C}$ and $a \in \mathcal{A}$

1. $\langle x, \lambda y + \mu z \rangle_{\mathcal{A}} = \lambda \langle x, y \rangle_{\mathcal{A}} + \mu \langle x, z \rangle_{\mathcal{A}}$,
2. $\langle x, y \cdot a \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} a$,
3. $\langle x, y \rangle_{\mathcal{A}}^* = \langle y, x \rangle_{\mathcal{A}}$,
4. $\langle x, x \rangle_{\mathcal{A}} \geq 0$ i.e. $\langle x, x \rangle_{\mathcal{A}}$ is a positive element of $\mathcal{A}$,
5. $\langle x, x \rangle_{\mathcal{A}} = 0$ implies $x = 0$.

Remark. If the last condition is not satisfied, we call X a *pre-inner product $\mathcal{A}$-module.*

As one can see, such a pairing looks very similar to an inner product, but instead of having values in $\mathbb{C}$ (more generally a field) it has values in $\mathcal{A}$. Therefore, it is not surprising that for right inner product $\mathcal{A}$-modules there exists a version of the Cauchy–Schwarz inequality, given in the following lemma.

Lemma 1.2.5. *Suppose X is a right inner product $\mathcal{A}$-module, then for all $x, y \in \mathsf{X}$ we have $\langle y, x \rangle_{\mathcal{A}} \langle x, y \rangle_{\mathcal{A}} \leq \|\langle x, x \rangle_{\mathcal{A}}\| \langle y, y \rangle_{\mathcal{A}}$.*

Proof. We can assume that $\|\langle x, x \rangle_{\mathcal{A}}\| \neq 0$, otherwise x would be 0 and then the statement is clearly true. Now use the fact that for positive elements $c \in \mathcal{A}$ $a^*ca \leq \|c\| a^*a$ and that $\|\langle x, x \rangle_{\mathcal{A}}\| \in \mathbb{R}_{\geq 0}$ to obtain

$$\begin{aligned} 0 \leq \langle x \cdot a - y, x \cdot a - y \rangle_{\mathcal{A}} &= \langle x \cdot a, x \cdot a \rangle_{\mathcal{A}} - \langle x \cdot a, y \rangle_{\mathcal{A}} - \langle y, x \cdot a \rangle_{\mathcal{A}} + \langle y, y \rangle_{\mathcal{A}} \\ &= a^* \langle x, x \rangle_{\mathcal{A}} a - a^* \langle x, y \rangle_{\mathcal{A}} - \langle y, x \rangle_{\mathcal{A}} a + \langle y, y \rangle_{\mathcal{A}} \\ &\leq \|\langle x, x \rangle_{\mathcal{A}}\| a^* a - a^* \langle x, y \rangle_{\mathcal{A}} - \langle y, x \rangle_{\mathcal{A}} a + \langle y, y \rangle_{\mathcal{A}}. \end{aligned}$$

Now take $a = \left\langle \frac{1}{\|\langle x,x\rangle_{\mathcal{A}}\|} x, y \right\rangle_{\mathcal{A}}$, and write $t = \|\langle x,x\rangle_{\mathcal{A}}\|$, then

$$\begin{aligned}0 &\le t \left\langle y, t^{-1}x\right\rangle_{\mathcal{A}} \left\langle t^{-1}x, y\right\rangle_{\mathcal{A}} - \left\langle y, t^{-1}x\right\rangle_{\mathcal{A}} \langle x,y\rangle_{\mathcal{A}} - \langle y,x\rangle_{\mathcal{A}} \left\langle t^{-1}x, y\right\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}} \\ &= \frac{t}{t^2}\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} - \frac{1}{t}\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} - \frac{1}{t}\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}} \\ &= \left(t^{-1} - t^{-1} - t^{-1}\right)\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}} \\ &= -\frac{1}{\|\langle x,x\rangle_{\mathcal{A}}\|}\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}}.\end{aligned}$$

Hence, $\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} \le \|\langle x,x\rangle_{\mathcal{A}}\|\langle y,y\rangle_{\mathcal{A}}$ for all $x, y \in \mathsf{X}$. ■

Remark. The Cauchy–Schwarz inequality also holds if X is a pre-inner product $\mathcal{A}$-module. The proof is as above, but we also need to consider the case where $\langle x,x\rangle_{\mathcal{A}} = 0$, with $x \neq 0$. Then we have

$$\begin{aligned}0 \le \langle x \cdot a - y, x \cdot a - y\rangle_{\mathcal{A}} &= a^*\langle x,x\rangle_{\mathcal{A}} a - a^*\langle x,y\rangle_{\mathcal{A}} - \langle y,x\rangle_{\mathcal{A}} a + \langle y,y\rangle_{\mathcal{A}} \\ &= -a^*\langle x,y\rangle_{\mathcal{A}} - \langle y,x\rangle_{\mathcal{A}} a + \langle y,y\rangle_{\mathcal{A}}.\end{aligned}$$

If we take $a = \langle nx, y\rangle_{\mathcal{A}}$ for $n \in \mathbb{N}$ we get $\langle x,y\rangle_{\mathcal{A}}^* \langle x,y\rangle_{\mathcal{A}} \le \frac{1}{2n}\langle y,y\rangle_{\mathcal{A}}$.
Thus for all $\epsilon > 0$, we get

$$\|\langle x,y\rangle_{\mathcal{A}}^*\langle x,y\rangle_{\mathcal{A}}\| \le \epsilon,$$

so $\|\langle x,y\rangle_{\mathcal{A}}\|^2 = \|\langle x,y\rangle_{\mathcal{A}}^*\langle x,y\rangle_{\mathcal{A}}\| = 0$, which implies $\langle x,y\rangle_{\mathcal{A}} = 0$. Hence

$$\langle y,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}} \le \|\langle x,x\rangle_{\mathcal{A}}\|\langle y,y\rangle_{\mathcal{A}}.$$

The following two examples are extensions of Examples 1.2.2 and 1.2.3.

Example 1.2.6. Every vector space V over $\mathbb{C}$ with an inner product that is conjugate linear in the first variable is a right inner product $\mathbb{C}$-module.

Example 1.2.7. Every C^*-algebra $\mathcal{A}$ is a right inner product $\mathcal{A}$-module, with $a \cdot b = ab$ and $\langle a,b\rangle_{\mathcal{A}} = a^*b$.

As we can see, the role of $\mathbb{C}$ is not completely replaced by $\mathcal{A}$ because we can still multiply by the complex numbers, which makes sense because $\mathcal{A}$ and X are vector spaces over $\mathbb{C}$.

Recall the definition of a Hilbert space $\mathcal{H}$ in Definition 1.1.6. It states that $\mathcal{H}$ needs to be complete in the norm $\|v\| = \langle v,v\rangle^{\frac{1}{2}}$. This leads to the following definition of a Hilbert $\mathcal{A}$-module.

Definition 1.2.8. A *Hilbert $\mathcal{A}$-module* is a right inner product $\mathcal{A}$-module X, that is complete in the norm $\|.\|_{\mathcal{A}}$ defined by

$$\|x\|_{\mathcal{A}} = \|\langle x,x\rangle_{\mathcal{A}}\|^{\frac{1}{2}} \text{ for all } x \in \mathsf{X}.$$

It is called *full* if the ideal $I = \operatorname{span}\{\langle x,y\rangle_{\mathcal{A}} | x, y \in \mathsf{X}\}$ is dense in $\mathcal{A}$.

Now we look at our two examples again, but in Example 1.2.6, instead of an inner product space V, we take a Hilbert space $\mathcal{H}$.

Example 1.2.9. Every Hilbert space with inner product that is conjugate linear in the first variable is a full Hilbert $\mathbb{C}$-module. Every Hilbert space with inner product that is conjugate linear in the second variable is a full Hilbert $\mathbb{C}$-module with inner product $\langle h, k\rangle_{\mathbb{C}} = (k|h)$.

Example 1.2.10. Every C^*-algebra $\mathcal{A}$ is a full Hilbert $\mathcal{A}$-module, with

$$\langle a, b\rangle_{\mathcal{A}} = a^* b.$$

Example 1.2.11. A Hilbert space $\mathcal{H}$ is a left Hilbert $K(\mathcal{H})$-module, with

$$T \cdot h = T(h) \text{ and } \langle h, k\rangle_{K(\mathcal{H})} = h \otimes \bar{k} : l \mapsto (l|k)h.$$

Furthermore, it is full as $\operatorname{span}\{\langle h, k\rangle_{K(\mathcal{H})} \,|\, h, k \in \mathcal{H}\} = \operatorname{span}\{h \otimes \bar{k} \,|\, h, k \in \mathcal{H}\}$ is dense in $K(\mathcal{H})$ by [33, Proposition 1.1].

Just as with Hilbert spaces, the direct sum of two Hilbert $\mathcal{A}$-modules X and Y is also a Hilbert $\mathcal{A}$-module.

Example 1.2.12. Let X and Y be Hilbert $\mathcal{A}$-modules, then define

$$\mathsf{Z} = \mathsf{X} \oplus \mathsf{Y} = \{(x, y) : x \in \mathsf{X}, y \in \mathsf{Y}\},$$

with

$$((x, y), a) \to (x \cdot a, y \cdot a) \text{ and } \langle (x, y), (x', y')\rangle_{\mathcal{A}} = \langle x, x'\rangle_{\mathcal{A}} + \langle y, y'\rangle_{\mathcal{A}}.$$

It is easy to check that Z is a right inner product $\mathcal{A}$-module. To see that it is a Hilbert $\mathcal{A}$-module, look at the norms on X, Y and Z and compute

$$\|x\|_{\mathcal{A}}^2 = \|\langle x, x\rangle_{\mathcal{A}}\| \le \|\langle x, x\rangle_{\mathcal{A}} + \langle y, y\rangle_{\mathcal{A}}\| = \|(x, y)\|_{\mathcal{A}}^2 \le \|x\|_{\mathcal{A}}^2 + \|y\|_{\mathcal{A}}^2,$$

then we obtain

$$\max\{\|x\|_{\mathcal{A}}, \|y\|_{\mathcal{A}}\} \le \|(x, y)\|_{\mathcal{A}} \le \sqrt{\|x\|_{\mathcal{A}}^2 + \|y\|_{\mathcal{A}}^2}.$$

Thus Z is complete because X and Y are complete, so Z is a Hilbert $\mathcal{A}$-module.

We can also take infinite direct sums. Let $\{\mathsf{X}_i\}_{i\in I}$ be an infinite set of Hilbert $\mathcal{A}$-modules, then define

$$\bigoplus_{i\in I} \mathsf{X}_i = \left\{(x_i) \in \prod_{i\in I} \mathsf{X}_i : \sum_{i\in I} \langle x_i, x_i\rangle_{\mathcal{A}} \text{ converges in } \mathcal{A}\right\}, \text{ with}$$

$$\langle (x_i), (y_i)\rangle_{\mathcal{A}} = \sum_{i\in I} \langle x_i, y_i\rangle_{\mathcal{A}}.$$

Here, the fact that the sum of inner products converges is needed to prove that the infinite direct sum is indeed a Hilbert $\mathcal{A}$-module.

Example 1.2.13. Let $\mathcal{A}$ be a C^*-algebra, then define

$$\mathsf{H}_{\mathcal{A}} = \bigoplus_{i=1}^{\infty} \mathcal{A} = \left\{(a_i) \in \prod_{i=1}^{\infty} \mathcal{A} : \sum a_i^* a_i \text{ converges in } \mathcal{A}\right\},$$

with

$$(a_i) \cdot a = (a_i a) \text{ and } \langle (a_i), (b_i)\rangle_{\mathcal{A}} = \sum_{i=1}^{\infty} a_i^* b_i.$$

Then $\mathsf{H}_{\mathcal{A}}$ is a Hilbert $\mathcal{A}$-module, as can be seen in the proof of [33, Proposition 2.15], and $\mathsf{H}_{\mathcal{A}}$ is called the *standard Hilbert module over* $\mathcal{A}$.

Another way of creating new Hilbert C^*-modules is by completing a pre-inner product module.

Lemma 1.2.14. *Suppose $\mathcal{A}_0$ is a dense $*$-subalgebra of a C^*-algebra $\mathcal{A}$ and suppose X_0 is a pre-inner product $\mathcal{A}_0$-module such that $\langle x, x\rangle_{\mathcal{A}_0} \geq 0$ in the completion $\mathcal{A}$. Then there exist a Hilbert $\mathcal{A}$-module X and a linear map $q : \mathsf{X}_0 \to \mathsf{X}$ such that $q(\mathsf{X}_0)$ is dense in X, $q(x \cdot a) = q(x) \cdot a$ for all $x \in \mathsf{X}_0$ and $a \in \mathcal{A}_0$, and $\langle q(x), q(y)\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}_0}$ for all $x, y \in \mathsf{X}_0$.*

We call X the completion of the pre-inner product module X_0.

Proof. Let $N = \{x \in \mathsf{X}_0 : \langle x, x\rangle_{\mathcal{A}_0} = 0\}$ and let $q : \mathsf{X}_0 \to \mathsf{X}_0/N$ be the quotient map. From [33, Lemma 2.5] we know that the Cauchy–Schwarz inequality also holds in X_0, so we have

$$\langle x, y\rangle_{\mathsf{X}_0} = 0 = \langle y, x\rangle_{\mathsf{X}_0} \text{ for all } y \in \mathsf{X}_0 \text{ and } x \in N.$$

Hence N is a $\mathcal{A}_0$-submodule. It also follows that $\langle q(x), q(y)\rangle_{\mathcal{A}} := \langle x, y\rangle_{\mathcal{A}_0}$ and $q(x \cdot a) := q(x) \cdot a$ give a well-defined pairing and module structure on X_0/N making it an inner product $\mathcal{A}_0$-module. Then $\|q(x)\| = \|\langle x, x\rangle_{\mathcal{A}_0}\|^{\frac{1}{2}}$ is a norm on X_0/N, and we can form the completion on X. From the inequality $a^*b^*ba \leq \|b\|^2 a^*a$, we can deduce that

$$\|q(x) \cdot a\|^2 = \|\langle x \cdot a, x \cdot a\rangle_{\mathcal{A}_0}\| = \|a^*\langle x, x\rangle_{\mathcal{A}_0} a\| \leq \|a\|^2 \|q(x)\|^2.$$

Thus right multiplication by $a \in \mathcal{A}_0$ is a bounded operator on X_0/N, and hence we can extend it to an operator on X such that $\|x \cdot a\| \leq \|x\|\|a\|$. Then we can extend it again such that we can multiply by every $a \in \mathcal{A}$.

In a similar way the inner product can be extended to X. If $\{q(x_n)\}$ converges to x and $\{q(y_n)\}$ converges to y, then we can define

$$\langle x, y\rangle_{\mathcal{A}} := \lim_{n\to\infty} \langle q(x_n), q(y_n)\rangle_{\mathcal{A}}.$$

The first three properties of Definition 1.2.4 are easy to check, the other properties follow from the fact that the positive cone $\mathcal{A}^+$ is closed, and if $\langle x, x\rangle_{\mathcal{A}} = 0$, then there exists a sequence $q(x_n) \to x$ with $\|q(x_n)\| \to 0$, which means that x must be the zero element of X. ■

1.2.1 Operators on Hilbert modules

As mentioned before, the operators on Hilbert modules are more interesting to us than the Hilbert modules themselves. They are very similar to operators on Hilbert spaces, but there are some important differences, as we will see in this subsection. The reason for these differences is that orthogonal complements of Hilbert modules behave differently from orthogonal complements of Hilbert spaces, as explained in [3].

Definition 1.2.15. Suppose X and Y are Hilbert $\mathcal{A}$-modules. A map $T : \mathsf{X} \to \mathsf{Y}$ is called *adjointable* if there exists a function $T^* : \mathsf{Y} \to \mathsf{X}$ such that

$$\langle T(x), y\rangle_{\mathcal{A}} = \langle x, T^*(y)\rangle_{\mathcal{A}} \text{ for all } x \in \mathsf{X} \text{ and } y \in \mathsf{Y}.$$

Remark. Note that the inner products in the definition above are not the same even though they have the same subscript. The first one is an inner product on Y and the second one is an inner product on X.

As we know, every operator on a Hilbert space is adjointable, but this is not true for operators on a Hilbert module. Consider the following example.

Example 1.2.16. Let $\mathcal{A} = C([0,1])$ and let $J = \{f \in \mathcal{A} : f(0) = 0\}$. Then $\mathcal{A}$ and J are Hilbert $\mathcal{A}$-modules. Take $\mathsf{X} = \mathcal{A} \oplus J$ and define $T(f,g) = (g,0)$. Then T is bounded and $\mathcal{A}$-linear. Now suppose T has an adjoint T^* and write $T^*(\mathbb{1},0) = (f,g)$. Then by Definition 1.2.15, we would have for all $(h,k) \in \mathsf{X}$

$$\bar{k} = \langle (k,0), (\mathbb{1},0) \rangle_{\mathcal{A}} = \langle (h,k), (f,g) \rangle_{\mathcal{A}} = \bar{h}f + \bar{k}g,$$

but this gives us $f = 0$ and $g = 1$, which contradicts $g(0) = 0$. Thus T is not adjointable.

Lemma 1.2.17. *Let X, Y be Hilbert $\mathcal{A}$-modules and let $T : \mathsf{X} \to \mathsf{Y}$ be an adjointable map. Then T is a bounded linear $\mathcal{A}$-module map from X to Y.*

Proof. By the Cauchy–Schwarz inequality in Lemma 1.2.5, we have for any Hilbert $\mathcal{A}$-module Z and all $z \in \mathsf{Z}$

$$\|z\| = \sup\{\|\langle z, w\rangle_{\mathcal{A}}\| : w \in \mathsf{Z} \text{ and } \|w\|_{\mathcal{A}} \leq 1\}.$$

This implies that $z = w$ in Z if and only if $\langle z, v\rangle_{\mathcal{A}} = \langle w, v\rangle_{\mathcal{A}}$ for all $v \in \mathsf{Z}$. With this fact it is easy to prove that every adjointable map $T : \mathsf{X} \to \mathsf{Y}$ is a linear $\mathcal{A}$-module map from X to Y. Indeed, we have

$$\langle T(x \cdot a), y\rangle_{\mathcal{A}} = \langle x \cdot a, T^*(y)\rangle_{\mathcal{A}} = a^* \langle x, T^*(y)\rangle_{\mathcal{A}} = a^* \langle T(x), y\rangle_{\mathcal{A}} = \langle T(x) \cdot a, y\rangle_{\mathcal{A}},$$

thus $T(x \cdot a) = T(x) \cdot a$ and

$$\begin{aligned} \langle T(\lambda x + \mu z), y\rangle_{\mathcal{A}} &= \langle \lambda x + \mu z, T^*(y)\rangle_{\mathcal{A}} \\ &= \bar{\lambda}\langle x, T^*(y)\rangle_{\mathcal{A}} + \bar{\mu}\langle z, T^*(y)\rangle_{\mathcal{A}} \\ &= \bar{\lambda}\langle T(x), y\rangle_{\mathcal{A}} + \bar{\mu}\langle T(z), y\rangle_{\mathcal{A}} \\ &= \langle \lambda T(x) + \mu T(z), y\rangle_{\mathcal{A}}, \end{aligned}$$

thus $T(\lambda x + \mu z) = \lambda T(x) + \mu T(z)$.

Next we show that $T : \mathsf{X} \to \mathsf{Y}$ is also bounded, by using the closed graph theorem. Suppose $x_n \to x$ in X and $T(x_n) \to z$ in Y. Then for all $y \in \mathsf{Y}$

$$\langle T(x_n), y\rangle_{\mathcal{A}} \to \langle z, y\rangle_{\mathcal{A}} \text{ and } \langle x_n, T^*(y)\rangle_{\mathcal{A}} \to \langle x, T^*(y)\rangle_{\mathcal{A}} = \langle T(x), y\rangle_{\mathcal{A}},$$

so $\langle T(x), y\rangle_{\mathcal{A}} = \langle z, y\rangle_{\mathcal{A}}$, thus $T(x) = z$. This means that the graph of T is closed and hence T is bounded. Thus T is a bounded linear $\mathcal{A}$-module map. ■

Definition 1.2.18. We denote the *set of all adjointable operators* from X to Y by $\mathcal{L}(\mathsf{X}, \mathsf{Y})$ and write $\mathcal{L}(\mathsf{X}) = \mathcal{L}(\mathsf{X}_{\mathcal{A}}) = \mathcal{L}(\mathsf{X}, \mathsf{X})$.

Lemma 1.2.19. *The set $\mathcal{L}(\mathsf{X})$ is a C^*-algebra with respect to the operator norm.*

Proof. It is easy to see that T^* is unique for every $T \in \mathcal{L}(\mathsf{X})$ and that $T^{**} = T$ and $T^* \in \mathcal{L}(\mathsf{X})$. It is also clear that $\mathcal{L}(\mathsf{X})$ is a subalgebra of the Banach algebra of bounded operators, which we denote by $B(\mathsf{X})$. Thus we have $\|T^*T\| \leq \|T^*\|\|T\|$. Also, from the Cauchy–Schwarz inequality we obtain

$$\|T^*T\| \geq \sup_{\|x\| \leq 1} \|\langle T^*T(x), x\rangle_{\mathcal{A}}\| = \sup_{\|x\| \leq 1} \|\langle T(x), T(x)\rangle_{\mathcal{A}}\| = \|T\|^2.$$

Hence $\|T^*\|\|T\| \geq \|T^*T\| \geq \|T\|^2$, thus $\|T^*\| \geq \|T\|$.

This also shows that $\|T\| = \|T^{**}\| \geq \|T^*\| \geq \|T\|$, and thus $\|T\| = \|T^*\|$. Also,

$$\|T\|\|T\| = \|T^*\|\|T\| \geq \|T^*T\| \geq \|T\|^2,$$

thus $\|T^*T\| = \|T\|^2$.

Thus if we define the involution on $\mathcal{L}(\mathsf{X})$ to be $T \mapsto T^*$, we get from continuity that $\mathcal{L}(\mathsf{X})$ is closed in $B(\mathsf{X})$. Therefore, we have proven that $\mathcal{L}(\mathsf{X})$ is a C^*-algebra. ∎

Example 1.2.20. If $\mathsf{X} = \mathcal{H}$ is a Hilbert $\mathbb{C}$-module (that is, X is a Hilbert space), then $\mathcal{L}(\mathsf{X}) = B(\mathcal{H})$.

We conclude this chapter by looking at a specific C^*-subalgebra of $\mathcal{L}(\mathsf{X})$, called the set of compact operators on X. This generalizes the definition of compact operators on Hilbert spaces.

Definition 1.2.21. Suppose X and Y are Hilbert $\mathcal{A}$-modules. Define

$$\Theta_{y,x} : \mathsf{X} \to \mathsf{Y}, z \mapsto y \cdot \langle x, z\rangle_{\mathcal{A}},$$

and

$$\mathcal{K}(\mathsf{X}, \mathsf{Y}) = \overline{\operatorname{span}}\{\Theta_{y,x} : y \in \mathsf{Y} \text{ and } x \in \mathsf{X}\}.$$

We call $\mathcal{K}(\mathsf{X}, \mathsf{X}) = \mathcal{K}(\mathsf{X})$ *the algebra of compact operators on* X.

Remark. By computing

$$\begin{aligned}
\langle \Theta_{y,x}(z), w\rangle_{\mathcal{A}} &= \langle y \cdot \langle x, z\rangle_{\mathcal{A}}, w\rangle_{\mathcal{A}} \\
&= \langle x, z\rangle_{\mathcal{A}}^* \langle y, w\rangle_{\mathcal{A}} \\
&= \langle z, x\rangle_{\mathcal{A}} \langle y, w\rangle_{\mathcal{A}} \\
&= \langle z, x \cdot \langle y, w\rangle_{\mathcal{A}}\rangle_{\mathcal{A}} \\
&= \langle z, \Theta_{x,y}(w)\rangle_{\mathcal{A}},
\end{aligned}$$

we see that $\Theta_{y,x}^* = \Theta_{x,y}$.

Lemma 1.2.22. *If* X *is a right Hilbert* $\mathcal{A}$*-module, then* X *is a full left Hilbert* $\mathcal{K}(\mathsf{X})$*-module.*

Proof. Take ${}_{\mathcal{K}(\mathsf{X})}\langle x, y\rangle = \Theta_{x,y}$. Linearity in the first variable is easy to check and we already saw that $\Theta_{y,x}^* = \Theta_{x,y}$. Also, ${}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle = \Theta_{x,x}$ is positive if $\langle \Theta_{x,x} y, y\rangle_{\mathcal{A}} \geq 0$, and this is the case since

$$\langle \Theta_{x,x}(y), y\rangle_{\mathcal{A}} = \langle x\langle x, y\rangle_{\mathcal{A}}, y\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}}^* \langle x, y\rangle_{\mathcal{A}} \geq 0.$$

Furthermore, if ${}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle = \Theta_{x,x} = 0$, then $\Theta_{x,x}(x) = x\langle x, x\rangle_{\mathcal{A}} = 0$, which means that $x = 0$ or $\langle x, x\rangle_{\mathcal{A}} = 0$, which implies that $x = 0$. Thus ${}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle = 0$ implies that $x = 0$. Lastly, we see that for all $T \in \mathcal{K}(\mathsf{X})$,

$$\begin{aligned} {}_{\mathcal{K}(\mathsf{X})}\langle Tx, y\rangle(z) &= \Theta_{T(x),y}(z) \\ &= T(x)\langle y, z\rangle_{\mathcal{A}} \\ &= T(x\langle y, z\rangle_{\mathcal{A}}) \\ &= T\Theta_{x,y}(z) \\ &= T\,{}_{\mathcal{K}(\mathsf{X})}\langle x, y\rangle(z), \end{aligned}$$

so ${}_{\mathcal{K}(\mathsf{X})}\langle Tx, y\rangle = T\,{}_{\mathcal{K}(\mathsf{X})}\langle x, y\rangle$. Thus X is a left inner product $\mathcal{K}(\mathsf{X})$-module.

$\mathcal{K}(\mathsf{X})$ is clearly full by definition, so the only thing left to prove is that X is complete in the norm ${}_{\mathcal{K}(\mathsf{X})}\|x\| = \|\Theta_{x,x}\|^{\frac{1}{2}}$. Using the Cauchy–Schwarz inequality we see that

$$\left\langle {}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle y, y\right\rangle_{\mathcal{A}} = \langle x, y\rangle_{\mathcal{A}}^{*}\langle x, y\rangle_{\mathcal{A}} \leq \|\langle x, x\rangle_{\mathcal{A}}\|\langle y, y\rangle_{\mathcal{A}},$$

so $\|{}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle\| \leq \|\langle x, x\rangle_{\mathcal{A}}\|$. Taking $y = x$, we obtain

$$\|\langle {}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle x, x\rangle_{\mathcal{A}}\| = \|\langle x, x\rangle_{\mathcal{A}}\|^2,$$

which implies

$$\|{}_{\mathcal{K}(\mathsf{X})}\langle x, x\rangle\| \geq \|\langle x, x\rangle_{\mathcal{A}}\|,$$

so

$${}_{\mathcal{K}(\mathsf{X})}\|.\| = \|.\|_{\mathcal{A}}.$$

Thus X is also complete in the norm ${}_{\mathcal{K}(\mathsf{X})}\|.\|$, hence X is a full left Hilbert $\mathcal{K}(\mathsf{X})$-module. ■

This is an interesting result, as this means that every right Hilbert C^*-module is also always a full left Hilbert C^*-module, where the left action is given by compact operators. We will use this idea of left actions in Section 4.1.

2. Graph C^*-algebras

In Chapter 1, we have seen the basics of C^*-algebras and Hilbert C^*-modules. In this chapter we will use this knowledge to create graph C^*-algebras.

Before we do this we need to discuss graph theory.

2.1 Graphs

A graph is one of the most intuitive structures in mathematics. Normally when introducing graphs one starts with undirected finite graphs $G = (V, E)$, but we will look at directed graphs that are even allowed to be infinite, subject to some given extra condition. We will use the definition of a directed graph by Raeburn in [32]. Note that we will not be adopting his definition of paths, row-finite graphs and Cuntz–Krieger families, as defined in [32], in the following Section 2.2, because this is counterintuitive and seems only practical when looking at higher rank graphs.

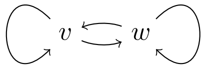
Definition 2.1.1. A *directed graph* E is a quadruple (E^0, E^1, r, s), where E^0 and E^1 are countable sets and $r, s : E^1 \to E^0$ are two functions.

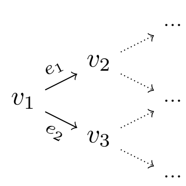
We call E^0 the set of *vertices*, E^1 the set of *edges* and r and s the *range* and *source* maps respectively.

$$s(e) \bullet \xrightarrow{\ e\ } \bullet\, r(e)$$

Definition 2.1.2. A graph E is called *row-finite* if $s^{-1}(v)$ is a finite set for all $v \in E^0$. If in addition $r^{-1}(v)$ is finite for all $v \in E^0$, then we call E *locally finite*. A vertex $v \in E^0$ is called a sink if $s^{-1}(v)$ is empty, and v is called a source if $r^{-1}(v)$ is empty.

Example 2.1.3. The left picture shows an example of a graph where both E^0 and E^1 are finite, that is, a finite graph. The right picture shows an example of an infinite locally finite graph without sinks, that is, $s^{-1}(v) \neq \emptyset$ for all $v \in E^0$.





Definition 2.1.4. The *adjacency matrix* A_E of E is the $E^0 \times E^0$ matrix with entries $A_E(v, w) = \#\{e \in E^1 : s(e) = v, r(e) = w\}$.

Example 2.1.5. The adjacency matrix of the left most graph in Example 2.1.3 is given below.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The term row-finite comes from this matrix, as a graph E is row-finite if and only if each row of A_E has a finite sum, that is,

$$\sum_{w \in E^0} A_E(v, w) < \infty \text{ for all } v \in E^0.$$

Definition 2.1.6. The *edge matrix* B_E of E is the $E^1 \times E^1$ matrix with entries

$$B_E(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.1.7. The edge matrix of the left most graph in Example 2.1.3 is given below.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The edge matrix and the adjacency matrix are linked in the following way.

Example 2.1.8. Let E be a directed graph. Define the *dual graph* $\widehat{E}$ by $\widehat{E}^0 = E^1$ and $\widehat{E}^1 = E^2 = \left\{ef \,|\, e, f \in E^1, r(e) = s(f)\right\}$ with $\hat{r}(ef) = f$ and $\hat{s}(ef) = e$. Then the edge matrix of E is the adjacency matrix of $\widehat{E}$, since

$$A_{\widehat{E}}(e, f) = \#\left\{gh \in \widehat{E}^1 : s(gh) = e, r(gh) = f\right\} = \#\left\{ef \in E^2\right\},$$

as $s(gh) = e$ and $r(gh) = f$ if and only if $g = e$ and $h = f$. Furthermore, $ef \in E^2$ if and only if $r(e) = s(f)$, so

$$\begin{aligned} A_{\widehat{E}}(e, f) &= \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{otherwise,} \end{cases} \\ &= B_E(e, f). \end{aligned}$$

2.1.1 The path space

In Example 2.1.8 we defined $E^2 = \{ef \,|\, e, f \in E^1, r(e) = s(f)\}$, that is, a sequence of two edges ef is contained in E^2 if these edges are adjacent. We can extend this definition to sequences of any length, even infinite length. We will see that the set of those infinite sequences can be equipped with a topology, making it into a locally compact Hausdorff space.

Definition 2.1.9. A *finite path* in E is a sequence $\mu = (\mu_1, ..., \mu_k)$ of edges with $s(\mu_{i+1}) = r(\mu_i)$ for $1 \leq i \leq k - 1$. We extend the source and range maps by defining $s(\mu) = s(\mu_1)$ and $r(\mu) = r(\mu_k)$ and we will denote the *length* of μ by $|\mu| = k$. An *infinite path* is an infinite sequence of such edges.

If we denote by E^n the set of paths of length n in E, then the elements of E^0 (the vertices of E) can be regarded as paths of length 0.

Definition 2.1.10. Define $E^* = \bigcup_{n\geq 0} E^n$, the set of finite paths in E and define

$$E^\infty = \left\{(x_1, x_2, ...) : x_i \in E^1, r(x_i) = s(x_{i+1})\, \forall i \in \mathbb{N}\right\}$$

to be the set of infinite paths in E.

The infinite path space is a subset of the product space $\prod_{i=1}^{\infty} E^1$ and thus inherits the product topology for which the *cylinder sets*

$$Z(\mu) = \{x \in E^\infty : x_1 = \mu_1, ..., x_{|\mu|} = \mu_{|\mu|}\}$$

with $\mu \in E^*$, form a basis of open sets. The cylinder sets are also closed, since

$$E^\infty \setminus Z(\mu) = E^\infty \cap \left(\bigcup_{i=1}^{|\mu|} \pi_i^{-1}\left(E^1 \setminus \{\mu_i\}\right)\right)$$

and $\bigcup_{i=1}^{|\mu|} \pi_i^{-1}\left(E^1 \setminus \{\mu_i\}\right)$ is open in $\prod_{i=1}^{\infty} E^1$.

To see that E^∞ is Hausdorff, we give a description of the intersection of cylinder sets.

Lemma 2.1.11. *For $\alpha, \beta \in E^*$, we have*

$$Z(\alpha) \cap Z(\beta) = \begin{cases} Z(\alpha) & \text{if there exists } \epsilon \in E^* \text{ such that } \alpha = \beta\epsilon, \\ Z(\beta) & \text{if there exists } \epsilon \in E^* \text{ such that } \beta = \alpha\epsilon, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Suppose $Z(\alpha) \cap Z(\beta)$ is non-empty and let $x \in Z(\alpha) \cap Z(\beta)$. Then $x_1 = \alpha_1, ..., x_{|\alpha|} = \alpha_{|\alpha|}$ and $x_1 = \beta_1, ..., x_{|\beta|} = \beta_{|\beta|}$. We consider the two cases $|\alpha| \leq |\beta|$ and $|\alpha| > |\beta|$.

In the case of $|\alpha| \leq |\beta|$, we have $x_1 = \alpha_1 = \beta_1, ..., x_{|\alpha|} = \alpha_{|\alpha|} = \beta_{|\alpha|}$, so we must have $\beta = \alpha\epsilon$ for some $\epsilon \in E^*$. We also have $x_1 = \beta_1, ..., x_{|\beta|} = \beta_{|\beta|}$, thus, in this case, $Z(\alpha) \cap Z(\beta) = Z(\beta)$. If $|\alpha| > |\beta|$, we can interchange α and β in the proof of the first case, which proves that $Z(\alpha) \cap Z(\beta)$ must be equal to $Z(\alpha)$, if there exists $\epsilon \in E^*$ such that $\alpha = \beta\epsilon$. ■

For a given vertex $v \in E^0$, let $E_k(v)$ be the set of edges that can be reached by paths of length k starting in v. When E is row-finite these sets are finite and hence compact in E^1. Thus by Tychonoff's theorem [38, Theorem 3.3.21], $\prod_{k=1}^{\infty} E_k(r(\alpha))$ is compact for every $\alpha \in E^*$ and the closed set $Z(\alpha)$ is homeomorphic to a subset of $\prod_{k=1}^{\infty} E_k(r(\alpha))$, thus it is compact. This shows that these cylinder sets form a basis for a locally compact Hausdorff topology on E^∞, whenever E is a row-finite directed graph.

The topology on E^∞ is metrizable with respect to the following metric.

Definition 2.1.12. Define $d : E^\infty \times E^\infty \to \mathbb{R}_{\geq 0}$ by

$$d(x, y) = \begin{cases} \frac{1}{1+\min\{j \in \mathbb{Z}^+ : x_j \neq y_j\}} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Remark. The topology generated by the metric d coincides with the product topology on E^∞.

To conclude this section, we show that there is a link with symbolic dynamics.

Definition 2.1.13. Let E be a row-finite directed graph. Then the *shift map* $\sigma : E^\infty \to E^\infty$ is defined by $(\sigma x)_i = x_{i+1}$ for all $i \in \mathbb{Z}^+$.

This shift map is clearly a local homeomorphism and hence, as explained in [41, Chapter 5], (E^∞, σ) is a one-sided subshift of finite type over the alphabet E^1.

2.2 The Cuntz–Krieger model

In this section we will show how to associate a C^*-algebra $C^*(E)$ to any row-finite directed graph E. The basic idea of this construction is to represent the vertices by orthogonal projections on a Hilbert space and the edges by partial isometries on the same Hilbert space. We start by recalling some interesting facts about orthogonal projections and partial isometries.

Definition 2.2.1. an element $P \in B(\mathcal{H})$ is called an orthogonal projection if $P^2 = P^* = P$.
an element p in a C^*-algebra $\mathcal{A}$ is called a projection if $p^2 = p = p^*$.

Lemma 2.2.2 ([32, Proposition A.1]). *Suppose that P and Q are orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $P\mathcal{H} \subset Q\mathcal{H}$*;*

(b) $QP = P = PQ$*;*

(c) $Q - P$ *is a projection;*

(d) $P \leq Q$ *(in the sense that* $(Ph|h) \leq (Qh|h)$ *for all* $h \in \mathcal{H}$*).*

Lemma 2.2.3 ([32, Corollary A.3]). *Suppose that $\{p_i : 1 \leq i \leq n\}$ are projections in a C^*-algebra $\mathcal{A}$. Then $\sum_{i=1}^n p_i$ is a projection if and only if $p_i p_j = 0$ for $i \neq j$, in which case we say that the projections are mutually orthogonal.*

Definition 2.2.4. Let E be a row-finite directed graph and $\mathcal{H}$ a Hilbert space. A *Cuntz–Krieger E-family $\{S, P\}$ on $\mathcal{H}$* consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$, such that the following two conditions, called the Cuntz–Krieger relations, hold:

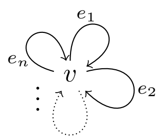
(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$

(CK2) $P_v = \displaystyle\sum_{\{e \in E^1 : s(e) = v\}} S_e S_e^*$ whenever v is not a sink.

Remark. It is easy to see that $P_{s(e)} S_e = S_e = S_e P_{r(e)}$, because $S_e S_e^*$ is the projection on the range of S_e, and the second Cuntz–Krieger relation implies that the projection $S_e S_e^*$ is dominated by the projection $P_{s(e)}$, thus by Lemma 2.2.2 we get $S_e \mathcal{H} \subset P_{s(e)} \mathcal{H}$, so $P_{s(e)} S_e = S_e$.

In general, we can create a Hilbert space $\mathcal{H}$ for every row-finite directed graph E, such that there exists a Cuntz–Krieger E-family with every P_v and S_e nonzero, as explained in [32, Chapter 1].

Example 2.2.5. The Cuntz–Krieger algebras are generalizations of the Cuntz algebras $\mathcal{O}_n$ for $n \geq 2$. Recall the definition of the Cuntz algebra $\mathcal{O}_n$ in 1.1.20. It is defined to be the universal C^*-algebra generated by n isometries $S_1, \ldots, S_n$ subject to the relations $S_i^* S_i = 1$ for all $i \leq n$ and $\sum_{i=1}^n S_i S_i^* = 1$. Let E be the graph pictured below, defined by $E^0 = \{v\}$, $E^1 = \{e_i : 1 \leq i \leq n\}$ and $r(e_i) = s(e_i) = v$ for all $i \leq n$.



Then $P_v = 1$, $S_{e_i} = S_i$ for all i, defines a Cuntz–Krieger E-family, because $S_{e_i}^* S_{e_i} = S_i^* S_i = 1 = P_v = P_{r(e_i)}$ for all i and

$$P_v = 1 = \sum_{i=1}^{n} S_i S_i^* = \sum_{i=1}^{n} S_{e_i} S_{e_i}^* = \sum_{e \in E^1 : s(e) = v} S_e S_e^*.$$

This shows how properties of the graph are reflected as properties of the elements of the algebra.

To get a different characterization of the algebra generated by a Cuntz–Krieger E-family, we consider the following relations between the isometries.

Lemma 2.2.6. *Let E be a row-finite directed graph and $\{S, P\}$ a Cuntz–Krieger E-family. Then*

1. *The projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal.*
2. *$S_e S_f \neq 0$ implies $r(e) = s(f)$.*
3. *$S_e S_f^* \neq 0$ implies $r(e) = r(f)$.*
4. *$S_e^* S_f \neq 0$ implies $e = f$.*

Proof. Let $e, f \in E^1$ and suppose first that $s(e) = s(f)$. Then the projection $P_{s(e)}$ is the sum of $S_e S_e^*$, $S_f S_f^*$ and other projections. Thus by Lemma 2.2.3, $S_e S_e^*$ and $S_f S_f^*$ are mutually orthogonal.

Next suppose that $s(e) \neq s(f)$. Then

$$(S_e S_e^*)(S_f S_f^*) = (S_e S_e^* P_{s(e)})(P_{s(f)} S_f S_f^*) = (S_e S_e^*) 0 (S_f S_f^*) = 0.$$

The other statements can be proven by the following simple computations.

If $S_e S_f = S_e P_{r(e)} P_{s(f)} S_f \neq 0$, then $P_{r(e)} P_{s(f)} \neq 0$, thus $r(e) = s(f)$.

If

$$S_e S_f^* = S_e P_{r(e)} (S_f P_{r(f)})^* = S_e P_{r(e)} P_{r(f)} S_f^* \neq 0,$$

then $P_{r(e)} P_{r(f)} \neq 0$, thus $r(e) = r(f)$.

If $e \neq f$, then

$$S_e^* S_f = S_e^* (S_e S_e^*)(S_f S_f^*) S_f = 0.$$

■

The previous lemma implies that $S_{e_1}S_{e_2}\cdots S_{e_k}=0$ unless $r(e_i)=s(e_{i+1})$ for $0<i<k$. Thus $S_{e_1}S_{e_2}\cdots S_{e_k}=0$ unless $(e_1,e_2,...,e_k)$ is a path in E.

Definition 2.2.7. For a path $\mu=(\mu_1,...,\mu_k)\in E^k$ define $S_\mu=S_{\mu_1}S_{\mu_2}\cdots S_{\mu_k}$ and for $v\in E^0$ define $S_v=P_v$.

Then $P_{s(\mu)}S_\mu S_\mu^*=P_{s(\mu_1)}S_\mu S_\mu^*=S_\mu S_\mu^*$ and

$$\begin{aligned}
S_\mu^*S_\mu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_k})^*(S_{\mu_1}S_{\mu_2}\cdots S_{\mu_k})\\
&= S_{\mu_k}^*\cdots S_{\mu_2}^*S_{\mu_1}^*S_{\mu_1}S_{\mu_2}\cdots S_{\mu_k}\\
&= S_{\mu_k}^*\cdots S_{\mu_2}^*P_{r(\mu_1)}S_{\mu_2}\cdots S_{\mu_k}\\
&= S_{\mu_k}^*\cdots S_{\mu_2}^*P_{s(\mu_2)}S_{\mu_2}\cdots S_{\mu_k}\\
&= S_{\mu_k}^*\cdots S_{\mu_2}^*S_{\mu_2}\cdots S_{\mu_k}\\
&\vdots\\
&= S_{\mu_k}^*S_{\mu_k}\\
&= P_{r(\mu_k)}\\
&= P_{r(\mu)}
\end{aligned}$$

Hence we can extend Lemma 2.2.6 to the partial isometries S_μ, as follows.

Lemma 2.2.8. *Let E be a row-finite directed graph, $\{S,P\}$ a Cuntz–Krieger E-family and $\mu,\nu\in E^*$. Then*

1. *If $|\mu|=|\nu|$ and $\mu\neq\nu$ then $(S_\mu S_\mu^*)(S_\nu S_\nu^*)=0$.*
2. $S_\mu^*S_\nu=\begin{cases}S_{\mu'}^* & \text{if } \mu=\nu\mu' \text{ for some } \mu'\in E^*,\\ S_{\nu'} & \text{if } \nu=\mu\nu' \text{ for some } \nu'\in E^*,\\ 0 & \text{otherwise.}\end{cases}$
3. *If $S_\mu S_\nu\neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu=S_{\mu\nu}$.*
4. *If $S_\mu S_\nu^*\neq 0$, then $r(\mu)=r(\nu)$.*

Proof. If $\mu\neq\nu$, then there exists a smallest integer $i\leq|\nu|=|\mu|$ such that $\mu_i\neq\nu_i$. Then $(\mu_1,..,\mu_{i-1})=(\nu_1,...,\nu_{i-1})$, so

$$\begin{aligned}
S_\mu^*S_\nu &= \left(S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{|\mu|}}\right)^*\left(S_{\nu_1}S_{\nu_2}\cdots S_{\nu_{|\mu|}}\right)\\
&= S_{\mu_{|\mu|}}^*\cdots S_{\mu_2}^*S_{\mu_1}^*S_{\nu_1}S_{\nu_2}\cdots S_{\nu_{|\mu|}}\\
&= S_{\mu_{|\mu|}}^*\cdots S_{\mu_i}^*P_{r(\mu_{i-1})}S_{\nu_i}\cdots S_{\nu_{|\mu|}}\\
&= S_{\mu_{|\mu|}}^*\cdots S_{\mu_i}^*P_{s(\mu_i)}S_{\nu_i}\cdots S_{\nu_{|\mu|}}\\
&= S_{\mu_{|\mu|}}^*\cdots S_{\mu_i}^*S_{\nu_i}\cdots S_{\nu_{|\mu|}}\\
&= 0,
\end{aligned}$$

because $\mu_i\neq\nu_i$, so by Lemma 2.2.6 $S_{\mu_i}^*S_{\nu_i}=0$.

Next assume $|\mu|\leq|\nu|$, then by our argument above, if $\mu=\nu_1\cdots\nu_{|\mu|}$,

$$S_\mu^*S_\nu=S_\mu^*\left(S_{\nu_1}\cdots S_{\nu_{|\mu|}}\right)S_{\nu_{|\mu|+1}}\cdots S_{\nu_{|\nu|}}\neq 0.$$

Hence, in this case we have $\nu = \mu\nu'$ with $\nu' = \nu_{|\mu|+1} \cdots \nu_{|\nu|} \in E^*$ and

$$S_\mu^* S_\nu = S_\mu^* S_\mu S_{\nu'} = P_{r(\mu)} S_{\nu'} = P_{s(\nu')} S_{\nu'} = S_{\nu'}.$$

Analogously, we get for $|\nu| \leq |\mu|$, that $S_\mu^* S_\nu = 0$ or $S_\mu^* S_\nu = S_{\mu'}^*$, when $\mu = \nu\mu'$ for some $\mu' \in E^*$.

Now suppose $S_\mu S_\nu \neq 0$. Then $S_\mu P_{r(\mu)} P_{s(\nu)} S_\nu \neq 0$ and thus $P_{r(\mu)} P_{s(\nu)} \neq 0$, which means that $r(\mu) = s(\nu)$. Therefore $\mu\nu$ is a path in E and by Definition 2.2.7 $S_\mu S_\nu = S_{\mu\nu}$.

In a similar way, we get that if $S_\mu S_\nu^* \neq 0$, then $S_\mu P_{r(\mu)} P_{r(\nu)} S_\nu^* \neq 0$ and thus $P_{r(\mu)} P_{r(\nu)} \neq 0$, which means that $r(\mu) = r(\nu)$. ∎

Lemma 2.2.9. *Let E be a row-finite directed graph, $\{S, P\}$ a Cuntz–Krieger E-family, and $\alpha, \beta, \mu, \nu \in E^*$ with $r(\alpha) = r(\beta)$ and $r(\mu) = r(\nu)$. Then we have*

$$(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = \begin{cases} S_\alpha S_{\nu\beta'}^* & \text{if } \beta = \mu\beta', \\ S_{\alpha\mu'} S_\nu^* & \text{if } \mu = \beta\mu', \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 2.2.8, we have $S_\beta^* S_\mu = S_{\beta'}^*$ if $\beta = \mu\beta'$, $S_\beta^* S_\mu = S_{\mu'}$ if $\mu = \beta\mu'$ and $S_\beta^* S_\mu = 0$ otherwise. Therefore, if $\beta = \mu\beta'$, we get

$$(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = S_\alpha S_{\beta'}^* S_\nu^* = S_\alpha (S_\nu S_{\beta'})^* = S_\alpha S_{\nu\beta'}^*.$$

If $\mu = \beta\mu'$, we get $(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = S_\alpha S_{\mu'} S_\nu^* = S_{\alpha\mu'} S_\nu^*$, and otherwise we have $(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = S_\alpha (S_\beta^* S_\mu) S_\nu^* = 0$, which proves the lemma. ∎

We can write $S_e = S_e S_{r(e)}^*$ and $P_v = S_v S_v^*$, so eventually all elements of $C^*(S, P)$ are of the form $S_\alpha S_\beta^*$ with $r(\alpha) = r(\beta)$. Hence we can prove the following theorem.

Theorem 2.2.10. *Let $\{S, P\}$ be a Cuntz–Krieger E-family for a row-finite graph E. Then*

$$C^*(S, P) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}.$$

Proof. Lemma 2.2.9 implies that $\operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}$ is a subalgebra of $C^*(S, P)$, and since $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$, it is a $*$-subalgebra. Thus its closure is a C^*-subalgebra of $C^*(S, P)$ that contains all generators S_e and P_v, so it is all of $C^*(S, P)$. ∎

In the following we will write $C^*(E)$ for $C^*(S, P)$, because this C^*-algebra is universal for E in a certain sense described by the following Lemmas from [32].

Lemma 2.2.11 ([32, Proposition 1.21])**.** *For any row-finite directed graph E there is a C^*-algebra $C^*(E)$ generated by a Cuntz–Krieger E-family $\{s, p\}$ such that, for every Cuntz–Krieger E-family $\{T, Q\}$ in a C^*-algebra B, there is a homomorphism $\pi_{T,Q} : C^*(E) \to B$ satisfying $\pi_{T,Q}(s_e) = T_e$ for all $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for all $v \in E^0$.*

Lemma 2.2.12 ([32, Corollary 1.22]). *Suppose E is a row-finite directed graph and C is a C^*-algebra generated by a Cuntz–Krieger E-family $\{w, r\}$ such that, for every Cuntz–Krieger E-family $\{T, Q\}$ in a C^*-algebra B, there is a homomorphism $\rho_{T,Q} : C \to B$ satisfying*

$$\rho_{T,Q}(w_e) = T_e \text{ for all } e \in E^1 \text{ and } \rho_{T,Q}(r_v) = Q_v \text{ for all } v \in E^0.$$

Then there is an isomorphism $\phi : C^*(E) \to C$ such that $\phi(s_e) = w_e$ for all $e \in E^1$ and $\phi(p_v) = r_v$ for all $v \in E^0$.

This means that we can prove that a C^*-algebra $\mathcal{B}$ is isomorphic to $C^*(E)$ by finding a Cuntz–Krieger E-family $\{T, Q\}$ that generates $\mathcal{B}$ and has the universal property described in Lemma 2.2.12.

Another theorem we can use to prove that a C^*-algebra $\mathcal{B}$ is isomorphic to $C^*(E)$ is the gauge invariant uniqueness theorem.

Lemma 2.2.13 ([32, Proposition 2.2]). *Let E be a row-finite directed graph. Then there is an action γ of $\mathbb{T}$ (the circle group) on $C^*(E)$ such that $\gamma_z(S_e) = zS_e$ for every $e \in E^1$ and $\gamma_z(P_v) = P_v$ for every $v \in E^0$.*

Theorem 2.2.14 (Gauge-Invariant Uniqueness Theorem, [32, Theorem 2.2]). *Let E be a row-finite directed graph, and suppose that $\{T, Q\}$ is a Cuntz–Krieger E-family in a C^*-algebra $\mathcal{B}$ with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to \operatorname{Aut} \mathcal{B}$ such that $\beta_z(T_e) = zT_e$ for every $e \in E^1$ and $\beta_z(Q_v) = Q_v$ for all $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

2.2.1 Examples

In this subsection we will look at some examples of directed graphs and their associated graph C^*-algebras. The following lemma provides an application of the previous theorem.

Lemma 2.2.15. *Suppose E is a row-finite directed graph without sinks, and let $\widehat{E}$ be the dual graph of E as defined in Example 2.1.8. Then $\widehat{E}$ is a row-finite directed graph without sinks and $C^*(E) \cong C^*(\widehat{E})$.*

Proof. for every $e \in \widehat{E}^0 = E^1$, we have

$$\begin{aligned} |\widehat{s}^{-1}(e)| &= |\{hf \in E^2 : h = \widehat{s}(hf) = e\}| \\ &= |\{f \in E^1 : s(f) = r(e)\}| \\ &= |s^{-1}(r(e))| \\ &< \infty, \end{aligned}$$

because E is row-finite. Therefore $\widehat{E}$ is row-finite. Furthermore, the adjacency matrix of $\widehat{E}$ only has $\{0, 1\}$-entries.

Suppose for the sake of contradiction, that $\widehat{E}$ has a sink $e \in \widehat{E}^0 = E^1$ and suppose $r(e) = w \in E^0$. Since E has no sinks, we know that there exists an edge $f \in E^1$ such that $s(f) = w$. That means that $ef \in E^2 \in \widehat{E}^1$ with $\hat{s}(ef) = e$. This contradicts our assumption that e is a sink in $\widehat{E}$. Thus $\widehat{E}$ has no sinks. (Note that $f = e$ is allowed; this would give us a loop based at e in $\widehat{E}$). Let $\{S, P\}$ be the universal Cuntz–Krieger family generating $C^*(E)$ and define $Q_e = S_eS_e^*$ and $T_{ef} = S_eS_fS_f^*$. Then

$$\begin{aligned}
T_{ef}^*T_{ef} &= S_fS_f^*S_e^*S_eS_fS_f^* \\
&= S_fS_f^*P_{r(e)}S_fS_f^* \\
&= S_fS_f^*P_{s(f)}S_fS_f^* \\
&= S_fS_f^*S_fS_f^* \\
&= S_fS_f^* \\
&= Q_{\hat{r}(ef)},
\end{aligned}$$

and

$$\begin{aligned}
Q_e = &\quad S_eS_e^* \\
= &\quad S_eP_{r(e)}S_e^* \\
&= S_e\left(\sum_{\{f\in E^1:s(f)=r(e)\}} S_fS_f^*\right)S_e^* \\
&= S_e\left(\sum_{\{f\in E^1:s(f)=r(e)\}} S_fS_f^*S_fS_f^*\right)S_e^* \\
&= \sum_{\{f\in E^1:s(f)=r(e)\}} S_eS_fS_f^*S_fS_f^*S_e^* \\
&= \sum_{\{f\in E^1:s(f)=r(e)\}} T_{ef}T_{ef}^* \\
&= \sum_{\{ef\in E^2:\hat{s}(ef)=e\}} T_{ef}T_{ef}^*.
\end{aligned}$$

Thus $\{T, Q\}$ is a Cuntz–Krieger $\widehat{E}$-family, and the universal property of $C^*(\hat{E})$ gives us a homomorphism $\pi_{T,Q} : C^*(\hat{E}) \to C^*(E)$. It is easy to check that this homomorphism intertwines the gauge actions, and since all the isometries S_e are non-zero, so are the projections Q_e. Hence the gauge-invariant uniqueness theorem implies that $\pi_{T,Q}$ is an isomorphism from $C^*(\hat{E})$ onto $C^*(T, Q) \subseteq C^*(E)$. Since the generators $P_v = \sum_{\{e\in E^1:s(e)=v\}} Q_e$ and $S_e = \sum_{\{f\ \in E^1:r(e)=s(f)\}} T_{ef}$ of $C^*(E)$ lie in $C^*(T, Q)$, we see that $C^*(T, Q) \cong C^*(E)$.

Thus $C^*(E) \cong C^*(\hat{E})$. ■

By Lemma 2.2.15 we see that for a row-finite directed graph E without sinks, $\widehat{E}$ is also a row-finite directed graph without sinks, so by applying the Lemma again with $\widehat{E}$ instead of E, we get

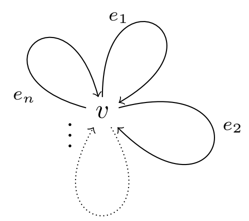
$$C^*(E) \cong C^*(\widehat{E}) \cong C^*(\widehat{E}^2) \cong \ldots \cong C^*(\widehat{E}^n)$$

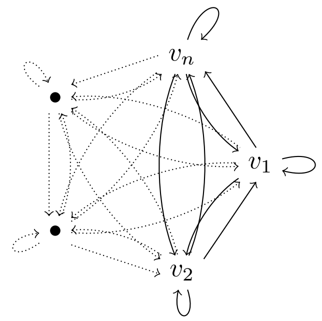
for all $n \in \mathbb{N}$. Here $\widehat{E}^n$ is the dual graph of $\widehat{E}^{n-1}$, that is, the graph that arises from repeating the dualization n times, starting from E. Note that $\widehat{E}^2$ is the dual graph of $\widehat{E}$ and not the edge set $\widehat{E}^2$ of $\widehat{E}$.

Example 2.2.16. By Lemma 2.2.15 the Cuntz algebras $\mathcal{O}_n$ described in 2.2.5, are graph C^*-algebras for E as well as $\widehat{E}$ pictured below on the left and right side, respectively, where

$E^0 = \{v\}$, $E^1 = \{e_i : 1 \leq i \leq n\}$ and $r(e_i) = s(e_i) = v$ for all $i \leq n$, and

$\widehat{E}^0 = \{1, \ldots, n\}$, $\widehat{E}^1 = \{e_{ij} : 1 \leq i, j \leq n\}$ and $s(e_{ij}) = i$ and $r(e_{ij}) = j$.

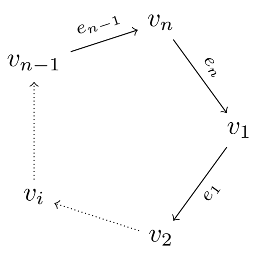




The graph $\widehat{E}$ is called the complete graph on n vertices. As we just saw, $\mathcal{O}_n$ is also isomorphic to the graph C^*-algebras associated to $\widehat{E}^m$ for all $m \in \mathbb{N}$, but we will not draw these graphs as they grow in size as m gets larger.

Next, we will give another interesting and more concrete example of a graph C^*-algebra, where we use Theorem 2.2.14 to show that it is isomorphic to a certain algebra of continuous functions. We will use this example in the proof of Theorem 5.2.3.

Example 2.2.17 ([32, Example 2.14])**.** Consider the graph C_n consisting of a simple loop of n edges. The graph is defined by $E^0 = \{v_i \mid 1 \leq i \leq n\}$ and $E^1 = \{e_i \mid 1 \leq i \leq n\}$, where $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for all $i < n$ and $r(e_n) = v_1$.



We claim that $C^*(C_n)$ is isomorphic to the C^*-algebra $C(\mathbb{T}, M_n(\mathbb{C}))$ with pointwise multiplication and norm given by $\|f\| = \sup_{z \in \mathbb{T}} \|f(z)\|$.

To see this, define $P_{v_i}(z) = e_{ii}$, $S_{e_i}(z) = e_{i(i+1)}$ for all $i < n$, and lastly $S_{e_n}(z) = z e_{n1}$. Then one can check that $\{S, P\}$ is a Cuntz–Krieger C_n-family in $C(\mathbb{T}, M_n(\mathbb{C}))$.

Furthermore, every function of the form $z \mapsto z^m e_{ij}$ for $m \geq 0$, is generated by $\{S, P\}$, because for all $i \leq n$ we have

$$(S_{e_i} S_{e_{i+1}} \cdots S_{e_n} S_{e_1} \cdots S_{e_{i-1}})^m : z \mapsto z^m e_{ii}.$$

If we take adjoints, we see that this is also true for $m < 0$. Thus $\pi_{S,P}(C^*(C_n))$ contains all functions of the form $z \mapsto \sum_{i,j=1}^n f_{ij}(z)e_{ij}$, where each f_{ij} is a trigonometric polynomial. These functions are dense in $C(\mathbb{T}, M_n(\mathbb{C}))$, thus

$$\pi_{S,P}(C^*(C_n)) = C(\mathbb{T}, M_n(\mathbb{C})),$$

and $\{S, P\}$ is a generating Cuntz–Krieger C_n-family in $C(\mathbb{T}, M_n(\mathbb{C}))$.

Now we will define a gauge action on $C(\mathbb{T}, M_n(\mathbb{C}))$ by

$$\beta_w(f)(z) = U_w^* f(w^n z) U_w,$$

where $U_w = \sum_{j=1}^n w^j e_{jj}$ for all $w \in \mathbb{T}$. Then $U_w^* = \sum_{j=1}^n w^{-j} e_{jj}$, so

$$\beta_w(P_{v_i})(z) = U_w^* P_{v_i}(w^n z) U_w = U_w^* e_{ii} U_w = w^{-i} e_{ii} w^i = e_{ii} = P_{v_i}(z).$$

Furthermore, for all $i < n$ we have, $S_{e_i}(z) = e_{i(i+1)}$ for all $z \in \mathbb{T}$, so

$$\beta_w(S_{e_i})(z) = U_w^* e_{i(i+1)} U_w = w^{-i} e_{i(i+1)} w^{i+1} = w e_{i(i+1)} = w S_{e_i}(z).$$

Now the only generator left to check is S_{e_n}, which sends z to ze_{n1}, thus

$$\begin{aligned}\beta_w(S_{e_n})(z) &= U_w^* S_{e_n}(w^n z) U_w = U_w^* w^n z e_{n1} U_w \\ &= w^{-n} w^n z e_{n1} w^1 = wze_{n1} = wS_{e_n}(z).\end{aligned}$$

Therefore, $\beta : \mathbb{T} \to \operatorname{Aut} C(\mathbb{T}, M_n(\mathbb{C}))$ is a continuous action, such that

$$\pi_{S,P} \circ \gamma_w = \beta_w \circ \pi_{S,P}$$

for every $w \in \mathbb{T}$. Hence by Theorem 2.2.14 the map $\pi_{S,P}$ is an isomorphism, thus $C^*(C_n) \cong C(\mathbb{T}, M_n(\mathbb{C}))$.

Next we look at some other interesting examples, where we will not use the Gauge-Invariant Uniqueness Theorem. We start by recalling the following results about matrix algebras and direct sums.

Lemma 2.2.18 ([32, Proposition A.5]). *Suppose $\mathcal{B}$ is a C^*-algebra and $\{e_{ij} : 1 \leq i, j \leq n\}$ is a set of elements of $\mathcal{B}$ such that*

$$e_{ij}^* = e_{ji} \quad \text{and} \quad e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Lemma 2.2.19 ([32, Proposition A.7 (a)]). *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-subalgebras of a C^*-algebra $\mathcal{C}$ such that $ab = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then* $\operatorname{span}\{\mathcal{A} \cup \mathcal{B}\}$ *is a C^*-subalgebra of $\mathcal{C}$, and the map $(a, b) \mapsto a + b$ is an isomorphism of $\mathcal{A} \oplus \mathcal{B}$ onto* $\operatorname{span}\{\mathcal{A} \cup \mathcal{B}\}$.

Lemma 2.2.20. *Suppose E is a finite directed graph with no cycles. Let $w_1, \ldots w_n$ be the sinks in E, and denote $n(w_i) = |\{\alpha \in E^* : r(\alpha) = w_i\}|$. Then*

$$C^*(E) \cong \bigoplus_{i=1}^n M_{n(w_i)}(\mathbb{C}).$$

Proof. Recall that $C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu)\}$ and suppose we have $S_\mu S_\nu^* \in C^*(E)$ with $r(\mu) \neq w_i$ for all i. Then $r(\mu)$ is not a sink, hence we can write

$$S_\mu S_\nu^* = \sum_{\{e \in E^1 \,:\, s(e) = r(\alpha)\}} S_\mu S_e S_e^* S_\nu^*.$$

Since the graph is finite and there are no loops, we can repeat this process until $S_\mu S_\nu^*$ is a sum of terms of the form $S_{\mu\gamma} S_{\nu\gamma}^*$ where $r(\gamma) = w_i$ for some i. Thus

$$C^*(E) \cong \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) = w_i \text{ for some } i\}.$$

Furthermore, we see that

$$A_i = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) = w_i\}$$

is isomorphic to $M_{n(w_i)}(\mathbb{C})$ as for all $S_\mu S_\nu^*, S_\alpha S_\beta^* \in A_i$, we have

$$(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = \begin{cases} S_\alpha S_\nu & \text{if } \mu = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the family $S_\alpha S_\beta^*$ is a family of matrix units and by Lemma 2.2.18 we get $A_i \cong M_{n(w_i)}(\mathbb{C})$. This, together with the fact that $(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = 0$ whenever $r(\mu) = w_i \neq w_j = r(\beta)$, shows that $A_i A_j = 0$ for all $i \neq j$. Hence by Lemma 2.2.19 we get $C^*(E) \cong \bigoplus_{i=1}^n M_{n(w_i)}(\mathbb{C})$. ■

Example 2.2.21. Let E be the graph as pictured below, defined by $E^0 = \{v_1, \ldots, v_n\}$ and $E^1 = \{e_1, \ldots, e_n\}$, where $s(e_i) = v_i$ for all $i \leq n$ and $r(e_i) = v_{i+1}$ for all $i < n$.

$$v_1 \xrightarrow{e_1} v_2 \dashrightarrow v_i \dashrightarrow v_{n-1} \xrightarrow{e_{n-1}} v_n$$

Then by the Lemma above, $C^*(E) \cong M_n(\mathbb{C})$.

Example 2.2.22 ([32, Example 1.19])**.** Let E be the graph pictured below, defined by $E^0 = \{v, w\}$, $E^1 = \{e, f\}$ with $s(e) = r(e) = v = s(f)$ and $r(f) = w$.

$$\begin{tikzcd} v \arrow[loop left, "e"] \arrow[r, "f"] & w \end{tikzcd}$$

Then $S_e^* S_e = P_v$, $S_f^* S_f = w$ and $P_v = S_e S_e^* + S_f S_f^*$.

Clearly $P_v + P_w$ is the identity for $C^*(E)$ and $S_e + S_f$ satisfies

$$(S_e + S_f)^*(S_e + S_f) = S_e^* S_e + S_f^* S_e + S_e^* S_f + S_f^* S_f = P_v + 0 + 0 + P_w,$$

thus $S_e + S_f$ is an isometry in $C^*(E)$.

Furthermore,

$$\begin{aligned} (S_e + S_f)(S_e + S_f)^* = S_e S_e^* + S_f S_e^* + S_e S_f^* + S_f S_f^* &= P_v, \\ P_w &= (S_e + S_f)^*(S_e + S_f) - P_v, \\ S_f &= (S_e + S_f) P_w \\ \text{and } S_e &= (S_e + S_f) P_v. \end{aligned}$$

Therefore, $C^*(E)$ is generated by the isometry $S_e + S_f$.

Conversely, if V is an isometry, then $P_w = 1 - VV^*$, $P_v = VV^*$, $S_e = VP_v$, and $S_f = VP_w$ defines a Cuntz–Krieger E-family in $C^*(V)$. We have thus shown that $C^*(E)$ is isomorphic to the Toeplitz algebra defined in Example 1.1.19.

2.2.2 Adding tails and heads

Recall from Lemma 1.1.24 that all full corners $p\mathcal{A}p$ of $\mathcal{A}$, have the same ideal structure as $\mathcal{A}$. We will show that for every row-finite directed graph E with sinks and sources, there is a row-finite directed graph F without sinks and sources, such that $C^*(E)$ is a full corner of $C^*(F)$.

Definition 2.2.23. For a row-finite directed graph E and a sink $v \in E^0$, define a *tail* by

$$\begin{aligned} \mathrm{T}^0 &= \{t_i \ : \ i \in \mathbb{N}\}, \\ \mathrm{T}^1 &= \{f_i \ : \ i \in \mathbb{N} \cup \{0\}\}, \end{aligned}$$

with $s(f_i) = t_i$ and $r(f_i) = t_{i+1}$, where $t_0 = v$.

For a source $w \in E^0$, define a *head* by

$$\begin{aligned} \mathrm{H}^0 &= \{h_i \ : \ i \in \mathbb{N}\}, \\ \mathrm{H}^1 &= \{g_i \ : \ i \in \mathbb{N} \cup \{0\}\}, \end{aligned}$$

with $s(g_i) = h_{i+1}$ and $r(g_i) = h_i$, where $h_0 = w$.

Lemma 2.2.24 ([32, Lemma 2.10])**.** *Let E be a row-finite directed graph and let V be an infinite set of vertices in E. Then there is a projection p_V in $M(C^*(E))$ such that*

$$P_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & \text{if } r(\mu) \in V, \\ 0 & \text{if } r(\mu) \notin V. \end{cases}$$

Theorem 2.2.25 ([32, Corolarry 2.11])**.** *Suppose that E is a row-finite directed graph and let F be the row-finite directed graph obtained by adding a head to every source and a tail to every sink. Denote by $\{S, P\}$ and $\{T, Q\}$ the canonical Cuntz–Krieger families generating $C^*(E)$ and $C^*(F)$. Let*

$$Q_E = \sum_{v \in E^0} Q_v \in M(C^*(F))$$

as defined in Lemma 2.2.24, where $E^0 \subseteq F^0$. Then $Q_E C^*(F) Q_E$ is a full corner in $C^*(F)$ and there exists an isomorphism

$$\phi : C^*(E) \to Q_E C^*(F) Q_E$$

such that $\phi(S_e) = T_e$ for $e \in E^1$ and $\phi(P_v) = Q_v$ for $v \in E^0$.

Proof. Recall from Theorem 2.2.10, that

$$C^*(F) = C^*(T, Q) = \overline{\mathrm{span}}\{T_\mu T_\nu^* : \mu, \nu \in F^*, r(\mu) = r(\nu)\}.$$

For every $e \in E^1$, $T_e^* T_e = Q_{r(e)}$, where $r(e) \in E^0$ and if v is not a sink in E, then it does not have a tail in F, so

$$\{e \in E^1 : s(e) = v\} = \{e \in F^1 : s(e) = v\}.$$

Thus, the Cuntz–Krieger relations hold and $\{T_e, Q_v \mid e \in E^1, v \in E^0\}$ forms a Cuntz–Krieger E-family in $C^*(F)$. The homomorphism $\pi_{T,Q} : C^*(E) \to C^*(F)$ intertwines the gauge actions, and thus by Theorem 2.2.14, $\pi_{T,Q}$ is injective. Now for $\mu, \nu \in F^*$ with $r(\mu) = r(\nu)$, we have

$$Q_E T_\mu T_\nu^* Q_E = T_\mu T_\nu^*$$

if $s(\mu), s(\nu) \in E^0$, otherwise $Q_E T_\mu T_\nu^* Q_E = 0$. In case $s(\mu), s(\nu) \in E^0$, we either have $r(\mu) = r(\nu) \in E^0$ or $r(\mu) = r(\nu)$ lies on a tail. Suppose this tail starts at $v \in E^0$. Then $\mu = \mu'\alpha$ and $\nu = \nu'\alpha$ with $s(\alpha) = v$. Therefore,

$$T_\mu T_\nu^* = T_{\mu'} T_\alpha T_\alpha^* T_{\nu'} = T_{\mu'} Q_v T_{\nu'} = T_{\mu'} T_{\nu'},$$

as we have for all $e \in F^1$ on a tail or head, $s(e) \neq s(e')$ for all $e' \in F^1 \setminus \{e\}$. Thus for $v = s(e)$ we have

$$Q_v = \sum_{f \in s^{-1}(v)} T_f T_f^* = T_e T_e^*.$$

Thus, every non-zero element of the form $Q_E T_\mu T_\nu^* Q_E$ is equal to an element of the form $T_{\mu'} T_{\nu'}^*$, where $s(\mu'), s(\nu') \in E^0$ and $r(\mu') = r(\nu') \in E^0$. Then using that $a \mapsto Q_E a Q_E$ is continuous and linear, we see that

$$\begin{aligned} Q_E C^*(F) Q_E &= \overline{\operatorname{span}}\{T_\mu T_\nu^* : \mu, \nu \in E^*, r(\mu) = r(\nu) \in E^0, s(\mu), s(\nu) \in E^0\} \\ &= \pi_{T,Q}(C^*(E)). \end{aligned}$$

Thus $\pi_{T,Q}$ is an isomorphism from $C^*(E)$ to $Q_E C^*(F) Q_E$. The only thing left to show is that $Q_E C^*(F) Q_E$ is a full corner.

To prove this, suppose that J is an ideal in $C^*(F)$ containing $Q_E C^*(F) Q_E$, then J certainly contains $\{Q_v \mid v \in E^0\}$. Now suppose $w \in F^0$ lies on a tail starting at v. Then there exists a unique path α, with $s(\alpha) = v$ and $r(\alpha) = w$, from which we get

$$Q_v \in J \Rightarrow T_\alpha = Q_v T_\alpha \in J \Rightarrow Q_w = T_\alpha^* T_\alpha \in J.$$

Hence $Q_v, T_e \in J$ for all vertices v and edges e on tails in F. For $w \in F^0$ on a head ending in v, we similarly have a unique path α, with $s(\alpha) = w$ and $r(\alpha) = v$. Therefore,

$$Q_v \in J \Rightarrow T_\alpha = T_\alpha Q_v \in J \Rightarrow Q_w = T_\alpha T_\alpha^* \in J,$$

where we use the fact that for all $e \in F^1$ on a head or tail, $Q_{s(e)} = T_e T_e^*$. Hence all the generators $\{T_e, Q_v\}$ of $C^*(F)$ lie in J, thus $J = C^*(F)$, which means that $C^*(E) \cong Q_E C^*(F) Q_E$ is a full corner of $C^*(F)$. ■

3. Graph C^*-algebras as groupoid C^*-algebras

In this section we describe another model for the C^*-algebra $C^*(E)$ of a directed graph, namely the groupoid model. We first discuss what groupoids are and then we show how we can construct a groupoid for every directed graph E without sinks, called the path groupoid $\mathcal{G}_E$. Secondly we show how we can construct a C^*-algebra $C^*(G)$ for certain groupoids G, and finally we show that $C^*(\mathcal{G}_E) \cong C^*(E)$ for all row-finite directed graphs without sinks.

3.1 Groupoids and equivalence relations

A groupoid can be equivalently defined as a small category[1] in which every morphism is invertible. Keeping this in mind, the following definitions and corollaries are intuitively clear. We will use Renault's notation [35].

Definition 3.1.1. A *groupoid* is a set G endowed with a product map

$$(\ ,\ ) : G^{(2)} \to G, (x, y) \mapsto xy,$$

where $G^{(2)} \subset G \times G$ is the set of *composable pairs*, and an inverse map

$$\cdot^{-1} : G \to G, x \mapsto x^{-1}$$

such that the following four conditions hold:

1. If $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$, then $(xy, z), (x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$.
2. $(x^{-1})^{-1} = x$ for all $x \in G$.
3. $(x, x^{-1}) \in G^{(2)}$ for all $x \in G$ and if $(z, x) \in G^{(2)}$ then $(zx)x^{-1} = z$.
4. $(x^{-1}, x) \in G^{(2)}$ for all $x \in G$ and if $(x, y) \in G^{(2)}$ then $x^{-1}(xy) = y$.

Definition 3.1.2. The *range* and *domain* maps are defined by

$$r : G \to G, x \mapsto xx^{-1} \quad \text{and} \quad d : G \to G, x \mapsto x^{-1}x,$$

[1] A category is small if it has a set of objects and a set of morphisms, i.e. a small category is an internal category in the category Set.

respectively. The set $G^{(0)} = r(G) = d(G)$ is called the *unit space* of G; its elements are units in the sense that $xd(x) = r(x)x = x$. Having defined the source and domain maps, it follows that the set of *composable pairs* satisfies

$$G^{(2)} = \{(x, y) \in G \times G : d(x) = r(y)\}.$$

Thinking of G as a category again, we see that $(x, y) \mapsto xy$ is simply the composition of the morphisms x and y, $d(x)$ is the identity map on the domain of x, and $r(x)$ is the identity map on the range of x. Thus $G = G^{(1)}$, $G^{(0)}$ are the objects of the category, and $G^{(2)}$ are the composable morphisms.

Example 3.1.3. As the name suggests, groupoids are related to groups. A group G is actually a groupoid with $G^{(2)} = G \times G$ and $G^{(0)} = \{e\}$ (the unit element).

Example 3.1.4. Let Γ be a discrete group acting on a set X. For every $x \in X$ and $g \in \Gamma$, let xg denote the transform of x by g. Let

$$\begin{aligned} G &= X \times \Gamma, \\ G^{(2)} &= \{((x, g), (y, h)) : y = xg\}, \end{aligned}$$

and define the product and the inverse map as $(x, g)(xg, h) = (x, gh)$ and $(x, g)^{-1} = (xg, g^{-1})$. Then G is a groupoid and $G^{(0)}$ can be identified with X. This groupoid is called the transformation groupoid.

Example 3.1.5. Let $E \subset X \times X$ be the graph of an equivalence relation on X. Then we can define a groupoid structure on E by setting

$$E^{(2)} = \{((x_1, y_1), (x_2, y_2)) \in E \times E : y_1 = x_2\}$$

with $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$. It is easy to see that $E^{(0)}$ can then be identified with X. Thus every equivalence relation can be viewed as a groupoid.

Definition 3.1.6. For $u, v \in G^{(0)}$ we define $G^u = r^{-1}(u)$, $G_v = d^{-1}(v)$ and $G^u_v = G^u \cap G_v = \{x \in G : r(x) = u,\ d(x) = v\}$.

This means that G^u consists of all morphisms that have the same range as u, whereas G_v consists of all morphisms that have the same domain as v, so G^u_v consists of all the morphisms from the domain of v to the range of u. In terms of the category, G^u_v is the set of morphisms from the object associated with v to the object associated with u.

In Example 3.1.4 we see that the *stabilizer subgroup*

$$\Gamma_x = \{g \in \Gamma : xg = x\}$$

of Γ with respect to x is isomorphic to G^u_u, where $u = (x, id_\Gamma)$. Indeed,

$$G^u_u = \{(y, g) \in G : (y, id_\Gamma) = (x, id_\Gamma)$$

and

$$(yg, id_\Gamma) = (x, id_\Gamma)\} = \{(x, g) \in G : xg = x\}.$$

Another interesting fact about the sets G_v^u, is that it gives us an equivalence relation on $G^{(0)}$:

$$u \sim v \iff G_v^u \neq \emptyset,$$

that is, u is equivalent to v if and only if there exists an arrow from v to u. The graph of this equivalence relation is $R = \{(r(x), d(x)) : x \in G\}$. The equivalence classes of this relation are called *orbits* and we denote the orbit of a unit u by $[u]$. This means that the orbit of u contains all units whose ranges can be reached via a morphism $x \in G$ with the same domain as u. In the language of category theory, the orbit of an object u contains all objects v such that a morphism from v to u exists.

In Example 3.1.4, we see that the orbit $\Gamma \cdot x = \{xg : g \in \Gamma\}$ of an element $x \in X$ is isomorphic to the orbit $[(x, id_\Gamma)]$ in G, because $(y, id_\Gamma) \in [(x, id_\Gamma)]$ if and only if $G_{(y,id_\Gamma)}^{(x,id_\Gamma)}$ is non-empty, which is equivalent to the existence of an element $g \in \Gamma$ such that $xg = y$, which by definition means that $y \in \Gamma \cdot x$.

In Example 3.1.5, we see that the orbit of an element $(x, x) \in E^{(0)}$ is isomorphic to the equivalence class of x in E because

$$(y, y) \in [(x, x)] \iff E_{(y,y)}^{(x,x)} \neq \emptyset \iff (x, y) \in E \iff y \in [x].$$

As we saw in Example 3.1.5, we can endow every equivalence relation with a groupoid structure. So it is natural to consider the equivalence relation $R = \{(r(x), d(x)) : x \in G\}$ that was defined. We call the equivalence relation R viewed as a groupoid the *principal groupoid associated with* G. Thus we may go from a groupoid to an equivalence relation, and then back to a groupoid. The term principal comes from the following definition.

Definition 3.1.7. A groupoid is called *principal* if the map

$$\theta : G \to G^{(0)} \times G^{(0)}, x \mapsto (r(x), d(x))$$

is injective.

In the language of category theory, this means that two morphims with the same domain and range must be equal.

It is easy to see that the principal groupoid associated with G is principal. Let $(a, b), (u, v) \in R$ with $\theta(a, b) = \theta(u, v)$. Then $((a, a), (b, b)) = ((u, u), (v, v))$, so $a = u$ and $b = v$, thus $(a, b) = (u, v)$. Hence θ is injective, thus R is a principal groupoid. In fact, this even shows that every equivalence relation is a principal groupoid. The following lemma shows an even stronger result.

Lemma 3.1.8. *Suppose that G is prinicipal. Then $R \cong G$.*

Proof. If G is principal, we have $\theta(G) = \{(r(x), d(x)) : x \in G\} = R$ and $\theta : G \to \theta(G)$ is bijective. Furthermore, it is easy to check that $(x, y) \in G^{(2)}$ implies $(\theta(x), \theta(y)) \in R^{(2)}$, $\theta(x)\theta(y) = \theta(xy)$ and $\theta(x^{-1}) = \theta(x)^{-1}$. Thus θ also preserves the groupoid structure, which means that $\theta : G \to \theta(G)$ is an isomorphism and $G \cong R$. ∎

This shows us that a principal groupoid is an equivalence relation on its unit space.

Consider the groupoid G in Example 3.1.4. Now let $(x,g),(y,h) \in G$. Then $\theta(x,g) = \theta(y,h)$ if and only if $((x, id_\Gamma),(xg, id_\Gamma)) = ((y, id_\Gamma),(yh, id_\Gamma))$, which is equivalent to $x = y$ and $xg = xh$. Thus G is principal if and only if for all $x \in X$ and $g, h \in \Gamma$, $xg = xh$ implies $g = h$. In other words, G is principal if the action of Γ on X is a free action.

In order to define a C^*-algebra from a groupoid, as we will explain in section 3.2, it is convenient to have a topology on the groupoid.

Definition 3.1.9. A *topological groupoid* is a groupoid G endowed with a topology with respect to which

1. The inverse map $x \mapsto x^{-1}$, $G \to G$ is continuous.
2. The multiplication $(x,y) \mapsto xy$, $G^{(2)} \to G$ is continuous, where $G^{(2)}$ has the induced topology from $G \times G$.

It follows that the inverse map is a homeomorphism, that r and d are continuous, and that $G^{(0)}$ is closed when G is Hausdorff.

Consider the groupoid G in Example 3.1.4, where Γ and X are finite and endowed with the discrete topology. Then G is a topological groupoid. Furthermore, because all subsets of $\Gamma \times X$ and $X \times X$ are compact, the map $\Gamma \times X \to X \times X, (g,x) \mapsto (gx, x)$ is proper. Thus the action is proper. Recall that a group action is called principal if it is free and proper. Thus in this case, G is principal if and only if the action of Γ on X is principal.

The following definitions all apply to locally compact Hausdorff groupoids and are of great importance in the construction of the groupoid C^*-algebra.

Definition 3.1.10. A locally compact Hausdorff groupoid is *r-discrete* if its unit space $G^{(0)}$ is an open subset of G.

Remark. Let $u \in G^{(0)}$, then G^u and G_u are discrete spaces in an r-discrete groupoid. Indeed, for all $x \in G^u$ with $v = d(x)$ we have that $\{v\} = G^{(0)} \cap G^v$ is an open set in G^v, and $y \mapsto xy$, $G^v \to G^u$ is a homeomorphism, so $\{xv\} = \{x\}$ is open in G^u.

Definition 3.1.11. A *bisection* is a subset $S \subseteq G$ such that the restrictions of r and d to S are homeomorphisms onto open subsets of G.

Definition 3.1.12. A locally compact Hausdorff groupoid is *étale* if r is a local homeomorphism.

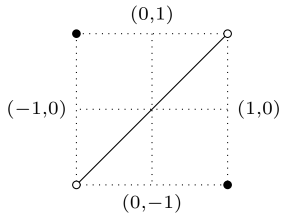
This is equivalent to G having a base of open bisections, as proven in [35, Proposition 2.8].

Remark. If a locally compact Hausdorff groupoid is étale, then it is r-discrete, because for all $x \in G$ there exists an open set U_x with $x \in U_x$ such that $r(U_x)$ is open, thus $\cup_{x \in G} r(U_x) = G^{(0)}$ is open.

Hence for locally compact Hausdorff groupoids, we see that being étale implies being r-discrete, but being r-discrete does not imply being étale, as the following example demonstrates.

Example 3.1.13. Let us consider the following equivalence relation

$$E = \{(-1,1),(1,-1)\} \cup \{(x,x) : x \in [-1,1]\} \subset [-1,1] \times [-1,1].$$



With the induced topology from $\mathbb{R}^2$, E becomes a topological groupoid that is locally compact Hausdorff and such that $E^{(0)} = r(E) = \{(x,x) : x \in [-1,1]\}$ is open in E. Thus E is r-discrete. However, r is not a local homeomorphism. Looking at the definition, $r : E \to E^{(0)}$ is a local homeomorphism if for every point x in E there exists an open set U_x containing x such that the image $r(U_x)$ is open in $E^{(0)}$ and the restriction $r_{|U_x} : U_x \to r(U_x)$ is a homeomorphism.

Now take $x = (-1,1)$. In order for $r(U_x)$ to be open in $E^{(0)}$, it needs to contain the set $\{(x,x) : x \in [-1,-1+\epsilon]\}$ for some $\epsilon > 0$. Otherwise $r(-1,1) = (-1,-1)$ would be a point disconnected from the rest of $r(U_x)$.

This means that U_x must contain the set $\{(x,x) : x \in (-1,-1+\epsilon]\}$. Now let $V = \{(-1,1)\}$. This set is open in U_x, but $r(V) = \{(-1,-1)\}$ is not open in $r(U_x) \supset \{(x,x) : x \in [-1,-1+\epsilon]\}$, which means that $r_{|U_x}$ is not an open mapping. Thus r cannot be a local homeomorphism, so E is r-discrete but not étale.

Definition 3.1.14. A topological groupoid G is called *ample* if it has a base of compact open bisections.

Remark. If G is an étale groupoid, then G is ample if and only if $G^{(0)}$ is totally disconnected.

3.1.1 The path groupoid

Now that we have discussed the basics of groupoid theory we can construct the link with directed graphs by constructing a topological groupoid from a directed graph.

Definition 3.1.15. For a directed graph E, define the *path groupoid* (as in [23]) by

$$\mathcal{G}_E = \{(x,k,y) : x,y \in E^\infty, k \in \mathbb{Z}, \exists N \in \mathbb{N} \text{ with } x_i = y_{i+k}\ \forall i \geq N\},$$

where $\mathcal{G}_E^{(2)} = \{((x,k,y),(w,l,z)) : w = y\}$ and the composition and inverse maps are given by

$$(x,k,y)(y,l,z) = (x,k+l,z) \text{ and } (x,k,y)^{-1} = (y,-k,x).$$

This means that $(x, k, y) \in \mathcal{G}_E$ if and only if $x = wz$ and $y = vz$ where w, v are finite paths and $|w| + k = |v|$.

It is easy to check that the conditions in Definition 3.1.1 hold, which means that the path groupoid is indeed a groupoid.

Remark. Recall from Subsection 2.1.1 the dynamical system (E^∞, σ) given by a one-sided subshift of finite type over the alphabet E^1. This is linked to $\mathcal{G}_E$ in the following way:

$$\begin{aligned} \mathcal{G}_E^{(0)} &= \{(x, 0, x) : x \in E^\infty\} \cong E^\infty, \\ \mathcal{G}_E &= \{(x, k, y) : x, y \in E^\infty, k \in \mathbb{Z}, \exists N \in \mathbb{N} \text{ with } x_i = y_{i+k} \ \forall i \geq N\} \\ &= \{(x, k, y) : x, y \in E^\infty, k \in \mathbb{Z}, \exists N \in \mathbb{N} \text{ with } \sigma^N(x) = \sigma^{N+k}(y)\}. \end{aligned}$$

Now, for reasons that will become clear in Section 3.2, we need to define a topology on $\mathcal{G}_E$ such that $\mathcal{G}_E$ becomes a locally compact étale Hausdorff groupoid. In order to make this happen we will use the following sets.

Definition 3.1.16. For all (possibly empty) paths $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$, define

$$Z(\alpha, \beta) = \{(x, k, y) : x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|, x_i = y_{i+k} \text{ for } i > |\alpha|\}.$$

Lemma 3.1.17. *For $\alpha, \beta, \gamma, \delta \in E^*$ with $r(\alpha) = r(\beta)$ and $r(\gamma) = r(\delta)$, we have*

$$Z(\alpha, \beta) \cap Z(\gamma, \delta) = \begin{cases} Z(\alpha, \beta) & \text{if there exists } \epsilon \in E^* \text{ such that } \alpha = \gamma\epsilon, \beta = \delta\epsilon, \\ Z(\gamma, \delta) & \text{if there exists } \epsilon \in E^* \text{ such that } \gamma = \alpha\epsilon, \delta = \beta\epsilon, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let $(x, k, z) \in Z(\alpha, \beta) \cap Z(\gamma, \delta)$. Then $k = |\beta| - |\alpha| = |\delta| - |\gamma|$, $x \in Z(\alpha) \cap Z(\gamma)$, and $y \in Z(\beta) \cap Z(\delta)$. By Lemma 2.1.11 there exists $\epsilon \in E^*$ such that $\alpha = \gamma\epsilon$ or such that $\gamma = \alpha\epsilon$ and there exists $\epsilon' \in E^*$ such that $\beta = \delta\epsilon'$ or such that $\delta = \beta\epsilon'$.

Let us first look at the case $\alpha = \gamma\epsilon$. Here

$$|\beta| = k + |\alpha| = |\delta| - |\gamma| + |\alpha| = |\delta| + |\epsilon| > |\delta|.$$

Then we know from Lemma 2.1.11 that $\beta = \delta\epsilon'$. We also have $x_i = y_{i+k}$ for all $i > |\gamma|$, thus for all $|\epsilon| \geq j > 0$ we have

$$\epsilon_j = x_{|\gamma|+j} = y_{|\gamma|+k+j} = y_{|\delta|+j} = \epsilon'_j.$$

Therefore, $\epsilon = \epsilon'$, hence we have $x \in Z(\alpha)$, $y \in Z(\beta)$, $k = |\beta| - |\alpha|$, and $x_i = y_{i+k}$ for $i > |\alpha|$, which means that $(x, k, y) \in Z(\alpha, \beta)$ and there exists $\epsilon \in E^*$ such that $\alpha = \gamma\epsilon, \beta = \delta\epsilon$. This gives us the first case of the statement.

The second case can be proven analogously, and it is clear that in any other case $Z(\alpha, \beta) \cap Z(\gamma, \delta)$ must be the empty set. ∎

Theorem 3.1.18. *Let E be a row-finite directed graph. The sets $Z(\alpha, \beta)$ as above form a basis for a locally compact Hausdorff topology on $\mathcal{G}_E$. With this topology, $\mathcal{G}_E$ is a second countable étale locally compact Hausdorff groupoid.*

Proof. By Lemma 3.1.17 each finite intersection of sets of the form $Z(\alpha, \beta)$ is another one of those sets. Since E is row-finite, E^* is countable. Thus the sets form a countable basis.

Since for all $(x, k, y) \in \mathcal{G}_E$ we have $(x, k, y) \in Z(x^n, y^{n+k})$ for sufficiently large n, where $x^n = (x_1, ..., x_n)$, we have for $(x, k, y) \neq (u, l, v)$ in $\mathcal{G}_E$ either $l \neq k$, $x \neq u$ or $y \neq v$. Thus there exist $n, m \in \mathbb{N}$ large enough such that $(x, k, y) \in Z(x^n, y^{n+k})$, $(u, l, v) \in Z(u^m, v^{m+l})$ and $Z(x^n, y^{n+k}) \cap Z(u^m, v^{m+l})$ is empty, so the topology is Hausdorff.

Now let $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ and define

$$h_{\alpha,\beta} : \{x \in E^\infty : s(x) = r(\alpha)\} \to Z(\alpha, \beta), x \mapsto (\alpha x, |\beta| - |\alpha|, \beta x).$$

This is clearly a bijection, which is continuous because the sets $Z(\alpha\epsilon, \beta\epsilon)$ form a basis of the induced topology in $Z(\alpha, \beta)$ and $h^{-1}_{\alpha,\beta}(Z(\alpha\epsilon, \beta\epsilon)) = Z(\epsilon)$ is open in E^∞.

Since E is row-finite,

$$\{x \in E^\infty : s(x) = r(\alpha)\} = E^\infty \cap \prod_{k=1}^{\infty} E_k(r(\alpha))$$

is compact, and the continuous bijection $h_{\alpha,\beta}$ is automatically a homeomorphism by [38, Theorem 3.3.11]. Thus the sets $Z(\alpha, \beta)$ are compact and open, so the topology is locally compact.

The inversion map is continuous, because it maps $Z(\alpha, \beta)$ to $Z(\beta, \alpha)$. The product map is continuous by the following argument. Let $((x, k, y), (y, l, z)) \in \mathcal{G}_E^{(2)}$ and suppose $(x, k, y)(y, l, z) = (x, k + l, z) \in Z(\alpha, \beta)$. Then we can choose $N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for $i \geq N$ and $y_j = z_{j+l}$ for $j \geq N + k$ and also such that $N \geq |\alpha|$ and $N \geq |\beta| - k - l$. Let

$$\begin{aligned}
\alpha' = x^n &= (x_1, ..., x_N) = (\alpha_1, ..., \alpha_{|\alpha|}, x_{|\alpha|+1}, ..., x_N), \\
\gamma = y^{N+k} &= (y_1, ..., y_{N+k}) \text{ and} \\
\beta' = z^{N+k+l} &= (z_1, ..., z_{N+k+l}) = (\beta_1, ..., \beta_{|\beta|}, z_{|\beta|+1}, ..., z_{N+k+l}).
\end{aligned}$$

Then $(x, k, y) \in Z(\alpha', \gamma)$, $(y, l, z) \in Z(\gamma, \beta')$ and the set

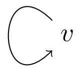
$$\mathcal{G}_E^{(2)} \cap Z(\alpha', \gamma) \times Z(\gamma, \beta'),$$

which is open, is mapped into $Z(\alpha, \beta)$. Thus the product map is continuous at $((x, k, y), (y, l, z))$, but since we chose this point arbitrarily, the product map is continuous at every point and hence continuous. Thus $\mathcal{G}_E$ is a topological groupoid.

Finally, the range map r is a homeomorphism of $Z(\alpha, \beta)$ onto $Z(\alpha)(\cong Z(\alpha, \alpha))$, which means that r is a local homeomorphism. Thus $\mathcal{G}_E$ is étale. ■

To get a better understanding of the path groupoid, we conclude this section with two interesting examples.

Example 3.1.19. Recall from Example 3.1.3 that a group is a special case of a groupoid. Now let us check if there is a graph E for which the path groupoid $\mathcal{G}_E$ is a group. We need $\mathcal{G}_E^{(0)} \cong E^\infty$ to only have one element, so suppose $x \in E^\infty$. Then for all $n \in \mathbb{N}$, if $x_1 \neq x_n$ then $y = (x_n, x_{n+1}, \dots) \in E^\infty$ and $y \neq x$, thus for all $n \in \mathbb{N}$, $x_1 = x_n$. Hence we can assume that $x = (e, e, \dots)$, so we know that $r(e) = s(e)$. Thus for the graph E with $E^0 = \{v\}$, $E^1 = \{e\}$ and $r(e) = s(e) = v$ as pictured below, the path groupoid $\mathcal{G}_E$ is also a group.



Furthermore, $(x, k, y) \in \mathcal{G}_E$ if $x, y \in E^\infty$ and $\exists N \in \mathbb{N}$ such that $x_i = y_{i+k}$ for all $i \geq N$, but x and y must be equal to $(e, e, \dots)$. Thus $x_i = y_{i+k}$ holds for all $k \in \mathbb{Z}$ and $i \in \mathbb{N}$. Hence $\mathcal{G}_E$ is isomorphic to $\mathbb{Z}$ as a set, with composition given by $+$ and inverse given by $-$, so it is isomorphic to the group of integers.

Example 3.1.20. Let E be the graph as pictured below with, $E^0 = \mathbb{Z}$ and $E^1 = \{e_i \mid i \in \mathbb{Z}, s(e_i) = i, r(e_i) = i+1\}$. Obviously every infinite path is determined by its source, thus $\mathcal{G}_E^{(0)} \cong E^\infty \cong \mathbb{Z}$ and for all $n, m \in \mathbb{Z}$, we have $(n, n-m, m) \in \mathcal{G}_E$. It is easy to see that $\mathcal{G}_E$ is principal and that this groupoid is isomorphic to the groupoid of the full equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

$$\cdots\cdots \to v_{-1} \xrightarrow{e_{-1}} v_0 \xrightarrow{e_0} v_1 \cdots\cdots \to$$

3.2 The C^*-algebra of an étale groupoid

In Subsection 3.1.1 we defined the path groupoid $\mathcal{G}_E$ of a graph E and we saw that it is always étale. It was also mentioned that this property would be of great importance in constructing the groupoid C^*-algebra, and indeed we will see in Example 3.2.3 that it is actually required in our construction.

In this section we will discuss the construction of groupoid C^*-algebras due to Renault [35, Chapter 2].

Let G be a locally compact Hausdorff étale groupoid.

Definition 3.2.1. *The groupoid algebra* $C_c(G)$ associated to G, is the algebra of all continuous compactly supported functions $f : G \to \mathbb{C}$ with involution and composition for $f, g \in C_c(G)$ defined by:

$$f^*(z) = \overline{f(z^{-1})} \quad \text{and} \quad (f * g)(z) = \sum_{xy=z} f(x)g(y) = \sum_{y \in G_{d(z)}} f(zy^{-1})g(y).$$

Theorem 3.2.2. *If G is a locally compact Hausdorff étale groupoid, then $C_c(G)$ is a $*$-algebra.*

Proof. First of all, the convolution is well defined because the sum is finite. To see this, recall the definition of étale from Definition 3.1.12, according to which r is a local homeomorphism, which implies that d is also a local homeomorphism.

Therefore, we can cover any compact subset $K \subset G$ with finitely many open subsets on which d is a homeomorphism. Thus for $K = U_1 \cup \cdots \cup U_n$, we have for all $x \in G^{(0)}$ and $y \in U_i$, if $d(y) = x$, then $d(z) \neq x$ for all other $z \in U_i$. Hence $|U_i \cap G_x| \leq 1$. This implies that

$$\sup_{x \in G^{(0)}} |K \cap G_x| = \sup_{x \in G^{(0)}} |(\bigcup_{i=1}^{n} U_i) \cap G_x| = \sup_{x \in G^{(0)}} |(U_1 \cap G_x) \cup \cdots \cup (U_n \cap G_x)| \leq n.$$

Now let $K = \mathrm{supp}(g)$ and $x = d(z)$. Then we see that $\mathrm{supp}(g) \cap G_{d(z)}$ is finite so the sum in the convolution is finite.

To check that the convolution is associative and the involution is its own inverse, one computes

$$\begin{aligned}
((f * g) * h)(z) &= \sum_{xy=z} (f * g)(x) h(y) \\
&= \sum_{xy=z} (\sum_{vw=x} f(v) g(w)) h(y) \\
&= \sum_{vwy=z} f(v) g(w) h(y) \\
&= \sum_{vu=z} f(v) (\sum_{wy=u} g(w) h(y)) \\
&= \sum_{vu=z} f(v) (g * h)(u) \\
&= (f * (g * h))(z).
\end{aligned}$$

Therefore, $(f * g) * h = f * (g * h)$ and $f^{**}(x) = \overline{f^*(x^{-1})} = \overline{\overline{f((x^{-1})^{-1})}} = f(x)$, hence $f^{**} = f$. Furthermore,

$$\begin{aligned}
(f * g)^*(z) &= \overline{(f * g)(z^{-1})} \\
&= \overline{\sum_{xy=z^{-1}} f(x) g(y)} \\
&= \sum_{y^{-1}x^{-1}=z} \overline{f(x)}\, \overline{g(y)} \\
&= \sum_{vw=z} \overline{g(v^{-1})}\, \overline{f(w^{-1})} \\
&= \sum_{vw=z} g^*(v) f^*(w) \\
&= (g^* * f^*)(z),
\end{aligned}$$

so that $(f * g)^* = g^* * f^*$. We also need to check that $C_c(G)$ is closed under convolution and involution.

First of all, $f^* \in C_c(G)$, because $x \mapsto x^{-1}$ is continuous and $z \mapsto \bar{z}$ is continuous, so $x \mapsto \overline{f(x^{-1})}$ is continuous. Moreover, since the map $x \mapsto x^{-1}$ is continuous, $\mathrm{supp}(f^*) = \{x^{-1} : x \in \mathrm{supp}(f)\} = (\mathrm{supp}(f))^{-1}$ is compact.

Furthermore, $\mathrm{supp}(f * g) \subseteq \left\{ xy : x \in \mathrm{supp}(f), y \in \mathrm{supp}(g) \text{ and } (x, y) \in G^{(2)} \right\}$ is compact because multiplication is continuous and $(\mathrm{supp}(f) \times \mathrm{supp}(g)) \cap G^{(2)}$

is compact, as it is the product of two compact spaces.

The only thing left to show is that $f * g$ is continuous, and to prove this we need the fact that G is étale. Let $\gamma \in G$. We will show that there exists a neighborhood U of γ for which $f * g|_U$ is continuous. First of all, we see that $\mathrm{supp}(g) \cap G_{d(\gamma)} = \{\gamma_1, ..., \gamma_m\}$ is finite because it is an intersection of a compact set and a discrete set. Since G is Hausdorff and étale we have open sets U_i for all $1 \leq i \leq m$, such that $\gamma_i \in U_i$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$, and d is a homeomorphism on these sets. Now define $E = \mathrm{supp}(g) \setminus (\bigcup_{i=1}^m U_i)$. This is a compact set, thus $d(E)$ is compact as well, and $d(\gamma) \notin d(E)$, for otherwise there would be $x \in E$ with $d(x) = d(\gamma)$ and $g(x) \neq 0$, but then $x \in \bigcup_{i=1}^m U_i$, so $x \notin E$. Since $G^{(0)}$ is Hausdorff, we can find an open set U, with $d(\gamma) \in U$ and $U \cap d(E)$ empty. Then define

$$W = U \cap \left(\bigcap_{i=1}^{m} d(U_i) \right),$$

which is a neighbourhood of $d(\gamma)$. In addition, define

$$W_i = U_i \cap d^{-1}(W),$$

which is a neighbourhood of γ_i. Then $d^{-1}(W) \cap \mathrm{supp}(g) \subset \bigcup_{i=1}^m W_i$.

As for $x \in d^{-1}(W) \cap \mathrm{supp}(g)$, we have $x \in \mathrm{supp}(g)$ and $d(x) \in W \subset U$ so $d(x) \notin d(E)$, which implies $x \notin E = \mathrm{supp}(g) \setminus (\bigcup_{i=1}^m U_i)$, thus $x \in \bigcup_{i=1}^m U_i$. Hence

$$x \in d^{-1}(W) \cap \bigcup_{i=1}^{m} U_i = \bigcup_{i=1}^{m} (d^{-1}(W) \cap U_i) = \bigcup_{i=1}^{m} W_i.$$

Also, d is a homeomorphism from W_i to W for all $1 \leq i \leq m$.

Now let V be an open set containing γ such that $d|_V$ is a homeomorphism. Then $d|_{V \cap W_i}$ is a homeomorphism from $V \cap W_i$ to $d(V) \cap W$ for all $1 \leq i \leq m$. For every $1 \leq i \leq m$, let $h_i : d(V) \cap W \to V \cap W_i$ be the inverse map of $d|_{V \cap W_i}$ and define for all $1 \leq i \leq m$

$$\begin{aligned} \phi_i : V \cap d^{-1}(W) &\to \mathbb{C}, \\ \alpha &\mapsto f\left(\alpha \big(h_i(d(\alpha)) \big)^{-1} \right) g\left(h_i\left(d(\alpha) \right) \right), \end{aligned}$$

and

$$\begin{aligned} \Phi : V \cap d^{-1}(W) &\to \mathbb{C}^m, \\ \alpha &\mapsto \left(\phi_1(\alpha), \ldots, \phi_m(\alpha) \right). \end{aligned}$$

These functions are continuous since they are compositions of continuous functions. Let $F : \mathbb{C}^m \to \mathbb{C}$ denote component-wise addition, which is also continuous. Then we have that $(f * g)|_{V \cap d^{-1}(W)} = F \circ \Phi$. To see this, take $\alpha \in V \cap d^{-1}(W)$. Then

$$(f * g)(\alpha) = \sum_{\beta \in G_{d(\alpha)}} f\left(\alpha \beta^{-1} \right) g(\beta) \text{ and}$$

$$F \circ \Phi(\alpha) = \sum_{i=1}^{m} f\left(\alpha \big(h_i(d(\alpha)) \big)^{-1} \right) g(h_i(d(\alpha))).$$

Thus, to show that $f * g$ is continuous on $V \cap d^{-1}(W)$, we need to check that $h_i(d(\alpha)) \in G_{d(\alpha)}$, and that for all $\beta \in G_{d(\alpha)}$ there exists $1 \leq i \leq m$ such that $\beta \in V \cap W_i$. First of all, $d(h_i(d(\alpha))) = d(\alpha)$, thus $h_i(d(\alpha)) \in G_{d(\alpha)}$. Now let $\alpha \in V \cap d^{-1}(W)$ and $\beta \in G_{d(\alpha)} \cap \operatorname{supp}(g)$, then we have $d(\beta) = d(\alpha) \in W \subset U$, then $d(\beta) \notin d(E)$, so $\beta \notin E$, thus $\beta \in \bigcup_{i=1}^{m} U_i$. Hence there exists an $1 \leq i \leq m$ with $\beta \in U_i$.
We know that d is a homeomorphism on U_i, and we have

$$d(\beta) = d(\alpha) = d(h_i(d(\alpha))),$$

with $h_i(d(\alpha)) \in V \cap W_i \subset U_i$. Therefore, by injectivity on U_i we can see that $d(\beta) = d(h_i(d(\alpha)))$ and $\beta, h_i(d(\alpha)) \in U_i$ implies $\beta = h_i(d(\alpha)) \in V \cap W_i$. Thus $f * g|_{V \cap d^{-1}(W)} = F \circ \Phi$. This shows that $f * g$ is continuous on the neighborhood $V \cap d^{-1}(W)$ of γ, so for every $\gamma \in G$ there exists a neighborhood U for which $f * g|_U$ is continuous. Thus $f * g$ is continuous on G. ■

One may think that assuming r-discreteness would be enough to show that convolution is continuous, but that is not the case, as this counterexample shows.

Example 3.2.3. Recall that the groupoid E in Example 3.1.13 is an r-discrete locally compact Hausdorff groupoid, but is not étale. Define the functions f and g on E by

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,-1), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x,y) = \begin{cases} 1 & \text{if } (x,y) = (-1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then f and g are continuous compactly supported functions on E, and for the convolution $f * g$ given by $(f * g)(x,y) = \sum_{z \in [-1,1]} f(x,z)g(z,y)$, we obtain $f(x,z)g(z,y) \neq 0$ if and only if $x = 1, y = 1, z = -1$. Hence we have

$$(f * g)(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f*g$ is not continuous, because $\lim_{x \to 1}(f*g)(x,x) = 0$, but $(f*g)(1,1) = 1$.

Hence, from now on we shall assume that any groupoid G is étale so that the groupoid algebra $C_c(G)$ is indeed a $*$-algebra.

As we saw in Example 3.1.3 a group G is a special case of a groupoid where the unit space consists of just one element e and $G^{(2)} = G \times G$. For every locally compact Hausdorff group there exists a Haar measure, which is used to construct the full and reduced group C^*-algebras. As it turns out, the notion of Haar measure can be extended to locally compact Hausdorff groupoids, in which case it is called a *(left) Haar system.* It follows from [35, Lemma 2.7] and [35, Lemma 2.8] that a Haar system for an étale groupoid always exists and that it consists of the counting measures. Hence our underlying groupoid algebra is the algebra $C_c(G)$ from Definition 3.2.1.

Theorem 3.2.4. *The algebra $(C_c(G)$ is a normed $*$-algebra with respect to the norm*

$$\|f\|_I = \max\left\{ \sup_{u \in G^{(0)}} \sum_{r(x)=u} |f(x)|, \sup_{u \in G^{(0)}} \sum_{d(x)=u} |f(x)| \right\}.$$

Definition 3.2.5. A *representation* of $C_c(G)$ is a $*$-homomorphism L from $C_c(G)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $B(\mathcal{H})$. A representation L is called *bounded* if $\|L(f)\| \leq \|f\|_I$ for all $f \in C_c(G)$, it is called *non-degenerate* if $\overline{\text{span}}\{L(f)v : f \in C_c(G), v \in \mathcal{H}\} = \mathcal{H}$.

Definition 3.2.6. The full norm is defined by $\|f\|_{\text{full}} = \sup_L \|L(f)\|$, where L runs over all bounded non-degenerate representations of $C_c(G)$.

Definition 3.2.7. *The full C^*-algebra* $C^*(G)$ is the C^*-algebra obtained by completing $C_c(G)$ in the full norm described in definition 3.2.6.

Definition 3.2.8. For $u \in G^{(0)}$ and $f \in C_c(G)$, let $\text{Ind}_u(f)$ be the operator on $L^2(G)$ defined by

$$\text{Ind}_u(f)\xi(x) = \sum_{d(y)=u, r(y)=d(x)} f(xy)\xi(y^{-1}).$$

Definition 3.2.9. The *reduced norm* is defined by $\|f\|_{\text{red}} = \sup\limits_{u \in G^{(0)}} \|\text{Ind}_u(f)\|$.

Definition 3.2.10. *The reduced C^*-algebra* $C^*(G)$ is the C^*-algebra obtained by completing $C_c(G)$ in the reduced norm described in definition 3.2.9.

To get a better understanding of these C^*-algebras, we look at the following example.

Example 3.2.11. In Example 3.1.19 we saw that the group of integers $\mathbb{Z}$ is also a path groupoid. It is easy to see that the topology on this path groupoid is discrete.That means that every function on $\mathbb{Z}$ is continuous and the compact sets are the finite sets in $\mathbb{Z}$, so the groupoid algebra $C_c(\mathbb{Z})$ is the set of functions on $\mathbb{Z}$ with finite support.

In general, we see that if the groupoid $\mathcal{G}$ is a group G, then the construction of the groupoid C^*-algebras $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$ is the same as the construction of the group C^*-algebras $C^*(G)$ and $C^*_r(G)$. Therefore, we can use our knowledge of group C^*-algebras in this case.

The group $\mathbb{Z}$ is a discrete abelian group, and hence by [30, Theorem 2.5.5] $C^*(\mathbb{Z})$ is isomorphic to $C(\hat{\mathbb{Z}})$, where $\hat{\mathbb{Z}}$ is the dual group of $\mathbb{Z}$, which consists of all group homomorphisms $\chi : \mathbb{Z} \to \mathbb{T}$. Also, by [30, Proposition 2.5.4] the map that sends χ to $\chi(1)$ is a group isomorphism from $\hat{\mathbb{Z}}$ to $\mathbb{T}$ and therefore $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, the universal C^*-algebra generated by a single unitary.

Definition 3.2.12. An étale locally compact groupoid G is said to be *amenable* if there exists a net of compactly supported nonnegative functions $\mu_i : G \to \mathbb{C}$ such that

$$\sum_{\beta \in G_{r(\gamma)}} \mu_i(\beta) \to 1 \quad \text{and} \quad \sum_{\beta \in G_{r(\gamma)}} |\mu_i(\beta) - \mu_i(\beta\gamma)| \to 0,$$

for $\gamma \in G$, uniformly on compact subsets of G.

Notice that we only looked at the full groupoid C^*-algebra in our example. This is because Paterson proved in [28] that path groupoids are always amenable, even for graphs which are not necessarily row-finite, and it is known that for amenable groupoids the full groupoid C^*-algebra and the reduced groupoid C^*-algebra coincide, as is shown for example in [9, Corollary 5.6.17].

3.2.1 Graph C^*-algebras as groupoid C^*-algebras

Now that we have seen two ways to construct a C^*-algebra from a row-finite directed graph, we can construct the path groupoid $\mathcal{G}_E$ and the corresponding groupoid C^*-algebra $C^*(\mathcal{G}_E)$. Also, recall from Section 2.2 that we can construct $C^*(E)$ from the Cuntz–Krieger relations. As the title of this subsection suggests, these C^*-algebras are actually isomorphic for row-finite directed graphs without sinks. To prove that $C^*(E) \cong C^*(\mathcal{G}_E)$ we need two theorems.

Remark. Note that the assumption that E has no sinks has the important consequence that for all $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$, the sets $Z(\alpha)$, $Z(\beta)$ and $Z(\alpha, \beta)$ are non-empty. This will be needed in the next theorem.

Theorem 3.2.13. *Let E be a row-finite directed graph without sinks, then $C^*(\mathcal{G}_E)$ is generated by a Cuntz–Krieger E-family.*

Proof. We start by defining $S_e = \mathbb{1}_{Z(e,r(e))}$ in $C_c(\mathcal{G}_E)$, where

$$Z(e, r(e)) = \{(x, -1, y) : x_0 = e, x_i = y_{i-1} \text{ for } i \geq 1\}$$

and $P_v = \mathbb{1}_{D_v}$ where

$$D_v = \{(x, 0, x) : s(x) = v\}.$$

These are characteristic function on bisections of the path groupoid. Then

$$\overline{S_e(z, -l, x)} S_e(z, k - l, y) = \begin{cases} 1 \text{ if } l = 1, k = 0, z = ex, y = x, \\ 0 \text{ otherwise,} \end{cases}$$

so

$$\begin{aligned} S_e^* S_e(x, k, y) &= \sum_{l,z:(x,l,z)\in\mathcal{G}_E} \overline{S_e(z, -l, x)} S_e(z, k - l, y) \\ &= \begin{cases} 1 \text{ if } (x, k, y) = (x, 0, x) \text{ and } r(e) = s(x), \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Thus we have $S_e^* S_e = \mathbb{1}_{D_{r(e)}} = P_{r(e)}$, which gives us the first Cuntz–Krieger relation from Definition 2.2.4. In a similar way we get

$$S_e(x, l, z) \overline{S_e(y, l - k, z)} = \begin{cases} 1 \text{ if } l = -1, k = 0, x_0 = y_0 = e, y = x, \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} S_e S_e^*(x, k, y) &= \sum_{l,z:(x,l,z)\in\mathcal{G}_E} S_e(x, l, z) \overline{S_e(y, l - k, z)} \\ &= \begin{cases} 1 \text{ if } (x, k, y) = (x, 0, x) \text{ with } x \in Z(e), \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

Thus $S_eS_e^* = \mathbb{1}_{Z(e,e)}$, which also shows that S_e is a partial isometry for all $e \in E^1$. Since E has no sinks, $s^{-1}(v) \neq \emptyset$ for all $v \in E^0$, and if $s^{-1}(v) \neq \emptyset$, then we have

$$\emptyset \neq D_v = \{(x,0,x) : s(x) = v\} = \bigcup_{\{e \in E^1 : s(e)=v\}} Z(e,e),$$

where $Z(e,e) \cap Z(f,f)$ is empty if $e \neq f$ by Lemma 3.1.17. Thus we have

$$P_v = \mathbb{1}_{D_v} = \sum_{\{e \in E^1 : s(e)=v\}} \mathbb{1}_{Z(e,e)} = \sum_{\{e \in E^1 : s(e)=v\}} S_eS_e^*,$$

which gives us the second Cuntz–Krieger relation from Definition 2.2.4. Thus $\{S, P\}$ is a Cuntz–Krieger E-family in $C^*(\mathcal{G}_E)$.

Now we have to prove that $C^*(\mathcal{G}_E)$ is generated by $\{S, P\}$. Using Theorem 2.2.10, we only have to prove that $\operatorname{span}\{\mathbb{1}_{Z(\alpha,\beta)} : \alpha, \beta \in E^*\}$ is dense in $C^*(\mathcal{G}_E)$. We also know that $C_c(\mathcal{G}_E)$ is dense in $C^*(\mathcal{G}_E)$, and that the C^*-norm is dominated by the I-norm, thus it is enough to show that $\operatorname{span}\{\mathbb{1}_{Z(\alpha,\beta)} : \alpha, \beta \in E^*\}$ is $\|\cdot\|_I$-dense in $C_c(\mathcal{G}_E)$. Now suppose $f \in C_c(\mathcal{G}_E)$. Then $\operatorname{supp}(f)$ is compact, so by Lemma 3.1.17 we can assume that $\operatorname{supp}(f)$ is a disjoint union of some compact open base sets of the form $Z(\gamma, \delta)$. Therefore, f is a sum of functions in $C(Z(\gamma_i, \delta_i))$ for a finite number of γ_i, δ_i.

This further simplifies our proof, because now we only have to show that $f \in C(Z(\gamma,\delta))$ is in $\overline{\operatorname{span}}\{\mathbb{1}_{Z(\alpha,\beta)} : \alpha, \beta \in E^*\}$. This is easy to prove, because r and d are injective on $Z(\gamma,\delta)$, so that the uniform norm on $C(Z(\gamma,\delta))$ dominates the $\|\cdot\|_I$-norm. Thus by the Stone–Weierstrass Theorem [38, Theorem 4.3.13]

$$\begin{aligned}\operatorname{span}\{\mathbb{1}_{Z(\alpha,\beta)} &: \alpha, \beta \in E^*\} \cap C(Z(\gamma,\delta)) \\ &= \operatorname{span}\{\mathbb{1}_{Z(\gamma\mu,\delta\mu)} : \mu \in E^*, r(\gamma) = r(\delta) = s(\mu)\}\end{aligned}$$

is uniformly dense in $C(Z(\gamma,\delta))$. Hence $C^*(\mathcal{G}_E)$ is generated by the Cuntz–Krieger E-family $\{S, P\}$. Note that $P_v = \mathbb{1}_{D_v} \in C_c(\mathcal{G}_E) \subset C^*(\mathcal{G}_E)$, since

$$\operatorname{supp}(\mathbb{1}_{D_v}) = D_v = \{(x,0,x) : s(x) = v\} = \bigcup_{e \in s^{-1}(v)} Z(e,e),$$

and $s^{-1}(v)$ is finite and $Z(e,e)$ are compact open sets. ■

To show that $C^*(E)$ is isomorphic to $C^*(\mathcal{G}_E)$, we also need to prove that this family has the universal property described in Lemma 2.2.12.

Theorem 3.2.14. *Let E be a row-finite directed graph without sinks. If $\{S, P\}$ is a Cuntz–Krieger E-family on a Hilbert space $\mathcal{H}$, then there exists a representation π of $C^*(\mathcal{G}_E)$ on $\mathcal{H}$ such that $\pi(\mathbb{1}_{Z(e,r(e))}) = S_e$ for all $e \in E^1$ and $\pi(\mathbb{1}_{D_v}) = P_v$ for all $v \in E^0$.*

Before we can prove this theorem, we need the following lemmas.

Lemma 3.2.15 ([32, Proposition A.8])**.** *Suppose $\mathcal{A}$ is a C^*-algebra and there are C^*-subalgebras $\mathcal{A}_n$ of $\mathcal{A}$, such that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ and $\mathcal{A} = \overline{\bigcup_{n=0}^\infty \mathcal{A}_n}$. If we have injective homomorphisms ϕ_n of each $\mathcal{A}_n$ into the same C^*-algebra $\mathcal{B}$ such that $\phi_{n+1}|_{\mathcal{A}_n} = \phi_n$ for all n, then there is an injective homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi|_{\mathcal{A}_n} = \phi_n$ for all n.*

Lemma 3.2.16. *For each $v \in E^0$, there exists a representation π_v of*

$$C(\{\mu \in E^\infty : s(\mu) = v\})$$

on $\mathcal{H}$ such that $\pi_v(\mathbb{1}_{Z(\gamma)}) = S_\gamma S_\gamma^*$ for each $\gamma \in E^*$ with $s(\gamma) = v$.

Proof. For every $k \in \mathbb{N}$, define

$$C_k = \operatorname{span}\{\mathbb{1}_{Z(\gamma)} : \gamma \in E^*, s(\gamma) = v, |\gamma| = k\}.$$

Since E is a row-finite graph, C_k is a finite-dimensional C^*-subalgebra of the algebra $C(\{\mu \in E^\infty : s(\mu) = v\})$ spanned by the mutually orthogonal projections $\mathbb{1}_{Z(\gamma)}$, for $\gamma \in E^*$ with $s(\gamma) = v$. Furthermore, $Z(\gamma)$ is the disjoint union of the sets $Z(\gamma e)$ with $e \in E^1$ and $r(\gamma) = s(e)$, thus $\mathbb{1}_{Z(\gamma)} = \sum_{e \in E^1, s(e) = r(\gamma)} \mathbb{1}_{Z(\gamma e)}$, so $C_k \subset C_{k+1}$.

Since the sets $Z(\gamma)$ with $s(\gamma) = v$ form a basis for the topology on

$$\{\mu \in E^\infty : s(\mu) = v\},$$

the Stone–Weierstrass Theorem [38, Theorem 4.3.13] implies that

$$C(\{\mu \in E^\infty : s(\mu) = v\}) = \overline{\cup C_k}.$$

For a fixed $k \in \mathbb{N}$ the projections $S_\gamma S_\gamma^*$ with $|\gamma| = k$ are mutually orthogonal, so that there exists an injective $*$-homomorphism $\phi_k : C_k \to B(\mathcal{H})$ such that $\phi_k(\mathbb{1}_{Z(\gamma)}) = S_\gamma S_\gamma^*$. Furthermore,

$$\begin{aligned}
\phi_k\left(\mathbb{1}_{Z(\gamma)}\right) &= S_\gamma S_\gamma^* \\
&= S_\gamma \left(\sum_{e \in E^1} S_e S_e^* \right) S_\gamma^* \\
&= \sum_{e \in E^1, r(\gamma) = s(e)} S_\gamma S_e S_e^* S_\gamma^* \\
&= \sum_{e \in E^1, r(\gamma) = s(e)} S_{\gamma e} S_{\gamma e}^* \\
&= \sum_{e \in E^1, r(\gamma) = s(e)} \phi_{k+1}\left(\mathbb{1}_{Z(\gamma e)}\right) \\
&= \phi_{k+1}\left(\sum_{e \in E^1, r(\gamma) = s(e)} \mathbb{1}_{Z(\gamma e)} \right) \\
&= \phi_{k+1}\left(\mathbb{1}_{Z(\gamma e)}\right),
\end{aligned}$$

so $\phi_{k+1}|_{C_k} = \phi_k$.

Hence by Lemma 3.2.15, there exists an injective $*$-homomorphism

$$\phi : C(\{\mu \in E^\infty : s(\mu) = v\}) \to B(\mathcal{H})$$

with $\phi|_{C_k} = \phi_k$. Taking $\pi_v = \phi$, we get the representation we were looking for. ∎

In view of the previous lemma, we get a representation $\pi_{r(\alpha)} \circ \phi^{-1}_{\alpha,\beta}$ from $C(Z(\alpha,\beta))$ on $\mathcal{H}$, where

$$\phi_{\alpha,\beta} : C(\{\mu \in E^\infty : s(\mu) = r(\alpha)\}) \to C(Z(\alpha,\beta))$$

is the isomorphism induced by

$$h_{\alpha,\beta} : \{\mu \in E^\infty : s(\mu) = r(\alpha)\} \to Z(\alpha,\beta)$$

as defined in the proof of Theorem 3.1.18. Note that there is more than one way to write a compact set as a union of sets of the form $Z(\alpha,\beta)$, thus we have to check that the representations $\pi_{r(\alpha)} \circ \phi^{-1}_{\alpha,\beta}$ are consistent. We will prove this in the following two lemmas.

Lemma 3.2.17. *Let $\alpha,\beta \in E^*$ such that $r(\alpha) = r(\beta)$ and let $f \in C(Z(\alpha,\beta))$. Then for $k \geq 1$, we have*

$$\pi_{r(\alpha)} \circ \phi^{-1}_{\alpha,\beta}(f) = \sum_{\gamma \in E^*, |\gamma|=k, r(\alpha)=s(\gamma)} S_\gamma \pi_{r(\gamma)} \circ \phi^{-1}_{\alpha\gamma,\beta\gamma} \left(f|_{Z(\alpha\gamma,\beta\gamma)} \right) S^*_\gamma \tag{3.1}$$

Proof. Both sides of (3.1) are continuous and linear in $f \in C(Z(\alpha,\beta))$. Hence we can just look at the functions of the form $f = \phi_{\alpha,\beta}(\mathbb{1}_{Z(\mu)}) = \mathbb{1}_{Z(\alpha\mu,\beta\mu)}$ for some $\mu \in E^*$ with $r(\alpha) = s(\mu)$. The left-hand side of (3.1) is then

$$\pi_{r(\alpha)} \circ \phi^{-1}_{\alpha,\beta} \left(\phi_{\alpha,\beta}(\mathbb{1}_{Z(\mu)}) \right) = \pi_{r(\alpha)} \left(\mathbb{1}_{Z(\mu)} \right) = S_\mu S^*_\mu.$$

If $k < |\mu|$, then $f|_{Z(\alpha\gamma,\beta\gamma)} \neq 0$ if and only if $\mu = \gamma\mu'$, so only $\gamma = \mu^k$ gives a non-zero term in the sum on the right-hand side. The right-hand side then becomes

$$S_\gamma \pi_{r(\gamma)} \circ \phi^{-1}_{\alpha\gamma,\beta\gamma} \left(\mathbb{1}_{Z(\alpha\gamma\mu',\beta\gamma\mu')} \right) S^*_\gamma = S_\gamma \pi_{r(\gamma)} \left(\mathbb{1}_{Z(\mu')} \right) S^*_\gamma = S_\gamma S_{\mu'} S^*_{\mu'} S^*_\gamma = S_\mu S^*_\mu.$$

If $k \geq |\mu|$, we only get a non-zero term in the sum on the right-hand side of (3.1) if $\gamma = \mu\gamma'$ for some $\gamma' \in E^*$ with $r(\mu) = s(\gamma')$. Hence, the right-hand side of (3.1) now becomes

$$\sum_{\gamma' \in E^*, r(\mu)=s(\gamma'), |\gamma'|=k-|\mu|} S_{\mu\gamma'} \pi_{r(\gamma')} \circ \phi^{-1}_{\alpha\mu\gamma',\beta\mu\gamma'} \left(\mathbb{1}_{Z(\alpha\mu\gamma',\beta\mu\gamma')} \right) S^*_{\mu\gamma'},$$

which is equal to

$$\sum_{\gamma' \in E^*} S_{\mu\gamma'} \pi_{r(\gamma')} \left(\mathbb{1}_{\{x \in E^\infty : r(\gamma')=s(x)\}} \right) S^*_{\mu\gamma'} = \sum_{\gamma'} S_{\mu\gamma'} S^*_{\mu\gamma'} = S_\mu S^*_\mu.$$

As we saw earlier, this is exactly the left-hand side, and therefore equation (3.1) is correct. ■

To conclude the proof of our main result, we need to define a representation of $C_c(\mathcal{G}_E)$ on $\mathcal{H}$. For $f \in C_c(\mathcal{G}_E)$, we can write $\mathrm{supp}(f) = \bigcup_i Z(\alpha_i,\beta_i)$, which is a disjoint union of open compact sets, and we can define π on f as:

$$\pi(f) = \sum_i S_{\alpha_i} \pi_{r(\alpha_i)} \circ \phi^{-1}_{\alpha_i,\beta_i} \left(f|_{Z(\alpha_i,\beta_i)} \right) S^*_{\beta_i} \tag{3.2}$$

As explained before we need to show that this is independent of the description of $\mathrm{supp}(f)$.

Lemma 3.2.18. *The map* $\pi : C_c(\mathcal{G}_E) \to B(\mathcal{H})$ *in (3.2) is a well-defined linear map, such that it is continuous in the inductive limit topology and such that equation (3.2) holds whenever* $\text{supp}(f) \subset \bigcup_i Z(\alpha_i, \beta_i)$ *with* $Z(\alpha_i, \beta_i)$ *disjoint.*

Proof. Recall that the functions $\phi_{\alpha,\beta}$ are homeomorphisms in the uniform topology and the inductive limit topology on $\{\mu \in E^\infty : r(\alpha) = s(\mu)\}$. Also, the functions π_v are uniformly continuous, thus we only have to prove that π is well-defined, that is, show that equation (3.2) holds for every description of $\text{supp}(f)$. Suppose that we can also write $\text{supp}(f)$ as $\bigcup_l Z(\gamma_l, \delta_l)$. We know that $Z(\alpha, \beta) \cap Z(\gamma, \delta) \neq \emptyset$ only when $Z(\alpha, \beta) \subset Z(\gamma, \delta)$ or $Z(\gamma, \delta) \subset Z(\alpha, \beta)$ by Lemma 3.1.17. Therefore, without loss of generality we can assume that $Z(\alpha_i, \beta_i) = \bigcup_{j=1}^n Z(\gamma_j, \delta_j)$, so each (γ_j, δ_j) has the form $(\alpha_i\mu_j, \beta_i\mu_j)$. We also know that for every $k \in \mathbb{N}$,

$$Z(\alpha_i, \beta_i) = \bigcup_{\epsilon \in E^*, r(\alpha_i) = s(\epsilon), |\epsilon| = k} Z(\alpha_i\epsilon, \beta_i\epsilon).$$

Let $k = \max |\mu_j|$ and define $F_j = \{\mu_j\nu \in E^* : |\nu| = k - |\mu_j|\}$. Then we get $Z(\gamma_j, \delta_j) = Z(\alpha_i\mu_j, \beta_i\mu_j) = \bigcup_{\mu_j\nu \in F_j} Z(\alpha_i\mu_j\nu, \beta_i\mu_j\nu)$ and hence

$$Z(\alpha_i, \beta_i) = \bigcup_j \left(\bigcup_{\mu_j\nu \in F_j} Z(\alpha_i\mu_j\nu, \beta_i\mu_j\nu) \right) = \bigcup_j \left(\bigcup_{\mu_j\nu \in F_j} Z(\gamma_j\nu, \delta_j\nu) \right).$$

By applying Lemma 3.2.17 multiple times, we obtain

$$\begin{aligned}
S_{\alpha_i}\pi_{r(\alpha_i)} \circ \phi^{-1}_{\alpha_i,\beta_i}(f|_{Z(\alpha_i,\beta_i)})S^*_{\beta_i} &= \sum_{\substack{\epsilon \in E^*, |\epsilon| = k, \\ r(\alpha_i) = s(\epsilon)}} S_{\alpha_i\epsilon}\pi_{r(\epsilon)} \circ \phi^{-1}_{\alpha_i\epsilon,\beta_i\epsilon}(f|_{Z(\alpha_i\epsilon,\beta_i\epsilon)})S^*_{\beta_i\epsilon} \\
&= \sum_j \sum_{\mu_j\nu \in F_j} S_{\gamma_j\nu}\pi_{r(\nu)} \circ \phi^{-1}_{\gamma_j\nu,\delta_j\nu}(f|_{Z(\gamma_j\nu,\delta_j\nu)})S^*_{\delta_j\nu} \\
&= \sum_j S_{\gamma_j}\pi_{r(\gamma_j)} \circ \phi^{-1}_{\gamma_j,\delta_j}(f|_{Z(\gamma_j,\delta_j)})S^*_{\delta_j}.
\end{aligned}$$

Using this argument for the other sets $Z(\alpha_i, \beta_i)$ and $Z(\gamma_j, \delta_j)$ in the description of $\text{supp}(f)$, we see that our definition of $\pi(f)$ in equation (3.2) is independent of the choice of description of $\text{supp}(f)$, so that π is well-defined. ■

In order to prove Theorem 3.2.14, we need to show that π is a $*$-homomorphism, that it extends to $C^*(\mathcal{G}_E)$, that $\pi(\mathbb{1}_{Z(e,r(e))}) = S_e$ for all $e \in E^1$ and that $\pi(\mathbb{1}_{D_v}) = P_v$ for all $v \in E^0$.

Proof of Theorem 3.2.14. If $\text{supp}(f) \subset Z(\alpha, \beta)$, then $f^* = \phi_{\beta,\alpha} \circ \overline{\phi^{-1}_{\alpha,\beta}(f)}$. Thus

$$\begin{aligned}
\pi(f)^* &= \left(S_\alpha\pi_{r(\alpha)}\phi^{-1}_{\alpha,\beta}(f)S^*_\beta\right)^* \\
&= S_\beta\pi_{r(\alpha)}\left(\overline{\phi^{-1}_{\alpha,\beta}(f)}\right)S^*_\alpha \\
&= S_\beta\pi_{r(\alpha)}\left(\phi^{-1}_{\beta,\alpha}(f^*)\right)S^*_\alpha \\
&= \pi(f^*).
\end{aligned}$$

By the previous lemmas, this extends to arbitrary $f \in C_c(\mathcal{G}_E)$ by linearity. Let $f, g \in C_c(\mathcal{G}_E)$ with $\operatorname{supp}(f) \subset Z(\alpha, \beta)$ and $\operatorname{supp}(g) \subset Z(\gamma, \delta)$. If $Z(\beta) \cap Z(\gamma) = \emptyset$ then $f * g = 0$ and $S_\beta^* S_\gamma = 0$, which gives us $\pi(f)\pi(g) = 0$, so in this case $\pi(f)\pi(g) = 0 = \pi(f * g)$. Thus we can assume that $Z(\beta) \cap Z(\gamma) \neq \emptyset$, which means that either $\beta = \gamma\beta'$ or $\gamma = \beta\gamma'$. Without loss of generality, assume $\gamma = \beta\gamma'$. Then we can write

$$f = \sum_{\nu \in E^*, r(\beta) = s(\nu), |\nu| = |\gamma| - |\beta|} f|_{Z(\alpha\nu, \beta\nu)}.$$

By the same reasoning as before, we have $f|_{Z(\alpha\nu,\beta\nu)} * g = 0$ and $S_{\beta\nu}^* S_\gamma = 0$ unless $\nu = \gamma'$. Therefore, $f * g = f|_{Z(\alpha\gamma', \beta\gamma')} * g$, thus in general we can assume $\operatorname{supp}(f) \subset Z(\alpha, \gamma)$ and $\operatorname{supp}(g) \subset Z(\gamma, \delta)$. But in this case $\operatorname{supp}(f * g) \subset Z(\alpha, \delta)$, so

$$\begin{aligned}
\pi(f)\pi(g) &= S_\alpha \pi_{r(\gamma)} \circ \phi_{\alpha,\gamma}^{-1}(f) S_\gamma^* S_\gamma \pi_{r(\gamma)} \circ \phi_{\gamma,\delta}^{-1}(f) S_\delta^* \\
&= S_\alpha \pi_{r(\gamma)} \left(\phi_{\alpha,\gamma}^{-1}(f) \phi_{\gamma,\delta}^{-1}(g) \right) S_\delta^*, \\
\pi(f * g) &= S_\alpha \pi_{r(\alpha)} \left(\phi_{\alpha,\delta}^{-1}(f * g) \right) S_\delta^*,
\end{aligned}$$

and $r(\alpha) = r(\gamma)$. This means that we only have to check that $\phi_{\alpha,\gamma}^{-1}(f)\phi_{\gamma,\delta}^{-1}(g) = \phi_{\alpha,\delta}^{-1}(f * g)$, in order to prove that $\pi(f * g) = \pi(f)\pi(g)$. By linearity and continuity in the sup-norm, we only have to show this for functions of the form $f = \mathbb{1}_{Z(\alpha\nu,\beta\nu)}$ and $g = \mathbb{1}_{Z(\beta\mu,\delta\mu)}$. In this case,

$$\phi_{\alpha,\gamma}^{-1}(f)\phi_{\gamma,\delta}^{-1}(g) = \mathbb{1}_{Z(\nu)} \mathbb{1}_{Z(\mu)} = \begin{cases} 0 & \text{if } Z(\nu) \cap Z(\mu) = \emptyset, \\ \mathbb{1}_{Z(\nu)} & \text{if } Z(\nu) \subset Z(\mu), \\ \mathbb{1}_{Z(\mu)} & \text{if } Z(\mu) \subset Z(\nu), \end{cases}$$

and

$$f * g = \mathbb{1}_{Z(\alpha\nu,\beta\nu)} * \mathbb{1}_{Z(\beta\mu,\delta\mu)} = \begin{cases} 0 & \text{if } Z(\nu) \cap Z(\mu) = \emptyset, \\ \mathbb{1}_{Z(\alpha\nu,\delta\nu)} & \text{if } Z(\nu) \subset Z(\mu), \\ \mathbb{1}_{Z(\alpha\mu,\delta\mu)} & \text{if } Z(\mu) \subset Z(\nu), \end{cases}$$

Furthermore, $\phi_{\alpha,\delta}^{-1}\left(\mathbb{1}_{Z(\alpha\nu,\delta\nu)}\right) = \mathbb{1}_{Z(\nu)}$ and $\phi_{\alpha,\delta}^{-1}\left(\mathbb{1}_{Z(\alpha\mu,\delta\mu)}\right) = \mathbb{1}_{Z(\mu)}$. Therefore, $\phi_{\alpha,\gamma}^{-1}(f)\phi_{\gamma,\delta}^{-1}(g) = \phi_{\alpha,\delta}^{-1}(f * g)$, which means that $\pi(f * g) = \pi(f)\pi(g)$, thus π is a $*$-homomorphism. Since $\mathcal{G}_E$ is étale, π is automatically bounded in the norm $\|\cdot\|_I$ by [35, Corollary II.1.22] and hence π extends to $C^*(\mathcal{G}_E)$.

Now replacing f by $\mathbb{1}_{Z(e,r(e))}$ and $\mathbb{1}_{D_v}$ in equation (3.2), we obtain

$$\begin{aligned}
\pi(\mathbb{1}_{Z(e,r(e))}) &= \sum_{f\in E^1, s(f)=r(e)} S_{ef}\pi_{r(ef)} \circ \phi_{ef,f}^{-1}\left(\mathbb{1}_{Z(ef,f)}\right) S_f^* \\
&= \sum_{f\in E^1, s(f)=r(e)} S_e S_f \pi_{r(f)}\left(\mathbb{1}_{D_{r(f)}}\right) S_f^* \\
&= \sum_{f\in E^1, s(f)=r(e)} S_e S_f P_{r(f)} S_f^* \\
&= S_e\Big(\sum_{f\in E^1, s(f)=r(e)} S_f S_f^*\Big) \\
&= S_e P_{r(e)} \\
&= S_e
\end{aligned}$$

for all $e \in E^1$, whereas for all $v \in E^0$ we have

$$\begin{aligned}
\pi(\mathbb{1}_{D_v}) &= \sum_{e\in E^1, s(e)=v} S_e \pi_{r(e)} \circ \phi_{e,e}^{-1}(\mathbb{1}_{Z(e,e)}) S_e^* \\
&= \sum_{e\in E^1, s(e)=v} S_e \pi_{r(e)}(\mathbb{1}_{D_{r(e)}}) S_e^* \\
&= \sum_{e\in E^1, s(e)=v} S_e P_{r(e)} S_e^* \\
&= \sum_{e\in E^1, s(e)=v} S_e S_e^* \\
&= P_v.
\end{aligned}$$

Thus π is a representation of $C^*(\mathcal{G}_E)$ on $\mathcal{H}$ such that $\pi(\mathbb{1}_{Z(e,r(e))}) = S_e$ for all $e \in E^1$ and $\pi(\mathbb{1}_{D_v}) = P_v$ for all $v \in E^0$. ■

Now we can prove the main result of this chapter.

Theorem 3.2.19. *Let E be a row-finite directed graph without sinks, and let $\mathcal{G}_E$ be the corresponding path groupoid. Then $C^*(\mathcal{G}_E) \cong C^*(E)$.*

Proof. By the proof of Theorem 3.2.13 we know that $C^*(\mathcal{G}_E)$ is generated by the Cuntz–Krieger E-family $\{\mathbb{1}_{Z(e,r(e))}, \mathbb{1}_{D_v}\}$ and by Theorem 3.2.14 we know that this family has the universal property described in Lemma 2.2.12. Thus by Lemma 2.2.12 there exists an isomorphism $\phi : C^*(E) \to C^*(\mathcal{G}_E)$ such that $\phi(s_e) = \mathbb{1}_{Z(e,r(e))}$ for all $e \in E^1$ and $\phi(p_v) = \mathbb{1}_{D_v}$ for all $v \in E^0$. ■

In the case that E does have sinks we cannot find a groupoid model for $C^*(E)$ this way. However, we can in this case define a groupoid such that $C^*(E)$ is a full corner in the corresponding groupoid C^*-algebra. Recall from Theorem 2.2.25 that for a row-finite directed graph E with sinks we can construct a row-finite directed graph F without sinks such that $C^*(E)$ is a full corner of $C^*(F)$. Since F has no sinks we know that $C^*(\mathcal{G}_F) \cong C^*(F)$, and therefore we know that $C^*(E)$ is a full corner in $C^*(\mathcal{G}_F)$ and thus has the same ideal structure. Hence, we have the following corollary.

Corollary. *For every row-finite directed graph E with sinks, there exists a groupoid $\mathcal{G}_F$ such that $C^*(E)$ is a full corner in $C^*(\mathcal{G}_F)$.*

4. Graph C^*-algebras as Cuntz–Pimsner algebras

In this chapter we will look at another construction of C^*-algebras. This construction uses a Hilbert C^*-module X equipped with a left action given by adjointable operators. In the first section we discuss the general construction of this class of C^*-algebras known as Cuntz–Pimsner algebras. In the second section we create two such Hilbert modules for every row-finite directed graph without sinks and sources and prove that their corresponding Cuntz–Pimsner algebras are isomorphic to the graph C^*-algebra $C^*(E)$.

4.1 Cuntz–Pimsner algebras

In Section 1.1 we discussed some simple constructions of C^*-algebras. In this section we will look at another construction due to Pimsner [29], where we use the fact that for X a Hilbert module, the set of adjointable operators $\mathcal{L}(\mathsf{X})$ is a C^*-algebra as we saw in Lemma 1.2.19.

This construction is rather intricate and only applies to certain Hilbert modules X, called correspondences. From each such correspondence X, we will construct another correspondence $\mathcal{X}_+$, called the Fock correspondences associated to X. Using Lemma 1.2.19 we then define certain C^*-subalgebras of $\mathcal{L}(\mathcal{X}_+)$, called the Toeplitz and Cuntz–Pimsner algebras associated to X.

Definition 4.1.1. A *correspondence over a C^*-algebra $\mathcal{A}$* is a right Hilbert $\mathcal{A}$-module X together with a left action of $\mathcal{A}$ by adjointable operators on X given by a $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$.

It is called *essential* if for all $x \in \mathsf{X}$, $\phi(a)(x) = 0$ for all $a \in \mathcal{A}$, implies that $x = 0$.

Remark. Note that a notion of $(\mathcal{A}, \mathcal{B})$-correspondence also exist. In this thesis, however, we are only interested in correspondences over a single algebra.

Definition 4.1.2. Two C^*-correspondences (X, ϕ) and (Y, ψ) over $\mathcal{A}$ are isomorphic, if there exists a unitary $U : \mathsf{X} \to \mathsf{Y}$, such that for all $a \in \mathcal{A}$ and $\xi, \eta \in \mathsf{X}$:

$$\begin{aligned} \phi(a)U(\xi) &= U(\psi(a)x), \\ U(\xi) \cdot a &= U(x \cdot a), \\ \langle U(\xi), U(\eta) \rangle &= U(\langle \xi, \eta \rangle). \end{aligned}$$

Example 4.1.3. Recall from Example 1.2.10 that $\mathcal{A}$ is a right Hilbert $\mathcal{A}$-module with $\langle a, b \rangle_{\mathcal{A}} = a^*b$. It is easy to see that $\mathcal{A}$ is a correspondence over $\mathcal{A}$, with $\phi(a)(b) = a \cdot b$.

Given two C^*-correspondences, one can take their tensor product. We will use this fact in the construction of the Fock correspondences.

Definition 4.1.4. Let $\mathcal{A}$ be a C^*-algebra and suppose X and Y are correspondences over $\mathcal{A}$, with $\phi : \mathcal{A} \to \mathcal{L}(\mathsf{X})$ and $\psi : \mathcal{A} \to \mathcal{L}(\mathsf{Y})$. Let $\mathsf{X} \odot \mathsf{Y}$ denote the algebraic tensor product of X and Y. Then the *balanced tensor product*, denoted by $\mathsf{X} \otimes \mathsf{Y}$, is the quotient of $\mathsf{X} \odot \mathsf{Y}$ by the subspace

$$N = \operatorname{span}\{(\xi \cdot a) \otimes \eta - \xi \otimes (\psi(a)\eta) : \xi \in \mathsf{X}, \eta \in \mathsf{Y} \text{ and } a \in \mathcal{A}\}.$$

If we define $(\xi \otimes \eta)a = \xi \otimes (\eta a)$ and $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \psi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle$ we can see that $\mathsf{X} \otimes \mathsf{Y}$ is a pre-inner product $\mathcal{A}$-module and by Lemma 1.2.14 we define $\mathsf{X}\widehat{\otimes}_\psi\mathsf{Y}$ to be its completion.

To turn $\mathsf{X}\widehat{\otimes}_\psi\mathsf{Y}$ in to a correspondence over $\mathcal{A}$, we need a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{L}(\mathsf{X}\widehat{\otimes}_\psi\mathsf{Y})$. We can define such a $*$-homomorphism quite easily, as for every $T \in \mathcal{L}(\mathsf{X})$ we can define an operator on simple tensors by

$$\xi \otimes \eta \mapsto T(\xi) \otimes \eta,$$

which extends to a well-defined operator $T \otimes \mathbb{1}$ on $\mathsf{X}\widehat{\otimes}_\psi\mathsf{Y}$ and is adjointable with adjoint given by $T^* \otimes \mathbb{1}$. Thus for all $a \in \mathcal{A}$ we can define initially on simple tensors,

$$(\phi \otimes_\psi \mathbb{1})(a)(\xi \otimes \eta) = \phi(a)\xi \otimes \eta,$$

and extend it by linearity to a $*$-homomorphism $\phi \otimes_\psi \mathbb{1} : \mathcal{A} \to \mathcal{L}(\mathsf{X}\widehat{\otimes}_\psi\mathsf{Y})$.

Hence, the tensor product of two correspondences over the same C^*-algebra is also a correspondence over that C^*-algebra. This will be used in the following construction.

Definition 4.1.5. Let (X, ϕ) be a correspondence over $\mathcal{A}$ with ϕ isometric. Then define the *infinite direct summodule* as

$$\mathcal{X}_+ = \bigoplus_{k=0}^{\infty} \mathsf{X}^{(k)},$$

where $\mathsf{X}^{(k)} = \mathsf{X}^{\widehat{\otimes}_\phi^k}$ is the k-fold tensor product for $k > 0$ and $\mathsf{X}^{(0)} = \mathcal{A}$, with inner product on $\mathcal{X}_+$ defined as in Example 1.2.12, where the inner products on $\mathsf{X}^{(k)}$ are defined recursively, using the fact that $\mathsf{X}^{(k)} = \mathsf{X}^{(k-1)}\widehat{\otimes}_\phi\mathsf{X}$.

Define for all $a \in \mathcal{A}$ and all $\xi_1 \otimes \cdots \otimes \xi_k \in \mathsf{X}^{(k)}$ with $k \geq 1$,

$$\phi_+(a)(\xi_1 \otimes \cdots \otimes \xi_k) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_k,$$

and define $\phi_+(a)a' = aa'$ for all $a' \in \mathsf{X}^{(0)} = \mathcal{A}$. Then $(\mathcal{X}_+, \phi_+)$ is called the *(positive) Fock correspondence* associated to (X, ϕ).

Definition 4.1.6. For every $\xi \in \mathsf{X}$ and all elementary tensors $\mu \in \mathsf{X}^{(k)}$ $(k \geq 1)$, define

$$T_\xi(\mu) = \xi \otimes \mu \in \mathsf{X}^{(k+1)},$$

and

$$T_\xi^*(\mu) = \phi\left(\langle \xi, \mu_1 \rangle_{\mathcal{A}}\right) \mu_2 \otimes \mu_3 \otimes \cdots \otimes \mu_k \in \mathsf{X}^{(k-1)}.$$

Also, for all $a \in \mathsf{X}^{(0)} = \mathcal{A}$, define $T_\xi(a) = \xi \cdot a$ and $T_\xi^*(a) = 0$. This gives us operators $T_\xi \in \mathcal{L}(\mathcal{X}_+)$ with adjoints $T_\xi^* \in \mathcal{L}(\mathcal{X}_+)$, for all $\xi \in \mathsf{X}$.

Before we give the definitions of the Toeplitz algebra and the Cuntz–Pimsner algebra, it is important to mention that we make three assumptions that are not necessary in general. We assume that

1. ϕ is isometric (that is, injective),
2. X is full, and
3. $\phi(\mathcal{A}) \subseteq \mathcal{K}(\mathsf{X})$.

Definition 4.1.7. The *Toeplitz algebra* $\mathcal{T}_\mathsf{X}$ of the C^*-correspondence (X, ϕ) is the smallest C^*-subalgebra of $\mathcal{L}(\mathcal{X}_+)$ containing $\{T_\xi \mid \xi \in \mathsf{X}\}$.

This C^*-algebra has some interesting properties that are listed in the theorem below.

Theorem 4.1.8 ([9, Theorem 4.6.6])**.** *Let (X, ϕ) be a full C^*-correspondence over $\mathcal{A}$ and let $\mathcal{T}_\mathsf{X}$ be the corresponding Toeplitz algebra. Then the following relations hold:*

1. *$\alpha T_\xi + \beta T_\eta = T_{\alpha\xi+\beta\eta}$ for all $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in \mathsf{X}$;*
2. *$T_\xi \phi(a) = T_{\xi a}$ and $\phi(a) T_\xi = T_{\phi(a)\xi}$ for all $\xi \in \mathsf{X}$ and $a \in \mathcal{A}$;*
3. *$T_\xi^* T_\eta = \phi(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathsf{X}$;*
4. *there is an action γ, called the gauge action, of $\mathbb{T}$ on $\mathcal{T}_\mathsf{X}$ such that $\gamma_z(a) = a$ and $\gamma_z(T_\xi) = z T_\xi$ for all $z \in \mathbb{T}, a \in \mathcal{A}$ and $\xi \in \mathsf{X}$.*

Since $\mathcal{K}_\mathcal{A}(\mathcal{X}_+)$ is an ideal of $\mathcal{T}_\mathsf{X}$, we can define the *Cuntz–Pimsner algebra* of the C^*-correspondence (X, ϕ), $\mathcal{O}_\mathsf{X}$, by the exact sequence:

$$0 \longrightarrow \mathcal{K}_\mathcal{A}(\mathcal{X}_+) \longrightarrow \mathcal{T}_\mathsf{X} \xrightarrow{\pi} \mathcal{O}_\mathsf{X} \longrightarrow 0.$$

The image of an element $T_\xi \in \mathcal{T}_\mathsf{X}$ under the quotient map π will be denoted S_ξ. The C^*-algebra $\mathcal{O}_\mathsf{X}$ has the same properties as those described in Theorem 4.1.8.

Theorem 4.1.9. *Let (X, ϕ) be a full C^*-correspondence over $\mathcal{A}$ and let $\mathcal{O}_\mathsf{X}$ be the corresponding Cuntz–Pimsner algebra, then the following relations hold:*

1. *$\alpha S_\xi + \beta S_\eta = S_{\alpha\xi+\beta\eta}$ for all $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in \mathsf{X}$;*
2. *$S_\xi \phi(a) = S_{\xi a}$ and $\phi(a) S_\xi = S_{\phi(a)\xi}$ for all $\xi \in \mathsf{X}$ and $a \in \mathcal{A}$;*
3. *$S_\xi^* S_\eta = \phi(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathsf{X}$;*

4. There is an action γ, called the gauge action, of $\mathbb{T}$ on $\mathcal{O}_{\mathsf{X}}$ such that; $\gamma_z(a) = a$ and $\gamma_z(S_\xi) = zS_\xi$ for all $z \in \mathbb{T}, a \in \mathcal{A}$ and $\xi \in \mathsf{X}$.

Now we look at some examples. We first consider the Cuntz–Pimsner algebra $\mathcal{O}_{\mathbb{C}^n}$, (for $\mathbb{C}^n$ seen as a model over $\mathbb{C}$,) which we will prove to be equal to the Cuntz algebra $\mathcal{O}_n$, defined in Example 1.1.20.

Example 4.1.10. Let $\mathsf{X} = \mathbb{C}^n$ be a correspondence over $\mathbb{C}$ with the usual inner product and left action by multiplication. Then if we look at the operators corresponding to the base elements $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$, we see that $\mathcal{T}_{\mathsf{X}}$ is generated by n isometries $T_1, \dots, T_n$ with $\sum_{i=1}^n T_iT_i^* \leq 1$ and $T_i^*T_i = 1$.

Also,

$$1 - \sum_{i=1}^n T_iT_i^* = \Theta_{\mathbb{1}_{\mathbb{C}},\mathbb{1}_{\mathbb{C}}} \in \mathcal{K}_{\mathcal{A}}(\mathcal{X}_+),$$

as this sends λ to $1 \cdot \langle 1, \lambda\rangle_{\mathbb{C}} = \lambda$ and maps $\xi_1 \otimes \cdots \otimes \xi_n$ to

$$1 \cdot \langle 1, \xi_1 \otimes \cdots \otimes \xi_n\rangle_{\mathbb{C}} = 1 \cdot 0 = 0,$$

since the last inner product is the inner product defined on $\mathcal{X}_+$.

Thus $\sum_{i=1}^n \pi(T_i)\pi(T_i^*) = \sum_{i=1}^n S_iS_i^* = 1$ in $\mathcal{O}_{\mathsf{X}}$, hence $\mathcal{O}_{\mathsf{X}} \cong \mathcal{O}_n$.

The next two examples are more general.

Example 4.1.11 ([19, Section 2])**.** Let X be a finitely generated projective $\mathcal{A}$-module, that is, suppose $\mathbb{1}_{\mathsf{X}} \in \mathcal{K}(\mathsf{X})$ [2, Theorem 2.1.6]. Then we know that X admits a finite frame $\{\xi_i\}_{i=1}^n$, and, as explained in [2, Example 3.3], we get for all $a \in \mathcal{A}$

$$\phi(a)\xi_j = \sum_{i=1}^n \xi_i \langle \xi_i, \phi(a)\xi_j\rangle .$$

Thus we see that we get n generators of $\mathcal{O}_{\mathsf{X}}$, satisfying the explicit formulas

$$S_i^*S_j = \langle \xi_i, \xi_j\rangle,$$
$$\sum_{i=1}^n S_iS_i^* = 1,$$

and

$$aS_j = \sum_{i=1}^n S_i\langle \xi_i, \phi(a)\xi_j\rangle.$$

The generators are partial isometries if and only if $\langle \xi_i, \xi_j\rangle = 0$ for $i \neq j$.

Example 4.1.12 ([29, Example 3])**.** Let $\mathcal{B}$ be a C^*-algebra and $\alpha : \mathcal{B} \to \mathcal{B}$ an automorphism of $\mathcal{B}$. Then we can view $\mathcal{B}$ as a correspondence over itself with inner product $\langle a, b\rangle = a^*b$ and $\phi : \mathcal{B} \to \mathcal{L}(\mathcal{B})$ defined by $\phi(a)(b) = \alpha(a)b$.

Then every module $\mathcal{B}^{(k)}$ is isomorphic to $\mathcal{B}$ as a right module, with

$$\phi(a)(b_1 \otimes \cdots \otimes b_k) = \alpha^k(a)\alpha^{k-1}(b_1)\cdots\alpha(b_{k-1})b_k.$$

The C^*-algebra $\mathcal{O}_{\mathcal{B}}$ then coincides with the crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{Z}$.

Remark. Recall the three assumptions we made before defining the Toeplitz and Cuntz–Pimsner algebras. The first two assumptions are not necessary and as explained in [29, Remark 1.2 (1)] and [29, Remark 1.2 (3)] are made for the sake of convenience. The third assumption, however, is not redundant, but as explained in [29, Remark 1.2 (4)], allows for a simpler definition of $\mathcal{O}_{\mathsf{X}}$. Since we will only be looking at examples where this assumption holds, it makes sense for us to assume it in the definition. A construction of the algebras without the last two assumptions is given in [9, Paragraph 4.6].

4.2 Graph algebras as Cuntz–Pimsner algebras

In the previous section we saw the construction of Cuntz–Pimsner algebras from correspondences. In this section we will first define, for a directed graph E, two different correspondences and check when these correspondences satisfy the three conditions we assumed in our construction. Then we shall prove that for a row-finite directed graph E without sinks and sources, the corresponding Cuntz–Pimsner algebras are isomorphic to the graph C^*-algebra $C^*(E)$ and the groupoid C^*-algebra $C^*(\mathcal{G}_E)$.

The graph correspondence

We start by defining the graph correspondence and proving some helpful relations between the graph and the correspondence.

Example 4.2.1. Let E be a directed graph, and let $\mathcal{A} = c_0(E^0)$ be the C^*-algebra of functions $a : E^0 \to \mathbb{C}$ that vanish at infinity. Define the following right action and inner product on the space $c_c(E^1)$ of functions of finite support:

$$(x \cdot a) = x(e)a(r(e)) \quad \text{and} \quad \langle x, y \rangle_{\mathcal{A}}(v) = \sum_{\{e \in E^1 : r(e) = v\}} \overline{x(e)} y(e).$$

Then the completion of $c_c(E^1)/\{x \in c_c(E^1) : \langle x, x \rangle_{\mathcal{A}} = 0\}$ as described in Lemma 1.2.14 is a Hilbert $\mathcal{A}$-module. We will denote this module by $\mathsf{X}(E)$. Now define the left action $\phi : c_0(E^0) \to \mathcal{L}(\mathsf{X}(E))$ by setting

$$(\phi(a)(x))(e) = a(s(e))x(e),$$

for x in the dense subset $c_c(E^1) \subset \mathsf{X}(E)$, and extending by continuity. Then $\mathsf{X}(E)$ is a correspondence over $c_0(E^0)$ called the *graph correspondence* associated to E.

The next lemma shows how the graph correspondence reflects underlying properties of the graph, including those conditions we assumed in Section 4.1.

Lemma 4.2.2. *Let E be a directed graph and $\mathsf{X}(E)$ the associated graph correspondence. For $v \in E^0$ define the Kronecker delta $\delta_v \in c_0(E^0)$ by $\delta_v(w) = \delta_{v,w}$. Then*

1. $\phi(\delta_v) \in \mathcal{K}(\mathsf{X}(E))$ *if and only if* $|s^{-1}(v)| < \infty$;
2. $\phi(\delta_v) = 0$ *if and only if v is a sink;*

3. $\{\langle x, y\rangle : x, y \in \mathsf{X}(E)\}$ *generates* $C_0(E^0)$ *if and only if* E *has no sources;*

4. $\mathsf{X}(E)$ *is essential.*

Proof. First of all, we have

$$\phi(\delta_v)(x)(e) = \delta_v(s(e))x(e) = \begin{cases} 0 & \text{if } s(e) \neq v, \\ x(e) & \text{if } s(e) = v, \end{cases}$$

thus

$$\phi(\delta_v)(x) = \sum_{\{e \in E^1 : s(e) = v\}} \delta_e \langle \delta_e, x\rangle = \sum_{\{e \in E^1 : s(e) = v\}} \Theta_{\delta_e, \delta_e}(x).$$

Therefore,

$$\phi(\delta_v) = \sum_{\{e \in E^1 : s(e) = v\}} \Theta_{\delta_e, \delta_e} \in \mathcal{K}(\mathsf{X}(E))$$

if and only if $|\{e \in E^1 : s(e) = v\}| < \infty$, which is true if and only if $|s^{-1}(v)| < \infty$. Hence statement 1 is proven.

The second statement is also easy to prove because $\phi(\delta_v) = 0$ if and only if for all $x \in \mathsf{X}(E)$ we have $\phi(\delta_v)(x) = 0$. This implies that v is a sink, since

$$\sum_{\{e \in E^1 : s(e) = v\}} \Theta_{\delta_e, \delta_e}(\delta_f) = 0$$

for all $f \in E^1$ if and only if $s(f) \neq v$ for all $f \in E^1$, which is true if and only if v is a sink. Furthermore, if v is a sink, then $\{e \in E^1 : s(e) = v\}$ is empty, thus

$$\phi(\delta_v) = \sum_{\{e \in E^1 : s(e) = v\}} \Theta_{\delta_e, \delta_e} = 0.$$

To prove statement 3 we first observe that $\langle \delta_f, \delta_f\rangle(v) = \delta_{r(f),v}$. Thus if E has no sources, then for all $v \in E^0$ there exists an edge $e \in E^1$ such that $r(e) = v$. Therefore $\delta_v = \delta_{r(f)} = \langle \delta_f, \delta_f\rangle \in \{\langle x, y\rangle : x, y \in \mathsf{X}(E)\}$, thus $\{\langle x, y\rangle : x, y \in \mathsf{X}(E)\}$ generates $c_0(E^0)$.

Conversly, suppose E has a source $v \in E^0$, then we want to show that δ_v is not generated by $\{\langle x, y\rangle : x, y \in \mathsf{X}(E)\}$. We know that v is a source, so $r^{-1}(v) = \{e \in E^1 : r(e) = v\}$ is empty, thus

$$\langle x, y\rangle_{\mathcal{A}}(v) = \sum_{\{e \in E^1 : r(e) = v\}} \overline{x(e)}y(e) = 0 \text{ for all } x, y \in \mathsf{X}(E).$$

But $\delta_v(v) = 1$, thus δ_v can not be generated by $\{\langle x, y\rangle : x, y \in \mathsf{X}(E)\}$. This proves statement 3.

Finally, to prove the last statement, suppose $\phi(\delta_v)(x) = 0$ for all $v \in E^0$, then $\phi(\delta_{s(e)})(x)(e) = x(e) = 0$ for all $e \in E^1$, so $x = 0$. Thus $\mathsf{X}(E)$ is essential. ∎

The shift correspondence

We can also create another correspondence for graphs, called the shift correspondence.

Example 4.2.3. Let E be a row-finite directed graph. As we saw in Section 2.1, E^∞ is a locally compact space and the usual shift map σ is a local homeomorphism, because $\sigma(Z(\alpha_1 \cdots \alpha_n)) = Z(\alpha_2 \cdots \alpha_n)$. Then we can create another correspondence X_σ by defining the actions and inner product on X_σ by

$$\begin{aligned} \xi \cdot f(x) &= \xi(x) f(\sigma(x)), \\ \langle \xi, \eta \rangle (y) &= \sum_{\sigma(z)=y} \overline{\xi(z)} \eta(z), \end{aligned}$$

and

$$\phi(f)(\xi)(x) = f(x)\xi(x).$$

The fact that E^∞ is locally compact and σ is a local homeomorphism, implies that the inner product is well-defined. We call X_σ the *shift correspondence* associated to E.

Remark. Note that this is a special case of a general construction of a correspondence X_σ for locally compact spaces X with a local homeomorphism σ, as explained in [13, Section 1].

For this correspondence we can state a lemma similar to Lemma 4.2.2.

Lemma 4.2.4. *Let E be a row-finite directed graph and X_σ the correspondence described above. Then*

1. $\phi(\mathbb{1}_{Z(\alpha)}) \in \mathcal{K}(\mathsf{X}_\sigma)$ *for all* $\alpha \in E^*$*;*
2. ϕ *is injective;*
3. $\{\langle x, y \rangle : x, y \in \mathsf{X}_\sigma\}$ *generates* $C_0(E^\infty)$ *if and only if* σ *is surjective;*
4. X_σ *is essential.*

Proof. First of all, take $x \in E^\infty$, $\xi \in C_c(E^\infty)$ and for $\alpha, \beta \in E^*$ take the characteristic functions $\mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\beta)} \in C_c(E^\infty)$. We compute

$$\Theta_{\mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\beta)}}(\xi)(x) = \mathbb{1}_{Z(\alpha)}(x) \langle \mathbb{1}_{Z(\beta)}, \xi \rangle (\sigma(x)),$$

which is clearly 0 if $x \notin Z(\alpha)$. Therefore, we assume $x \in Z(\alpha)$. Then

$$\Theta_{\mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\beta)}}(\xi)(x) = \mathbb{1}_{Z(\alpha)}(x) \langle \mathbb{1}_{Z(\beta)}, \xi \rangle (\sigma(x)) = \sum_{\sigma(z)=\sigma(x)} \overline{\mathbb{1}_{Z(\beta)}(z)} \xi(z),$$

and this is clearly 0 if $\sigma(x) \notin Z(\sigma(\beta))$. Thus we can assume $\sigma(\alpha) = \sigma(\beta)$ and we see that

$$\Theta_{\mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\alpha)}}(\xi)(x) = \phi(\mathbb{1}_{Z(\alpha)})(\xi)(x),$$

so $\phi(\mathbb{1}_{Z(\alpha)}) = \Theta_{\mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\alpha)}} \in \mathcal{K}(\mathsf{X}_\sigma)$, which proves statement 1.

Suppose $\phi(f) = 0$. We want to prove that $f(x) = 0$ for all $x \in E^\infty$. Let $x \in E^\infty$ be arbitrary and let $\alpha = x_1x_2$. Then

$$\phi(f)(\mathbb{1}_{Z(\alpha)})(x) = f(x)\mathbb{1}_{Z(\alpha)}(x) = f(x) = 0,$$

hence $f = 0$. Hence ϕ is injective, which proves part 2.

Now suppose that σ is surjective. Then for all $x \in E^\infty$ there exists a $z \in E^\infty$ such that $\sigma(z) = x$. Thus for all $\beta \in E^*$, there exists an $\alpha \in E^*$ with $\sigma(\alpha) = \beta$. Note that this is not true if $Z(\beta)$ is empty, however we can ignore this case as that would mean that $\mathbb{1}_{Z(\beta)} = 0$. Then,for all $\beta \in E^*$ we get

$$\left\langle \mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\alpha)} \right\rangle (y) = \sum_{\sigma(z)=y} \mathbb{1}_{Z(\alpha)}(z) = \mathbb{1}_{Z(\sigma(\alpha))}(y) = \mathbb{1}_{Z(\beta)}(y),$$

so

$$\mathbb{1}_{Z(\beta)} = \left\langle \mathbb{1}_{Z(\alpha)}, \mathbb{1}_{Z(\alpha)} \right\rangle \in \{\langle x, y\rangle : x, y \in \mathsf{X}_\sigma\}.$$

Now suppose that σ is not surjective, which means that there exists a path $y \in E^\infty$ with $\sigma(z) \neq y$ for all $z \in E^\infty$. Then we have for all $\xi, \eta \in \mathsf{X}_\sigma$

$$1 = \mathbb{1}_{Z(y_1 \cdot y_n)}(y) \neq \langle \xi, \eta\rangle(y) = \sum_{\sigma(z)=y} \overline{\xi(z)}\eta(z) = 0.$$

Hence $\mathbb{1}_{Z(y_1 \cdot y_n)}$ cannot be generated by $\{\langle x, y\rangle : x, y \in \mathsf{X}_\sigma\}$. Thus we have proven statement 3.

To prove the last statement, suppose $\phi(\mathbb{1}_{Z(\alpha)})(\xi) = 0$ for all $\alpha \in E^*$. Then $\mathbb{1}_{Z(\alpha)}(x)\xi(x) = 0$ for all $x \in E^\infty$, so $\xi(x) = 0$ for all $x \in Z(\alpha)$. Since we chose α arbitrarily, we have $\xi(x) = 0$ for all $x \in E^\infty$. Thus $f \cdot \xi = 0$ for all $f \in C_0(E^\infty)$ implies that $\xi = 0$. Hence X_σ is always essential. ■

Remark. The results of Lemma 4.2.2 and Lemma 4.2.4 seem to differ at first sight, but we need to keep in mind that in the construction of X_σ we assume that E is row-finite in order for E^∞ to be locally compact. Furthermore, if E has sinks, the infinite path space E^∞ loses information about the underlying graph. Therefore we assume that E is row-finite and has no sinks, and in this case it is easy to see that σ is surjective if and only if E has no sources. Thus for row-finite graphs without sinks, these statements agree. Recall from Theorem 3.2.19 that for these graphs E, we have $C^*(E) \cong C^*(\mathcal{G}_E)$.

Now that we have defined these correspondences, we can create the corresponding Cuntz–Pimsner algebras. Recall from Section 4.1 that we require the correspondences to be full, ϕ to be faithful and $\phi(\mathcal{A}) \subseteq \mathcal{K}(\mathsf{X})$. It follows from Lemma 4.2.2 and Lemma 4.2.4 that $\mathsf{X}(E)$ and X_σ satisfy these conditions whenever E is row-finite and has no sinks and sources.

Theorem 4.2.5. *Suppose E is a row-finite directed graph without sinks and sources. Then $\mathcal{O}_{\mathsf{X}(E)} \cong C^*(E)$.*

Proof. Let $\xi = \delta_e \in \mathsf{X}(E)$ for $e \in E^1$. Then

$$\begin{aligned} T_\xi^* T_\xi(\mu_1 \otimes \cdots \otimes \mu_k) &= T_\xi^*(\xi \otimes \mu_1 \otimes \cdots \otimes \mu_k) \\ &= \phi(\langle \xi, \xi\rangle)\mu_1 \otimes \cdots \otimes \mu_k \\ &= \phi(\delta_{r(e)})\mu_1 \otimes \cdots \otimes \mu_k, \end{aligned}$$

so $T_\xi^* T_\xi = \phi(\delta_{r(e)})$, which means that $S_\xi^* S_\xi = \phi(\delta_{r(e)}) = P_{r(e)}$.

Furthermore,

$$\begin{aligned} T_\xi T_\xi^*(\mu_1 \otimes \cdots \otimes \mu_k) &= T_\xi(\phi(\langle \xi, \mu_1 \rangle)\mu_2 \otimes \cdots \otimes \mu_k) \\ &= \xi \otimes \phi(\langle \xi, \mu_1 \rangle)\mu_2 \otimes \cdots \otimes \mu_k \\ &= \xi \cdot \langle \xi, \mu_1 \rangle \otimes \mu_2 \otimes \cdots \otimes \mu_k, \end{aligned}$$

and $T_\xi T_\xi^*(\delta_w) = 0$ for all $w \in E^0$. This, together with statement 1 of Lemma 4.2.2 and the fact that for all $v \in E^0$, $\Theta_{\delta_v,\delta_v} \in \mathcal{K}(\mathcal{X}_+)$ is 0 on X^k and equal to δ_v on $\mathsf{X}^0 = C_0(E^0)$, gives us

$$\sum_{e \in E^1 : s(e) = v} T_{\delta_e} T_{\delta_e}^* = \phi(\delta_v) - \Theta_{\delta_v,\delta_v},$$

so

$$\sum_{e \in E^1 : s(e) = v} S_{\delta_e} S_{\delta_e}^* = \phi(\delta_v).$$

This shows that the Cuntz–Krieger relations hold. Furthermore it is quite easy to check that $\{P_v = \phi(\delta_v) : v \in E^0\}$ is a set of mutually orthogonal projections and that $\{S_e = S_{\delta_e} : e \in E^1\}$ is a set of partial isometries, thus we have a Cuntz–Krieger E-family generating $\mathcal{O}_{\mathsf{X}(E)}$.

The fourth property of Theorem 4.1.9 tells us that $\mathcal{O}_{\mathsf{X}(E)}$ has a gauge action that satisfies the conditions of the gauge invariant uniqueness theorem [Theorem 2.2.14], therefore we obtain $\mathcal{O}_{\mathsf{X}(E)} \cong C^*(E)$. ■

Theorem 4.2.6. *Suppose E is a row-finite directed graph without sinks and sources. Then $\mathcal{O}_{\mathsf{X}_\sigma} \cong C^*(E)$.*

Proof. Let $\xi = \mathbb{1}_{Z(e)} \in \mathsf{X}_\sigma$ for $e \in E^1$. Then

$$\begin{aligned} T_\xi^* T_\xi(\mu_1 \otimes \cdots \otimes \mu_k) &= T_\xi^*(\xi \otimes \mu_1 \otimes \cdots \otimes \mu_k) \\ &= \phi(\langle \xi, \xi \rangle)\mu_1 \otimes \cdots \otimes \mu_k \\ &= \phi(\mathbb{1}_{Z(\sigma(e))})\mu_1 \otimes \cdots \otimes \mu_k, \end{aligned}$$

so $T_\xi^* T_\xi = \phi((\mathbb{1}_{Z(r(e))}) = P_{r(e)}$, which means that $S_\xi^* S_\xi = \phi((\mathbb{1}_{Z(r(e))}) = P_{r(e)}$.

Furthermore,

$$\begin{aligned} T_\xi T_\xi^*(\mu_1 \otimes \cdots \otimes \mu_k) &= T_\xi(\phi(\langle \xi, \mu_1 \rangle)\mu_2 \otimes \cdots \otimes \mu_k) \\ &= \xi \otimes \phi(\langle \xi, \mu_1 \rangle)\mu_2 \otimes \cdots \otimes \mu_k \\ &= \xi \cdot \langle \xi, \mu_1 \rangle \otimes \mu_2 \otimes \cdots \otimes \mu_k \end{aligned}$$

and $T_\xi T_\xi^*(\mathbb{1}_{Z(v)}) = 0$ for all $v \in E^0$. This, together with statement 1 of Lemma 4.2.4 and the fact that $\Theta_{\mathbb{1}_{Z(v)},\mathbb{1}_{Z(v)}} \in \mathcal{K}(\mathcal{X}_+)$ is 0 on X^k and equal to $\mathbb{1}_{Z(v)}$ on $\mathsf{X}^0 = C_0(E^\infty)$, gives us

$$\sum_{e \in E^1 : s(e) = v} T_{\mathbb{1}_{Z(e)}} T_{\mathbb{1}_{Z(e)}}^* = \phi(\mathbb{1}_{Z(v)}) - \Theta_{\mathbb{1}_{Z(v)},\mathbb{1}_{Z(v)}},$$

so

$$\sum_{e \in E^1 : s(e) = v} S_{\mathbb{1}_{Z(e)}} S_{\mathbb{1}_{Z(e)}}^* = \phi(\mathbb{1}_{Z(v)}) = P_v.$$

This shows that the Cuntz–Krieger relations hold.

Furthermore, it is quite easy to check that $\{P_v = \phi(\mathbb{1}_{Z(v)}) : v \in E^0\}$ is a set of mutually orthogonal projections and that $\{S_e = S_{\mathbb{1}_{Z(e)}} : e \in E^1\}$ is a set of partial isometries. The only thing left to prove is that this generates $\mathcal{O}_{\mathsf{X}_\sigma}$.

Denote $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_{|\alpha|}}$ for $\alpha \in E^*$, as before. We just proved that for $|\alpha| = 1$, $S_\alpha S_\alpha^* = \phi(\mathbb{1}_{Z(\alpha)})$. Now we can prove by induction that this holds for all $\alpha \in E^*$. Suppose for all $\alpha \in E^n$ we have $S_\alpha S_\alpha^* = \phi(\mathbb{1}_{Z(\alpha)})$. Then for $\beta \in E^{n+1}$ we have $\beta = e\alpha$ for some $e \in E^1$ and $\alpha \in E^n$. Then

$$S_\beta S_\beta^* = S_e S_\alpha S_\alpha^* S_e^* = S_e \phi(\mathbb{1}_{Z(\alpha)}) S_e^*$$

and

$$\begin{aligned}
S_e \phi(\mathbb{1}_{Z(\alpha)}) S_e^*(\mu_1 \otimes \cdots \otimes \mu_k) &= S_e \phi(\mathbb{1}_{Z(\alpha)}) \left(\phi(\langle \mathbb{1}_{Z(e)}, \mu_1\rangle)\mu_2 \otimes \cdots \otimes \mu_k\right) \\
&= S_e \phi\left(\mathbb{1}_{Z(\alpha)}\right) \left((\phi(\langle \mathbb{1}_{Z(e)}, \mu_1\rangle)\mu_2) \otimes \cdots \otimes \mu_k\right) \\
&= \mathbb{1}_{Z(e)} \otimes \phi\left(\mathbb{1}_{Z(\alpha)}\right) \left(\phi(\langle \mathbb{1}_{Z(e)}, \mu_1\rangle)\mu_2\right) \otimes \cdots \otimes \mu_k \\
&= \mathbb{1}_{Z(e)} \cdot \mathbb{1}_{Z(\alpha)} \otimes \left(\phi\left(\langle \mathbb{1}_{Z(e)}, \mu_1\rangle\right) \mu_2\right) \otimes \cdots \otimes \mu_k \\
&= \left(\mathbb{1}_{Z(e)} \cdot \mathbb{1}_{Z(\alpha)}\right) \cdot \langle \mathbb{1}_{Z(e)}, \mu_1\rangle \otimes \mu_2 \otimes \cdots \otimes \mu_k \\
&= \phi\left(\mathbb{1}_{Z(e\alpha)}\right) \mu_1 \otimes \cdots \otimes \mu_k,
\end{aligned}$$

because

$$\begin{aligned}
\left(\mathbb{1}_{Z(e)} \cdot \mathbb{1}_{Z(\alpha)}\right) \cdot \langle \mathbb{1}_{Z(e)}, \mu_1\rangle(x) &= \left(\mathbb{1}_{Z(e)} \cdot \mathbb{1}_{Z(\alpha)}\right)(x) \langle \mathbb{1}_{Z(e)}, \mu_1\rangle(\sigma(x)) \\
&= \mathbb{1}_{Z(e)}(x) \mathbb{1}_{Z(\alpha)}(\sigma(x)) \sum_{\sigma(z)=\sigma(x)} \overline{\mathbb{1}_{Z(e)}(z)} \mu_1(z) \\
&= \begin{cases} 0 & \text{if } x \notin Z(e\alpha) \\ \mu_1(x) & \text{otherwise} \end{cases} \\
&= \phi(\mathbb{1}_{Z(e\alpha)})(\mu_1)(x).
\end{aligned}$$

Furthermore, for all $\beta \in E^*$, we have $\mathbb{1}_{Z(\beta)} \in C_0(E^\infty)$, so $\Theta_{\mathbb{1}_{Z(\beta)}, \mathbb{1}_{Z(\beta)}} \in \mathcal{K}(\mathcal{X}_+)$ is 0 on X^k and equal to $\mathbb{1}_{Z(\beta)}$ on $\mathsf{X}^0 = C_0(E^\infty)$. Thus for $\beta \in E^{n+1}$, we have $S_\beta S_\beta^* = \phi(\mathbb{1}_{Z(\beta)})$. It follows by induction that we have for all $\alpha \in E^*$, $S_\alpha S_\alpha^* = \phi(\mathbb{1}_{Z(\alpha)})$. This, together with the fact that $S_{\mathbb{1}_{Z(\alpha)}} = S_{\mathbb{1}_{Z(\alpha_1)}} \phi(\mathbb{1}_{Z(\sigma(\alpha))})$, shows that the Cuntz–Krieger E-family generates $\mathcal{O}_{\mathsf{X}_\sigma}$.

As in the previous proof, the fourth property of Theorem 4.1.9 tells us that $\mathcal{O}_{\mathsf{X}_\sigma}$ has a gauge action that satisfies the conditions of the gauge invariant uniqueness theorem [Theorem 2.2.14], therefore we get that $\mathcal{O}_{\mathsf{X}_\sigma} \cong C^*(E)$. ■

In the case that E does have sinks and sources we can obtain interesting results similar to the remarks in Subsection 3.2.1. Recall from Theorem 2.2.25 that from a row-finite directed graph E with sinks and sources we can construct a row-finite directed graph F without sinks and sources such that $C^*(E)$ is a full corner of $C^*(F)$. Because F has no sinks and sources we know that $C^*(\mathcal{G}_F) \cong C^*(F) \cong \mathcal{O}_{\mathsf{X}(F)} \cong \mathcal{O}_{\mathsf{X}_\sigma}$ and therefore we know that $C^*(E)$ is a full corner in $\mathcal{O}_{\mathsf{X}(F)}$ and $\mathcal{O}_{\mathsf{X}_\sigma}$ and thus has the same ideal structure. Hence we have the following corollary

Corollary. *For every row-finite directed graph E with sinks and sources, there exist correspondences $\mathsf{X}(F)$ and X_σ such that $C^*(E)$ is a full corner in the isomorphic Cuntz–Pimsner algebras $\mathcal{O}_{\mathsf{X}(F)}$ and $\mathcal{O}_{\mathsf{X}_\sigma}$.*

5. Structural properties of graph C^*-algebras

Now that we have seen that the different constructions of C^*-algebras coming from a row-finite graph E without sinks and sources lead to isomorphic C^*-algebras, we can look at the structure of $C^*(E)$ from different points of view.

In this chapter we look at the ideal structure and the properties of simplicity and pure infiniteness of $C^*(E)$ in the Cuntz–Krieger model, the groupoid model, and the two Cuntz–Pimsner models.

5.1 Ideal structure

We start this chapter by looking at the ideal structure in the different models. For graph C^*-algebras the ideal structure is completely known, as was shown by several authors using the Cuntz–Krieger model [4], [32]. We can translate this to the groupoid and Cuntz–Pimsner models. Recall the definition of an ideal.

Definition 5.1.1. Let $\mathcal{A}$ be a C^*-algebra. A subset $I \subseteq \mathcal{A}$ is an *ideal* of $\mathcal{A}$ if for all $a \in \mathcal{A}$ and $b \in I$, $ab, ba \in I$. In this thesis, we also require ideals to be closed and linear.

5.1.1 The ideal structure in the Cuntz–Krieger model

We start our analysis of the ideal structure by looking at the Cuntz–Krieger model.

Definition 5.1.2. For all vertices $v, w \in E^0$, we write $v \geq w$ if there exists a finite path $\alpha \in E^*$ such that $s(\alpha) = v$ and $r(\alpha) = w$.

Obviously for all $v \in E^0$ we have $v \geq v$ (letting α be the empty path) and for all $u, v, w \in E^0$ it follows that $u \geq v$ and $v \geq w$ imply $u \geq w$. Thus $\geq$ is a preorder, however it is not a partial order as there might be two vertices $v \neq w$ on a cycle, such that $v \geq w \geq v$.

The following condition is needed to describe the ideal structure.

Definition 5.1.3. A directed graph (E^0, E^1, r, s) is said to satisfy Condition (L) if the following holds: For all $\alpha \in E^*$ with $r(\alpha) = s(\alpha)$, there exists $1 \leq i \leq |\alpha|$ and $e \in E^1$ such that $s(\alpha_i) = s(e)$ and $\alpha_i \neq e$, that is, every cycle has an exit.

This condition is not only of importance in the ideal structure, but also more generally, as shown by the following theorem.

Theorem 5.1.4 ([32, Theorem 2.4]). *Suppose E is a row-finite directed graph satisfying condition (L) and suppose $\{T, Q\}$ is a Cuntz–Krieger E-family in a C^*-algebra $\mathcal{B}$ such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to \mathcal{B}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Now we describe the ideals of $C^*(E)$ in terms of subgraphs of E. In order to do so we need the following definitions:

Definition 5.1.5. A subset H of E^0 is called *hereditary* if $v \in H$ and $v \geq w$ imply $w \in H$, that is, every vertex that can be reached from H belongs to H.

Definition 5.1.6. A subset H of E^0 is called *saturated* if for all $v \in E^0$ with

$$s^{-1}(v) \neq \emptyset \quad \text{and} \quad r(e) \in H \text{ for all } e \in E^1 \text{ with } s(e) = v,$$

we have $v \in H$, that is, if a vertex only emits edges whose range is in H, then it belongs to H. The *saturation* of a hereditary set H is the smallest saturated subset $\overline{H}$ of E^0 containing H. Note that $\overline{H}$ is itself hereditary.

Lemma 5.1.7. *Let E be a row-finite graph. Suppose I is a non-zero ideal in $C^*(E)$ and define $H(I) = \{v \in E^0 \mid P_v \in I\}$. Then $H(I)$ is saturated and hereditary.*

Proof. First we will show that $H(I)$ is hereditary. Suppose $v \in H(I)$ and $v \geq w$. That means that $P_v \in I$ and there exists $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Therefore, $P_v = P_{s(\alpha)}$ and thus $S_\alpha = P_{s(\alpha)} S_\alpha \in I$ and $P_w = P_{r(\alpha)} = S_\alpha^* S_\alpha \in I$, hence $w \in H(I)$.

Now we prove that $H(I)$ is saturated. Let $v \in E^0$ such that $s^{-1}(v) \neq \emptyset$ and $r(e) \in H(I)$ for all $e \in E^1$ with $s(e) = v$. Then for those $e \in E^1$ we have $P_{r(e)} \in I$, thus $S_e = S_e P_{r(e)} \in I$ and

$$P_v = \sum_{\{e \in E^1 : s(e) = v\}} S_e S_e^* \in I,$$

hence $v \in H(I)$. ■

For a graph E and an ideal I of $C^*(E)$, consider the subgraph

$$E \setminus H(I) := \left(E^0 \setminus H(I), \{e \in E^1 : r(e) \notin H(I)\}, r, s\right).$$

Since $H(I)$ is hereditary we have that $s(e) \in H(I)$ implies $r(e) \in H(I)$ for all $e \in E^1$. Thus $r(e) \notin H(I)$ implies $s(e), r(e) \in E^0 \setminus H(I)$, so $E \setminus H(I)$ is indeed a subgraph of E.

Consider the quotient map $q : C^*(E) \to C^*(E)/I$. Then

$$\{q(P_v), q(S_e) : r(e) \notin H(I), v \notin H(I)\}$$

is a Cuntz–Krieger family for $E \setminus H(I)$ where every vertex projection is non-zero.

If condition (L) would hold for this graph $E \setminus H(I)$, then by Theorem 5.1.4 $C^*(E)/I$ and $C^*(E \setminus H(I))$ would be isomorphic. Therefore, we would like to find out when condition (L) holds for these graphs.

Definition 5.1.8. A graph E satisfies Condition (K) if for every vertex $v \in E^0$, there is either no cycle based at v or there are two distinct cycles μ, ν based at v that only pass through v once, that is, $\mu \neq \nu$, $r(\mu_i) \neq v$ for all $i < |\mu|$ and $r(\nu_j) \neq v$ for all $j < |\nu|$.

In general, we denote the set of such vertices by V_2 and the set of vertices that do not lie on any cycle by V_0, so (K) holds if and only if $E^0 = V_0 \cup V_2$.

Remark. It is easy to see that (K) implies (L), as for every cycle μ based at v there must be another cycle ν based at v such that $\mu \neq \nu$. Thus ν_j, where j is the minimal natural number such that $\mu_j \neq \nu_j$, is an 'exit edge' for μ.

However, (L) does not imply (K), as we can see in this easy example where $s(f) = s(e) = r(e) = v$ and $r(f) = w$.

$$\begin{tikzcd} v \arrow[loop left, "e"] \arrow[r, "f"] & w \end{tikzcd}$$

Lemma 5.1.9. *A directed graph E satisfies condition (K) if and only if for every saturated hereditary subset H of E^0, the graph*

$$E \setminus H = (E^0 \setminus H, \{e \in E^1 : r(e) \notin H\}, r, s)$$

satisfies condition (L).

Proof. $(\Rightarrow)$: Suppose E satisfies condition (K) and let $H \subseteq E^0$ be a hereditary saturated subset. Suppose μ is a cycle in $E \setminus H$. Then μ is also a cycle in E and $s(\mu_i) \notin H$ for all $i \leq |\mu|$. Since condition (K) holds, we know that there exists another cycle ν based at $s(\mu)$, so for all $j \leq |\nu|$ we have $s(\nu_j) \geq s(\mu)$ and $s(\mu) \notin H$. Consequently, because H is hereditary, we get $r(\nu_{j-1}) = s(\nu_j) \notin H$. Therefore, ν is also in $E \setminus H$, and by the same reasoning as before, ν_j (with $j = \min_{i \in \mathbb{N}}\{\mu_i \neq \nu_i\}$) is an 'exit edge' for μ in $E \setminus H$. Hence $E \setminus H$ satisfies condition (L) for every saturated hereditary subset H of E^0.

$(\Leftarrow)$: Now suppose that for every saturated hereditary subset H of E^0, $E \setminus H$ satisfies condition (L). We want to prove that E satisfies condition (K), so suppose μ is a cycle based at $s(\mu) = v \in E^0$. Then we want to find another cycle ν based at v. Defining $H = \{w : w \ngeq v\} \subseteq E^0$, we will show that this is a saturated hereditary set.

Let $z \in H$ with $z \geq w$ and suppose for the sake of contradiction that $w \notin H$. Then $w \geq v$, thus $z \geq w \geq v$ which contradicts $z \in H$, so we must have $w \in H$, which proves that H is hereditary.

Now suppose that $r(e) \in H$ for all $e \in E^1$ with $s(e) = w$. Again, for the sake of contradiction, suppose that $w \notin H$. Then $w \geq v$, so there exists a finite path starting in w and ending in v. This path must start in some e with $s(e) = w$, so $r(e) \geq v$ as well, but by assumption $r(e) \ngeq w$. Thus we must have $w \in H$.

It is easy to see that μ lies in $E \setminus H$, and because we just showed that H is a hereditary saturated subset, we know that $E \setminus H$ satisfies (L) and thus μ has an exit in $E \setminus H$. Let e be the 'exit edge', so there exists $i \leq |\mu|$ with $s(\mu_i) = s(e)$

but $e \neq \mu_i$. Then $r(e) \notin H$, so $r(e) \geq v$. Let $\alpha \in E^*$ be a finite path such that $s(\alpha) = r(e)$ and $r(\alpha) = v$. Then $\nu = \mu_1 \cdots \mu_{i-1} e\alpha$ is a cycle based at v such that $\nu \neq \mu$, so E satisfies condition (K). ■

Lemma 5.1.10. *If H is a hereditary subset of E^0, let I_H be the ideal generated by the projections $\{P_v \mid v \in H\}$. Then*

$$I_H = \overline{span}\{S_\alpha S_\beta^* \mid \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{H}\}.$$

Furthermore, $I_H = I_{\overline{H}}$ and I_H is gauge-invariant.

Proof. First we prove that the right hand side lies in I_H. Look at the set $\{v \in E^0 \mid P_v \in I_H\}$. This set obviously contains H and is saturated because for all $w \in E^0$ we have:

If for all $e \in s^{-1}(w)$ we have $r(e) \in \{v \in E^0 \mid P_v \in I_H\}$, then

$$P_w = \sum_{s(e)=w} S_e P_{r(e)} S_e^* \in I_H.$$

Therefore, $w \in \{v \in E^0 \mid P_v \in I_H\}$. Thus $\{v \in E^0 \mid p_v \in I_H\}$ is a saturated set that contains H, so it contains the saturation $\overline{H}$. But then we have for all $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta) \in \overline{H}$ that $r(\alpha) \in \{v \in E^0 \mid P_v \in I_H\}$, so $S_\alpha S_\beta^* = S_\alpha P_{r(\alpha)} S_\beta^* \in I_H$. Thus the right hand side is contained in I_H.

From Lemma 2.2.9 and the fact that $\overline{H}$ is hereditary, it follows that the righthand side is an ideal. Furthermore, it contains P_v for all $v \in \overline{H}$, so we have proven that $I_H = \overline{\text{span}}\{S_\alpha S_\beta^* \mid \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{H}\}$.

Obviously $I_H = I_{\overline{H}}$ and $\gamma_z(S_\alpha S_\beta^*) = z^{|\alpha|-|\beta|} S_\alpha S_\beta^*$, so $\gamma_z(I_H) \subseteq I_H$ for all $z \in \mathbb{T}$, thus I_H is gauge-invariant. ■

Theorem 5.1.11. *Suppose E is a row-finite directed graph. For $H \subseteq E^0$, let I_H be the ideal in $C^*(E)$ generated by $\{P_v \mid v \in H\}$. Then:*

1. *$H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$, with inverse given by $I \mapsto H(I)$.*
2. *If H is saturated and hereditary, then $C^*(E)/I_H \cong C^*(E \setminus H)$.*
3. *If H is a hereditary subset of E^0 and $G = (H, \{e \mid s(e) \in H\}, r, s)$, then $C^*(G)$ is isomorphic to a subalgebra of $C^*(E)$ and this subalgebra is a full corner in I_H.*

Proof. 1. Let I be a non-zero gauge-invariant ideal in $C^*(E)$. Then $H(I)$ is saturated and hereditary by Lemma 5.1.7. Clearly $I_{H(I)} \subseteq I$, so $P_v \notin I$ implies $P_v \notin I_{H(I)}$ and I and $I_{H(I)}$ both contain exactly the same set of projections $\{P_v \mid v \in H(I)\}$. Now consider the quotients $C^*(E)/I$ and $C^*(E)/I_{H(I)}$. Both are generated by Cuntz–Krieger $(E \setminus H(I))$-families where all the projections are non-zero and both I and $I_{H(I)}$ are gauge-invariant, so both quotients carry gauge actions. Thus by Theorem 2.2.14 we have two isomorphisms

$$\phi : C^*(E \setminus H(I)) \to C^*(E)/I$$

and

$$\psi : C^*(E \setminus H(I)) \to C^*(E)/I_{H(I)},$$

such that

$$\phi(\{t_e, q_v\}) = \{S_e + I, P_v + I\}$$

and

$$\psi(\{t_e, q_v\}) = \{S_e + I_{H(I)}, P_v + I_{H(I)}\},$$

where $\{t_e, q_v\}$ is the canonical Cuntz–Krieger $(E \setminus H(I))$-family.

Hence $\phi \circ \psi^{-1} : C^*(E)/I_{H(I)} \to C^*(E)/I$ is an isomorphism, which agrees with the quotient map $q : C^*(E)/I_{H(I)} \to C^*(E)/I$ on generators. Therefore, q is also an isomorphism, which implies that $I = I_{H(I)}$, so $H \to I_H$ is surjective. Now we have to prove that $H(I_H) = H$. Obviously $H \subseteq H(I_H)$. To prove the other inclusion, look at the graph $E \setminus H$ again and let $\{S_e, P_v\}$ be a Cuntz–Krieger $(E \setminus H)$-family with $P_v \neq 0$ for all $v \in E \setminus H$. We claim this can be extended to a Cuntz–Krieger E-family by setting $P_v = 0$ for all $v \in H$ and $S_e = 0$, whenever $r(e) \in H$. To see that this is true, we check the Cuntz–Krieger relations. First of all, if $r(e) \in H$, then $S_e^* S_e = 0 = P_{r(e)}$. If $r(e) \notin H$ we already knew that $S_e^* S_e = P_{r(e)}$.

For the second Cuntz–Krieger relation, suppose that $v \in H$, then since H is hereditary, for all $e \in E^1$ with $s(e) = v$, we have $r(e) \in H$. Thus $S_e = 0$, and

$$P_v = \sum_{\{e \in E^1 : s(e) = v\}} S_e S_e^* = 0.$$

Now suppose v is not a sink in $E \setminus H$. Then

$$P_v = \sum_{\{e \in E^1 : r(e) \notin H, s(e) = v\}} S_e S_e^*$$

and for all $e \in E^1$ with $r(e) \in H$ we have $S_e = 0$. So

$$P_v = \sum_{\{e \in E^1 : r(e) \notin H, s(e) = v\}} S_e S_e^* = \sum_{\{e \in E^1 : s(e) = v\}} S_e S_e^*.$$

The only case left to check is if v is a sink in $E \setminus H$. This means that for all $e \in E^1$ with $s(e) = v$, we have $r(e) \in H$.

By saturation, that means that $v \in H$ or $s^{-1}(v) = \emptyset$, but $v \in E^0 \setminus H$ so $s^{-1}(v) = \emptyset$, which implies that v is a sink in E as well.

Hence all the Cuntz–Krieger relations hold and $\{S, P\}$ is indeed a Cuntz–Krieger E-family, proving the claim. Furthermore, by Lemma 2.2.11, we have a homomorphism $\pi : C^*(E) \to C^*(S_e, P_v)$ with $\pi(p_v) = 0$ for all $v \in H$. Thus $\{p_v \mid v \in H\} \subseteq \ker \pi$, so π vanishes on I_H. But $\pi(p_v) = P_v \neq 0$ for all $v \notin H$, so $v \notin H$ implies $p_v \notin I_H$. Thus $H(I_H) \subseteq H$.

Now we have shown that $H \mapsto I_H$ is bijective, and this together with the fact that $H_1 \subset H_2$ if and only if $I_{H_1} \subset I_{H_2}$, proves that $H \mapsto I_H$ is a lattice isomorphism.

2. Let H be saturated and hereditary. We saw that in this case $H = H(I_H)$. Taking $I = I_H$ in the first part of the proof gives us

$$C^*(E)/I_H \cong C^*(E \setminus H(I_H)) \cong C^*(E \setminus H).$$

3. Let H be a hereditary (not necessarily saturated) subset of E^0, and using [5, Lemma 1.1] define $q_H = \sum_{v \in H} P_v \in M(C^*(E))$. We prove that $q_H I_{\overline{H}} q_H$ is generated by the Cuntz–Krieger G-family $\{S_e, P_v \mid s(e), v \in H\}$. First of all,

$$q_H I_{\overline{H}} q_H = \overline{\operatorname{span}}\{S_\alpha S_\beta^* \mid \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in \overline{H}, s(\alpha), s(\beta) \in H\},$$

so since H is hereditary the family lies in the corner and generates $q_H I_{\overline{H}} q_H$. Now Theorem 2.2.14 implies that $q_H I_{\overline{H}} q_H$ is isomorphic to $C^*(G)$, and in turn $C^*(G)$ is canonically isomorphic to the subalgebra $C^*(S_e, P_v : s(e), v \in H)$ of $C^*(E)$. The only thing left to show is that $q_H I_{\overline{H}} q_H$ is a full corner in I_H. So, suppose that J is a proper ideal in $I_H = I_{\overline{H}}$ containing $q_H I_{\overline{H}} q_H$. Then Lemma 5.1.7 implies that $H(J)$ is a saturated and hereditary set that contains H, so it also contains $\overline{H}$. But then J contains all the generators $\{p_v : v \in \overline{H}\}$ of $I_{\overline{H}}$, so that $J = I_{\overline{H}}$, which contradicts the fact that J is proper. Hence $q_H I_{\overline{H}} q_H$ is a full corner in I_H. ■

Remark. If we assume that E satisfies condition (K), then Lemma 5.1.9 and replacing Theorem 2.2.14 with Theorem 5.1.4 in the proof of Theorem 5.1.11, gives us an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of all ideals of $C^*(E)$.

5.1.2 The ideal structure in the groupoid model

We have seen that in the Cuntz–Krieger model, if E satisfies condition (K) the ideals correspond to saturated hereditary subsets of E^0. We will now state the analogous results for the groupoid model proven in [23, Section 6], and see that it all ties in nicely.

First, we define some properties of groupoids in general, then we will consider these properties for the path groupoid. This will give us the results we want.

Definition 5.1.12. A groupoid G is called *essentially free* or *topologically principal* if the set of points with trivial isotropy is dense in $G^{(0)}$, that is,

$$\overline{\{u \in G^{(0)} : G_u^u = \{u\}\}} = G^{(0)}.$$

Remark. A unit $u = (x, 0, x) \in \mathcal{G}_E^{(0)} \cong E^\infty$ has non-trivial isotropy if and only if x is eventually periodic. This is because $u = (x, 0, x)$ is eventually periodic if and only if there exists $k \in \mathbb{N}$ with $(x, k, x) \in \mathcal{G}_E$ if and only if there exists $k \in \mathbb{N}$ with $(x, k, x), (x, 0, x) \in \mathcal{G}_E{}_u^u$.

Lemma 5.1.13 ([22, Lemma 3.4])**.** *The groupoid $\mathcal{G}_E$ is essentially free if and only if E satisfies condition (L).*

Proof. $(\Leftarrow)$: We write G instead of $\mathcal{G}_E$ for notational purposes. Suppose that E satisfies (L). First of all, we always have

$$\overline{\{u \in G^{(0)} : G_u^u = \{u\}\}} \subseteq \overline{G^{(0)}} = G^{(0)},$$

so we only need to show that $G^{(0)} \subseteq \overline{\{u \in G^{(0)} : G_u^u = \{u\}\}}$.

Let $(x, 0, x)$ be an eventually periodic point. Then there exists $k \in \mathbb{N}$ such that $(x, k, x) \in G$, which means that there exists $N \in \mathbb{N}$ with $x_i = x_{i+k}$ for all $i \geq N$. This means that $\mu = (x_{N+1}, \ldots, x_{N+k})$ is a cycle, so $x = x_1 \cdots x_N \mu\mu \cdots$.

Now we use the fact that μ has an exit. Suppose this exit appears at x_{N+j} for some $1 \leq j \leq k$, so there exists $e \in E^1$ with $s(e) = s(x_{N+j})$ but $e \neq x_{N+j}$.

We want to construct an infinite path $\alpha \in E^\infty$ starting in $v = r(e)$ that is not eventually periodic. First suppose that v is connected via some finite path $\beta \in E^*$ to a vertex in V_2 and let γ and δ be two distinct cycles based at $r(\beta)$. Then

$$\alpha = \beta\gamma\delta\gamma\gamma\delta\delta \cdots \underbrace{\gamma \cdots \gamma}_{m \text{ times}} \overbrace{\delta \cdots \delta}^{m \text{ times}} \cdots$$

is an aperiodic path starting at v.

Now suppose v is not connected to V_2. Then the construction of α becomes more difficult. Let $W \subseteq E^0$ be the set of all vertices reachable from v, including v itself. Then W consists of vertices in V_0 and V_1. Define $F : W \to E^1$ as follows. For $w \in V_0$, choose $F(w) \in s^{-1}(w)$ arbitrarily. For $w \in V_1$, if w emits only one edge e, choose $F(w) = e$; if w emits more than one edge, choose an edge f which is not on a return path for w and define $F(w) = f$ (we know that such an edge exists as $w \notin V_2$).

Now define α recursively by setting $\alpha_1 = F(v)$ and $\alpha_i = F(r(\alpha_{i-1}))$ for $i \geq 2$. Suppose α is eventually periodic. Then there exists a cycle $(\alpha_n, \dots, \alpha_m)$ for some $n, m \in \mathbb{N}$ with $n \leq m$, but then every vertex u on this cycle is in V_1, and if there were an edge emitted by $u = r(\alpha_i)$ that did not lie on a return path to u, we would have chosen $F(u) \neq \alpha_{i+1}$. Thus this cycle does not have an exit, which contradicts condition (L). Hence α is not eventually periodic.

It follows that in every case we can construct an infinite path $\alpha \in E^\infty$ starting at $r(e)$ which is not eventually periodic, and we define

$$x^m = x_1 \cdots x_N \underbrace{\mu \cdots \mu}_{m \text{ times}} x_{N+1} \cdots x_{N+j-1} e\alpha.$$

Then for any open neighbourhood U of $(x, 0, x)$, there exists $m \in \mathbb{N}$ such that $(x^m, 0, x^m) \in U$ and clearly x^m is not eventually periodic, thus $(x^m, 0, x^m)$ has trivial isotropy, so $(x, 0, x) \in \overline{\{u \in G^{(0)} : G_u^u = \{u\}\}}$. Hence $\mathcal{G}_E$ is essentially free.

$(\Rightarrow)$: Now suppose that E does not satisfy condition (L). Then there must be a cycle μ based at v, without an exit. Then the only path starting at v is $x = \mu\mu\cdots$, and hence $Z(\mu) = \{x\}$ and $G_{(x,0,x)}^{(x,0,x)} \cong \mathbb{Z}$.

Thus $Z(\mu) \cap \{u \in G^{(0)} : G_u^u = \{u\}\} = \emptyset$, so $\mathcal{G}_E$ is not essentially principal. ■

Definition 5.1.14. A subset $D \subseteq G^{(0)}$ is called *invariant* if for all $g \in G$, $d(g) \in D$ implies $r(g) \in D$.

Definition 5.1.15. A groupoid G is called *essentially principal* if the closed subgroupoid $G_D = \{g \in G \mid d(g) \in D\}$ is topologically principal for every closed invariant subset $D \subseteq G^{(0)}$.

Remark. If $D \subseteq G^{(0)}$ is invariant, then $G_D = G^D = \{g \in G \mid r(g) \in D\}$ and $G_D^{(0)} = D$.

Lemma 5.1.16. *Let E be a row-finite directed graph without sinks. Then E satisfies condition (K) if and only if $\mathcal{G}_E$ is essentially principal.*

Proof. To enhance readability we will for the duration of the proof write G instead of $\mathcal{G}_E$.
$(\Rightarrow)$: Suppose E satisfies condition (K) and let D be a closed invariant subset of $G^{(0)}$. We will prove that G_D is topologically principal. Suppose there exists $u = (x, 0, x) \in D$ such that u has non-trivial isotropy in G_D, which also implies $G_u^u \neq \{u\}$. Then there exists $k \in \mathbb{Z}$ such that $(x, k, x) \in G$, thus x is of the form $x = \alpha\mu\mu\cdots$, where $\alpha \in E^*$ and μ a cycle. First suppose μ is a simple cycle based at $r(\alpha)$. Then there exists another cycle ν based at $r(\alpha)$ in E and we define

$$y^{n,m} = \alpha \underbrace{\mu\cdots\mu}_{n \text{ times}} \nu\mu\nu\nu\mu\mu\cdots \underbrace{\nu\cdots\nu}_{m \text{ times}} \mu\mu\cdots .$$

Then, for all $n, m \in \mathbb{N}$, we have $y^{n,m} \in D$ and for all fixed $n \in \mathbb{N}$, $(y^{n,m})_{m\in\mathbb{N}}$ converges to y^n, where

$$y^n = \alpha \underbrace{\mu\cdots\mu}_{n \text{ times}} \nu\mu\nu\nu\mu\mu\cdots \underbrace{\nu\cdots\nu}_{m \text{ times}} \overbrace{\mu\cdots\mu}^{m \text{ times}} \cdots .$$

So $y^n \in D = G_D^{(0)}$ for all $n \in \mathbb{N}$ and most importantly y^n is not eventually periodic. Thus $y^n \in \{v \in G_D^{(0)} \mid v \text{ has trivial isotropy in } G_D\}$ and $(y^n)_{n\in\mathbb{N}}$ converges to x, so

$$u = (x, 0, x) \in \overline{\{v \in G_D^{(0)} \mid v \text{ has trivial isotropy in } G_D\}}.$$

Hence the set $\{v \in G_D^{(0)} \mid v \text{ has trivial isotropy in } G_D\}$ is dense in D, from which we conclude that G_D is topologically principal, so $G = \mathcal{G}_E$ is essentially principal.

$(\Leftarrow)$: Suppose G is essentially principal and let $\alpha \in E^*$ be a simple cycle based at $v \in E^0$. We will prove that there is another cycle based at v.
First define $x = \alpha\alpha\cdots$ and define the set

$$D(x) = \{(z, 0, z) \mid \exists k \in \mathbb{Z}\,;\ (z, k, x) \in G\} = r(d^{-1}((x, 0, x)))$$

as before. Then $D = \overline{D(x)}$ is a closed invariant subset of $G^{(0)}$, so G_D is topologically principal. Therefore, we can find $v^n \in D$ with trivial isotropy in G_D such that the v^n converge to $u = (x, 0, x)$. Since $v^n \in D = \overline{D(x)}$, we can find $w^{n,m} \in D(x)$ with $w^{n,m}$ converging to v^n. Then there exists $N \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $v^N \in Z(\underbrace{\alpha\cdots\alpha}_{k \text{ times}})$ but $v^N \notin Z(\underbrace{\alpha\cdots\alpha}_{k+1 \text{ times}})$, otherwise v^N would simply be x, which does not have trivial isotropy. Hence we can write $v^N = \underbrace{\alpha\cdots\alpha}_{k \text{ times}} y_1 y_2 \cdots$, where $y_i \in E^1$. Now define

$$j = \min_{i\in\mathbb{N}}\{\alpha_i \neq y_i\}.$$

Then $y_j \neq \alpha_i$ for all $i \leq |\alpha|$ as α is a simple cycle. Using the fact that $w^{N,m} \in D(x)$ converges to v^N, there exists $M \in \mathbb{N}$ such that

$$w^{N,M} \in Z(\underbrace{\alpha \cdots \alpha}_{k \text{ times}} y_1 \cdots y_j).$$

Therefore, $w^{N,M} = \underbrace{\alpha \cdots \alpha}_{k \text{ times}} y_1 \cdots y_j \cdots \in D(x)$, so by definition of $D(x)$ there exists $l \in \mathbb{N}$ such that $\sigma^l(w^{N,M}) = x$. However, $y_j \neq \alpha_j$ so there must be a different cycle μ based at v starting with $y_1 \cdots y_j$. Thus E satisfies condition (K). ∎

Now that we have seen what these properties mean for path groupoids, we will apply our knowledge of the ideal structure of groupoid C^*-algebras in general to the path groupoid. Let G be a locally compact Hausdorff étale groupoid. Then for each open invariant subset U of $G^{(0)}$, the set

$$C_c(G_U^U) = \{f \in C_c(G) : \operatorname{supp} f \subset G_U^U\}$$

is a self adjoint two-sided ideal in $C_c(G)$, and hence its closure, which we will denote by $I(U)$, is an ideal in $C^*(G)$.

Proposition 5.1.17 ([35, II Proposition 4.6])**.** *Let G be an étale essentially principal groupoid. Then the correspondence $U \mapsto I(U)$ is an order preserving bijection between the lattice of open invariant subsets of G^0 and the lattice of ideals of the reduced groupoid C^*-algebra $C_r^*(G)$.*

By Proposition 5.1.17 we have that $U \mapsto I(U)$ is an isomorphism of the lattice of open invariant subsets of $\mathcal{G}_E^{(0)}$ onto the lattice of closed ideals of $C_r^*(\mathcal{G}_E)$, whenever $\mathcal{G}_E$ is essentially principal. Recall that $C_r^*(\mathcal{G}_E) = C^*(\mathcal{G}_E)$, because $\mathcal{G}_E$ is amenable. Hence we see that for a row-finite graph E where (K) holds, the ideals of $C^*(\mathcal{G}_E)$ correspond to open invariant subsets of $\mathcal{G}_E^{(0)} \cong E^\infty$. We show that these sets relate to hereditary saturated subsets of E^0.

Definition 5.1.18. For each non-empty open subset U of E^∞, define

$$H(U) = \{v \in E^0 \;:\; \exists \alpha \in E^* \text{ such that } r(\alpha) = v \text{ and } Z(\alpha) \subseteq U\}.$$

For each subset H of E^0, define

$$U(H) = \{x \in E^\infty \;:\; r(x_n) \in H \text{ for some } n \in \mathbb{N}\}.$$

Lemma 5.1.19 ([23, Lemma 6.5])**.** *For any directed graph E, the map $H \mapsto U(H)$ is an isomorphism between the lattice of saturated hereditary subsets of E^0 and the lattice of open invariant subsets of E^∞, with inverse given by $U \mapsto H(U)$.*

Proof. We begin by proving that each subset $H(U)$ is hereditary and saturated. First of all, if $v \in H(U)$, then there exists a path $\alpha \in E^*$ with $r(\alpha) = v$ and $Z(\alpha) \subseteq U$. Thus $Z(\alpha\epsilon) \subseteq Z(\alpha) \subseteq U$ for all $\epsilon \in E^*$ with $s(\epsilon) = v$ and so $r(\epsilon) \in H(U)$. Hence for all $v \geq w$, we have $w \in H(U)$. Now let $v \in H(U)$ and let $s^{-1}(v)$ be non-empty with $r(e_i) \in H(U)$ for all $e_i \in s^{-1}(v)$. Then there exists $\alpha_i \in E^*$ with $r(\alpha_i) = r(e_i)$ and $Z(\alpha_i) \subseteq U$. Since U is invariant, for all $x \in E^\infty$ with $s(x) = r(e_i)$, we have $r((\alpha_i x, 1 - |\alpha_i|, e_i x)) \in U$ so

$d((\alpha_i x, 1-|\alpha_i|, e_i x)) \in U$. Thus for all $e_i \in s^{-1}(v)$, we have $Z(e_i) \subseteq U$, so every path starting at v is in U, hence by invariance we have $Z(\beta) \subseteq U$, whenever $r(\beta) = v$, and therefore $v \in H(U)$.

Now we prove that $U(H)$ is open and invariant for every saturated hereditary subset of E^0. First of all, $U(H)$ is open, because $x \in U(H)$ implies $r(x_n) \in H$, so $Z(x_1 x_2 \dots x_n) \subseteq U$. Secondly, if we have $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta) \in H$, then $Z(\alpha) \subseteq U$ implies $Z(\beta) \subseteq U$, whenever H is hereditary. Hence $U(H)$ is open and invariant. To show that $H \mapsto U(H)$ is an isomorphism, we need to prove that $U(H(U)) = U$ whenever U is open and invariant and $H(U(H)) = H$ whenever H is saturated and hereditary.

We trivially have $H \subset H(U(H))$, so the only thing we need to show is that for all $v \in H(U(H))$, we have $v \in H$. Thus let $v \in H(U(H))$, that is, there exists $\alpha \in E^*$ with $r(\alpha) = v$ and $Z(\alpha) \subseteq U(H)$. Thus for all $\alpha\gamma \in Z(\alpha)$ there exists some $n \in \mathbb{N}$, with $r((\alpha\gamma)_n) \in H$. If, for the sake of contradiction, $v \notin H$ we would have $\gamma \in E^\infty$ with $s(\gamma) = v$ and $r(\gamma_k) \notin H$ for all $k \in \mathbb{N}$, because H is saturated. But then $\alpha\gamma \in Z(\alpha) \setminus U(H)$, which is impossible. Thus we must have $v \in H$, which proves our claim that $H = H(U(H))$.

We also trivially have $U \subseteq U(H(U))$, so we just need to prove the inclusion $U(H(U)) \subseteq U$. Suppose $x \in U(H(U))$. Then we have $r(x_n) \in H(U)$ for some n. Thus there exists a finite path $\alpha \in E^*$ with $s(\alpha) = r(x_n)$ and $Z(\alpha) \subseteq U$. By invariance of U and by the fact that $(\alpha x_{n+1} \cdots, n - |\alpha|, x) \in \mathcal{G}_E$, we see that $x \in U$, and hence $U(H(U)) = U$.
It is easy to check that $H \mapsto H(U)$ preserves lattice operations, and hence $H \mapsto U(H)$ is a lattice isomorphism with inverse $U \mapsto H(U)$. ■

This gives us the result we were looking for and we can state a theorem analogous to Theorem 5.1.11:

Theorem 5.1.20 ([23, Theorem 6.6])**.** *Let E be a locally finite directed graph that satisfies condition (K). For $H \subseteq E^0$, let*

$$I(H) = \overline{\operatorname{span}}\{\mathbb{1}_{Z(\alpha,\beta)} \;:\; \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}.$$

Then:

1. *The map $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of ideals in $C^*(\mathcal{G}_E)$.*

2. *If H is saturated and hereditary, the quotient $C^*(\mathcal{G}_E)/I(H)$ is naturally isomorphic to $C^*(\mathcal{G}_{E\setminus H})$.*

3. *Let*
$$G = (H, \{e \mid s(e) \in H\}, r, s).$$
If H is saturated and hereditary, then $C^*(\mathcal{G}_G)$ is isomorphic to a full corner in the ideal $I(H)$.

The proof of this theorem can be found in [23]; it relies on techniques that can be found in Renault's book [35, Section II.4].

5.1.3 The ideal structure in the Cuntz–Pimsner models

In the Cuntz–Krieger and groupoid models, we can get explicit results. Not much is known about the structure of Cuntz–Pimsner algebras in general, however we do have a characterization of all gauge-invariant ideals.

Definition 5.1.21. Let X be a C^*-correspondence over $\mathcal{A}$.

- We say that an ideal I of $\mathcal{A}$ is X-*invariant* if $\phi(I)\mathsf{X} \subseteq \mathsf{X}I$.
- We say that an ideal I of $\mathcal{A}$ is X-*saturated* if for all $a \in \mathcal{A}$,

$$\phi(a)\mathsf{X} \subseteq \mathsf{X}I \Rightarrow a \in I.$$

The graph correspondence

We first look at this from the perspective of the graph correspondence $\mathsf{X}(E)$. We see that subsets of E^0 correspond to ideals in $C_0(E^0)$ through

$$H \longleftrightarrow \overline{\operatorname{span}}\{\delta_v : v \in H\},$$

and it is easy to see that this is a lattice isomorphism.

Lemma 5.1.22. *Let H be a subset of E^0, then:*
H is hereditary if and only if $\overline{\operatorname{span}}\{\delta_v : v \in H\}$ is $\mathsf{X}(E)$-invariant and H is saturated if and only if $\overline{\operatorname{span}}\{\delta_v : v \in H\}$ is $\mathsf{X}(E)$-saturated.

Proof. Suppose H is hereditary, so for all $v \in H$ and $w \in E^0$ we have that $v \geq w$ implies $\in H$. Defining $I_H = \overline{\operatorname{span}}\{\delta_v : v \in H\}$, we see that

$$\phi(I_H)\mathsf{X}(E) = \{\phi(\delta_v)(x) | v \in H, x \in \mathsf{X}(E)\}$$

and

$$\mathsf{X}(E)I_H = \{x \cdot \delta_v | v \in H, x \in \mathsf{X}(E)\}.$$

Let $\phi(\delta_v)(\delta_f) \in \phi(I_H)\mathsf{X}(E)$. We will show that $\phi(I_H)\mathsf{X}(E) = y \cdot \delta_w$ for some $y \in \mathsf{X}(E)$ and $w \in H$. Computing

$$\phi(\delta_v)(\delta_f)(e) = \delta_v(s(e))\delta_f(e) = \begin{cases} 1 & \text{if } s(f) = v \text{ and } e = f, \\ 0 & \text{otherwise,} \end{cases}$$

we see that

$$\phi(\delta_v)(\delta_f) = \begin{cases} \delta_f = \delta_f \delta_{r(f)} & \text{if } s(f) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Since H is hereditary and $v \in H$, we have that if $s(f) = v$, then $r(f) \in H$, so $\delta_f \delta_{r(f)} \in \mathsf{X}(E)I_H$. Thus for all $v \in H$ and $f \in E^1$ we have $\phi(\delta_v)(\delta_f) \in \mathsf{X}(E)I_H$. Thus I_H is $\mathsf{X}(E)$-invariant.

Now suppose I_H is $\mathsf{X}(E)$-invariant. Then for all $v \in H$ and $f \in E^1$, we have

$$\phi(\delta_v)(\delta_f)(e) = y \cdot \delta_w(e) = y(e)\delta_w(r(e)) = \begin{cases} y(e) & \text{if } r(e) = w, \\ 0 & \text{otherwise,} \end{cases}$$

for some $y \in \mathsf{X}(E)$ and $w \in H$. If $s(f) \neq v$, then we can chose $y = 0$, so suppose $s(f) = v$. Then we need $\delta_f = y \cdot \delta_w$ and $\delta_f(f) = 1$, thus we must have $r(f) = w$. Hence for all $f \in E^1$ if $s(f) \in H$, then $r(f) \in H$. Induction gives us that for all $v \in H$ and $w \in E^0$, $v \geq w$ implies $w \in H$. Thus H is hereditary if and only if I_H is $\mathsf{X}(E)$-invariant.

Suppose $\phi(\delta_v)(\delta_f) \in \mathsf{X}(E)I_H$ for all $f \in E^1$. Then using the formulae above, we see that this is equivalent to $r(f) \in H$ for all $f \in E^1$ with $s(f) = v$.

Therefore, if I_H is $\mathsf{X}(E)$-saturated, then for $v \in E^0$ we have, if $r(f) \in H$ for all $f \in E^1$ with $s(f) = v$, and then $\phi(\delta_v)(\delta_f) \in \mathsf{X}(E)I_H$ for all $f \in E^1$, so that $\delta_v \in I_H$, which means that $v \in H$. Thus H is saturated.

Now suppose H is saturated and that for $v \in E^0$ we have $\phi(\delta_v)(\delta_f) \in \mathsf{X}(E)I_H$ for all $f \in E^1$. Then $r(f) \in H$ for all $f \in E^1$ with $s(f) = v$, so $v \in H$, thus $\delta_v \in I_H$. Hence H is saturated if and only if I_H is $\mathsf{X}(E)$-saturated. ∎

This shows that the gauge-invariant ideals of $C^*(E)$ correspond to $\mathsf{X}(E)$-invariant and $\mathsf{X}(E)$-saturated ideals of $C_0(E^0)$. However, there is an even stronger result from [26, Theorem 6.4], which is an analogue of Theorem 5.1.11. It states that for all C^*-correspondences X over $\mathcal{A}$, there exists a lattice isomorphism from the lattice of X-invariant X-saturated ideals onto the lattice of gauge invariant ideals of $\mathcal{O}_{\mathsf{X}}$, given by

$$I \mapsto \mathcal{I}(I) = \overline{\operatorname{span}}\{S_{\xi_1} \cdots S_{\xi_n}\phi_+(a)S^*_{\eta_1} \cdots S^*_{\eta_m} \mid a \in I, \xi_i, \eta_j \in \mathsf{X}, m, n \geq 0\}.$$

Furthermore, $\mathcal{O}_{\mathsf{X}}/\mathcal{I}(I) \cong \mathcal{O}_{\mathsf{X}/\mathsf{X}I}$, and $\mathcal{O}_{\mathsf{X}I}$ is isomorphic to a full corner in the ideal $\mathcal{I}(I)$, where $\mathsf{X}I$ is a C^*-correspondence over $\mathcal{A}$ and $\mathsf{X}/\mathsf{X}I$ is a C^*-correspondence over $\mathcal{A}/I$, as explained in [26, Chapter 6].

The shift correspondence

In the case of the shift correspondence X_σ we also have a complete description of the ideals of $C_0(E^\infty)$. As for all locally compact Hausdorff spaces X, we know that there is a bijection from the open subsets of X to the ideals of $C_0(X)$.

Similarly, for all open sets $U \subseteq E^\infty$, the set

$$I_U = \{f \in C_0(E^\infty) : f(x) = 0 \text{ if } x \notin U\}$$

is an ideal of $C_0(E^\infty)$.

Lemma 5.1.23. *Let $U \subseteq E^\infty$ be an open subset and I_U as above. Then I_U is X_σ-invariant and X_σ-saturated if and only if U is invariant.*

Proof. First, we can write any open subset as

$$U = \bigsqcup_{i \in I} Z(\alpha_i).$$

So $\mathbb{1}_{Z(\alpha_i)} \in I_U$ and every $f \in I_U$ is generated by these characteristic functions.

($\Rightarrow$): Using this characterization, we see that I_U is X_σ-invariant implies that for all $\epsilon \in E^*$ with $s(\epsilon) = r(\alpha_i)$, we have $\mathbb{1}_{Z(\alpha_i\epsilon)} = \phi(\mathbb{1}_{Z(\alpha_i)})\mathbb{1}_{Z(\alpha_i\epsilon)} \in \mathsf{X}_\sigma I_U$. That is,

$$\mathbb{1}_{Z(\alpha_i\epsilon)}(x) = \xi(x)\sum_{j \in J}\lambda_j \mathbb{1}_{Z(\alpha_j)}(\sigma(x)) = \xi(x)\sum_{j \in J}\lambda_j \mathbb{1}_{Z(\alpha_j)}(\sigma(x))$$

for some $J \subset I$. Thus for all $x \in Z(\alpha_i \epsilon)$ we must have $\sigma(x) \in Z(\alpha_j)$ for some $j \in I$. Hence for all $x \in U$, we have $\sigma(x) \in U$.

Now suppose that I_U is X_σ-saturated. Then for all $\beta \in E^*$, if for all $f \in \mathsf{X}_\sigma$ there exists $g \in \mathsf{X}_\sigma$ such that $\mathbb{1}_{Z(\beta)}(x)f(x) = g(x)\mathbb{1}_{Z(\alpha_i)}(\sigma(x))$, then $\mathbb{1}_{Z(\beta)} \in I_U$.

Take $i \in I$ and $e \in E^1$ such that $r(e) = s(\alpha_i)$, and define $\beta = e\alpha_i$. Then clearly for all $f \in X_\sigma$,

$$\mathbb{1}_{Z(\beta)}(x)f(x) = g(x)\mathbb{1}_{Z(\alpha_i)}(\sigma(x)),$$

where $g = \phi(\mathbb{1}_{Z(\beta)})(f)$. Thus $\mathbb{1}_{Z(\beta)} \in I_U$, so $Z(\beta) \subseteq U$. Hence for all $x \in U$ and $y \in E^\infty$ with $\sigma(y) = x$, we have $y \in U$. This together with the fact that $\sigma(x) \in U$ for all $x \in U$, shows that U is invariant because, for all $(x, k, y) \in \mathcal{G}_E$, there exists $N, k \in \mathbb{N}$ such that $\sigma^{N+k}(x) = \sigma^N(y)$.

Therefore, if $x \in U$, then by induction $\sigma^{N+k}(x) = \sigma^N(y) \in U$ and again by induction $y \in U$.

($\Leftarrow$): Let U be invariant. Then for all $f = \sum_{j \in J} \lambda_j \mathbb{1}_{Z(\alpha_j)} \in I_U$ and $g \in \mathsf{X}_\sigma$,

$$(\phi(f)g)(x) = (\phi(f)(g))(x) \sum_{j \in J} \lambda_j \mathbb{1}_{Z(\sigma(\alpha_j))}(\sigma(x)) \in \mathsf{X}_\sigma I_U.$$

Hence I_U is X_σ-invariant. Furthermore, suppose $\beta \in E^*$ satisfies that for all $f \in \mathsf{X}_\sigma$ there exists $g \in \mathsf{X}_\sigma$ with

$$\mathbb{1}_{Z(\beta)}(x)f(x) = g(x) \sum_{j \in J} \lambda_j \mathbb{1}_{Z(\sigma(\alpha_j))}(\sigma(x)).$$

Then

$$\mathbb{1}_{Z(\beta)}(x) = g(x) \sum_{j \in J} \lambda_j \mathbb{1}_{Z(\sigma(\alpha_j))}(\sigma(x))$$

for some $g \in \mathsf{X}_\sigma$ and $J \subseteq I$. Thus for all $x \in Z(\beta)$ there exists an $i \in J$ with $\sigma(x) \in Z(\alpha_i) \subseteq U$, so $Z(\beta) \subseteq U$ by invariance. Hence $\mathbb{1}_{Z(\beta)} \in I_U$, which means that I_U is X_σ-saturated. Thus I_U is X_σ-invariant and X_σ-saturated if and only if U is invariant. ■

5.2 Simplicity

Knowing what the ideal structure looks like, the next logical step would be to look at simplicity.

Definition 5.2.1. A C^*-algebra $\mathcal{A}$ is called *simple* if $\mathcal{A}$ and 0 are its only closed ideals.

In the case of graph C^*-algebras, a sufficient and necessary condition on the graph E for $C^*(E)$ to be simple is known. As with the ideal structure, we will first look at the Cuntz–Krieger model and then translate the condition to the other models, at least to some extent.

5.2.1 Simplicity in the Cuntz–Krieger model

We have already seen one of the necessary conditions for simplicity, namely condition (L). We will show that condition (L), together with the following property, is not only necessary but also sufficient.

Definition 5.2.2. Let $E^{\leq\infty} = E^{\infty} \cup \{\alpha \in E^* \mid r(\alpha) \text{ is a sink}\}$. We say that E is *cofinal* if, for all $\mu \in E^{\leq\infty}$ and $v \in E^0$, there exists $\alpha \in E^*$ such that $s(\alpha) = v$ and $r(\alpha) = r(\mu_n)$ for some $n \in \mathbb{N}$ with $n \leq |\mu|$. This means that in a cofinal graph we can reach any path in $E^{\leq\infty}$ from any given vertex.

Theorem 5.2.3 ([32, Proposition 4.2])**.** *Suppose E is a row-finite graph. Then $C^*(E)$ is simple if and only if E is cofinal and satisfies condition (L).*

Proof. ($\Leftarrow$): Recall that every closed ideal in a C^*-algebra is the kernel of a representation. This means that we only have to prove that every non-zero representation $\pi_{S,P}$ of $C^*(E)$ is faithful. So suppose that $\{S, P\}$ is a Cuntz–Krieger E-family such that $\pi_{S,P} \neq 0$. Then there exists a vertex $v \in E^0$ such that $P_v \neq 0$, otherwise we would have $P_v = 0$ for all $v \in E^0$, which together with the first Cuntz–Krieger relation $P_{r(e)} = S_e^* S_e$, implies $S_e = 0$ for all $e \in E^1$. This implies $\pi_{S,P} = 0$ contradicting our assumption.

Thus let $v \in E^0$ be a vertex such that $P_v \neq 0$. If v is not a sink, then the second Cuntz–Krieger relation implies that there exists an $e \in E^1$ with $s(e) = v$ and $S_e S_e^* \neq 0$. But that implies that $P_{r(e)} = S_e^* S_e \neq 0$.

If $r(e)$ is not a sink, we can repeat this argument with $r(e)$ instead of v. If we continue this process we either obtain an infinite path or a path that ends in a sink. Call this path μ. Then $\mu \in E^{\leq\infty}$, $s(\mu) = v$ and $P_u \neq 0$ for all vertices u on μ.

Now consider an arbitrary $w \in E^0$. By cofinality there exists $\alpha \in E^*$ such that $s(\alpha) = w$ and $r(\alpha) = \mu_n$ for some $n \in \mathbb{N}$ with $n \leq |\mu|$. Then from the calculations in Section 2.2 we get $0 \neq P_{\mu_n} = P_{r(\alpha)} = S_\alpha^* S_\alpha$ and thus $P_w = P_{s(\alpha)} S_\alpha S_\alpha^* = S_\alpha S_\alpha^* \neq 0$, which implies $P_w \neq 0$. Thus $P_w \neq 0$ for all $w \in E^0$ and since condition (L) holds by assumption, we can apply [32, Theorem 2.4] which implies that $\pi_{S,P}$ is faithful. Thus $C^*(E)$ is simple.

($\Rightarrow$): Suppose $C^*(E)$ is simple. For $\lambda \in E^{\leq\infty}$, let

$$H_\lambda = \{w \in E^0 \mid w \not\geq r(\lambda_n) \text{ for all } n \leq |\lambda|\}.$$

Then H_λ is clearly hereditary and saturated, and $r(\lambda_n) \notin H_\lambda$, so $H_\lambda \subsetneq E^0$.

Thus if $H_\lambda \neq \emptyset$, then I_{H_λ} would be a proper ideal of $C^*(E)$ by Theorem 5.1.11, but $C^*(E)$ is simple, so $H_\lambda = \emptyset$. Thus for all $\lambda \in E^{\leq\infty}$ and $v \in E^0$ we have $v \notin H_\lambda$, so $v \geq r(\lambda_n)$ for some $n \leq |\lambda|$. Hence E is cofinal.

Now suppose α is a loop with no exit. Then $H = \{s(\alpha_i) \mid 1 \leq i \leq |\alpha|\}$ is a non-empty hereditary set, so we must have $\overline{H} = E^0$, such that $I_{\overline{H}} = C^*(E)$.

Therefore, by Theorem 5.1.11, we have $C^*(G) \cong q_H I_{\overline{H}} q_H = q_H C^*(E) q_H$, where $G = (H, \{e \mid s(e) \in H\}, r, s)$, which is just the simple loop α. Hence by Example 2.2.17 we have $C^*(G) \cong C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$.

Now let $J \subsetneq C^*(G)$ be the proper ideal of functions that vanish at 1. Since by Theorem 5.1.11, $C^*(G)$ is isomorphic to a subalgebra of $C^*(E)$, we also have that $C^*(E)JC^*(E) = \overline{\text{span}}\{ajb : a, b \in C^*(E), j \in J\}$ is a non-zero ideal in $C^*(E)$. But $C^*(E)$ is simple, so we must have $C^*(E)JC^*(E) = C^*(E)$. Thus $q_H C^*(E) q_H = q_H C^*(E)JC^*(E) q_H = q_H C^*(E) q_H J q_H C^*(E) q_H = J$, so $J = C^*(G)$, but this contradicts the fact that J is proper. Thus every loop must have an exit. Hence E is cofinal and satisfies condition (L). ■

Remark. Recall that if E satisfies condition (K), then there exists an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of all ideals of $C^*(E)$. We could have used this to prove the ($\Leftarrow$) direction of Theorem 5.2.3, as cofinality together with condition (L) implies condition (K) and cofinality of E implies that every saturated hereditary subset H of E^0 is either empty or all of E^0, so the only non-zero ideal in $C^*(E)$ would be $C^*(E)$ itself, which implies the simplicity of $C^*(E)$.

5.2.2 Simplicity in the groupoid model

We can check now what being cofinal and satisfying condition (L) means for our path groupoid. As we will see, this is in accordance with what is already known about simplicity for groupoid C^*-algebras.

Definition 5.2.4. A groupoid G is called *minimal* if there are no proper non-trivial closed invariant subsets in $G^{(0)}$.

Lemma 5.2.5. *Let E be a row-finite directed graph without sinks. Then E is cofinal if and only if $\mathcal{G}_E$ is minimal.*

Proof. ($\Rightarrow$) : Let E be cofinal and suppose D is a non-empty closed invariant subset of $\mathcal{G}_E^{(0)}$. Then there exists an infinite path x such that $(x, 0, x) \in D$. Let $(y, 0, y) \in \mathcal{G}_E^{(0)}$ arbitrarily. We will prove that $(y, 0, y) \in D$. Since E is cofinal, there exists $\alpha^1 \in E^*$ such that $s(\alpha^1) = r(y_1)$ and $r(\alpha^1) = r(x_m)$ for some $m \in \mathbb{N}$. Denote $f(1) = m$ and define

$$z^1 = y_1 \alpha x_{f(1)+1} \cdots .$$

Now recursively define, for $n \in \mathbb{N}$,

$$z^n = y_1 \cdots y_n \alpha^n x_{f(n)+1} \cdots ,$$

where $\alpha^n \in E^*$ such that $s(\alpha^n) = r(y_n)$ and $r(\alpha^n) = r(x_{f(n)})$ for some $f(n) \in \mathbb{N}$.

Then $(z^n, f(n) - n - |\alpha^n|, x) \in \mathcal{G}_E$, thus $(z^n, 0, z^n) \in D$ and $z^n, y \in Z(y_1 \cdots y_n)$ for all $n \in \mathbb{N}$. Therefore, $(z^n)_{n \in \mathbb{N}} \in D$ converges to y and hence $(y, 0, y) \in D$. Thus $D = \mathcal{G}_E^{(0)}$ and $\mathcal{G}_E$ is minimal.
($\Leftarrow$) : Suppose E is not cofinal. Then there exists an infinite path x and a vertex $v \in E^0$ such that there is no path from v to some point of x. Define

$$D(x) = \{(z, 0, z) \mid \exists k \in \mathbb{Z} \,;\, (z, k, x) \in \mathcal{G}_E\} = r(d^{-1}((x, 0, x))).$$

We will prove that $\overline{D(x)}$ is invariant. Suppose $(u, l, w) \in \mathcal{G}_E$ and $(w, 0, w) \in \overline{D(x)}$, then either there exists $k \in \mathbb{N}$ such that $(w, k, x) \in \mathcal{G}_E$, in which case $(u, l, w)(w, k, x) = (u, l + k, x) \in \mathcal{G}_E$, and hence $(u, 0, u) \in D(x) \subseteq \overline{D(x)}$, or there exists a sequence $(z^n)_{n \in \mathbb{N}}$ in $D(x)$ that converges to w.

Since $(u, l, w) \in \mathcal{G}_E$, there exist $\alpha, \beta \in E^*$ and $\gamma \in E^\infty$ such that $\alpha\gamma = u$ and $\beta\gamma = w$. Thus there exists $M \in \mathbb{N}$, such that $z^n \in Z(\beta)$ for all $n \geq M$. Therefore, we can write $z^n = \beta c^n$ for all $n \geq M$, and then define $f(z^n) = \alpha c^n$ for all $n \geq M$. Then $f(z^n) \in D(x) \cap Z(\alpha)$ for all $n \geq M$ and $(f(z^n))_{n \geq M}$ converges to $u = \alpha\gamma$ instead of $w = \beta\gamma$. Thus $(u, 0, u) \in \overline{D(x)}$, hence $\overline{D(x)}$ is a closed invariant subset of $\mathcal{G}_E^{(0)}$.

Now we prove that $\overline{D(x)}$ is non-trivial. We know $\overline{D(x)}$ is non-empty because $x \in D(x)$, so we need to prove that there exists an infinite path $y \in E^\infty$ such that $(y, 0, y) \notin \overline{D(x)}$. Let y be an infinite path starting in v. If $(y, 0, y) \in \overline{D(x)}$, then either $(y, k, x) \in \mathcal{G}_E$ for some $k \in \mathbb{Z}$, in which case there exists a path from v to some vertex on x, which contradicts our assumption, or there exists a $z \in Z(y_1 \cdots y_m) \cap D(x)$ for some $m \in \mathbb{N}$. But then z is a path starting in v that connects to x, which contradicts our assumption once again. Thus $(y, 0, y) \notin \overline{D(x)}$, so $\overline{D(x)} \neq \mathcal{G}_E^{(0)}$ is a proper non-trivial closed invariant subset of $\mathcal{G}_E^{(0)}$, hence $\mathcal{G}_E$ is not minimal. ∎

From Theorem 5.2.3 we know that $C^*(E)$ is simple if and only if E is cofinal and satisfies condition (L). This, together with Lemma 5.1.13 and Lemma 5.2.5, shows that $C^*(\mathcal{G}_E)$ is simple if and only if $\mathcal{G}_E$ is essentially free and minimal. The following result holds for groupoids in general.

Theorem 5.2.6 ([8, Theorem 1.1]). *Let $\mathcal{G}$ be a second countable, locally compact, Hausdorff, étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if all of the following conditions are satisfied:*

1. *$C^*(\mathcal{G}) = C_r^*(\mathcal{G})$;*
2. *$\mathcal{G}$ is topologically principal;*
3. *$\mathcal{G}$ is minimal.*

We see that this coincides with our result for the path groupoid $\mathcal{G}_E$, because $\mathcal{G}_E$ is a second countable, locally compact, Hausdorff, étale groupoid that satisfies $C^*(\mathcal{G}_E) = C_r^*(\mathcal{G}_E)$ for all row-finite directed graphs E.

5.2.3 Simplicity in the Cuntz–Pimsner models

For the sake of simplicity, we only focus here on the unital case. Note that recently, similar results have been obtained in the nonunital case in [11] and [24].

Theorem 5.2.7 ([39, Theorem 3.9]). *Let (X, ϕ) be a full C^*-correspondence over a unital C^*-algebra $\mathcal{A}$, with ϕ faithful. Then $\mathcal{O}_\mathsf{X}$ is simple if and only if X is minimal and nonperiodic.*

Definition 5.2.8. Let (X, ϕ) be a full C^*-correspondence over a C^*-algebra $\mathcal{A}$. Then

- X is called *minimal* if $\mathcal{A}$ contains no proper ideal J with $\langle \mathsf{X}, J\mathsf{X}\rangle_\mathcal{A} \subseteq J$.
- X is called *nonperiodic* if $\mathsf{X}^{\otimes k} \cong \mathcal{A}$ implies $k = 0$.

Recall from Section 4.2 that X_σ and $\mathsf{X}(E)$ are full and faithful if and only if E has no sinks and sources. Furthermore, $C_0(E^0)$ and $C_0(E^\infty)$ are unital if and only if E^0 and E^∞ are compact. This clearly implies that E^0 must be finite.

The graph correspondence

Lemma 5.2.9. *Let E be a finite directed graph without sinks and sources and let $\mathsf{X}(E)$ be the graph correspondence. Then $\mathsf{X}(E)$ is minimal if and only if E is cofinal.*

Proof. Let $J = \overline{\operatorname{span}}\{\delta_v : v \in H\}$ with $H \subseteq E^0$ an arbitrary subset. Then

$$\langle \delta_e, \phi(\delta_v)\delta_f\rangle(w) = \sum_{\{g \in E^1 : r(g)=w\}} \delta_e(g)\delta_v(s(g))\delta_f(g) = 1$$

if and only if $e = f$, $r(e) = w$ and $s(e) = v$. Therefore,

$$\langle \delta_e, \phi(\delta_{s(e)})\delta_e\rangle(w) = \delta_{r(e)}(w).$$

Thus for all $e \in s^{-1}(H)$, we have $\delta_{r(e)} \in \langle \mathsf{X}, J\mathsf{X}\rangle_{\mathcal{A}}$.

Now suppose that there exists a proper ideal J of $C_0(E^0)$ with $\langle \mathsf{X}, J\mathsf{X}\rangle_{\mathcal{A}} \subseteq J$. Then by induction for all $w \in E^0$ such that there exists $v \in H$ with $v \geq w$, we have $\delta_w \in J$.

Suppose $v \notin H$. Since v is not a source, we can find an edge $e \in E^1$ with $r(e) = v$. We must have $s(e) \notin H$, for otherwise $v \in H$ and since $E^0 \setminus H$ is finite, we get a cycle α in $E^0 \setminus H$. Then for all $w \in H$ there is no path from w to α, otherwise $s(\alpha) \in H$, so E is not cofinal. Now suppose E is not cofinal, then there exists a path $\mu \in E^\infty$ and a vertex v, such that v is not connected to μ. Let

$$H = \{w \in E^0 : v \geq w\}.$$

Then H is a proper subset of E^0 as $r(\mu_i) \notin H$ for all $i \in \mathbb{N}$ and $v \in H$. Thus $J = \overline{\operatorname{span}}\{\delta_v : v \in H\}$ is a proper ideal and by the first part of this proof we have $\langle \mathsf{X}, J\mathsf{X}\rangle_{\mathcal{A}} \subseteq J$, so $\mathsf{X}(E)$ is not minimal. ■

Recall that C_k is the graph that consists of a simple loop of k edges. Define the direct sum of two graph to be

$$(E^0, E^1, r_E, s_E) \oplus (F^0, F^1, r_F, s_F) = (E^0 \sqcup F^0, E^1 \sqcup F^1, r, s),$$

where $r, s : E^1 \sqcup F^1 \to E^0 \sqcup F^0$ are defined by $r(e) = r_E(e)$ and $s(e) = S_E(e)$ for all $e \in E^1$ and $r(e) = r_F(e)$ and $s(e) = S_F(e)$ for all $e \in F^1$.

Lemma 5.2.10. *Let E be a finite directed graph without sinks and sources. Then $\mathsf{X}(E)$ is periodic if and only if $E \cong \bigoplus_{i=1}^m C_{k_i}$ for some $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$.*

Proof. Suppose $\mathsf{X}(E)^{\otimes k}$ is isomorphic to $\mathcal{A}$ as a C^*-correspondence. This means that there exists a unitary $U : \mathsf{X}(E)^{\otimes k} \to \mathcal{A}$ that intertwines the left actions and preserves the inner product. Then for all $\mu \in E^k$, we have

$$\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k} = S_\mu(\mathrm{Id}_{\mathcal{A}}) \neq 0,$$

so

$$\begin{aligned} U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) &= \delta_{s(\mu)} U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) \\ &= U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) \cdot \delta_{r(\mu)}, \end{aligned}$$

and $U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) \in \mathcal{A}$ is of the form

$$U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) = \sum_{v \in E^0} \lambda_v \delta_v,$$

so that we must have $U(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_k}) = \lambda_{r(\mu)}\delta_{r(\mu)} = \lambda_{s(\mu)}\delta_{s(\mu)}$. Hence all paths $\mu \in E^k$ are cycles of length k. Furthermore, if we would have $v \in E^0$ and $e, f \in E^1$ with $r(e) = r(f) = v$, then for all $\mu \in E^{k-1}$ with $s(\mu) = v$ we would obtain

$$U(\delta_e \otimes \delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_{k-1}}) = \lambda' \delta_{r(\mu)} = \lambda' \delta_{s(e)}$$

and

$$U(\delta_f \otimes \delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_{k-1}}) = \lambda \delta_{r(\mu)} = \lambda \delta_{s(f)}.$$

But U is bijective and linear, and therefore

$$\begin{aligned} 0 &= \lambda' U(\delta_f \otimes \delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_{k-1}}) - \lambda U(\delta_e \otimes \delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_{k-1}}) \\ &= U((\lambda' \delta_f - \lambda \delta_e) \otimes \delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_{k-1}}), \end{aligned}$$

so we must have $(\lambda'\delta_f - \lambda\delta_e) = 0$, which implies $e = f$.

In the same way, for all $e, f \in E^1$ with $s(e) = s(f) = v$, we find $e = f$. Thus for every vertex, there is only one incoming and one outgoing edge. Using induction we obtain $E \cong \bigoplus_{i=1}^m C_{k_i}$ and since every path of length k must be a cycle, k_i must divide k. Next, suppose that $E \cong \bigoplus_{i=1}^m C_{k_i}$ for some constants $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$ and define $k = \prod_{i=1}^m k_i$. For all elementary basis tensors $\delta_{e_1} \otimes \cdots \otimes \delta_{e_k} \in \mathsf{X}(E)^{\otimes k}$, we know that $\delta_{e_1} \otimes \cdots \otimes \delta_{e_k} = 0$ unless $(e_1, \ldots, e_k) \in E^k$. Therefore, for all $(e_1, \ldots, e_k) \in E^k$, we can define $U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k}) = \delta_{r(e_k)}$.

Since k_i divides k, we know that $s(e_1) = r(e_k)$, so clearly for all $v \in E^0$,

$$\delta_v U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k}) = U(\phi(\delta_v)\delta_{e_1} \otimes \cdots \otimes \delta_{e_k})$$

and

$$U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k} \cdot \delta_v) = U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k}) \cdot \delta_v.$$

Furthermore,

$$\langle U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k}), U(\delta_{e_1} \otimes \cdots \otimes \delta_{e_k})\rangle = \langle \delta_{e_1} \otimes \cdots \otimes \delta_{e_k}, \delta_{e_1} \otimes \cdots \otimes \delta_{e_k}\rangle.$$

Moreover, U is bijective (hence unitary), because for all $v \in E^0$ there exists a unique path $\mu \in E^k$ with $s(\mu) = v$, which μ must be a cycle of length k. Therefore, $\mathsf{X}(E)$ is periodic. ■

It is quite easy to see that if E is cofinal, then $E \ncong C_n$ for some $n \in \mathbb{N}$ if and only if every loop has an exit. This implies the following corollary.

Corollary. *Let E be a finite directed graph without sinks and sources and let the graph correspondence $\mathsf{X}(E)$ be minimal. Then $\mathsf{X}(E)$ is nonperiodic if and only if E satisfies condition (L).*

This also follows from the fact that $\mathsf{X}(E)$ is minimal and nonperiodic if and only if $C^*(E) \cong \mathcal{O}_{\mathsf{X}(E)}$ is simple, which is equivalent to E being cofinal and satisfying condition (L).

The shift correspondence

In the case of the shift correspondence X_σ, in the unital case $C_0(E^\infty) = C(E^\infty)$. Recall that all ideals of $C_0(E^\infty)$ correspond to open subsets $U \subseteq E^\infty$, by the map sending U to the ideal $I_U = \{f \in C_0(E^\infty) : f(x) = 0 \text{ if } x \notin U\}$.

Lemma 5.2.11. *Let E be a finite directed graph without sinks and sources and let X_σ be the shift correspondence. Then X_σ is minimal if and only if E is cofinal.*

Proof. Let I_U be an ideal with $U \subseteq E^\infty$ an arbitrary open subset. Suppose that $\langle \mathsf{X}, I_U\mathsf{X}\rangle_{\mathcal{A}} \subseteq I_U$. Then for all $Z(\alpha) \subset U$ and $\beta, \gamma \in E^*$ we must have $\left\langle \mathbb{1}_{Z(\beta)}, \phi(\mathbb{1}_{Z(\alpha)})(\mathbb{1}_{Z(\gamma)})\right\rangle(x) \in I_U$, where

$$\begin{aligned}\left\langle \mathbb{1}_{Z(\beta)}, \phi\left(\mathbb{1}_{Z(\alpha)}\right)\left(\mathbb{1}_{Z(\gamma)}\right)\right\rangle(x) &= \sum_{\sigma(y)=x} \mathbb{1}_{Z(\beta)}(y)\left(\phi\left(\mathbb{1}_{Z(\alpha)}\right)\left(\mathbb{1}_{Z(\gamma)}\right)\right)(y) \\ &= \sum_{\sigma(y)=x} \mathbb{1}_{Z(\beta)}(y)\mathbb{1}_{Z(\alpha)}(y)\mathbb{1}_{Z(\gamma)}(y).\end{aligned}$$

Therefore, $\langle \mathbb{1}_{Z(\beta)}, \mathbb{1}_{Z(\alpha)}\mathbb{1}_{Z(\gamma)}\rangle(x) \neq 0$ if and only if $x \in Z(\sigma(\beta)) \cap Z(\sigma(\alpha))$ and $x \in Z(\gamma)$. Then, for all $\epsilon \in E^*$ with $s(\epsilon) = r(\alpha)$, we can take $\beta = \alpha\epsilon$ and $\gamma = \sigma(\beta)$. This implies that

$$\left\langle \mathbb{1}_{Z(\beta)}, \mathbb{1}_{Z(\alpha)}\mathbb{1}_{Z(\gamma)}\right\rangle(x) = \mathbb{1}_{Z(\sigma(\alpha\epsilon))}(x)\mathbb{1}_Z(\gamma)(x) = \mathbb{1}_{Z(\sigma(\alpha\epsilon))}(x) \in I_U,$$

so for all $Z(\alpha) \subseteq U$, we have $Z(\sigma(\alpha\epsilon)) \subseteq U$.

It is clear that for all open invariant subsets $U \subseteq E^\infty$, one has $\langle \mathsf{X}, I_U\mathsf{X}\rangle_{\mathcal{A}} \subseteq I_U$.

($\Rightarrow$): Suppose X_σ is minimal. Then there is no proper ideal J with $\langle \mathsf{X}, J\mathsf{X}\rangle_{\mathcal{A}} \subseteq J$, so there cannot be a proper open invariant subset $U \subseteq E^\infty$, hence there cannot be a proper closed invariant subset of E^∞ either. Hence $\mathcal{G}_E$ is minimal, and so E is cofinal.

($\Leftarrow$): Suppose X_σ is not minimal, that is, suppose there exists a proper open subset $U \subset E^\infty$ such that $Z(\alpha) \subseteq U$ implies $Z(\sigma(\alpha)) \subseteq U$. Then there exists an infinite path $x \in E^\infty$, with $x \notin U$, thus $Z(x_1) \nsubseteq U$, which implies $Z(ex_1) \nsubseteq U$ for all $e \in E^1$ with $r(e) = s(x)$.

Knowing that E is finite and has no sources, there is a cycle $\mu \in E^*$ and a finite path $\nu \in E^*$ such that $s(\nu) = r(\mu)$, $r(\nu) = s(x)$ and $Z(\mu\nu x_1) \nsubseteq U$. Hence $Z(\mu) \nsubseteq U$. Now let $\alpha \in E^*$ such that $Z(\alpha) \subseteq U$ and let $y = \mu\mu\cdots \in E^\infty$. If there were an $n \in \mathbb{N}$ such that $r(\alpha) \geq r(y_n)$, that is, if there exists a finite path $\epsilon \in E^*$ with $r(\alpha) = s(\epsilon)$ and $r(\epsilon) = r(y_n)$, then using the fact that U is open, we have $Z(\alpha\epsilon y_{n+1}\cdots y_{k|\mu|}) \subseteq U$ for all $k \in \mathbb{N}$. Choosing k big enough, we can then apply induction on the fact that $Z(\beta) \subseteq U$ implies $Z(\sigma(\beta)) \subseteq U$. This gives us

$$Z(\mu) = Z\left(\mu_1\cdots\mu_{|\mu|}\right) = Z\left(y_{(k-1)|\mu|+1}\cdots y_{k|\mu|}\right) \subseteq U.$$

However, this contradicts our earlier result that $Z(\mu) \nsubseteq U$. Hence for all $n \in \mathbb{N}$ we cannot have $r(\alpha) \geq r(y_n)$, and hence E is not cofinal. ∎

Remark. We can also prove this in another way, by looking at two cases separately.

Suppose that E is a finite directed graph with n vertices. Consider the case $E \cong C_n$ first, where E is cofinal. Since the infinite paths in C_n are completely determined by their source, we can say that $E^\infty \cong \{1, \ldots, n\}$. The statement $Z(\alpha) \subseteq U$ implies $Z(\sigma(\alpha)) \subseteq U$ then corresponds to $i \in U$ implies $i \mod n \in U$. Therefore, it is clear that there are no proper subsets U with this property, so X_σ is minimal.

In case that $E \not\cong C_n$, one can show that cofinality implies condition (L). A loop in E without an exit edge cannot be C_n, so there must be another vertex v not in the loop. Using the fact that v is not a source and that E is finite, one obtains $w \geq v$ for some w on the loop, or there is another loop, but by cofinality w must then connect to that loop, which means that the first loop has an exit.

Thus in this case cofinality of E implies condition (L), which in turn implies simplicity of $C^*(E) \cong \mathcal{O}_{\mathsf{X}_\sigma}$, which finally implies minimality of X_σ by Theorem 5.2.7.

Lemma 5.2.12. *Let E be a finite directed graph without sinks and sources. Then X_σ is periodic if and only if $E \cong \bigoplus_{i=1}^m C_{k_i}$ for some $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$.*

Proof. ($\Rightarrow$): Suppose $\mathsf{X}_\sigma^{\otimes k}$ is isomorphic to $\mathcal{A}$ as a C^*-correspondence. This means that there exists a unitary $U : \mathsf{X}_\sigma^{\otimes k} \to \mathcal{A}$ that intertwines the left actions and preserves the inner product. Then for all $\mu \in E^k$, we have that

$$\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)} = S_\mu(\mathrm{Id}_{\mathcal{A}}) \neq 0,$$

so

$$\begin{aligned} U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) &= \mathbb{1}_{Z(s(\mu))} U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \\ &= U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \cdot \mathbb{1}_{Z(r(\mu))} \\ &\neq 0. \end{aligned}$$

Thus $\mathrm{supp}(U) \subseteq Z(s(\mu)) \cap Z(r(\mu))$ and $U \neq 0$, so we must have $r(\mu) = s(\mu)$. It follows that for all $\mu \in E^k$, μ is a cycle. Furthermore, we also have

$$\begin{aligned} U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) &= U\left(\phi(\mathbb{1}_{Z(\mu)})\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \\ &= \mathbb{1}_{Z(\mu)} U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \\ &= U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \mathbb{1}_{Z(\mu)} \\ &= U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)} \cdot \mathbb{1}_{Z(\mu)}\right) \\ &\vdots \\ &= U\left(\phi(\mathbb{1}_{Z(\mu\mu)})\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right) \\ &= \mathbb{1}_{Z(\mu\mu)} U\left(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right). \end{aligned}$$

Hence by induction, we find for all $n \in \mathbb{N}$,

$$U(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}) = \mathbb{1}_{Z(\mu^n)} U(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}).$$

Therefore, $\mathrm{supp}(U) \subseteq Z(\mu^n)$ for all $n \in \mathbb{N}$, and so $\mathrm{supp}(U) = \{\mu^\infty\}$.

Now writing $\chi = \mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}$ for notational purposes, we can compute

$$\begin{aligned}\overline{U(\chi)}U(\chi) &= \left\langle U(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}), U(\mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)})\right\rangle \\ &= \left\langle \mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}, \mathbb{1}_{Z(\mu_1)} \otimes \cdots \otimes \mathbb{1}_{Z(\mu_k)}\right\rangle = \mathbb{1}_{Z(r(\mu))},\end{aligned}$$

and we see that there cannot be an exit edge e at $r(\mu_i)$, because otherwise

$$0 = \overline{U(\chi)(\mu\mu_1 \cdots \mu_i e \cdots)}U(\chi)(\mu\mu_1 \cdots \mu_i e \cdots) \neq \mathbb{1}_{Z(r(\mu))}(\mu\mu_1 \cdots \mu_i e \cdots) = 1,$$

since $\mu\mu_1 \cdots \mu_i e \cdots \notin \operatorname{supp}(U)$.

Hence for all $\mu \in E^k$ we see that $s(\mu) = r(\mu)$ and there are no exit edges, that is, $E \cong \bigoplus_{i=1}^m C_{k_i}$ for some $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$ where all k_i divide k.
($\Leftarrow$): Suppose $E \cong \bigoplus_{i=1}^m C_{k_i}$ for some constants $m \in \mathbb{N}$ and $k_i \in \mathbb{N}$ and define $k = \prod_{i=1}^m k_i$. Then for all $(\alpha_1, \ldots, \alpha_{|\alpha|}) \in E^*$ we know that $Z(\alpha) = Z(\alpha_1)$, so all elementary basis tensors are of the form $\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)} \in \mathsf{X}(E)^{\otimes k}$ and we have $\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)} = 0$, unless $(e_1, \ldots, e_k) \in E^k$.

Then, for all $(e_1, \ldots, e_k) \in E^k$, define $U(\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)}) = \mathbb{1}_{Z(r(e_k))}$. Since k_i divides k, we know that $s(e_1) = r(e_k)$, so clearly

$$\mathbb{1}_{Z(\alpha)}U\left(\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)}\right) = U\left(\phi(\mathbb{1}_{Z(\alpha)})\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)}\right)$$

and

$$U\left(\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)} \cdot \mathbb{1}_{Z(\alpha)}\right) = U\left(\mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)}\right) \cdot \mathbb{1}_{Z(\alpha)}$$

for all $\alpha \in E^*$. Furthermore, writing $\chi = \mathbb{1}_{Z(e_1)} \otimes \cdots \otimes \mathbb{1}_{Z(e_k)}$ again, we see that $\langle U(\chi), U(\chi)\rangle = \langle \chi, \chi\rangle$ and U is bijective (hence unitary) because for all $v \in E^0$ there exists a unique path $\mu \in E^k$ with $r(\mu) = v$, and this path μ must be a cycle of length k. Therefore, X_σ is periodic. ∎

Again, if E is cofinal, then $E \ncong C_n$ for some $n \in \mathbb{N}$ if and only if every loop has an exit. This implies the following corollary.

Corollary. *Let E be a finite directed graph without sinks and sources and let X_σ be minimal. Then X_σ is nonperiodic if and only if E satisfies condition (L).*

This also corresponds to the fact that X_σ is minimal and nonperiodic if and only if $C^*(E) \cong \mathcal{O}_{\mathsf{X}_\sigma}$ is simple, which is equivalent to E being cofinal and satisfying condition (L).

Remark. We see that the notions of minimality and periodicity of X_σ correspond to the notions of minimality and periodicty of the path groupoid $\mathcal{G}_E$ in the following way. If E is a finite directed graph without sinks and sources, then $\mathcal{G}_E$ is minimal if and only if E is cofinal if and only if X_σ is minimal. Also, all $x \in \mathcal{G}_E^{(0)}$ are eventually periodic if and only if $E \cong \bigoplus_{i=1}^m C_{k_i}$, which is equivalent to X_σ being periodic.

5.3 Pure infiniteness

Now that we have shown what the ideal structure of $C^*(E)$ looks like and what makes $C^*(E)$ simple, the question arises of how other properties of the algebra are reflected in the different models. One of those interesting properties is pure infiniteness.

Pure infiniteness was first defined for Von Neumann-algebras and later generalized by Joachim Cuntz to simple C^*-algebras. Later, Eberhard Kirchberg and Mikael Rørdam extended this definition to also cover non-simple C^*-algebras in [20, Chapter 4].

Before we discuss this property, we will first look at approximately finite-dimensional (AF) algebras. We will see later that this is strongly related to the notion of pure infiniteness.

5.3.1 Approximately finite-dimensional algebras

In this subsection we look at approximately finite-dimensional (AF) algebras. We will see that the graph C^*-algebra $C^*(E)$ is AF if and only if E has no loops. This is extremely helpful in finding out if $C^*(E)$ is purely infinite, since in general a C^*-algebra cannot be both purely infinite and AF. In this subsections we only look at the Cuntz–Krieger model. We begin with some definitions.

Definition 5.3.1. Let $\mathcal{A}$ be a C^*-algebra and $p \in \mathcal{A}$ a projection. Then p is called *infinite* if it is equivalent to a proper sub-projection of itself, that is, there exists an isometry $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* \lneq p$. Furthermore, p is called *properly infinite* if it has mutually orthogonal sub-projections q_1 and q_2, such that there exist isometries $v, w \in \mathcal{A}$ with $v^*v = w^*w = p$, $vv^* = q_1$ and $ww^* = q_2$.

Definition 5.3.2. A C^*-algebra $\mathcal{A}$ is called an *approximately finite-dimensional (AF) C^*-algebra*, if there exists an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ of finite-dimensional sub-C^*-algebras of $\mathcal{A}$ such that $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ is dense in $\mathcal{A}$.

If $\mathcal{A}$ is seperable, then this is equivalent to the following statement [12, Theorem III.3.4]: For every finite subset $\{a_1, \dots, a_n\}$ of $\mathcal{A}$ and for every $\epsilon > 0$ there exists a finite-dimensional sub-C^*-algebra $\mathcal{B}$ of $\mathcal{A}$ and a subset $\{b_1, \dots, b_n\}$ of $\mathcal{B}$ with $\|a_j - b_j\| < \epsilon$ for all $j = 1, \dots, n$.

A sub-C^*-algebra $\mathcal{B}$ of $\mathcal{A}$ is called *hereditary* if for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $0 \leq a \leq b$, we have $a \in \mathcal{B}$.

An interesting fact about AF algebras is that every hereditary C^*-subalgebra of an AF C^*-algebra is also AF [15, Theorem 3.1]. Since every corner of a C^*-algebra $\mathcal{A}$ is a hereditary sub-C^*-algebra of $\mathcal{A}$ [27, Example 3.2.1], and every closed ideal I of a C^*-algebra $\mathcal{A}$ is a hereditary sub-C^*-algebra of $\mathcal{A}$ [27, Corollary 3.2.3], we know that every closed ideal and every corner of an AF algebra $\mathcal{A}$ are also AF. We will use these facts to prove the following results.

Proposition 5.3.3. *Every projection in an AF algebra is finite.*

Proof. First of all a projection in a finite-dimensional algebra cannot be infinite, this can be proven by looking at $\mathcal{A}$ as a direct sum of matrix algebras [12, Theorem III.1.1]. If p is a projection in an AF algebra, then we know by the second statement of Definition 5.3.2, that for all $\epsilon > 0$ there exists a finite-dimensional sub-C^*-algebra $\mathcal{B}$, with $\|p - b\| < \epsilon/24$. We can assume $\|b\| \leq 1$

and define $c = b^*b$. This is a selfadjoint element in $\mathcal{B}$ and

$$\begin{aligned}\|p - c\| &= \|p - b^*b\| \\ &= \|p^2 - b^*b + pb - pb\| \\ &\leq \|p(p-b)\| + \|(b^* - p)b\| \\ &\leq \|p\|\|p - b\| + \|(b^* - p^*)\|\|b\| \\ &\leq 2\|p - b\| \\ &< \frac{1}{12}\epsilon.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|c^2 - c\| &= \|c^2 - cp + cp - p^2 + p - c\| \\ &\leq \|c\|\|c - p\| + \|c - p\|\|p\| + \|p - c\| \\ &\leq 3\|c - p\| \\ &\leq \frac{1}{4}\epsilon.\end{aligned}$$

Thus by [27, Lemma 6.2.2] there is a projection $q \in \mathcal{B}$ such that $\|c - p\| < \epsilon/2$. Then we can choose ϵ such that $\|q - p\| < 1$, hence by [27, Lemma 6.2.1] p and q are equivalent. Furthermore, $q \in \mathcal{B}$, so q cannot be infinite. Hence p cannot be infinite. ■

Lemma 5.3.4. *Let $\mathcal{A}$ be an AF algebra. Then there are no non-zero hereditary sub-C^*-algebra of $\mathcal{A}$ which contain an infinite projection.*

Proof. If $\mathcal{A}$ is AF, then every projection is finite, and every hereditary C^*-subalgebra of an AF C^*-algebra is also AF. Therefore, none of the hereditary C^*-subalgebras of $\mathcal{A}$ contain an infinite projection. ■

Lemma 5.3.5. *The C^*-algebras $C(\mathbb{T})$ and $C(\mathbb{T}, M_n(\mathbb{C}))$ are not AF, for all $n \in \mathbb{N}$.*

Proof. We first prove that $C(\mathbb{T})$ is not AF. As explained in [27, Paragraph 6.2], a finite-dimensional C^*-algebra is the linear span of its projections, so an AF algebra is the closed linear span of its projections. As a consequence, an abelian AF-algebra has totally disconnected spectrum. $C(\mathbb{T})$ is an abelian algebra, but its spectrum $\mathbb{T}$ is not totally disconnected, hence it is not AF.

This means that $C(\mathbb{T}, M_n(\mathbb{C}))$ cannot be AF either, as $C(\mathbb{T})$ is a corner of $C(\mathbb{T}, M_n(\mathbb{C}))$, and corners of AF algebras are also AF (as we have just seen). ■

Theorem 5.3.6 ([22, Theorem 2.4])**.** *A directed graph E has no loops if and only if $C^*(E)$ is an AF algebra.*

Proof. ($\Rightarrow$): First suppose that E has no loops. Since

$$C^*(E) = \overline{\text{span}}\{S_\alpha S_\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta)\},$$

it is enough to show that every finite set of elements $S_\alpha S_\beta^*$ lies in a finite-dimensional subalgebra of $C^*(E)$. Therefore, let F be a finite set of pairs $(\alpha, \beta) \in E^* \times E^*$ satisfying $r(\alpha) = r(\beta)$ and let G be the finite subgraph of E consisting of all edges e occurring in the paths α, β with $(\alpha, \beta) \in F$ and all corresponding vertices $r(e), s(e)$. Let H be the subgraph obtained from G by adding all edges $f \in E^1$ with $s(e) = s(f)$ for some $e \in G$ as well as all corresponding vertices $r(f)$. Then for all vertices $v \in H^0$, either all edges $e \in E^1$ with $s(e) = v$ lie in H, or none do. Thus if v is not a sink in H, the second Cuntz–Krieger relation holds in H as well. Thus

$$\{P_v, S_e : e \in H^1, v \in H^0\}$$

is a Cuntz–Krieger H-family and $C^*(H)$ is isomorphic to the subalgebra of $C^*(E)$ generated by that family. By Lemma 2.2.20 this subalgebra is finite dimensional and since for all $(\alpha, \beta) \in F$, $S_\alpha S_\beta^*$ lies in this finite subalgebra, we see that $C^*(E)$ is AF.

($\Leftarrow$): Now suppose that E has a loop α at v. First suppose that α has an exit. We may assume without loss of generality that this exit edge f satisfies $s(f) = s(\alpha) = v$. Then $P_v = S_\alpha^* S_\alpha$ and

$$S_\alpha S_\alpha^* \leq S_{\alpha_1} S_{\alpha_1}^* < S_{\alpha_1} S_{\alpha_1}^* + S_f S_f^* \leq P_v,$$

and hence P_v is an infinite projection. As explained before, a projection in an AF algebra is equivalent to one in a finite-dimensional subalgebra, and hence cannot be infinite. Thus $C^*(E)$ cannot be AF.

Now suppose that α does not have an exit. Then by Theorem 5.1.11 and Example 2.2.17, $C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ is a full corner of an ideal of $C^*(E)$. Since $C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ is not AF, neither is $C^*(E)$. ■

5.3.2 Pure infiniteness in the Cuntz–Krieger model

Now that we know when $C^*(E)$ is AF, we consider pure infiniteness. We need the following definitions to define pure infiniteness, however we do not use these notions in our proofs.

Definition 5.3.7. If $a, b \in \mathcal{A}$ are positive elements, write $a \precsim b$ if there exists a sequence $\{x_k\}_{k=1}^\infty$ in $\mathcal{A}$ with $x_k^* b x_k \to a$ as $k \to \infty$.

Definition 5.3.8. A non-zero map $\tau : \mathcal{A} \to \mathbb{C}$, is called a *character* of $\mathcal{A}$ if it is linear and continuous and satisfies $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in \mathcal{A}$.

Definition 5.3.9. Let $\mathcal{A}$ be a simple C^*-algebra. Then $\mathcal{A}$ is called *purely infinite* if each of its non-zero hereditary sub-C^*-algebras contains an infinite projection.

We proved in Lemma 5.3.4, that AF algebras do not satisfy the property described in Definition 5.3.9. For (non-simple) C^*-algebras Kirchberg and Rørdam gave the following definition and sufficient condition.

Definition 5.3.10. Let $\mathcal{A}$ be a C^*-algebra, then $\mathcal{A}$ is called *purely infinite* if there are no characters on $\mathcal{A}$ and for every pair a, b of positive elements, we have $a \precsim b$ if and only if $a \in \overline{\mathcal{A}b\mathcal{A}}$.

Lemma 5.3.11 ([20, Propostion 4.17])**.** *Let $\mathcal{A}$ be a C^*-algebra such that every non-zero hereditary sub-C^*-algebras in every quotient of $\mathcal{A}$ contains an infinite projection. Then $\mathcal{A}$ is purely infinite.*

This lemma says that if every quotient of $\mathcal{A}$ satisfies the property described in Definition 5.3.9, then $\mathcal{A}$ is purely infinite. Since the only quotients of a simple C^*-algebra $\mathcal{A}$ are 0 and $\mathcal{A}$, this shows that on simple C^*-algebras the two definitions agree.

A necessary and sufficient condition for graph C^*-algebras to be purely infinite was initially found for simple graph C^*-algebras [22] and was later expanded to the non-simple case. However, this was done by using the groupoid model and therefore only applied to graphs without sinks. In the case of graphs with sinks, it was shown in [5] that this condition implies the property described in Definition 5.3.9.

Lemma 5.3.12 ([5, Proposition 5.3])**.** *Suppose E is a row-finite directed graph in which every vertex connects to a loop and every loop has an exit. Then every non-zero hereditary C^*-subalgebra of $C^*(E)$ contains an infinite projection.*

We will first look at the general case and then look at the simple case. We need the following definition to state the necessary and sufficient condition.

Definition 5.3.13. Let $M \subseteq E^0$ be a non-empty subset. Then we call M a *maximal tail* if the following three conditions hold:

1. If $v \in E^0$, $w \in M$, and $v \geq w$, then $v \in M$.
2. If $v \in M$ and $s^{-1}(v) \neq \emptyset$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.
3. For every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

Definition 5.3.14. For $X \subseteq E^0$, let $\Omega(X)$ be the set of vertices $w \in E^0 \setminus X$ such that there is no path from w to any vertex in X, that is,

$$\Omega(X) = \{w \in E^0 \setminus X \ : \ w \ngeq v \text{ for all } v \in X\}.$$

Remark. For all $X \subseteq E^0$, $\Omega(X)$ is a hereditary subset, as for all $w_1 \in \Omega(X)$ the condition $w_1 \geq w_2$, implies $w_2 \ngeq v$ for all $v \in X$, for otherwise $w_1 \geq w_2 \geq v$ for some $v \in X$. Thus $w_2 \in \Omega(X)$.

If M is a maximal tail, then, using the first two properties of maximal tails, it is easy to check that $\Omega(M) = E^0 \setminus M$ and that $\Omega(M)$ is also saturated.

Lemma 5.3.15. *If E is a row-finite directed graph, then the graph E satisfies Condition (K) if and only if all loops in each maximal tail M have exits in M (that is, any loop has an exit edge e, with $r(e) \in M$).*

Proof. ($\Leftarrow$): Suppose M is a maximal tail and μ is a loop in M. Since E satisfies condition (K), there is another loop α based at some vertex v of μ. Thus $r(\alpha_i) \in M$ by the first property, so μ has an exit in M.
($\Rightarrow$): Suppose E does not satisfy (K), then there exists a loop μ based at $s(\mu) = v$ with no return path. Now define

$$M = \{w \in E^0 \mid w \geq v\}.$$

Then the first and last property of Definition 5.3.13 hold, so we only need to check the second property. Obviously, for all $w \neq v$ with $w \in M$, there exists a nonempty path $\alpha \in E^*$, with $s(\alpha_1) = v$ and $r(\alpha_1) \geq v$, so $r(\alpha_1) \in M$. Also, for v itself, we know that μ is a nonempty path such that $s(\mu) = v = r(\mu)$, thus we also have $s(\mu_1) = v$ and $r(\mu_1) \geq v$, so $r(\mu_1) \in M$. Hence the second property also holds, and so M is a maximal tail.

Furthermore, μ is a loop in M without an exit in M. (Otherwise we would have an exit edge $e \neq \mu_1$ with $r(e) \in M$, so $r(e) \geq v$, which contradicts the fact that μ is the only loop based at v.) Thus we have proven that if (K) does not hold, then there exists a maximal tail M without an exit in M. ■

Recall that if E satisfies condition (K), then every ideal in $C^*(E)$ is gauge-invariant. We just saw that $\Omega(M)$ is saturated and hereditary for all maximal tails M, and we know that for all saturated and hereditary subsets H of E^0, the subset I_H is a closed gauge-invariant ideal. This already shows that maximal tails play an important role in the ideal theory of graph C^*-algebras. However, Hong and Szymański have proven even stronger results about the ideal structure of graph C^*-algebras, giving a complete description of the primitive ideal space of infinite graphs. We will not go into detail about primitive ideals, so we will just state the definition and an interesting result regarding maximal tails.

Definition 5.3.16. An ideal of a C^*-algebra $\mathcal{A}$ is called *primitive* if it is the kernel of an irreducible representation of $\mathcal{A}$.

Lemma 5.3.17 ([4, Lemma 4.1])**.** *If I is a primitive ideal, then $M = E^0 \setminus H(I)$ is a maximal tail.*

Knowing this, we can prove the main result of this subsection.

Theorem 5.3.18 ([18, Theorem 2.3])**.** *Let E be a row-finite directed graph. Then $C^*(E)$ is purely infinite if and only if all loops in each maximal tail M have exits in M and each vertex in every maximal tail of M connects to a loop in M.*

Proof. ($\Rightarrow$): Suppose that $C^*(E)$ is purely infinite. Then P_v is properly infinite for all $v \in E$ by [20, Theorem 4.16]. Now let M be a maximal tail. Then $\Omega(M) = E^0 \setminus M$ is a saturated hereditary subset, so by Theorem 5.1.11 we have $C^*(E)/I_{\Omega(M)} \cong C^*(M)$. For the sake of contradiction, suppose that M has a loop α without an exit. Then, just as in the proof of Theorem 5.2.3, $C(\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ is a full corner of the ideal of $C^*(M)$ generated by $P_{s(\alpha)}$.
Thus $P_{s(\alpha)}$ is not properly infinite in $C^*(E)/I_{\Omega(M)}$, so it is not properly infinite in $C^*(E)$ either, but this contradicts our first statement. Hence α must have an exit in M.

Again for the sake of contradiction, suppose that M is a maximal tail such that $v \in M$ does not connect to a loop in M. Then let

$$H = \{w \in M \ : \ v \geq w\}.$$

This is clearly a hereditary subset of M that does not contain a loop. Then by Theorem 5.1.11, $C^*(H)$ is a full corner of the ideal I_H of $C^*(M)$ generated by P_v, and hence by Theorem 5.3.6 it is an AF-algebra. Thus the projection P_v is not properly infinite in $C^*(E)/I_{\Omega(M)}$, so it is not properly infinite in $C^*(E)$ either, which is a contradiction. Hence v must be connected to a loop in M.

Thus if $C^*(E)$ is purely infinite, all loops in each maximal tail M have exits in M and each vertex in every maximal tail of M connects to a loop in M.

($\Leftarrow$): Now suppose that E satisfies the right-hand side. Then by Lemma 5.3.15, E satisfies condition (K). That means that every ideal in $C^*(E)$ is gauge-invariant and of the form I_H, where H is a saturated hereditary subset of E, and by Theorem 5.1.11, we have $C^*(E)/I_H \cong C^*(E \setminus H)$.

If we can show that for all hereditary saturated subsets H, $C^*(E \setminus H)$ satisfies the property described in Definition 5.3.9, then by Lemma 5.3.11, $C^*(E)$ is purely infinite. Thus, by Lemma 5.3.12 we need to show that every vertex in $E \setminus H$ connects to a loop, and every loop in $E \setminus H$ has an exit in $E \setminus H$.

Since E satisfies condition (K), we know by Lemma 5.1.9 that the last condition is true. To check the other condition suppose $v \in E \setminus H$ and let

$$X = \{w \in E^0 \setminus H \ : \ w \not\geq v\}.$$

This is a hereditary subset of $E^0 \setminus H$ and by definition $v \geq v$, so $v \notin X$.

Then $I_X = I_{\overline{X}}$ is a gauge-invariant ideal by Lemma 5.1.10 and $P_v \notin I_X$. Furthermore by [33, Proposition A.17] every closed ideal is the intersection of the primitive ideals containing it, so there exists a primitive ideal J with $P_v \notin J$. Thus by 5.3.17 there exists a maximal tail $M = (E^0 \setminus K) \setminus H(J)$ and $P_v \notin J$, so $v \notin H(J)$, thus $v \in M$. Therefore, v is a vertex in a maximal tail M in $E \setminus H$, and it is easy to check that M is also a maximal tail in E. Hence by assumption $v \in M$ connects to a loop in M.

Thus for all hereditary saturated subsets H, $C^*(E \setminus H)$ has the property described in Definition 5.3.9, which means that $C^*(E)$ is purely infinite. ■

This gives us a necessary and sufficient condition on graphs for the graph C^*-algebra to be purely infinite. We can now look at what this means in the simple case, where we will see we have an easier condition.

Theorem 5.3.19. *Let E be a row-finite directed graph. If E is cofinal, then $C^*(E)$ is purely infinite if and only if every vertex connects to a loop and every loop has an exit.*

Proof. Let E be cofinal and suppose we have a maximal tail $M \subseteq E^0$. By definition, M is non-empty, so choose $v \in M$. Then by the second property of Definition 5.3.13, we can find a path $\alpha \in E^{\leq\infty}$ with $s(\alpha) = v$ and $r(\alpha_i) \in M$ for all $i \leq |\alpha|$. Since E is cofinal, we know that for all $w \in E^0$ there exists $\beta \in E^*$ with $s(\beta) = w$ and $r(\beta) = r(\alpha_i)$ for some $i \leq |\alpha|$, so by the first property of Definition 5.3.13 we have $w \in M$. Hence $M = E^0$ is the only maximal tail of E. ■

Therefore, we have the following dichotomy for simple graph C^*-algebras. (See [22, Corollary 3.11] for the case of graphs without sinks.)

Theorem 5.3.20. *Let E be a row-finite graph that is cofinal and satisfies condition (L). Then $C^*(E)$ is simple and:*

1. *If E has no loops, then $C^*(E)$ is AF.*
2. *If E has a loop, then $C^*(E)$ is purely infinite.*

Proof. First of all, cofinality and condition (L) imply that $C^*(E)$ is simple.

If E has no loops, then by Theorem 5.3.6 $C^*(E)$ is AF. Now suppose E has a loop. Then by cofinality every vertex connects to that loop, and hence every vertex connects to a loop and every loop has an exit. Hence $C^*(E)$ is simple and satisfies the property described in Definition 5.3.9, so it is purely infinite by definition. ■

5.3.3 Pure infiniteness in the groupoid model

We will now look at pure infiniteness in the groupoid model. Since there was more known about groupoid C^*-algebras than about graph C^*-algebras, this model was actually used in [22] to find a necessary and sufficient condition for $C^*(E)$ to have the the property described in Definition 5.3.9. In this subsection we start with pure infiniteness in the simple case, and then we present some new results for the non-simple case.

Definition 5.3.21. A groupoid $\mathcal{G}$ is called *locally contracting* if for every non-empty open subset U of $\mathcal{G}^{(0)}$ there exists an open subset V of U and an open bisection S with $\overline{V} \subseteq d(S)$ and $\alpha_{S^{-1}}(\overline{V}) \subsetneq V$. Here, α_S is the homeomorphism from $r(S)$ to $d(S)$ defined by $x \mapsto d(xS)$.

In [1, Chapter 2] it is proven that if $\mathcal{G}$ is an étale, essentially free and locally contracting groupoid, then $C^*(\mathcal{G})$ has the property described in Definition 5.3.9.

We know that the path groupoid $\mathcal{G}_E$ is always étale and is essentially free if and only if E satisfies condition (L). The only thing left to investigate is when it is locally contracting.

Lemma 5.3.22 ([22, Lemma 3.8])**.** *Let E be a row-finite directed graph without sinks, and suppose that every vertex in E connects to a loop with an exit. Then $\mathcal{G}_E$ is locally contracting.*

Proof. Suppose U is a non-empty open subset of $\mathcal{G}_E^{(0)}$. Then there exists $\alpha \in E^*$ with $Z(\alpha) \subseteq U$. By assumption we know that there exists $\beta, \mu \in E^*$ with $r(\alpha) = s(\beta)$ and μ is a cycle with an exit based at $r(\beta)$.

Let $S = Z(\alpha\beta\mu, \alpha\beta)$ be the associated bisection and let $V = Z(\alpha\beta)$. Then $\overline{V} = V = d(S)$ and $\alpha_{S^{-1}}(Z(\alpha\beta)) = Z(\alpha\beta\mu)$. Since μ has an exit, we know that $Z(\alpha\beta\mu)$ is a proper subset of $Z(\alpha\beta)$. Thus $\alpha_{S^{-1}}(\overline{V}) \subsetneq V$, so $\mathcal{G}_E$ is locally contracting. ■

This gives us a sufficient condition for $C^*(E)$ to have the property described in Definition 5.3.9, but it is also necessary. We will show this by following the proof as in [22, Chapter 3].

Theorem 5.3.23. *Let E be a row-finite directed graph without sinks. Then every non-zero hereditary C^*-subalgebra of $C^*(E)$ contains an infinite projection if and only if every vertex connects to a loop and E satisfies condition (L).*

Proof. ($\Leftarrow$): If E satisfies condition (L), then $\mathcal{G}_E$ is essentially free by Lemma 5.1.13, and $\mathcal{G}_E$ is locally contracting by Lemma 5.3.22. Thus $C^*(\mathcal{G}_E) \cong C^*(E)$ has the property described in Definition 5.3.9.

($\Rightarrow$): Suppose that E does not satisfy condition (L). Then there exists a cycle

α without an exit, and then, just as in the proof of Theorem 5.2.3, $C(\mathbb{T})$ is a corner of $C((\mathbb{T}, M_{|\alpha|}(\mathbb{C}))$ which is a corner of an ideal of $C^*(E)$. But the only two projections in $C(\mathbb{T})$ are 0 and 1, so it cannot contain an infinite projection. Hence $C^*(E)$ does not satisfy the property described in Definition 5.3.9.

Now suppose that E does satisfy (L) and that there exists a vertex v that is not connected to a loop. Then

$$H = \{w \in E^0 : v \geq w\}$$

is a hereditary subset of E^0 and hence $C^*(H)$ is a hereditary subalgebra of $C^*(E)$. But $C^*(H)$ is AF, thus $C^*(E)$ does not contain an infinite projection, so $C^*(E)$ does not satisfy the property described in Definition 5.3.9. ■

Thus for simple graph C^*-algebras we see that $C^*(\mathcal{G}_E)$ is AF if it has no loops, and is purely infinite if it does have a loop.

We have seen that for graph C^*-algebras there is a sufficient and necessary condition on the graph for the C^*-algebra to be purely infinite. But for groupoid algebras, such a condition is not known. In search of this condition we look at the recent paper [7]. In this paper, the notion of paradoxical decompositions for actions of discrete groups on totally disconnected spaces is generalized to the setting of étale groupoids.

Definition 5.3.24 (cf. [7, Definition 4.5])**.** For a graph E and a finite path $\mu \in E^*$, we say that $Z(\mu)$ is $(G^a, 2, 1)$-*paradoxical*, if there exist numbers $m, n \in \mathbb{N}$ and compact open bisections

$$Z(\alpha_1, \beta_1), Z(\alpha_2, \beta_2), \ldots, Z(\alpha_n, \beta_n), Z(\gamma_1, \delta_1), Z(\gamma_2, \delta_2), \ldots, Z(\gamma_m, \delta_m),$$

such that:

$$\bigcup_{i=1}^{n} Z(\beta_i) = \bigcup_{j=1}^{m} Z(\delta_i) = Z(\mu),$$

and the sets $Z(\alpha_i), Z(\gamma_j) \subseteq Z(\mu)$ are pairwise disjoint.

Remark. We can assume that the compact open bisection $S \subseteq G_E$ that we use in the decomposition above is of the form $Z(\alpha, \beta)$ by the following reasoning.

We can write S as a union of base elements

$$S = \bigcup_{i \in I} Z(\alpha_i, \beta_i),$$

but S is compact, so that gives us

$$S = \bigcup_{i=1}^{n} Z(\alpha_i, \beta_i).$$

Now using the fact that S is a bisection, we know that $r : S \to \cup_{i=1}^{n} Z(\alpha_i)$ is a homeomorphism. Therefore, if $Z(\alpha_i) \cap Z(\alpha_j) \neq \emptyset$, then without loss of generality we have $\alpha_i \epsilon = \alpha_j$ and $r^{-1}(\alpha_i \epsilon x) = \{\beta_i \epsilon x, \beta_j x\}$. Thus if r is a homeomorphism, we get $\beta_i \epsilon = \beta_j$, so $Z(\alpha_j, \beta_j) \subseteq Z(\alpha_i, \beta_i)$, but then

$$S' = \bigcup_{i=1, i \neq j}^{n} Z(\alpha_i, \beta_i)$$

would also work in the decomposition. Thus we can assume $r(S) = \sqcup_{i=1}^n Z(\alpha_i)$ and lastly if $Z(\beta_i) \cap Z(\beta_j) \neq \emptyset$, then without loss of generality one has $\beta_i \epsilon = \beta_j$, and again we find that

$$S' = \bigcup_{i=1, i\neq j}^{n} Z(\alpha_i, \beta_i)$$

can replace S in the decomposition. Therefore, we can assume that

$$S = \bigsqcup_{i=1}^{n} Z(\alpha_i, \beta_i),$$

but then we can replace S by $Z(\alpha_1, \beta_1), Z(\alpha_2, \beta_2), \dots Z(\alpha_n, \beta_n)$ in the decomposition. Thus we can assume $S = Z(\alpha, \beta)$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ and μ is a subset of α and β.

It is easy to see that for all $\mu \in E^*$, $Z(\mu)$ is $(G^a, 2, 1)$-paradoxical if and only if $Z(r(\mu))$ is $(G^a, 2, 1)$-paradoxical.

Paradoxicality was used in [7] to prove the following lemma.

Lemma 5.3.25 ([7, Corollary 4.12])**.** *Let G be an ample groupoid which is essentially principal and inner exact. Let B be a basis for the topology of $G^{(0)}$ consisting of compact open sets. Suppose each $A \in B$ is $(G^a, 2, 1)$-paradoxical. Then $C_r^*(G)$ is purely infinite.*

In order to apply this result to the case at hand, we need to check what these conditions translate to in the case of path groupoids.

We know that $\mathcal{G}_E$ is always ample; it is essentially principal if and only if condition (K) holds. It is also easy to see that it is always inner exact, because by [7, Definition 3.5], $G = \mathcal{G}_E$ is inner exact if

$$0 \to C_r^*(G_{G^{(0)} \setminus D}) \to C_r^*(G) \to C_r^*(G_D) \to 0$$

is exact for all closed G-invariant subsets $D \subseteq G^{(0)}$. As explained in [8, Lemma 6.1] and [8, Remark 6.2] this is equivalent to G_D being amenable for all closed G-invariant subsets $D \subseteq G^{(0)}$, and we know that G_D is a path groupoid for a subgraph of E. Hence it is always amenable.

Therefore, we just need to find a condition on the graph that is equivalent to $Z(v)$ being $(G^a, 2, 1)$-paradoxical for all $v \in E^0$. We first consider some conditions we looked at before, starting with condition (K).

Suppose (K) does not hold. Then there exists a cycle μ in some v with no other return path. If $Z(v)$ has a paradoxical decomposition, then

$$x = \mu\mu\mu \cdots \in Z(\beta_i) \cap Z(\delta_j),$$

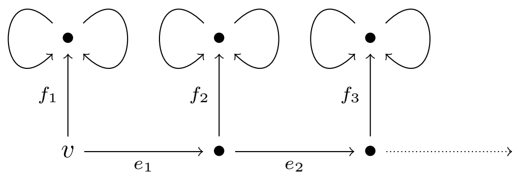
where

$$\beta_i = \underbrace{\mu \dots \mu}_{k \text{ times}} \mu_1 \dots \mu_l \quad \text{and} \quad \delta_j = \underbrace{\mu \dots \mu}_{n \text{ times}} \mu_1 \dots \mu_m,$$

for some k, l, n, m. Then α_i, γ_j must be paths from v to vertices on μ, but these paths must be subpaths of x, for otherwise we would have two distinct return paths in v. Thus $Z(\alpha_i) \cap Z(\gamma_j) = Z(\mu_1 \dots \mu_p) \neq \emptyset$ for some $p \in \mathbb{N}$. Hence $Z(v)$ cannot have a paradoxical decomposition.

Definition 5.3.26. Let E be a row-finite directed graph. We say that E satisfies condition (I) if for all $v \in E^0$ there exists a finite path $\alpha \in E^*$, with $s(\alpha) = v$ and $r(\alpha) \in V_2$.

Example 5.3.27. Consider the graph E pictured below.

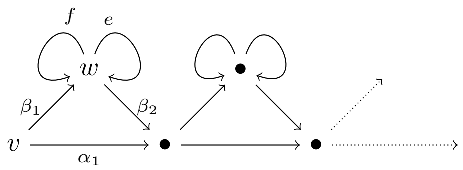


Clearly, condition (I) holds for this graph. Suppose now that $Z(v)$ has a paradoxical decomposition. Then

$$\bar{e} = e_1 e_2 \cdots \in Z(\beta_i) \cap Z(\delta_j)$$

for some i and j, but then we must have $\beta_i = e_1 \cdots e_n$ and $\delta_j = e_1 \cdots e_m$ for some $n, m \in \mathbb{N}$. Thus $\alpha_i = e_1 \cdots e_n$ and $\beta_j = e_1 \cdots e_m$ as well, so $Z(\alpha_i) \cap Z(\gamma_j) \neq \emptyset$. Hence $Z(v)$ has no paradoxical decomposition, so (I) is not a sufficient condition for paradoxicallity.

Example 5.3.28. Consider the graph E pictured below.



This graph has paradoxical decomposition in every point, since for all $u \in E^0$, $\{x \in E^{\leq\infty}; s(x) = u\}$ is either isomorphic to $\{x \in E^{\leq\infty}; s(x) = v\}$ or to $\{x \in E^{\leq\infty}; s(x) = w\}$, and for v and w we can take the following paradoxical decompositions:

For $Z(v)$, take $Z(\beta_1 e, \beta_1), Z(\alpha_1, \alpha_1)$ and $Z(\beta_1 f, \beta_1), Z(\beta_1\beta_2, \alpha_1)$.

For $Z(w)$, take $Z(ee, e), Z(ff, f), Z(\beta_2, \beta_2)$ and $Z(ef, e), Z(fe, f), Z(e\beta_2, \beta_2)$.

This shows that not every path needs to pass through V_2.

Because (K) must hold, we know that $V_1 = \emptyset$. If then V_2 would be empty, we would have $V = V_0$ and E would have no loops, so $C^*(\mathcal{G}_E)$ would be AF, and hence would not be purely infinite. Therefore, E does not have a paradoxical decomposition in every vertex, so there must be a cycle in E.

In the same way we find that if (I) does not hold, then there must be a vertex v that is not connected to V_2, and because (K) holds, it is not connected to V_1 either. Thus, as in the proof of Theorem 5.3.23, the graph $H(v)$ has no loops, so $C^*(H(v))$ is a hereditary subalgebra of $C^*(\mathcal{G}_E)$ which is AF, and so $C^*(\mathcal{G}_E)$ cannot be purely infinite. Thus E has no paradoxical decomposition in every vertex. Therefore condition (I) must hold.

We now know that (K) and (I) are sufficient conditions, but they are not necessary. We define a new condition (I'), which is stronger than both (K) and (I), and we will see that this is equivalent to $Z(v)$ being $(G^a, 2, 1)$-paradoxical for all $v \in E^0$.

Definition 5.3.29. We say that E satisfies condition (I') if for all $v \in E^0$ there exists a decomposition

$$Z(v) = \bigsqcup_{i=1}^{n} Z(\beta_i),$$

such that for all $i = 1, \ldots, n$ there exists a path $\alpha_i \in E^*$ with $r(\alpha_i) = r(\beta_i)$, $s(\alpha_i) = v$ and α_i passes through V_2.

Theorem 5.3.30. *Let E be a row-finite directed graph without sinks that satisfies (K). Then E satisfies condition (I') if and only if $Z(v)$ is $(G^a, 2, 1)$-paradoxical for all $v \in E^0$.*

Proof. Obviously, the left to right implication holds, as we can take decompositions $Z(\alpha_i^1, \beta_i)$ and $Z(\alpha_i^2, \beta_i)$ such that $Z(\alpha_i^k) \cap Z(\alpha_j^l) = \emptyset$ for all $i, j \leq n$ and $k, l = 1, 2$. Since all α_i pass through V_2 and for $w \in V_2$ we know that for all $m \in \mathbb{N}$ there exists $\mu_1, \ldots, \mu_m \in E^*$ with $s(\mu_i) = r(\mu_i) = w$ and $Z(\mu_i) \cap Z(\mu_j) = \emptyset$ for all $i \neq j$.

For the other direction, suppose E does not satisfy (I'). Then for all $n \in \mathbb{N}$ the set P_n defined by

$$\{\gamma \in E^n : s(\gamma) = v \text{ and } \nexists \alpha \text{ that passes through } V_2 \text{ with } s(\alpha) = v, r(\alpha) = r(\gamma)\}$$

is non-empty and for all $\mu \in P_{n+1}$, we have $\mu_1 \ldots \mu_n \in P_n$. Thus by Zorn's Lemma, there exists an infinite path $x \in E^\infty$ such that there is no path from v via a vertex in V_2 to $r(x_n)$ for some $n \in \mathbb{N}$. However, that means that

$$M = \{u \in E^0 : u \geq r(x_n) \text{ for some } n \in \mathbb{N}\}$$

is a maximal tail, where v does not connect to a loop in M. Therefore, $C^*(E) \cong C^*(\mathcal{G}_E)$ is not purely infinite, and hence by Lemma 5.3.25 $Z(v)$ is not $(G^a, 2, 1)$-paradoxical for all $v \in E^0$. ■

In this proof we used the notion of maximal tails. We already saw that the property all loops in each maximal tail M have exits in M is equivalent to condition (K), and (K) together with the property that each vertex in every maximal tail M connects to a loop in M is equivalent to $C^*(E)$ being purely infinite. This leads us to a slightly different condition, which is equivalent to the property that every vertex in a maximal tail M connects to a loop in M.

Definition 5.3.31. Let $v \in E^0$. We say that v satisfies condition (I") if for all $v \in E^0$ there exists a decomposition

$$Z(v) = \bigsqcup_{i=1}^{n} Z(\beta_i),$$

such that for all $i = 1, \ldots, n$ there exists a path $\alpha_i \in E^*$ with $r(\alpha_i) = r(\beta_i)$, $s(\alpha_i) = v$, and α_i passes through a loop.

Condition (I') implies that all $v \in E^0$ satisfy (I"), but not the other way around, as in the case of (I") we do not know if there are two loops or just one.

We will now prove the main result of this chapter, which leads us to a characterization of purely infinite graph C^*-algebras in terms of the associated path groupoids.

Lemma 5.3.32. *Let E be a row-finite directed graph without sinks, and let $v \in E^0$. Then v satisfies condition (I") if and only if for all maximal tails M with $v \in M$, v connects to a loop in M.*

Proof. ($\Rightarrow$): Suppose v satisfies condition (I") and let M be a maximal tail with $v \in M$. Then by condition 2 of Definition 5.3.13, there exists an infinite path $x \in E^\infty$ with $s(x) = v$ and $r(x_n) \in M$ for all $n \in \mathbb{N}$. We know that

$$Z(v) = \bigsqcup_{i=1}^{n} Z(\beta_i),$$

so $x \in Z(\beta_i)$ for some $1 \leq i \leq n$. Hence $r(\beta_i) = r(\alpha_i) \in M$, and then by condition 1 of Definition 5.3.13, every vertex on α_i is in M and α_i passes through a loop. So that loop is in M as well, and lastly v connects to that loop. Thus the property that v satisfies condition (I") implies that for all maximal tails M with $v \in M$, v connects to a loop in M.

($\Leftarrow$): Now suppose that v does not satisfy (I"). Then by the same reasoning as before, using Zorn's lemma, there exists an infinite path $x \in E^\infty$ with $s(x) = v$, such that there is no path from v via a loop to $r(x_n)$ for some $n \in \mathbb{N}$. Then

$$M = \{u \in E^0 : u \geq r(x_n) \text{ for some } n \in \mathbb{N}\}$$

is again a maximal tail, where v does not connect to a loop. Thus if for all maximal tails M with $v \in M$, v connects to a loop in M, then v must satisfy condition (I"). ∎

We also have the following obvious results:

Lemma 5.3.33. *Let E be a row-finite directed graph without sinks. If E satisfies condition (K) and each $v \in E^0$ satisfies condition (I"), then E satisfies condition (I').*

Corollary. *If E ia a row-finite directed graph without sinks that satisfies condition (K), then E satisfies condition (I') if and only if each $v \in E^0$ satisfies condition (I").*

It is easy to see that this gives us a complete description of pure infiniteness in the groupoid model.

Lemma 5.3.34. *Let E be a row-finite directed graph without sinks. Then the following are equivalent:*

1. *All loops in each maximal tail M have exits in M and each vertex in every maximal tail of M connects to a loop in M.*
2. *The path groupoid $\mathcal{G}_E$ is essentially principal and for every $\alpha \in E^*$, $Z(\alpha)$ is $(G^a, 2, 1)$-paradoxical.*

We see that paradoxicallity is not only a sufficient condition in the case of path groupoids, but also a necessary condition. Hence for path groupoids, we have the following theorem.

Theorem 5.3.35. *Let E be a row-finite directed graph without sinks. Then $C^*(\mathcal{G}_E)$ is purely infinite if and only if $\mathcal{G}_E$ is essentially principal and for every $\alpha \in E^*$, $Z(\alpha)$ is $(G^a, 2, 1)$-paradoxical.*

5.3.4 Pure infiniteness in the Cuntz–Pimsner models

As we saw in the case of simplicity, the structure of Cuntz–Pimnser algebras in general is not as well known as for groupoid algebras and Cuntz–Krieger algebras. Indeed, in the case of pure infiniteness there are barely any results for Cuntz–Pimnser algebras. That is why we will only look at the case where X is a correspondence over a unital C^*-algebra $\mathcal{A}$.

We recall that $C_0(E)$ and $C_0(E^\infty)$ are unital only if E^0 is finite and that the construction of $\mathcal{O}_{\mathsf{X}(E)}$ and $\mathcal{O}_{\mathsf{X}_\sigma}$ required no sinks and sources. This means that E always contains a loop. We know that for simple graph C^*-algebras, $C^*(E)$ is purely infinite if and only if E contains a loop. This gives us the following result.

Proposition 5.3.36. *Let E be a finite directed graph without sinks and sources, which satisfies condition (L) and is cofinal. Then $\mathcal{O}_{\mathsf{X}_\sigma}$ and $\mathcal{O}_{\mathsf{X}(E)}$ are simple and purely infinite.*

In other words we have the following two results.

Theorem 5.3.37. *Let E be a finite directed graph without sinks and sources. Then the following are equivalent:*

1. *$\mathsf{X}(E)$ is minimal and non-periodic.*
2. *$\mathcal{O}_{\mathsf{X}(E)}$ is simple.*
3. *$\mathcal{O}_{\mathsf{X}(E)}$ is purely infinite.*

Theorem 5.3.38. *Let E be a finite directed graph without sinks and sources. Then the following are equivalent:*

1. *X_σ is minimal and non-periodic.*
2. *$\mathcal{O}_{\mathsf{X}_\sigma}$ is simple.*
3. *$\mathcal{O}_{\mathsf{X}_\sigma}$ is purely infinite.*

After finishing this thesis, it was brought to my attention that conditions on the correspondences equivalent to condition (L) and (K), where found in [11] and [24]. These conditions don't require the C^*-algebra to be unital. This means that the only remaining question is: What does the condition, each vertex in every maximal tail of M connects to a loop in M, mean in the Cuntz–Pimsner model?

Apart from this, the necessary and sufficient conditions for pure infiniteness of the graph C^*-algebras were translated into the Cuntz–Pimsner and groupoid models, either in this thesis or in the papers [11] and [24].

Conclusion

We have seen that graph C^*-algebras $C^*(E)$ can be realized as groupoid C^*-algebras and as Cuntz–Pimsner algebras. In addition, we have seen how certain algebraic properties of $C^*(E)$ can be characterized in these different models.

We have discussed known results about the ideal structure and simplicity of $C^*(E)$, and we have found a new result regarding pure infiniteness of $C^*(E)$ in the groupoid model.

However, we did not discuss all the answers in the Cuntz–Pimsner cases. It would be interesting to further investigate these models, using the recent results of simplicity of Cuntz–Pimsner algebras in the non-unital setting [11], [24].

It would also be interesting to investigate what some of the other properties, such as condition (I'), condition (I"), maximal tails, and the path groupoid being locally contracting, translate into in the Cuntz–Pimsner models.

These properties all relate to pure infiniteness of the graph C^*-algebras, so if we could find conditions on the correspondences that associate to these conditions, it could possibly tell us more about pure infiniteness of Cuntz–Pimsner algebras.

Other than researching the Cuntz–Pimsner algebras, it would also be interesting to further explore the pure infiniteness of groupoid algebras in general. As we have now found a necessary and sufficient condition on the path groupoid, this could be a starting point in the search for a necessary and sufficient condition in general.

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