



RADBOD UNIVERSITEIT NIJMEGEN

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MASTER THESIS

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# Operator Algebras and Unbounded Self-Adjoint Operators

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*Author:*

Christian BUDDE

*Supervisor:*

Prof.Dr. N.P. LANDSMAN

*Second Reader:*

Dr. J. KAAD



# Introduction

This thesis is about mathematical techniques one uses in the context of quantum mechanics. We will discuss the theory of  $C^*$ -algebras and von Neumann algebras. These two special cases of algebras can in general be viewed as non-commutative function spaces and non-commutative measure spaces, respectively.  $C^*$ -algebras were first considered in quantum mechanics to build a model of physical observables. In particular, one can describe a physical system using a  $C^*$ -algebra. The self-adjoint elements of this  $C^*$ -algebra play the role of observables. The states on it describe the expected value of an observable. First, one defined  $C^*$ -algebras as a special kind of set consisting of linear bounded operators on a Hilbert space. In the 1930s, John von Neumann (1903 – 1957) and Francis J. Murray (1911 – 1996) wrote a series of papers on rings of operators. In particular, they described a special class of  $C^*$ -algebras, later called von Neumann algebras. In 1943, Israel M. Gelfand (1913 – 2009) and Mark A. Naimark (1909 – 1978) come up with an abstract definition of  $C^*$ -algebras, which does not use the notion of a Hilbert space.

The theory we develop in chapter 1 and 2 is mostly covered in courses on operator algebras, therefore most of the information can be found in a book like [13]. Another part of the theory we will develop is the theory of unbounded operators (chapter 3). Special examples one sees in quantum mechanics are the position and the momentum operator. The theory of unbounded operators arose between the 1920s and 1930s. In particular, von Neumann and Marshall H. Stone (1903 – 1989) developed this theory.

The research project of this thesis is placed in chapter 4. One knows from courses on functional analysis that there is a spectral theory for (possibly unbounded) self-adjoint operators on a Hilbert space. The main idea is to use projection-valued measures and the Cayley transform. In this thesis we want to introduce another approach to the spectral theorem, namely the so-called "bounded transform". We recall that the Cayley transform gives a bijective correspondence between unitary and self-adjoint operators. The bounded transform gives in general a bijective correspondence between closed operators and pure contractions. We will only consider the case of self-adjoint operators. The main research results for this topic are Theorem 4.2 and Theorem 4.3. Another approach to a continuous functional calculus via the bounded transform is given by Theorem 4.4. At the end we will connect unbounded operators to von Neumann algebras by the method of affiliated von Neumann algebras. The

main research results here are Theorem 4.8, Theorem 4.9 and Theorem 4.10. Moreover we will discuss the joint spectrum for unbounded self-adjoint operators (Theorem 4.14).

To follow this thesis we assume the reader have basis knowledge of functional analysis and operator algebras. Thus, one should know the notion of Banach algebras and properties of these. Moreover, bounded operators on a Hilbert space are also assumed to be well understood. Those who are unfamiliar with this part of theory we advise to consult the appendix of this thesis. Here we give the most important definitions and theorems (without proofs), which we will use during the text.

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# 1. $C^*$ -algebras

In this chapter we cover the most important properties of  $C^*$ -algebras in so far as they are necessary for our subsequent topic of von Neumann algebras and their connection with unbounded self-adjoint operators. We discuss the characterization of abelian  $C^*$ -algebras as function spaces, representations of  $C^*$ -algebras, and the realization of  $C^*$ -algebras as algebras of bounded operators given by the so-called GNS-construction. Basic references are [13] and [17].

## 1.1. Basic properties of $C^*$ -algebras

We begin with the definition of a  $C^*$ -algebra:

**Definition 1.1.** A  $C^*$ -algebra  $A$  is a Banach  $*$ -algebra with the following norm property:  $\|a^*a\| = \|a\|^2$ , for each  $a \in A$ .

The last property is sometimes called the  $C^*$ -condition. Note that we do not require  $A$  to have a unit. If  $A$  has a unit, we call  $A$  a *unital*  $C^*$ -algebra. We note that we can always adjoin a unit to  $A$ . This means that there always exists a (minimal) unitization  $\tilde{A}$  of a  $C^*$ -algebra  $A$ , namely  $\tilde{A} = A \oplus \mathbb{C}$  with algebraic operation given by  $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$ . The unit of  $\tilde{A}$  is  $(0, 1)$ . The following theorem, which is given without proof, guarantees that  $\tilde{A}$  really becomes a  $C^*$ -algebra.

**Theorem 1.1.** If  $A$  is a  $C^*$ -algebra, then there exists a unique norm on its unitization  $\tilde{A}$  extending the norm of  $A$ , and making  $\tilde{A}$  a  $C^*$ -algebra.

For  $B \subseteq A$  define  $B^* := \{b^* : b \in B\}$ . We say  $B$  is a *self-adjoint* subset if  $B^* = B$ , in other words if  $B \subseteq A$  is a self-adjoint subalgebra, then  $B$  is a  $*$ -subalgebra. If, furthermore,  $B$  is norm-closed,  $B$  is called a  $C^*$ -subalgebra of  $A$ .

**Definition 1.2.** A map  $\varphi : A \rightarrow B$  between two  $C^*$ -algebras is called a  *$*$ -homomorphism* if it is linear and satisfies:  $\varphi(aa') = \varphi(a)\varphi(a')$  and  $\varphi(a^*) = \varphi(a)^*$ , for each  $a, a' \in A$ . If  $\varphi$  is bijective,  $\varphi$  is called a  *$*$ -isomorphism*.

Well-known  $C^*$ -algebras are the complex numbers  $\mathbb{C}$ , the space  $C_0(X)$  consisting of continuous functions on a locally compact Hausdorff space  $X$  that vanish at infinity, and the space  $\mathcal{L}^\infty(\Omega, \Sigma, \mu)$ , where  $(\Omega, \Sigma, \mu)$  is a measure space. Since we have an involution we can characterize some special properties of elements of a  $C^*$ -algebra. An element  $a \in A$  is called:

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- *self-adjoint* if  $a = a^*$
- *normal* if  $a^*a = aa^*$
- a *projection* if  $a = a^2 = a^*$
- *unitary* if  $a^*a = \mathbf{1} = aa^*$

We will now introduce some basic properties of  $C^*$ -algebras. The first theorem gives us a connection between the spectral radius and the norm of a self-adjoint element. Let  $A$  be a  $C^*$ -algebra and let  $a \in A$  be self-adjoint. We know that in general the spectral radius  $r(a)$  satisfies:  $r(a) \leq \|a\|$  (see [17],[13]). The following theorem yields equality.

**Theorem 1.2.** *Let  $a \in A$  be a self-adjoint element of a  $C^*$ -algebra  $A$ . Then  $r(a) = \|a\|$ .*

*Proof.* The equality follows from the  $C^*$ -condition. If  $a \in A$  is self-adjoint, then  $\|a\|^2 = \|a^*a\| = \|a^2\|$ . By induction we get  $\|a^{2^n}\| = \|a\|^{2^n}$  for each  $n \in \mathbb{N}$ . Now we can apply the spectral radius formula (see [17]) to conclude:  $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \left(\|a\|^{2^n}\right)^{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \|a\| = \|a\|$ . This gives us equality.  $\square$

This theorem has the following interpretation: the norm in a  $C^*$ -algebra  $A$  is uniquely determined by the algebraic structure of  $A$  (which determines the spectrum  $\sigma(a)$  for  $a \in A$ ), in that  $\|a\| = \sqrt{r(a^*a)}$  for general  $a \in A$ .

If we talk about  $*$ -homomorphisms between  $C^*$ -algebras we naturally have continuity in mind. Actually, continuity is automatic by the following lemma.

**Lemma 1.1.** *A  $*$ -homomorphism  $\varphi : A \rightarrow B$  between two  $C^*$ -algebras  $A$  and  $B$  is norm-decreasing, i.e.  $\|\varphi(a)\| \leq \|a\|$  for each  $a \in A$ . In particular,  $\varphi$  is bounded with  $\|\varphi\| \leq 1$ .*

*Proof.* We may assume that  $A$  and  $B$  are both unital, since we can always make  $A$  and  $B$  unital by Theorem 1.1 and extend  $\varphi$ . Now let  $a \in A$  be arbitrary, then  $\sigma(\varphi(a)) \subset \sigma(a)$  almost by definition of the spectrum, hence

$$\|\varphi(a)\|^2 = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2$$

Thus,  $\|\varphi(a)\| \leq \|a\|$  for each  $a \in A$  and in particular  $\|\varphi\| \leq 1$ .  $\square$

## 1.2. Commutative $C^*$ -algebras

The Gelfand transform is a construction which already works for commutative Banach algebras. Thus, let  $A$  be a commutative Banach algebra (we do not assume that  $A$  is unital). A subset  $J \subseteq A$  is called an *ideal* if  $J$  is a linear subspace of  $A$  and if  $x \in A, y \in J$ , then  $xy \in J$  and  $yx \in J$ . Such an ideal is called *proper* if  $J \neq A$  and  $J \neq \{0\}$ . It is called *maximal* if it



is proper and is not contained in another proper ideal. An example of an ideal is the kernel of a map  $\varphi : A \rightarrow B$  between two commutative Banach algebras (in particular, the kernel is a closed ideal). The following lemma ([17]) gives us some information about maximal ideals.

**Lemma 1.2.** *Let  $A$  be a commutative Banach algebra. Then:*

1. *If  $A$  contains a unit, every proper ideal of  $A$  is contained in a maximal ideal of  $A$ .*
2. *No proper ideal of  $A$  contains an invertible element of  $A$ .*
3. *Every maximal ideal of  $A$  is closed.*

If we have a closed proper ideal  $J$  of  $A$ , we can look at the canonical quotient map  $\pi : A \rightarrow A/J$  and note that  $A/J$  is also a Banach algebra. This is the last ingredient we need to prove the following theorem ([17]):

**Theorem 1.3.** *Let  $A$  be an unital commutative Banach algebra and let  $\Omega(A)$  be the space of all non-zero homomorphisms  $\tau : A \rightarrow \mathbb{C}$  (also called **characters** of  $A$ ). Then:*

1. *Every maximal ideal of  $A$  is the kernel of some  $\tau \in \Omega(A)$ .*
2. *If  $\tau \in \Omega(A)$ , the kernel of  $\tau$  is a maximal ideal.*
3. *An element  $x \in A$  is invertible in  $A \iff \tau(x) \neq 0$  for every  $\tau \in \Omega(A)$ .*
4. *An element  $x \in A$  is invertible in  $A \iff x$  lies in no proper ideal of  $A$ .*
5.  *$\lambda \in \sigma(x) \iff \tau(x) = \lambda$  for some  $\tau \in \Omega(A)$ .*

*Proof.* 1. We use Lemma 1.2 to prove the first statement. Thus, suppose that  $J$  is a maximal ideal of  $A$ . By the previous lemma we conclude that  $J$  is closed and hence  $A/J$  is a Banach algebra, as stated before the theorem. Now let  $x \in A$  and  $x \notin J$  be arbitrary, which can be done since  $J$  is maximal and therefore proper. Define  $I := \{ax + y : a \in A, y \in J\}$ . Then  $I$  is an ideal of  $A$ , since  $I$  is clearly a vector subspace of  $A$  and  $b \cdot (ax + y) = bax + by$  for each  $b \in A$ . Moreover  $by \in J$  since  $J$  was assumed to be an ideal, thus the product lies in  $I$ , hence  $I$  is an ideal. By construction we have  $J \subset I$  and  $I \neq J$  since  $x \in I$  and  $x \notin J$ . Maximality implies that  $I = A$ . We assumed that  $A$  is unital, thus we can find  $a \in A$  and  $y \in J$  such that  $ax + y = e$ . Now we can use the quotient map  $\pi : A \rightarrow A/J$  to conclude that  $\pi(a)\pi(x) = e$ . Therefore  $\pi(x)$  is invertible, and hence every non-zero element of  $A/J$  is invertible since  $x$  was arbitrary. Therefore, we can apply the Gelfand-Mazur Theorem (see [13],[17]) to conclude that  $A/J \cong \mathbb{C}$  as Banach algebras. In particular we have an isomorphism  $\varphi : A/J \rightarrow \mathbb{C}$ . Now define  $\tau := \varphi \circ \pi : A \rightarrow \mathbb{C}$ . Then  $\tau \in \Omega(A)$  and  $\text{Ker}(\tau) = J$ .

2. To prove the second statement have a look at  $\tau^{-1}(0) = \{x \in A : \tau(x) = 0\}$ . This is a maximal ideal, since it has codimension 1.

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3. Suppose that  $x \in A$  is invertible and let  $\tau \in \Omega(A)$ . Then  $\tau(x)\tau(x^{-1}) = \tau(xx^{-1}) = \tau(e) = 1$ , hence  $\tau(x) \neq 0$ . If we assume that  $x \in A$  is not invertible, then  $e \notin \{ax : a \in A\}$ . Therefore, it is a proper ideal and by Lemma 1.2 it is contained in a maximal ideal  $M$ . Now we can use part (1) of the theorem to conclude that there exists  $\tau \in \Omega(A)$  such that  $M = \text{Ker}(\tau)$ .
4. Again by Lemma 1.2, we know that no invertible element lies in a proper ideal. The converse is already given in the proof of part 3.
5. If  $\lambda \in \sigma(x)$ , then  $x - \lambda e$  is not invertible, so by part 3. there exists  $\tau \in \Omega(A)$  such that  $\tau(x - \lambda e) = 0$ , from which we conclude  $\tau(x) = \lambda$ . The converse also follows by application of part 3. of this theorem.

□

This theorem gives us a bijective correspondence between maximal ideals of  $A$  and characters of  $A$ . In particular, we see that  $\Omega(A) \neq \emptyset$  since we assumed that  $A$  is unital and therefore admits at least one maximal ideal. But there is more structure on  $\Omega(A)$ , namely a topological one. To see this, recall the definition of the spectrum. Let  $A$  be a Banach algebra and  $a \in A$ . The spectrum of  $a$ , which is denoted by  $\sigma(a)$ , is the set of  $\lambda \in \mathbb{C}$  such that  $a - \lambda \mathbf{1}$  is invertible. If  $A$  is non-unital we set  $\sigma(a) = \sigma_{\tilde{A}}(a)$ , hence it is the set of all  $\lambda \in \mathbb{C}$  such that  $a - \lambda \mathbf{1}$  is invertible with respect to the unitization  $\tilde{A}$  of  $A$ . The following proposition gives the connection between the character space  $\Omega(A)$  and the spectrum ([13]).

**Proposition 1.1.** *Let  $A$  be a commutative Banach algebra and  $a \in A$ . If  $A$  is unital, then  $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$ . If  $A$  is non-unital, then  $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}$ .*

The following theorem describes the topological structure of  $\Omega(A)$  if one endows  $\Omega(A)$  with the relative weak\*-topology (see [13]).

**Theorem 1.4.** *Suppose that  $A$  is an abelian Banach algebra. Take the weak\*-topology for  $\Omega(A)$ . Then  $\Omega(A)$  is a locally compact Hausdorff space. If  $A$  is unital, then  $\Omega(A)$  is compact.*

*Proof.* During the proof let  $A^*$  be the set of all continuous linear functionals on  $A$ . Then notice that  $\Omega(A) \subset A^*$  by Lemma 1.1. Then observe that  $\Omega(A) \cup \{0\}$  is weak\*-closed in the closed unit ball  $K$  of  $A^*$  (notice that the union with  $\{0\}$  is not necessary if  $A$  is unital). We know by the Banach-Alaoglu Theorem that the closed unit ball of  $A^*$  is weak\*-compact. Thus since  $\Omega(A) \cup \{0\}$  is weak\*-closed we may conclude that this set is also compact, hence  $\Omega(A)$  is locally compact. If  $A$  is unital, then  $\Omega(A)$  itself is weak\*-closed in the closed unit ball of  $A^*$  and therefore compact. □

We are almost ready now to formulate the Gelfand Representation Theorem for commutative Banach algebras. But first we have to make some observations. If  $A$  is not unital it

may be the case that  $\Omega(A)$  is empty (consider  $A = \{0\}$ ). Assuming that  $A$  has non-empty character space  $\Omega(A)$ , for each  $a \in A$  we define a map  $\hat{a} : \Omega(A) \rightarrow \mathbb{C}$  by

$$\hat{a}(\tau) = \tau(a)$$

By definition of the weak\*-topology each  $\hat{a}$  is a continuous map and  $\hat{a} \in C_0(\Omega(A))$  ([13]). In particular, the space  $C_0(\Omega(A))$  is a commutative  $C^*$ -algebra under the pointwise operations and the sup-norm  $\|\cdot\|_\infty$ . Now we state the *Gelfand Representation Theorem*.

**Theorem 1.5.** *Suppose that  $A$  is a commutative Banach algebra such that  $\Omega(A) \neq \emptyset$ . Then the map  $\varphi : A \rightarrow C_0(\Omega(A))$  given by  $\varphi(a) = \hat{a}$  is a norm-decreasing homomorphism satisfying  $r(a) = \|\hat{a}\|_\infty$  (where  $r$  is the spectral radius). The map  $\varphi$  is called the **Gelfand transform**.*

*Proof.* The fact that this map is a homomorphism is left to the reader. It follows from Theorem 1.3 that for  $a \in A$  the spectrum is given by  $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\}$  if  $A$  is unital, and by  $\sigma(a) = \{\tau(a) : \tau \in \Omega(A)\} \cup \{0\}$  if  $A$  is not unital (see [13]). Therefore, we may conclude by definition of the spectral radius that  $r(a) = \|\hat{a}\|_\infty$ , whence  $\varphi$  is norm-decreasing.  $\square$

The theory we have so far holds for commutative Banach algebras and therefore also for commutative  $C^*$ -algebras. There is a refinement of Theorem 1.5 which holds only for commutative  $C^*$ -algebras and gives a characterization of these.

**Theorem 1.6.** *If  $A$  is a non-zero commutative  $C^*$ -algebra, then the map  $\varphi : A \rightarrow C_0(\Omega(A))$  defined by  $\varphi(a) = \hat{a}$  is an isometric isomorphism.*

*Proof.* We already know by the previous theorem that  $\varphi$  is norm-decreasing. Now let  $\tau \in \Omega(A)$ . Then  $\varphi(a^*)(\tau) = \tau(a^*) = \overline{\tau(a)} = \varphi(a)^*(\tau)$  (see Proposition 1.3), hence  $\varphi$  is a \*-homomorphism. Furthermore, by Theorem 1.2, we have:

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

hence  $\varphi$  is isometric. Now we can conclude by the Stone-Weierstrass Theorem that  $\varphi(A) = C_0(\Omega(A))$ , since  $\varphi(A)$  is a closed \*-subalgebra of  $C_0(\Omega(A))$  that separates points of  $\Omega(A)$ ; indeed, since  $\tau$  is non-zero, for each  $\tau \in \Omega(A)$  there is an  $a \in A$  such that  $\varphi(a)(\tau) \neq 0$ .  $\square$

This theorem tells us that every commutative  $C^*$ -algebra can be realized as the space of continuous functions on some locally compact Hausdorff space that vanish at infinity. In general this cannot be expected for commutative Banach algebras. Note that we omit the assumption that  $\Omega(A) \neq \emptyset$ , since this is automatic in a non-zero  $C^*$ -algebra (see [13]). Now we will give a definition which will be important later.

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**Definition 1.3.** Let  $A$  be a  $C^*$ -algebra and  $S \subseteq A$  some subset. Then the  $C^*$ -algebra generated by the set  $S$ , denoted by  $C^*(S)$ , is the smallest  $C^*$ -subalgebra of  $A$  containing  $S$ . If  $A$  is unital and  $a \in A$  normal, then  $C^*(a)$  is generated by the unit and  $a \in A$  (in particular  $C^*(a)$  is commutative).

Notice that  $C^*(S)$  exists as the intersection of all  $C^*$ -subalgebras containing the set  $S$ .

## 1.3. Positivity

### 1.3.1. Positive elements

Considering the  $C^*$ -algebra  $C_0(X)$  for some locally compact Hausdorff space  $X$  we say, for  $f, g \in C_0(X)$ , that  $f \leq g$  if and only if  $f$  and  $g$  both are real-valued and  $f(x) \leq g(x)$  for all  $x \in X$ . We call  $f \in C_0(X)$  *positive* if and only if  $f \geq 0$ . Since we have the Gelfand transform, we know that we can define positivity for commutative  $C^*$ -algebras, but we also want to generalize this to arbitrary  $C^*$ -algebras.

**Definition 1.4.** Let  $A$  be a  $C^*$ -algebra. Then  $a \in A$  is called **positive** if  $\sigma(a) \subseteq \mathbb{R}_+$  and  $a = a^*$ .

Returning to the example of  $C_0(X)$ , self-adjointness of  $a \in A$  is the same as being real-valued. The condition that  $\sigma(a) \subseteq \mathbb{R}_+$  comes from the fact that for  $f \in C_0(X)$  we have:  $\sigma(f) = \overline{f(X)} \subseteq \mathbb{R}_+$ . This motivates the definition of positivity in an arbitrary  $C^*$ -algebra. The set of all positive elements in a  $C^*$ -algebra  $A$  is denoted by  $A_+$ .

Let us stick to the example of the  $C^*$ -algebra  $C_0(X)$ . If  $f \in C_0(X)$  is real-valued and positive one can define  $g(x) := \sqrt{f(x)}$ , so that  $g \in C_0(X)$  and  $g \geq 0$ . In other words: we can take the square root of the positive elements of  $C_0(X)$ , and therefore, by the Gelfand transform, of positive elements of commutative  $C^*$ -algebras. But the existence of square roots can be generalized.

**Theorem 1.7.** Let  $A$  be a  $C^*$ -algebra and  $a \in A$  be positive. Then there exists a unique positive element  $b \in A$  such that  $a = b^2$ . Furthermore  $b = \sqrt{a}$  lies in the abelian  $C^*$ -subalgebra of  $A$  generated by  $a$ .

*Proof.* The existence of this element  $b \in A$  is not difficult, since we have the Gelfand transform. Indeed, since  $a \in A$  is positive it is by definition self-adjoint and therefore the  $C^*$ -algebra  $C^*(a)$  generated by  $a \in A$ , is commutative. Hence it is isometrically  $*$ -isomorphic to  $C_0(X)$ , for some locally compact Hausdorff space  $X$ . But we can apply the facts about  $C_0(X)$  we saw above and transform back to  $C^*(a)$ . We have to prove uniqueness of  $b = \sqrt{a}$ : assume that there is also another element  $c \in A$  such that  $c \in A_+$  and  $c^2 = a$ , and  $c$  commutes with  $a$ . If this is the case,  $c$  has also to commute with  $b$ , since  $b$  is the limit of a sequence of polynomials in  $a$  (for this argument we use the fact that  $\sqrt{a}$  can be defined by polynomials

via the continuous functional calculus). Now let  $B := C^*(b, c)$ , then by the observations before,  $B$  is a commutative  $C^*$ -algebra, and therefore we have an isometric  $*$ -isomorphism  $\varphi : B \rightarrow C_0(Y)$  (with  $Y$  a locally compact Hausdorff space) by the Gelfand transform. Hence  $\varphi(b)$  and  $\varphi(c)$  are both positive square roots of  $\varphi(a)$  (in  $C_0(Y)$ ). This means that  $\varphi(b) = \varphi(c)$  and hence  $b = c$ , so that the square root is unique.  $\square$

This proof was based on the Gelfand transform. One can also prove this by the methods of symbolic calculus from Functional Analysis as shown in [4],[10] and [17].

Remember, if  $f \in C_0(X)$ , then we can look at  $|f|$ , the absolute value of  $f$ . In particular, one has  $|f| = \sqrt{f^2}$ . Let  $a \in A$  be self-adjoint, then  $a^2$  is positive by applying the spectral mapping theorem, hence  $\sqrt{a^2}$  is well-defined. This leads to the following definition.

**Definition 1.5.** *Let  $A$  be a  $C^*$ -algebra and suppose  $a \in A$  is self-adjoint. Define the **absolute value** of  $a$  by  $|a| = \sqrt{a^2}$ . Moreover, define  $a_+ = \frac{1}{2}(|a| + a)$  and  $a_- = \frac{1}{2}(|a| - a)$ .*

By Gelfand's Theorem one obtains that  $|a|, a_+$  and  $a_-$  are all positive and that the following relations hold:  $a_+a_- = 0$  and  $a = a_+ - a_-$ .

**Proposition 1.2.** *Let  $A$  be a  $C^*$ -algebra and  $a \in A$  self-adjoint. Then  $a = a_+ - a_-$  for unique  $a_+, a_- \in A_+$  with  $a_+a_- = 0$ .*

Let us return to the commutative  $C^*$ -algebra  $C_0(X)$ . We then can express positivity in terms of the norm:  $f \geq 0$  if and only if  $\|f - t\| \leq t$  for at least one  $t \in \mathbb{R}_+$  with  $\|f\| \leq t$ . This, together with the Gelfand transform, can be used to derive the following other characterization of positivity in an arbitrary  $C^*$ -algebra ([4],[13]).

**Lemma 1.3.** *Let  $A$  be a  $C^*$ -algebra with unit. A self-adjoint element  $a \in A$  is positive if  $\|a - t\| \leq t$  for at least one  $t \in \mathbb{R}_+$ , with  $\|a\| \leq t$ . Conversely: if  $\|a\| \leq t$  and  $a \geq 0$ , then  $\|a - t\| \leq t$ .*

From this characterization of positivity we can derive other helpful properties. The first one follows from the previous lemma:  $A_+$  is a closed set in  $A$ . But this is not all. We want to show that  $A_+$  is closed under addition.

**Lemma 1.4.** *The sum of two positive elements in a  $C^*$ -algebra is positive.*

*Proof.* Let  $A$  be a  $C^*$ -algebra and  $a, b \in A_+$ . Since we can unitize every  $C^*$ -algebra we may assume that  $A$  is unital. Now we can apply Theorem 1.8. Since  $a \geq 0$  and  $b \geq 0$ , we get  $\|a - \|a\|\| \leq \|a\|$  and  $\|b - \|b\|\| \leq \|b\|$ . Therefore, we can make the following estimate:

$$\|a + b - \|a\| - \|b\|\| \leq \|a - \|a\|\| + \|b - \|b\|\| \leq \|a\| + \|b\|$$

which shows, by the same theorem, that  $a + b \geq 0$ .  $\square$

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The goal of the next discussion is to show that  $A_+ = \{a^*a : a \in A\}$ . Theorem 1.7 gives  $A_+ \subseteq \{a^*a : a \in A\}$ . But to show equality we have to show that  $a^*a \in A_+$ , for an arbitrary element  $a \in A$ . This is stated in the following lemma.

**Lemma 1.5.** *Let  $A$  be a  $C^*$ -algebra. Then  $a^*a \geq 0$  for each  $a \in A$ .*

*Proof.* First we want to show that  $a = 0$  if  $-a^*a \in A_+$ . To that effect, we use the fact that  $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$  for all  $a, b \in A$ . In particular, we obtain  $\sigma(-aa^*) \setminus \{0\} = \sigma(-a^*a) \setminus \{0\}$ , hence  $-aa^* \geq 0$  since  $-a^*a \geq 0$  by assumption. Now decompose  $a \in A$  by  $a = b + ic$  for some  $b, c \in A$  which are self-adjoint elements. In particular,  $b := \frac{1}{2}(a + a^*)$  and  $c := \frac{1}{2i}(a - a^*)$ . Then we find  $a^*a + aa^* = 2b^2 + 2c^2 \geq 0$ , hence  $a^*a = 2b^2 + 2c^2 - aa^*$ , and therefore  $a^*a \in A_+$ . Thus  $\sigma(a^*a) = \mathbb{R}_+ \cap -\mathbb{R} = \{0\}$ , thus  $\|a\|^2 = \|a^*a\| = r(a) = 0$  by the  $C^*$ -condition and Theorem 1.2. Since  $\|\cdot\|$  is a norm, we obtain  $a = 0$ .

Now let  $a \in A$  be arbitrary. Define  $b := a^*a$ , so that  $b$  is self-adjoint, whence  $b = b_+ - b_-$ . Moreover, let  $c := ab_-$ , then

$$-c^*c = -b_-a^*ab_- = -b_-(b_+ - b_-)b_- = (b_-)^3 \in A_+$$

Thus we may conclude that  $c = 0$ , since we have already proven this. But then  $b_- = 0$  and hence,  $b_+ = a^*a \in A_+$ , which completes the proof.  $\square$

**Theorem 1.8.** *Let  $A$  be a  $C^*$ -algebra. Then  $A_+ = \{a^*a : a \in A\}$ .*

*Proof.* We have to prove two inclusions:

$\subseteq$ : Let  $a \in A_+$  be arbitrary, then we know by Theorem 1.7, that there exists  $b \in A_+$  such that  $b^2 = a$ . But since  $b$  is also self-adjoint we may say  $a = b^2 = bb = b^*b$ . This proves the first inclusion.

$\supseteq$ : This inclusion is given by Lemma 1.5  $\square$

**Theorem 1.9.** *Let  $A$  be a  $C^*$ -algebra and  $a \in A$ . Then the following are equivalent:*

- $a \geq 0$
- $a^* = a$  and  $\sigma(a) \subset \mathbb{R}_+$
- $a = b^*b$  for some  $b \in A$
- $a = b^2$  for some self-adjoint  $b \in A$

### 1.3.2. Approximate Units

We already mentioned that an arbitrary  $C^*$ -algebra need not have a unit. We know that we can unitize every  $C^*$ -algebra, but as an alternative there is the concept of a so-called *approximate unit*.

**Definition 1.6.** Let  $A$  be a  $C^*$ -algebra. An **approximate unit** for  $A$  is an increasing net  $(u_\lambda)_{\lambda \in \Lambda}$  of positive elements in the closed unit ball of  $A$  such that  $a = \lim_\lambda au_\lambda$  for all  $a \in A$ , or  $\lim_{\lambda \in \Lambda} \|a - au_\lambda\| = 0$  for all  $a \in A$ . Equivalently,  $a = \lim_\lambda u_\lambda a$  for all  $a \in A$ .

The following theorem tells us that every  $C^*$ -algebra has an approximate unit.

**Theorem 1.10.** Every  $C^*$ -algebra  $A$  admits an approximate unit. If  $\Lambda$  is the upwards-directed set of all  $a \in A_+$  such that  $\|a\| < 1$  and  $u_\lambda = \lambda$  for all  $\lambda \in \Lambda$ , then  $(u_\lambda)_{\lambda \in \Lambda}$  is an approximate unit for  $A$ . In addition: when  $A$  is separable,  $\Lambda$  may be taken to be countable.

We are not going to prove this theorem here, but refer to [10] and [13]. The main idea of the proof is the following: if  $\Lambda$  is the upwards-directed set of all  $a \in A_+$  such that  $\|a\| < 1$  and  $u_\lambda = \lambda$  for all  $\lambda \in \Lambda$ , then  $(u_\lambda)_{\lambda \in \Lambda}$  is an approximate unit for  $A$ . In addition: when  $A$  is separable,  $\Lambda$  may be taken to be countable. Since an approximate unit is a rather technical concept, we will only state two important theorems and their corollaries (see [4],[10] and [13]).

**Theorem 1.11.** If  $I$  is a closed two-sided ideal in a  $C^*$ -algebra  $A$ , then  $I$  is self-adjoint and therefore is a  $C^*$ -subalgebra of  $A$ .

**Theorem 1.12.** If  $I$  is a closed two-sided ideal of a  $C^*$ -algebra  $A$ , then the quotient  $A/I$  is a  $C^*$ -algebra under its usual operations and the quotient norm.

**Corollary 1.1.** If  $\varphi : A \rightarrow B$  is an injective  $*$ -homomorphism between two  $C^*$ -algebras  $A$  and  $B$ , then  $\varphi$  is isometric.

**Corollary 1.2.** If  $\varphi : A \rightarrow B$  is a  $*$ -homomorphism between two  $C^*$ -algebras  $A$  and  $B$ , then  $\varphi(A)$  is a  $C^*$ -subalgebra of  $B$ .

### 1.3.3. Positive linear functionals

We know what it means for a function from a topological space  $X$  to  $\mathbb{C}$  to be positive, namely  $f(x) \geq 0$  for each  $x \in X$ . But we have the concept of positivity also for arbitrary  $C^*$ -algebras, so that we may also talk about positive maps between  $C^*$ -algebras.

**Definition 1.7.** A linear map  $\varphi : A \rightarrow B$  between two  $C^*$ -algebras  $A$  and  $B$  is called **positive** if  $\varphi(A_+) \subseteq B_+$ .

In particular, we say that a linear functional  $\tau : A \rightarrow \mathbb{C}$  is positive if  $\tau(a) \geq 0$  for all  $a \in A_+$ . Observe that every  $*$ -homomorphism is positive. Also, for positive linear functionals, as for  $*$ -homomorphisms the property of continuity is automatic.

**Theorem 1.13.** Let  $A$  be a  $C^*$ -algebra and  $\tau$  a positive linear functional on  $A$ . Then  $\tau$  is bounded (and therefore continuous).

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*Proof.* We prove by contradiction. Suppose  $\tau$  is not bounded. Then  $\sup_{a \in (A_+)_1} \tau(a) = \infty$ , where  $(A_+)_1$  is the closed unit ball of  $A$  intersected with  $A_+$ . The reason why this is true is the following: Suppose  $\tau$  is a positive linear functional such that there exists  $M \in \mathbb{R}_+$  such that  $|\tau(a)| \leq M$  for all  $a \in (A_+)_1$ . Then we show that  $\|\tau\| \leq 4M$ . First, we consider the case where  $a \in A$  is self-adjoint and  $\|a\| \leq 1$ . Then we have  $a_+, a_- \in (A_+)_1$  such that  $a = a_+ - a_-$  and therefore  $|\tau(a)| = |\tau(a_+) - \tau(a_-)| \leq 2M$ . Now, if  $a \in A_1$  is arbitrary, then we can write  $a = b + ic$  with  $b$  and  $c$  self-adjoint and both with norm at most 1. Thus  $|\tau(a)| = |\tau(b) + i\tau(c)| \leq 4M$ . This shows that  $\|\tau\| \leq 4M$ . The contrapositive of this implication gives us the fact we stated in the beginning. Since this is the case, one can find a sequence  $(a_n)_{n \in \mathbb{N}}$  in  $(A_+)_1$  such that  $2^n \leq \tau(a_n)$  for each  $n \in \mathbb{N}$ . Now define  $a := \sum_{n=0}^{\infty} 2^{-n} a_n$ . Then  $a \in A_+$ , hence  $1 \leq \tau(2^{-n} a_n)$  and therefore  $N \leq \sum_{n=0}^{N-1} \tau(2^{-n} a_n) = \tau\left(\sum_{n=0}^{N-1} 2^{-n} a_n\right) \leq \tau(a)$ . But this holds for each  $N \in \mathbb{N}$ , which is impossible. Thus  $\tau$  is bounded.  $\square$

**Proposition 1.3.** *Let  $\tau$  be a positive linear functional on a  $C^*$ -algebra  $A$ . Then  $\tau(a^*) = \overline{\tau(a)}$  and  $|\tau(a)|^2 \leq \|\tau\| \tau(a^*a)$  for each  $a \in A$ .*

*Proof.* The first equality can be proved as follows: let  $a \in A$  be self-adjoint, then  $a = a_+ - a_-$  by Proposition 1.2. Hence  $\tau(a) = \tau(a_+) - \tau(a_-)$  which means that  $\tau(a) \in \mathbb{R}$ . Now let  $a \in A$  be arbitrary and write  $a = b + ic$ , for unique self-adjoint elements  $b, c \in A$  (in particular  $b = \frac{1}{2}(a + a^*)$  and  $c = \frac{1}{2i}(a - a^*)$ ). Then  $\tau(a^*) = \tau(b - ic) = \tau(b) - i\tau(c) = \overline{\tau(b) + i\tau(c)} = \overline{\tau(a)}$

The second part we get by the existence of an approximate unit. Let  $(u_\lambda)_{\lambda \in \Lambda}$  be an approximate unit of the  $C^*$ -algebra  $A$ . Then we can make the following estimate:

$$|\tau(a)|^2 = \lim_{\lambda} |\tau(u_\lambda a)|^2 \leq \sup_{\lambda} \tau(u_\lambda^2) \tau(a^*a) \leq \|\tau\| \tau(a^*a)$$

where we use the Cauchy-Schwarz inequality  $|\tau(ba)|^2 \leq \tau(b^*b)\tau(a^*a)$  for each  $a, b \in A$ .  $\square$

The following theorem gives us a criterion to decide whether a bounded linear functional  $\tau$  is positive. Once again, we use an approximate unit.

**Theorem 1.14.** *Suppose that  $\tau$  is a bounded linear functional on a  $C^*$ -algebra  $A$ . Then the following three statements are equivalent:*

1.  $\tau$  is positive.
2. For each approximate unit  $(u_\lambda)_{\lambda \in \Lambda}$  of  $A$  one has  $\|\tau\| = \lim_{\lambda} \tau(u_\lambda)$ .
3. There exists an approximate unit  $(u_\lambda)_{\lambda \in \Lambda}$  of  $A$  such that  $\|\tau\| = \lim_{\lambda} \tau(u_\lambda)$ .

*Proof.* (1)  $\Rightarrow$  (2) : Without loss of generality, we may assume that  $\|\tau\| = 1$ , since we know that  $\tau$  is bounded. Now let  $(u_\lambda)_{\lambda \in \Lambda}$  be an arbitrary approximate unit of  $A$ . Then  $(\tau(u_\lambda))_{\lambda \in \Lambda}$  is an increasing net in  $\mathbb{R}$ , since  $\tau$  is positive. By the definition of an approximate unit, we



may infer that this net is bounded from above by 1. Therefore, we have convergence to its supremum, i.e.  $\lim_{\lambda} \tau(u_{\lambda}) \leq 1$ . Now let  $a \in A$  be such that  $\|a\| \leq 1$ . Then:

$$|\tau(u_{\lambda}a)|^2 \leq \tau(u_{\lambda}^2)\tau(a^*a) \leq \tau(u_{\lambda})\tau(a^*a) \leq \tau(u_{\lambda})$$

so  $|\tau(a)|^2 \leq \lim_{\lambda} \tau(u_{\lambda})$ , which means that  $1 \leq \lim_{\lambda} \tau(u_{\lambda})$ . This yields us the equality we wanted.

(2)  $\Rightarrow$  (3) : Trivial.

(3)  $\Rightarrow$  (1) : Let  $(u_{\lambda})_{\lambda \in \Lambda}$  be an approximate unit for the  $C^*$ -algebra  $A$ , such that  $1 = \lim_{\lambda} \tau(u_{\lambda})$ , which exists by assumption (3). We have to show that  $\tau$  is a positive linear functional. First assume that  $a \in A$  with  $a^* = a$  and  $\|a\| \leq 1$ , and let  $\tau(a) = \alpha + i\beta$ ,  $\alpha, \beta \in \mathbb{R}$ , with  $\alpha = \Re\tau(a)$  and  $\beta = \Im\tau(a)$ . We first want to show that  $\tau(a) \in \mathbb{R}$ . Without loss of generality we may assume that  $\beta \leq 0$ . Now let  $n \in \mathbb{N}$  be arbitrary. Then

$$\|a - inu_{\lambda}\|^2 = \|(a + inu_{\lambda})(a - inu_{\lambda})\| = \|a^2 + n^2u_{\lambda}^2 - in(au_{\lambda} - u_{\lambda}a)\| \leq 1 + n^2 + n\|au_{\lambda} - u_{\lambda}a\|$$

thus  $|\tau(a - inu_{\lambda})|^2 \leq 1 + n^2 + n\|au_{\lambda} - u_{\lambda}a\|$ . If we now take the limit of the nets  $(\tau(a - inu_{\lambda}))_{\lambda \in \Lambda}$  and  $(au_{\lambda} - u_{\lambda}a)_{\lambda \in \Lambda}$  we obtain  $\tau(a) - in$  and 0 respectively. Therefore, we find by taking these limits that  $|\alpha + i\beta - in|^2 \leq 1 + n^2$  which means in particular that  $\alpha^2 + \beta^2 - 2n\beta + n^2 \leq 1 + n^2$ , thus  $-2n\beta \leq 1 - \beta^2 - \alpha^2$ . Since we assumed that  $\beta \leq 0$  and that  $n \in \mathbb{N}$  was arbitrary, we may conclude that  $\beta = 0$ . Thus  $\tau(a) \in \mathbb{R}$  if  $a \in A$  is self-adjoint. Now let  $a \in (A_+)_1$  (i.e.  $a \geq 0$  and  $\|a\| \leq 1$ ), then  $u_{\lambda} - a = (u_{\lambda} - a)^*$  and  $\|u_{\lambda} - a\| \leq 1$ , hence  $\tau(u_{\lambda} - a) \leq 1$ . This means:  $1 - \tau(a) = \lim_{\lambda} \tau(u_{\lambda} - a) \leq 1$ , hence  $\tau(a) \geq 0$ . We therefore have shown that  $\tau$  is positive.  $\square$

From this theorem, we obtain two important properties:

**Theorem 1.15.** *Let  $\tau$  be a bounded linear functional on a unital  $C^*$ -algebra  $A$ . Then  $\tau$  is positive  $\iff \tau(1) = \|\tau\|$ .*

*Proof.* Since  $A$  is unital the net that is constantly 1 is an approximate unit. Now we can apply the previous theorem.  $\square$

**Corollary 1.3.** *If  $\tau$  and  $\omega$  are both positive linear functionals on a  $C^*$ -algebra  $A$ , then  $\|\tau + \omega\| = \|\tau\| + \|\omega\|$ .*

*Proof.* Let  $(u_{\lambda})_{\lambda \in \Lambda}$  be an approximate unit of the  $C^*$ -algebra  $A$ . Applying the previous theorem to  $\tau, \omega$  and  $\tau + \omega$  we obtain  $\|\tau\| = \lim_{\lambda} \tau(u_{\lambda})$ ,  $\|\omega\| = \lim_{\lambda} \omega(u_{\lambda})$   $\|\tau + \omega\| = \lim_{\lambda} (\tau + \omega)(u_{\lambda})$ . Thus we can conclude:

$$\|\tau + \omega\| = \lim_{\lambda} (\tau + \omega)(u_{\lambda}) = \lim_{\lambda} \tau(u_{\lambda}) + \lim_{\lambda} \omega(u_{\lambda}) = \|\tau\| + \|\omega\| \quad \square$$

## 1. $C^*$ -algebras

Now we come to the definition of a special class of positive linear functionals, which are important in the context of quantum mechanics.

**Definition 1.8.** Let  $\tau$  be a positive linear functional on a  $C^*$ -algebra  $A$ . Then  $\tau$  is called a **state** if  $\|\tau\| = 1$ . The set of all states on  $A$  is denoted by  $S(A)$ .

To close this subsection, we now discuss a theorem about states that gives us the information that we have "enough" states on an arbitrary  $C^*$ -algebra.

**Theorem 1.16.** Let  $A$  be a non-zero  $C^*$ -algebra, and let  $a \in A$  be normal. Then there exists a state  $\tau$  such that  $\|a\| = |\tau(a)|$ .

*Proof.* We may assume without loss of generality that  $a \neq 0$ . Let  $B := C^*(\mathbf{1}, a) \subseteq \tilde{A}$ . Then  $B$  is abelian, since  $a \in A$  was assumed to be normal. By the Gelfand transform we can conclude that  $\hat{a}$  is continuous on  $\Omega(B)$ . This implies that there exists a character  $\tau_2$  on  $B$  with  $\|a\| = \|\hat{a}\|_\infty = |\tau_2(a)|$ . Now we can apply the Hahn-Banach Theorem to find a bounded linear functional  $\tau_1$  on  $\tilde{A}$  that extends  $\tau_2$  and preserves the norm. Thus  $\|\tau_1\| = 1$ . Then  $\tau_1(\mathbf{1}) = \tau_2(\mathbf{1}) = 1$ , which means that  $\tau_1$  is positive. Define  $\tau := \tau_1|_A$ , which is positive. Then we have  $\|a\| = |\tau(a)|$ . To prove that  $\tau$  is a state we need  $\|\tau\| = 1$ . Now  $\|\tau\| \|a\| \geq |\tau(a)| \|a\|$ , which means that  $\|\tau\| \geq 1$ . The other inequality is obvious, since  $\|\tau\| \leq \|\tau_1\| = 1$ . Thus  $\tau$  is a state on  $A$ .  $\square$

## 1.4. The Gelfand-Naimark-Segal Representation

We saw in the previous section that we can realize commutative  $C^*$ -algebras as function spaces. The question is: can something similar be done for arbitrary  $C^*$ -algebras? The answer is given by the so-called *GNS-construction* (Gelfand-Naimark-Segal construction), which tells us that every  $C^*$ -algebra can be realized as a norm-closed self-adjoint subalgebra of  $B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . But first we have to build up some theory about representations of  $C^*$ -algebras.

**Definition 1.9.** A **representation** of a  $C^*$ -algebra  $A$  is a pair  $(\varphi, \mathcal{H})$ , where  $\mathcal{H}$  is a Hilbert space and  $\varphi$  is a  $*$ -homomorphism  $\varphi : A \rightarrow B(\mathcal{H})$ . We call a representation  $(\varphi, \mathcal{H})$  **faithful** if  $\varphi$  is injective.

If  $(\varphi_\iota, \mathcal{H}_\iota)_{\iota \in I}$  is a family of representations over some index set  $I$ , we can construct a new representation  $(\varphi, \mathcal{H})$  by  $\mathcal{H} = \bigoplus_{\iota \in I} \mathcal{H}_\iota$  and  $\varphi(a)(x_\iota)_{\iota \in I} = (\varphi_\iota(a)(x_\iota))_{\iota \in I}$  for each  $a \in A$  and all  $(x_\iota)_{\iota \in I} \in \mathcal{H}$ . By definition one has

$$\bigoplus_{\iota \in I} \mathcal{H}_\iota := \left\{ (x_\iota)_{\iota \in I} \in \prod_{\iota \in I} \mathcal{H}_\iota : \sum_{\iota \in I} \|x_\iota\|^2 < \infty \right\}$$

Now let  $\tau$  be a positive linear functional on  $A$ . We want to construct a representation from this functional. Define  $N_\tau := \{a \in A : \tau(a^*a) = 0\}$ . This is a closed left ideal of  $A$ , since we have the Cauchy-Schwarz inequality

$$\tau(b^*a^*ab) \leq \|a^*a\| \cdot \tau(b^*b)$$

for all  $a, b \in A$  (see [13]). Now we can take the quotient  $A/N_\tau$ . The bilinear map  $\langle \cdot, \cdot \rangle : A/N_\tau \times A/N_\tau \rightarrow \mathbb{C}$  given by  $\langle a + N_\tau, b + N_\tau \rangle := \tau(b^*a)$  is well defined and yields an inner product on  $A/N_\tau$ . Therefore we have constructed a pre-Hilbert space. Now we can take the completion of  $A/N_\tau$  with respect to the inner product. We write  $\mathcal{H}_\tau$  for the resulting space. Moreover: if  $a \in A$ , then we define  $\varphi : A \rightarrow B(A/N_\tau)$ ,  $\varphi(a)(b + N_\tau) = ab + N_\tau$ . This gives us a bounded operator:

$$\|\varphi(a)(b + N_\tau)\|^2 = \tau(b^*a^*ab) \leq \|a\|^2 \cdot \tau(b^*b) = \|a\|^2 \cdot \|b + N_\tau\|^2$$

Thus the operator  $\varphi(a)$  has a unique extension to a bounded operator on  $\mathcal{H}_\tau$ . Now we can define  $\varphi_\tau : A \rightarrow B(\mathcal{H}_\tau) : a \mapsto \varphi_\tau(a)$ , which is a  $*$ -homomorphism. Thus we obtain a representation  $(\varphi_\tau, \mathcal{H}_\tau)$ , which is called the *Gelfand-Naimark-Segal representation* (associated with  $\tau$ ). We can also define the so-called *universal representation*  $(\varphi, \mathcal{H})$  where  $\mathcal{H} = \bigoplus_{\tau \in S(A)} \mathcal{H}_\tau$  and  $\tau$  runs over the state space  $S(A)$ . The Gelfand-Naimark Theorem is the following.

**Theorem 1.17.** *If  $A$  is a  $C^*$ -algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

*Proof.* Let  $(\varphi, \mathcal{H})$  be the universal representation of  $A$  and suppose  $\varphi(a) = 0$  for some  $a \in A$ . Then we can find a state  $\tau$  on  $A$  such that  $\|a^*a\| = \tau(a^*a)$  (Theorem 1.18). Let  $b = (a^*a)^{\frac{1}{4}}$ , then  $\|a\|^2 = \tau(a^*a) = \tau(b^4) = \|\varphi_\tau(b)(b + N_\tau)\|^2 = 0$ . Thus  $a = 0$ , and therefore  $\varphi$  is injective.  $\square$

This theorem gives us the statement from the begin of this section, since the theorem implies that  $A$  is  $*$ -isomorphic to a norm-closed self-adjoint subalgebra of  $B(\mathcal{H})$ .

## 1.5. Representations

In the previous section about the GNS-construction we introduced representations of a  $C^*$ -algebra on a Hilbert space  $\mathcal{H}$ . We now want to discuss representations in some more detail.

**Definition 1.10.** *Let  $(\varphi, \mathcal{H})$  be a representation on a Hilbert space  $\mathcal{H}$  of a  $C^*$ -algebra  $A$ . We call  $x \in \mathcal{H}$  a cyclic vector if  $\overline{\varphi(A)x} = \mathcal{H}$ . If  $(\varphi, \mathcal{H})$  has a cyclic vector, then we call this representation *cyclic*.*

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**Definition 1.11.** Let  $(\varphi, \mathcal{H})$  be a representation on a Hilbert space  $\mathcal{H}$  of a  $C^*$ -algebra  $A$ . We say that  $A$  acts **non-degenerately** on  $\mathcal{H}$  if  $\overline{\varphi(A)\mathcal{H}} = \mathcal{H}$ .

This definition of a cyclic vector can also be used in the context of states and the GNS-construction.

**Theorem 1.18.** Let  $A$  be a  $C^*$ -algebra and  $\tau \in S(A)$ . Then there exists a unique vector  $x_\tau \in \mathcal{H}_\tau$  such that

$$\tau(a) = \langle a + N_\tau, x_\tau \rangle$$

for  $a \in A$ . Furthermore,  $x_\tau$  is a cyclic unit vector for the GNS-representation  $(\varphi_\tau, \mathcal{H}_\tau)$  and

$$\varphi_\tau(a)x_\tau = a + N_\tau$$

*Proof.* Define a function  $\rho_0 : A/N_\tau \rightarrow \mathbb{C}$  by  $\rho_0(a+N_\tau) = \tau(a)$ . This map is well defined, linear, and norm-decreasing, since  $\tau$  was assumed to be a state, i.e.  $|\rho_0(a+N_\tau)| = |\tau(a)| \leq \|a\|$ . By continuity this gives rise to a norm-decreasing extension  $\rho : \mathcal{H}_\tau \rightarrow \mathbb{C}$ . By the Riesz-Fréchet Theorem, we can find a unique  $x_\tau \in \mathcal{H}_\tau$  such that  $\rho(y) = \langle y, x_\tau \rangle$  for each  $y \in \mathcal{H}_\tau$ . By construction, this gives the unique existence of  $x_\tau \in \mathcal{H}_\tau$  with  $\tau(a) = \langle a + N_\tau, x_\tau \rangle$  for each  $a \in A$ . Now let  $a \in A$ . Then

$$\langle b + N_\tau, \varphi_\tau(a)x_\tau \rangle = \langle a^2b + N_\tau, x_\tau \rangle = \tau(a^*b) = \langle b + N_\tau, a + N_\tau \rangle$$

for each  $b \in A$ . This implies  $\varphi_\tau(a)x_\tau = a + N_\tau$ . Since  $\varphi_\tau(A)x_\tau$  is dense in  $A/N_\tau$ , it is also dense in  $\mathcal{H}_\tau$ . Conclusion:  $x_\tau \in \mathcal{H}_\tau$  is a cyclic vector for  $(\varphi_\tau, \mathcal{H}_\tau)$ . The last thing we have to show is that  $x_\tau$  is a vector of norm one. Since  $x_\tau$  is cyclic,  $\varphi_\tau(A)$  acts non-degenerately on  $\mathcal{H}_\tau$ . Now let  $(u_\lambda)_{\lambda \in \Lambda}$  be an approximate unit of  $A$ . Then  $(\varphi_\tau(u_\lambda))_{\lambda \in \Lambda}$  is an approximate unit for  $\varphi_\tau(A)$ , hence it converges strongly to  $id_{\mathcal{H}_\tau}$ , since

$$\|\varphi_\tau(u_\lambda)(x + N_\tau) - id_{\mathcal{H}_\tau}(x + N_\tau)\| = \|(u_\lambda x - x) + N_\tau\|$$

Thus

$$\|x_\tau\|^2 = \langle x_\tau, x_\tau \rangle = \lim_{\lambda} \langle \varphi_\tau(u_\lambda)(x_\tau), x_\tau \rangle = \lim_{\lambda} \tau(u_\lambda) = \|\tau\| = 1 \quad \square$$

**Definition 1.12.** The unique vector  $x_\tau \in \mathcal{H}_\tau$  from the previous theorem is called the **canonical cyclic vector**.

In what follows we want to compare various functionals: for positive linear functionals,  $\tau$  and  $\omega$ , we define  $\tau \leq \omega$  if and only if  $\omega - \tau$  is positive, hence if and only if  $(\omega - \tau)(a) \geq 0$  for each  $a \in A_+$ . If  $\tau \leq \omega$ , then we say  $\tau$  is majorised by  $\omega$ . We want to relate this to the canonical cyclic vector. For this we need a definition.

**Definition 1.13.** Let  $B$  be a subset of an algebra  $A$ . The **commutant** of  $B$  is the set of all elements of  $A$  that commute with every element of  $B$ . The commutant is denoted by  $B'$ . In symbols:

$$B' := \{a \in A : ab = ba \ \forall b \in B\}$$

**Theorem 1.19.** Let  $A$  be a  $C^*$ -algebra,  $\tau \in S(A)$ , and  $\omega$  a positive linear functional on  $A$ . If  $\omega \leq \tau$ , then there exists a unique operator  $v \in \varphi_\tau(A)'$  with  $0 \leq v \leq 1$  such that  $\omega(a) = \langle \varphi_\tau(a)v x_\tau, x_\tau \rangle$ .

*Proof.* Since the assumption gives  $\omega \leq \tau$ , we can define a sesquilinear form  $\sigma : A/N_\tau \times A/N_\tau \rightarrow \mathbb{C}$  by  $\sigma(a + N_\tau, b + N_\tau) = \omega(b^*a)$ . We see that

$$|\omega(b^*a)| \leq \sqrt{\omega(b^*b)}\sqrt{\omega(a^*a)} \leq \sqrt{\tau(b^*b)}\sqrt{\tau(a^*a)} = \|b + N_\tau\| \cdot \|a + N_\tau\|$$

which implies that  $\|\sigma\| \leq 1$ . Thus we can extend  $\sigma$  to  $\tilde{\sigma} : \mathcal{H}_\tau \rightarrow \mathbb{C}$ , also with  $\|\tilde{\sigma}\| \leq 1$ . This means that we can find  $v : \mathcal{H}_\tau \rightarrow \mathcal{H}_\tau$  such that  $\langle v(x), y \rangle = \tilde{\sigma}(x, y)$  for each  $x, y \in \mathcal{H}_\tau$  by Riesz-Fréchet. Moreover,  $\|v\| \leq 1$ . Thus we get

$$\omega(b^*a) = \sigma(a + N_\tau, b + N_\tau) = \langle v(a + N_\tau), b + N_\tau \rangle = \langle v\varphi_\tau(x)x_\tau, \varphi_\tau(b)x_\tau \rangle$$

by the previous theorem. Therefore,  $\langle v(a + N_\tau), a + N_\tau \rangle \geq 0$  for each  $a \in A$ , which means that  $v$  is a positive operator. Now suppose  $a, b, c \in A$ . Then we see that

$$\begin{aligned} \langle \varphi_\tau(a)v(b + N_\tau), c + N_\tau \rangle &= \langle v(b + N_\tau), a^*c + N_\tau \rangle \\ &= \omega(c^*ab) \\ &= \langle v(ab + N_\tau), c + N_\tau \rangle \\ &= \langle v\varphi_\tau(a)(b + N_\tau), c + N_\tau \rangle \end{aligned}$$

Thus  $\varphi_\tau(a)v = v\varphi_\tau(a)$  for each  $a \in A$ , which implies that  $v \in \varphi_\tau(A)'$ . Moreover, by using an approximate unit for  $A$  one can show that  $\omega(b) = \langle v\varphi_\tau(b)x_\tau, x_\tau \rangle$ . The proof of uniqueness is left to the reader.  $\square$

We already mentioned what it means for a representation  $(\varphi, \mathcal{H})$  to be non-degenerate. We also saw that the universal representation of a  $C^*$ -algebra is non-degenerate since it is cyclic. The following lemma gives us a special property of non-degenerate representations.

**Lemma 1.6.** Let  $(\varphi, \mathcal{H})$  be a non-degenerate representation of a  $C^*$ -algebra  $A$ . Then it is a direct sum of cyclic representations of  $A$ .

*Proof.* Define  $\mathcal{H}_x := \overline{\varphi(A)x}$  for all  $x \in \mathcal{H}$ . We can use Zorn's Lemma to show the existence of a maximal set  $\Lambda$  of non-zero elements of  $\mathcal{H}$  such that  $\mathcal{H}_x \perp \mathcal{H}_y$  for each  $x, y \in \Lambda$  with  $x \neq y$ . Now let  $y \in (\bigcup_{x \in \Lambda} \mathcal{H}_x)^\perp$ , then  $\langle y, \varphi(a^*b)x \rangle = 0$  for each  $x \in \Lambda$ , or in other words,

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$\langle \varphi(a)y, \varphi(b)x \rangle = 0$ . Thus  $\mathcal{H}_x$  and  $\mathcal{H}_y$  are orthogonal. By assumption  $(\varphi, \mathcal{H})$  is non-degenerate, hence  $y \in \mathcal{H}_y$ . By construction,  $\Lambda$  was maximal, which means that  $y = 0$ . Therefore,  $\mathcal{H}$  is the orthogonal direct sum of the family of Hilbert spaces  $(\mathcal{H}_x)_{x \in \Lambda}$ . One sees that  $\mathcal{H}_x$  is invariant under  $\varphi(A)$  for each  $x \in \Lambda$ , and that the restriction  $\varphi_x : A \rightarrow B(\mathcal{H}_x)$  has  $x$  as a cyclic vector. This proves the lemma.  $\square$

**Definition 1.14.** Let  $(\varphi_1, \mathcal{H}_1)$  and  $(\varphi_2, \mathcal{H}_2)$  be two representations of the  $C^*$ -algebra  $A$ . We call the representations **unitarily equivalent**, if there exists a unitary  $u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\varphi_2(a) = u\varphi_1(a)u^*$  for each  $a \in A$ .

**Theorem 1.20.** Let  $(\varphi_1, \mathcal{H}_1)$  and  $(\varphi_2, \mathcal{H}_2)$  be two representations of the  $C^*$ -algebra  $A$  with cyclic vectors  $x_1$  and  $x_2$ , respectively. Then the following are equivalent:

- (i) There exists a unitary  $u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $x_2 = u(x_1)$  and  $\varphi_2(a) = u\varphi_1(a)u^*$  for each  $a \in A$ .
- (ii)  $\forall a \in A : \langle \varphi_1(a)(x_1), x_1 \rangle = \langle \varphi_2(a)(x_2), x_2 \rangle$

*Proof.* The implication (i)  $\Rightarrow$  (ii) is a straightforward computation and is left to the reader. Hence, we only prove the implication (ii)  $\Rightarrow$  (i). Thus, suppose that  $\langle \varphi_1(a)(x_1), x_1 \rangle = \langle \varphi_2(a)(x_2), x_2 \rangle$  for each  $a \in A$ . Define  $u_0 : \varphi_1(A)x_1 \rightarrow \mathcal{H}_2$  by  $u_0(\varphi_1(a)x_1) = \varphi_2(a)x_2$ . This map is linear, well-defined and isometric, since:

$$\|\varphi_2(a)(x_2)\|^2 = \langle \varphi_2(a^*a)(x_2), x_2 \rangle = \langle \varphi_1(a^*a)(x_1), x_1 \rangle = \|\varphi_1(a)(x_1)\|^2$$

Now we can extend  $u_0$  to an isometry  $u : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , which is unitary, since  $u(\mathcal{H}_1) = \overline{\varphi_2(A)x_2} = \mathcal{H}_2$ . Let  $a, b \in A$ . Then  $u\varphi_1(a)\varphi_1(b)x_1 = \varphi_2(ab)(x_2) = \varphi_2(a)u\varphi_1(b)(x_1)$ , and therefore  $u\varphi_1(a) = \varphi_2(a)u$  for each  $a \in A$ . Now

$$\varphi_2(a)u(x_1) = u\varphi_1(a)(x_1) = \varphi_2(a)(x_2)$$

hence  $\varphi_2(a)(u(x_1) - x_2) = 0$ . Every cyclic representation is also non-degenerate, hence so is  $\varphi_2$ . Therefore,  $u(x_1) = x_2$ .  $\square$

**Definition 1.15.** A representation  $(\varphi, \mathcal{H})$  of a  $C^*$ -algebra  $A$  is called **irreducible** if  $0$  and  $\mathcal{H}$  are the only invariant closed subspaces of  $\mathcal{H}$  that are invariant under  $\varphi(A)$ .

In representation theory of groups and algebras, Schur's Lemma is well known. Such a statement also holds for representations of  $C^*$ -algebras.

**Theorem 1.21.** Let  $(\varphi, \mathcal{H})$  be a non-zero representation of a  $C^*$ -algebra  $A$ . Then:

- (i)  $(\varphi, \mathcal{H})$  is irreducible  $\iff \varphi(A)' = \mathbb{C}\mathbf{1}$ .
- (ii) If  $(\varphi, \mathcal{H})$  is irreducible, then every non-zero vector of  $\mathcal{H}$  is cyclic for  $(\varphi, \mathcal{H})$ .

*Proof.* Here we only prove part (ii) of the theorem, since part (i) is easier to prove if we know the theory of von Neumann algebras (see Theorem 2.5). Now suppose that  $(\varphi, \mathcal{H})$  is irreducible, and let  $x \in \mathcal{H}$  be a non-zero vector. Observe, that by construction,  $\overline{\varphi(A)x}$  is invariant for  $\varphi(A)$ , hence  $\overline{\varphi(A)x} = 0$  or  $\overline{\varphi(A)x} = \mathcal{H}$ . By assumption,  $\varphi$  is non-zero, hence there is an element  $y \in \mathcal{H}$  and an element  $a \in A$ , such that  $\varphi(a)(y) \neq 0$ . Therefore,  $\overline{\varphi(A)y} = \mathcal{H}$ . Thus the representation is non-degenerate. Therefore,  $\overline{\varphi(A)x}$  is not the zero space, so  $\overline{\varphi(A)x} = \mathcal{H}$  by irreducibility. This proves the theorem.  $\square$

**Definition 1.16.** Let  $A$  be a  $C^*$ -algebra and suppose  $\tau \in S(A)$ . Then  $\tau$  is called a **pure state** if it has the property that, whenever  $\rho$  is a positive linear functional on  $A$  such that  $\rho \leq \tau$ , there is a  $t \in [0, 1]$  such that  $\rho = t \cdot \tau$ . The set of pure states on  $A$  is denoted by  $PS(A)$ .

We remark that there is a very interesting structure on  $PS(A)$ , since we can equip  $PS(A)$  with a type of inner product such that  $PS(A)$  decomposes in so-called mutually disjoint and orthogonal sectors. Moreover, one can obtain the formula of transposition probability between two states in quantum mechanics from this. For more information consult [3]. Furthermore, one can show that pure states of a  $C^*$ -algebra are extreme points of the set of all positive linear functionals on  $A$ . Moreover, if  $A$  is unital, then  $S(A)$  is the weak\*-closed convex hull of all pure states of  $A$  by the Krein-Milman Theorem (see [3],[4],[13]).

Now we want to connect pure states with representations.

**Proposition 1.4.** Suppose that  $\tau$  is a state on a  $C^*$ -algebra  $A$ .

- (i)  $\tau$  is a pure state  $\iff (\varphi_\tau, \mathcal{H}_\tau)$  is irreducible.
- (ii) Assume that  $A$  is abelian. Then:  $\tau$  is pure  $\iff \tau$  is a character on  $A$ .

*Proof.* (i)  $\implies$ : Suppose  $\tau$  is pure. Let  $v \in \varphi_\tau(A)'$ , such that  $0 \leq v \leq 1$ . Construct a function  $\rho : A \rightarrow \mathbb{C}$  by  $\rho(a) = \langle \varphi_\tau(a)v x_\tau, x_\tau \rangle$ . This gives us a positive linear functional on  $A$  with  $\rho \leq \tau$ . Since  $\tau$  is pure, there exists  $t \in [0, 1]$  such that  $\rho = t\tau$ . Hence  $\langle \varphi_\tau(a)v(x_\tau), x_\tau \rangle = \langle t\varphi_\tau(a)(x_\tau), x_\tau \rangle$  for each  $a \in A$ . Therefore, for all  $a, b \in A$ :

$$\begin{aligned} \langle v(a + N_\tau), b + N_\tau \rangle &= \langle v\varphi_\tau(a)(x_\tau), \varphi_\tau(b)(x_\tau) \rangle \\ &= \langle v\varphi_\tau(b^*a)(x_\tau), x_\tau \rangle \\ &= \langle t\varphi_\tau(b^*a)(x_\tau), x_\tau \rangle \\ &= \langle t(a + N_\tau), b + N_\tau \rangle \end{aligned}$$

Now we can use the fact that  $A/N_\tau$  is dense in  $\mathcal{H}_\tau$  to conclude that  $v = t\mathbf{1}$ . Therefore,  $\varphi_\tau(A)' = \mathbb{C}\mathbf{1}$ . Then we can apply part (i) of the previous theorem to conclude that  $(\varphi_\tau, \mathcal{H}_\tau)$  is irreducible.

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$\Leftarrow$ : Suppose  $(\varphi_\tau, \mathcal{H}_\tau)$  is irreducible. Assume that there exists a positive linear functional  $\rho$  such that  $\rho \leq \tau$ . Therefore, we can find a unique operator  $v \in \varphi_\tau(A)'$  with  $0 \leq v \leq 1$  and  $\rho(a) = \langle \varphi_\tau(a)v(x_\tau), x_\tau \rangle$  for each  $a \in A$ . Since the representation is irreducible, we know again by the previous theorem that  $\varphi_\tau(A)' = \mathbb{C}\mathbf{1}$ . Thus  $v = t\mathbf{1}$  for some  $t \in [0, 1]$ . Hence  $\rho = t\tau$ . So  $\tau$  is pure.

(ii)  $\Rightarrow$ : Assume that  $\tau$  is pure. By the previous part, this implies that  $(\varphi_\tau, \mathcal{H}_\tau)$  is irreducible, which implies that  $\varphi_\tau(A)' = \mathbb{C}\mathbf{1}$ . But  $\varphi_\tau(A) \subseteq \varphi_\tau(A)'$ . So  $\varphi_\tau(A)$  consists of scalars. Therefore,  $B(\mathcal{H}_\tau) \subseteq \varphi_\tau(A)'$ . Hence  $B(\mathcal{H}_\tau) = \mathbb{C}\mathbf{1}$ . Now assume that  $u, v \in B(\mathcal{H}_\tau)$ . These are scalars. Therefore:

$$\langle uv(x_\tau), x_\tau \rangle = u \langle v(x_\tau), x_\tau \rangle = u \langle x_\tau, x_\tau \rangle \langle v(x_\tau), x_\tau \rangle = \langle u(x_\tau), x_\tau \rangle \langle v(x_\tau), x_\tau \rangle$$

Hence  $\tau(a) = \langle \varphi_\tau(a)x_\tau, x_\tau \rangle$  is multiplicative and therefore a character on  $A$ .

$\Leftarrow$ : Suppose that  $\tau$  is a character on  $A$  and assume that  $\rho$  is a positive linear functional such that  $\rho \leq \tau$ . If  $\tau(a) = 0$ , then  $\tau(a^*a) = 0$  by multiplicativity. Hence  $\rho(a^*a) = 0$ . Since  $|\rho(a)| \leq \sqrt{\rho(a^*a)}$ , we may conclude that  $\rho(a) = 0$ . Hence  $\text{Ker}(\tau) \subseteq \text{Ker}(\rho)$ . Therefore, we can find  $t \in \mathbb{R}$  such that  $\rho = t\tau$ . Now let  $a \in A$  such that  $\tau(a) = 1$ . Then  $\tau(a^*a) = 1$ , hence  $0 \leq \rho(a^*a) = t\tau(a^*a) = t \leq \tau(a^*a) = 1$ . Therefore,  $t \in [0, 1]$ . Hence  $\tau$  is pure.  $\square$

## 1.6. Multiplier algebra

If one talks about  $C^*$ -algebras  $A$ , one also has to consider the so called multiplier algebra  $\mathcal{M}(A)$ . We will give two different constructions of this algebra. References are [10] and [21].

### 1.6.1. The classical approach: Double centralizer

Let  $A$  be a  $C^*$ -algebra. A pair  $(L, R)$  of bounded linear maps on  $A$  is called a *double centralizer* for  $A$ , if for each  $a, b \in A$ :

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad R(a)b = aL(b)$$

Simple examples are  $(L_c, R_c)$ , where  $c \in A$  and  $L_c(a) = ca$  and  $R_c(a) = ac$  for each  $a \in A$ . In this case one has  $\|R_c\| = \|L_c\| = \|c\|$ . Equality of the norms holds in general.

**Lemma 1.7.** *Let  $(L, R)$  be a double centralizer of a  $C^*$ -algebra  $A$ . Then  $\|L\| = \|R\|$ .*

*Proof.* To prove the lemma we start with the estimate  $\|aL(b)\| = \|R(a)b\| \leq \|R\| \|a\| \|b\|$ . From this we obtain  $\|L(b)\| = \sup_{\|a\| \leq 1} \|aL(b)\| \leq \|R\| \|b\|$ . This gives us the first inequality:



$\|L\| \leq \|R\|$ . Now we can do the same sort of estimate by the observation  $\|R(a)b\| = \|aL(b)\| \leq \|L\| \|a\| \|b\|$  to obtain the second inequality,  $\|R\| \leq \|L\|$ .  $\square$

**Definition 1.17.** *Let  $A$  be a  $C^*$ -algebra. Then  $\mathcal{M}(A)$  denotes the set of all double centralizers of  $A$ . The norm of such an element  $(L, R) \in \mathcal{M}(A)$  is defined by  $\|(L, R)\| := \|L\| = \|R\|$ .*

Our goal is now to turn  $\mathcal{M}(A)$  into a  $C^*$ -algebra with respect to this norm. Therefore, we first show that we can turn  $\mathcal{M}(A)$  into a unital  $*$ -algebra. First we define the algebra structure. Let  $(L_1, R_1), (L_2, R_2) \in \mathcal{M}(A)$  and  $\lambda \in \mathbb{C}$  be arbitrary. Then we define the following operations:

$$\begin{aligned} \lambda(L_1, R_1) &:= (\lambda L_1, \lambda R_1) \\ (L_1, R_1) + (L_2, R_2) &:= (L_1 + L_2, R_1 + R_2) \\ (L_1, R_1) \cdot (L_2, R_2) &:= (L_1 L_2, R_2 R_1) \end{aligned}$$

Observe that, if the operations are well-defined, then we can obtain the zero-element in  $\mathcal{M}(A)$  by  $(0, 0)$  for  $0 \in B(A)$  (here  $B(A)$  is the space of all bounded linear maps on  $A$ ). Moreover,  $\mathcal{M}(A)$  will be unital, since  $(\mathbf{1}, \mathbf{1}) \in \mathcal{M}(A)$  for  $\mathbf{1} \in B(A)$  the identity operator. Furthermore, we want to have an involution on  $\mathcal{M}(A)$ . Therefore, for  $T \in B(A)$  define the operator

$$T^\#(a) := T(a^*)^*$$

for each  $a \in A$ . If  $(L, R) \in \mathcal{M}(A)$ , then define

$$(L, R)^* := (R^\#, L^\#)$$

. So far, we don't know whether all this operations are well-defined. But this is confirmed in the following lemma ([13],[21]).

**Lemma 1.8.** *Let  $A$  be a  $C^*$ -algebra. The operation  $T \mapsto T^\#$  is isometric, multiplicative, conjugate linear and idempotent on  $B(A)$ . Moreover, all the operations defined above turn  $\mathcal{M}(A)$  into a unital  $*$ -algebra.*

Now we are almost there. It remains to prove two things, namely that the norm on  $\mathcal{M}(A)$  satisfies the  $C^*$ -condition, and that  $\mathcal{M}(A)$  is complete with respect to the norm. That both hold is summarized in the following theorem ([6],[21]).

**Theorem 1.22.** *Let  $A$  be a  $C^*$ -algebra. If we equip  $\mathcal{M}(A)$  with the norm  $\|(L, R)\| := \|L\| = \|R\|$  for each  $(L, R) \in \mathcal{M}(A)$ , then  $\mathcal{M}(A)$  is a unital  $C^*$ -algebra.*

**Definition 1.18.** *Let  $A$  be a  $C^*$ -algebra. The  $C^*$ -algebra  $\mathcal{M}(A)$  is called the **multiplier algebra** of  $A$ .*

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Observe that we have a  $*$ -homomorphism  $\varphi : A \rightarrow \mathcal{M}(A)$  given by  $\varphi(a) = (L_a, R_a)$  for each  $a \in A$ . Suppose  $\varphi(a) = 0$ . Then  $R_a = 0$ , hence  $R_a(a^*) = a^*a = 0$ . By the  $C^*$ -condition, we obtain that  $a = 0$ , which shows that  $\varphi$  is an injective  $*$ -homomorphism. Furthermore, one can prove that  $A$  is a two-sided ideal in  $\mathcal{M}(A)$  and that  $A \cong \mathcal{M}(A)$  if and only if  $A$  is unital ([21],[6]). Typical examples of multiplier algebras are  $\mathcal{M}(C_0(X)) \cong C_b(X)$  for a locally compact Hausdorff space  $X$  and  $\mathcal{M}(K(\mathcal{H})) \cong B(\mathcal{H})$ , where  $K(\mathcal{H})$  is the set of all compact operators on the Hilbert space  $\mathcal{H}$  (see [13]). The following lemma gives more information about the ideal structure of  $A$  in  $\mathcal{M}(A)$  (for the proof see [13]).

**Lemma 1.9.** *If  $I$  is a closed two-sided ideal in a  $C^*$ -algebra  $A$ , then there exists a unique  $*$ -homomorphism  $\varphi : A \rightarrow \mathcal{M}(A)$  extending the inclusion  $I \rightarrow \mathcal{M}(I)$ . Moreover,  $\varphi$  is injective if  $I$  is essential in  $A$ .*

Here, *essential* means that, if  $I$  is a closed ideal of  $A$ , then  $aI = 0$  implies  $a = 0$ . Equivalently:  $I \cap J \neq 0$  for each non-zero closed ideal  $J$  in  $A$ . Notice that this lemma tells us that the multiplier algebra  $\mathcal{M}(I)$  of  $I$  is the largest unital  $C^*$ -algebra containing  $I$  as an essential ideal. Therefore, one can also define the multiplier algebra  $\mathcal{M}(A)$  of  $A$  as follows ([21]):

**Definition 1.19.** *Let  $A$  be a  $C^*$ -algebra. The multiplier algebra  $\mathcal{M}(A)$  is the universal  $C^*$ -algebra with the property that  $\mathcal{M}(A)$  contains  $A$  as an essential ideal and for any  $C^*$ -algebra  $B$  containing  $A$  as an essential ideal there exists a unique  $*$ -homomorphism  $\varphi : B \rightarrow \mathcal{M}(A)$  that is the identity on  $A$ .*

If we begin with this definition, it is not a priori clear that such a  $C^*$ -algebra  $\mathcal{M}(A)$  exists (by constructing double centralizers we already showed that it does). In what follows we want to study another possibility to establish the existence of  $\mathcal{M}(A)$ .

### 1.6.2. The modern approach: Hilbert $C^*$ -modules

We first want to give the definition of a Hilbert  $C^*$ -module (see [10]). Notice, that Hilbert  $C^*$ -modules are very important if one looks at the  $C^*$ -algebraic approach to quantum groups à la Woronowicz (see [26], [27]).

**Definition 1.20.** *Let  $A$  be an arbitrary  $C^*$ -algebra. An **inner product  $A$ -module** is a linear space  $E$  that is a right  $A$ -module, such that  $\lambda(xa) = (\lambda x)a = x(\lambda a)$  for each  $x \in E$ ,  $a \in A$  and  $\lambda \in \mathbb{C}$ , together with a map  $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$  such that:*

1.  $\forall x, y, z \in E \forall \alpha, \beta \in \mathbb{C} : \langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2.  $\forall x, y \in E \forall a \in A : \langle x, ya \rangle = \langle x, y \rangle a$
3.  $\forall x, y \in E : \langle y, x \rangle = \langle x, y \rangle^*$
4.  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$

For  $x \in E$ , we write  $\|x\| := \sqrt{\|\langle x, x \rangle\|}$ . This is a norm on  $E$  (see [10]). We are now prepared to state our definition of Hilbert  $C^*$ -modules.

**Definition 1.21.** *Let  $A$  be an arbitrary  $C^*$ -algebra. An inner product  $A$ -module that is complete with respect to the norm is called a **Hilbert  $A$ -module** or a **Hilbert  $C^*$ -module over the  $C^*$ -algebra  $A$***

Let  $A$  be a  $C^*$ -algebra. Then  $A$  itself is an example of a Hilbert  $A$ -module if we define  $\langle a, b \rangle := a^*b$  for each  $a, b \in A$ . This can now be used to construct the multiplier algebra  $\mathcal{M}(A)$ .

**Definition 1.22.** *Let  $A$  be a  $C^*$ -algebra and  $E$  a Hilbert  $A$ -module. Let  $L(E)$  be set of all maps  $t : E \rightarrow E$  such that there exists a map  $t^* : E \rightarrow E$  such that  $\langle tx, y \rangle = \langle x, t^*y \rangle$  for each  $x, y \in E$ . We call  $L(E)$  also the set of **adjointable maps**.*

Notice, that we do not require  $t \in L(E)$  be to  $A$ -linear or continuous, since this is automatic (see [9]).

**Definition 1.23.** *Let  $A$  be a  $C^*$ -algebra and  $E$  a Hilbert  $A$ -module. Define  $\vartheta_{x,y} : E \rightarrow E$  by  $\vartheta_{x,y}(z) = x \langle y, z \rangle$  for each  $z \in E$ .*

Observe that  $\vartheta_{x,y} \in L(E)$ , since one has  $(\vartheta_{x,y})^* = \vartheta_{y,x}$ . Moreover, one has the following relations for each  $x, y, u, v \in E$  and each  $s, t \in L(E)$ :

$$\vartheta_{x,y}\vartheta_{u,v} = \vartheta_{x\langle y,u \rangle, v} = \vartheta_{x,v\langle u,y \rangle}$$

$$t\vartheta_{x,y} = \vartheta_{tx,y}$$

$$\vartheta_{x,y}s = \vartheta_{x,s^*y}$$

**Definition 1.24.** *Let  $A$  be a  $C^*$ -algebra, and  $E$  a Hilbert  $A$ -module. We denote by  $K(E)$  the closed linear span of  $\{\vartheta_{x,y} : x, y \in E\}$  in  $L(E)$ .*

By construction,  $K(E)$  is a closed two-sided ideal in  $L(E)$ . Observe that, if  $E = A$ , then  $K(E) \cong A$  by identifying  $\vartheta_{x,y}$  with the operation of left multiplication by  $xy^*$ . To construct  $\mathcal{M}(A)$  we use representation theory. In particular, we want to identify  $L(A)$  with  $\mathcal{M}(A)$ . For that, we need some definitions:

**Definition 1.25.** *Let  $A$  be a  $C^*$ -algebra and  $E$  a Hilbert  $A$ -module. A  $*$ -homomorphism  $\alpha : A \rightarrow L(E)$  is called **non-degenerate**, if the set*

$$\alpha(A)E := \left\{ \sum_{i=1}^n \alpha(a_i)x_i : a_i \in A, x_i \in E \forall i \in \{1, \dots, n\} \right\}$$

*is dense in  $E$ .*

## 1. $C^*$ -algebras

We remark the *Cohen-Hewitt Factorization Theorem*, which states that if  $\alpha(A)E$  is non-degenerate, then for all  $x \in E$  there exists  $y \in E$  and  $a \in A$  such that  $x = \alpha(a)y$ . In particular, the sum in the definition above consists of one single term.

**Proposition 1.5.** *Let  $A, B$  and  $C$  be  $C^*$ -algebras such that  $A$  is an ideal in  $B$ . Suppose  $E$  is a Hilbert  $C$ -module. Assume that  $\alpha : A \rightarrow L(E)$  is a non-degenerate  $*$ -homomorphism. Then  $\alpha$  has a unique extension  $\hat{\alpha} : B \rightarrow L(E)$ . If  $\alpha$  is injective and  $A$  is essential in  $B$ , then  $\hat{\alpha}$  is injective.*

*Proof.* Let  $(u_\lambda)_{\lambda \in \Lambda}$  be an approximate unit of  $A$ . Assume that  $b \in B$  and  $a_1, \dots, a_n \in A$  and  $x_1, \dots, x_n \in E$ . Then

$$\left\| \sum_{i=1}^n \alpha(ba_i)x_i \right\| = \lim_{\lambda} \left\| \sum_{i=1}^n \alpha(bu_\lambda a_i)x_i \right\| = \lim_{\lambda} \left\| \alpha(bu_\lambda) \sum_{i=1}^n \alpha(a_i)x_i \right\| \leq \|b\| \left\| \sum_{i=1}^n \alpha(a_i)x_i \right\|$$

Therefore, we conclude that the map  $\sum_{i=1}^n \alpha(a_i)x_i \mapsto \sum_{i=1}^n \alpha(ba_i)x_i$  is well defined and continuous. By assumption,  $\alpha$  is non-degenerate, hence  $\alpha(A)E$  is dense in  $E$ . Therefore, we can extend the previous map  $\alpha(A)E \rightarrow \alpha(A)E$  to a bounded linear map  $\hat{\alpha}(b) : E \rightarrow E$ , such that  $(\hat{\alpha}(b))(\alpha(a)x) = \alpha(ba)x$  for each  $a \in A$ ,  $x \in E$ . One can also show, by using an approximate unit, that  $\hat{\alpha}(b^*)$  is the adjoint of  $\hat{\alpha}(b)$ , hence  $\hat{\alpha}(b) \in L(E)$ . That  $\hat{\alpha}$  is a  $*$ -homomorphism is left to the reader. Uniqueness follows from the fact that  $\alpha$  is non-degenerate. Assume that  $\alpha$  is injective. Then  $\text{Ker}(\hat{\alpha})$  is an ideal in  $B$ , which has zero-intersection with  $A$ . If  $A$  is essential, then this ideal has to be zero.  $\square$

This proposition can now be applied to show that  $L(A) = \mathcal{M}(A)$ . In particular, take  $C = E = A$  and let  $\alpha : A \rightarrow L(A)$  be the canonical embedding of  $A$  into  $L(A)$  (by identifying  $A$  with  $K(A)$ ). Now, the previous proposition tells us that each  $C^*$ -algebra  $B$  that contains  $A$  as essential ideal embeds in  $L(A)$ . Hence we obtain the required maximality property of  $\mathcal{M}(A)$ . We also have to verify the second condition. For that, suppose that  $A$  is an essential ideal in  $B$ . If  $A$  is also an essential ideal in another  $C^*$ -algebra  $C$ , then the identity map on  $A$  extends to an embedding  $\beta : C \rightarrow B$ . Since  $A$  is an essential ideal in  $L(A)$  we obtain an injection  $\beta : L(A) \rightarrow B$ , whose restriction is the identity on  $A$ . The previous proposition tells us that  $\alpha : A \rightarrow L(A)$  has an injective extension  $\hat{\alpha} : B \rightarrow L(A)$ . By a second application of the previous proposition we obtain a unique extension to a  $*$ -homomorphism  $\gamma : L(A) \rightarrow L(A)$  (since  $A$  is also essential in  $B$ ). But the identity on  $L(A)$  is such an extension, but also  $\hat{\alpha}\beta$ . Hence  $\hat{\alpha}\beta$  has to be the identity on  $L(A)$ , which means that  $\hat{\alpha}$  is surjective. Therefore,  $\hat{\alpha}$  is a  $*$ -isomorphism between  $B$  and  $L(A)$ . We may conclude that  $L(A)$  is the unique maximal essential extension of  $A$  (up to isomorphism). Therefore, we may write  $\mathcal{M}(A)$  instead of  $L(A)$ .

## 2. Von Neumann algebras

In the previous chapter we discussed the GNS-construction and saw that each  $C^*$ -algebra can be seen as a closed self-adjoint subalgebra of  $B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . In some books (for example [1]) this is expressed as: every "abstract"  $C^*$ -algebra is a "concrete"  $C^*$ -algebra. Here "concrete" means the following: A set  $A$  of bounded linear operators on a Hilbert space  $\mathcal{H}$  is called a concrete  $C^*$ -algebra if  $A$  is an operator norm-closed  $*$ -algebra. Observe that the operator norm defines a topology on such a concrete  $C^*$ -algebra. In what follows, we first want to introduce some other important topologies on  $B(\mathcal{H})$ .

### 2.1. Topologies on $B(\mathcal{H})$

**Strong Topology:** We first introduce the so-called *strong topology*. Let  $x \in \mathcal{H}$  and let  $p_x : B(\mathcal{H}) \rightarrow \mathbb{R}_+$  be the map defined by  $T \mapsto \|Tx\|$ . The family  $\{p_x\}_{x \in \mathcal{H}}$  is a separating family of seminorms on  $B(\mathcal{H})$  and therefore gives rise to a locally convex topology on  $B(\mathcal{H})$ , called the *strong topology*. Equivalently, we say that a net  $(u_\lambda)_{\lambda \in \Lambda}$  *converges strongly* to  $u \in B(\mathcal{H})$  if and only if  $\|u_\lambda x - ux\| \rightarrow 0$  for each  $x \in \mathcal{H}$ .

**Weak Topology:** Once again, we use a family of seminorms. For  $x, y \in \mathcal{H}$ , define the map  $p_{x,y} : B(\mathcal{H}) \rightarrow \mathbb{R}_+$  by  $p_{x,y}(T) = |\langle Tx, y \rangle|$ . This is again a separating family of seminorms, giving rise to a locally convex topology on  $B(\mathcal{H})$ . This is what we call the *weak topology*. In particular: a net  $(u_\lambda)_{\lambda \in \Lambda}$  *converges weakly* to  $u \in B(\mathcal{H})$  if and only if  $|\langle (u_\lambda - u)x, y \rangle| \rightarrow 0$  for each  $x, y \in \mathcal{H}$ .

**Ultraweak Topology:** To introduce the so-called *ultraweak* or  $\sigma$ -*weak* topology, consider the family of seminorms  $p_u : B(\mathcal{H}) \rightarrow \mathbb{R}_+$  given by  $v \mapsto |Tr(uv)|$ , for  $u$  a trace class operator (one can find more on this topic in [13],[4]). This gives us a third locally convex topology on  $B(\mathcal{H})$ . A net  $(u_\lambda)_{\lambda \in \Lambda}$  *converges ultraweakly* to  $u \in B(\mathcal{H})$  if and only if  $|Tr((u_\lambda - u)v)| \rightarrow 0$  for each trace class operator  $v$ .

Having all these topologies on the space  $B(\mathcal{H})$ , we may compare them and study their properties. Observe that the weak topology is weaker than the strong topology and that the weak topology is also weaker than the ultraweak topology. The following theorem indicates when some of the topologies coincide (for the proof see [4],[13]).

## 2. Von Neumann algebras

**Theorem 2.1.** *Let  $\mathcal{H}$  be a Hilbert space. Then the weak and ultraweak topology coincide on the closed unit ball of  $B(\mathcal{H})$ .*

As a consequence of this theorem, the closed unit ball is weakly compact in  $B(\mathcal{H})$ . This is because the ultraweak topology is the weak\* topology on  $B(\mathcal{H})$ , since  $B(\mathcal{H}) \cong B_1(\mathcal{H})^*$ . By the theorem of Banach and Alaoglu, the closed unit ball is compact in this topology.

The last theorem of this section is due to Vigier. Remember from Real Analysis, that every sequence that is bounded from above and increasing, is convergent (Bolzano-Weierstrass). You can compare this with Vigier's Theorem.

**Theorem 2.2.** *Assume that  $(u_\lambda)_{\lambda \in \Lambda}$  is a net of self-adjoint operators on a Hilbert space. If the net is increasing and bounded from above, then the net converges strongly.*

*Proof.* Suppose that  $(u_\lambda)_{\lambda \in \Lambda}$  is increasing and bounded from above. Without loss of generality, we may assume that the net is also bounded from below, since we can choose  $\lambda_0 \in \Lambda$  such that  $(u_\lambda)_{\lambda \geq \lambda_0}$  is bounded from below. Let  $v$  be this bound. We can also assume without loss of generality that the net consists only of positive elements (since we can always look at the net  $(u_\lambda - v)_{\lambda \in \Lambda}$ ). Granted this, we can find  $M \in \mathbb{R}$  such that  $\|u_\lambda\| \leq M$  for each  $\lambda \in \Lambda$ . Now consider the net  $(\langle u_\lambda(x), x \rangle)_{\lambda \in \Lambda}$ , which is also increasing and, by Cauchy-Schwarz, bounded by  $M \|x\|^2$ . We thus have convergence of this new net. Using the polarization identity we see that  $(\langle u_\lambda(x), y \rangle)_{\lambda \in \Lambda}$  is convergent for each  $x, y \in \mathcal{H}$ . Let  $\alpha(x, y)$  be this limit and define  $\alpha : \mathcal{H} \times \mathcal{H}$  by  $(x, y) \mapsto \alpha(x, y)$ . This is a sesquilinear form on  $\mathcal{H}$ . Observe that  $|\alpha(x, y)| = \lim_\lambda |\langle u_\lambda(x), y \rangle| \leq M \|x\| \|y\|$ , which means that  $\alpha$  is bounded. By the Riesz-Fréchet Theorem, there is an operator  $u$  on  $\mathcal{H}$  such that  $\langle u(x), y \rangle = \alpha(x, y)$  for each  $x, y \in \mathcal{H}$ . It is easy to check that  $u$  is self-adjoint, bounded by  $M$  (i.e.  $\|u\| \leq M$ ), and that  $u_\lambda \leq u$  for each  $\lambda \in \Lambda$ . Now we have the following estimate:

$$\|u(x) - u_\lambda(x)\|^2 = \left\| (u - u_\lambda)^{\frac{1}{2}} (u - u_\lambda)^{\frac{1}{2}}(x) \right\|^2 \leq \|u - u_\lambda\| \cdot \left\| (u - u_\lambda)^{\frac{1}{2}}(x) \right\|^2 \leq 2M \langle (u - u_\lambda)x, x \rangle$$

Since  $\lim_\lambda \langle (u - u_\lambda)x, x \rangle = 0$ , we see that  $(u_\lambda)_{\lambda \in \Lambda}$  converges strongly to  $u$ .  $\square$

## 2.2. The Double Commutant Theorem

We now give the definition of a *von Neumann algebra*. We already mentioned that every  $C^*$ -algebra is isomorphic to a norm-closed self-adjoint subalgebra of  $B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . But if we instead impose a stronger closedness condition we obtain the notion of a von Neumann algebra.

**Definition 2.1.** *Let  $\mathcal{H}$  be a Hilbert space. We say that  $A$  is a von Neumann algebra on  $\mathcal{H}$  if  $A$  is a strongly closed  $*$ -subalgebra of  $B(\mathcal{H})$ .*

## 2.2. The Double Commutant Theorem

Notice that every von Neumann algebra is also a  $C^*$ -algebra, because the strong topology is weaker than the norm topology. Hence a strongly closed set is also norm-closed. We now give an alternative characterization of a von Neumann algebra, called the *Double Commutant Theorem*. Therefore, we recall the definition of the commutant (see Definition 1.13). Let  $B$  be a subset of an algebra  $A$ . Then one can observe that  $B'$  is a subalgebra of  $A$ . Moreover, one can look at the commutant of the commutant, the so-called double commutant:  $(B')' = B''$ . The reader can verify that the following holds:  $B \subseteq B''$  and  $B' = B'''$ . Furthermore,  $B'$  is closed if  $A$  is a normed algebra. If one also has an involution, then  $B'$  is a  $*$ -subalgebra of  $A$ . The following lemma is the key to the Double Commutant Theorem.

**Lemma 2.1.** *Let  $\mathcal{H}$  be a Hilbert space. Assume that  $A$  is a  $*$ -subalgebra of  $B(\mathcal{H})$  containing the identity. Then  $A$  is strongly dense in  $A''$ .*

*Proof.* Suppose  $u \in A''$  and  $x \in \mathcal{H}$ . Define  $K := \overline{\{v(x) : v \in A\}}$ . By construction,  $K$  is a closed vector subspace of  $\mathcal{H}$ , which is invariant for all  $v \in A$ . Since  $A$  is self-adjoint, one also obtains invariance of  $K^\perp$ . Now let  $p$  be the orthogonal projection of  $\mathcal{H}$  on  $K$ . By the previous observations, one obtains  $p \in A'$ , hence  $pu = up$  for each  $u \in A''$ , hence  $u(x) \in K$ , which means that we can find a sequence  $(v_n)_{n \in \mathbb{N}}$  in  $A$  such that  $u(x) = \lim_{n \rightarrow \infty} v_n(x)$ .

Define for each  $n \in \mathbb{N}$ :

$$\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H}^n), \quad v \mapsto (\delta_{ij}v)$$

This is a unital  $*$ -homomorphism. Therefore,  $\varphi(A)$  is a  $*$ -subalgebra of  $B(\mathcal{H}^n)$ , containing the identity. Moreover  $\varphi(u) \in \varphi(A)''$ . To see this, let  $w \in \varphi(A)'$  and  $v \in A$ . Then  $\varphi(v)w = w\varphi(v)$ , hence  $vw_{ij} = w_{ij}v$ . Thus,  $w_{ij} \in A'$ , so  $uw_{ij} = w_{ij}u$ . Therefore,  $\varphi(u)w = w\varphi(u)$ . Now suppose  $x = (x_1, \dots, x_n) \in \mathcal{H}^n$  and look at the beginning of this proof: we know that there exists a sequence  $(v_m)_{m \in \mathbb{N}}$  in  $A$  such that  $\varphi(u)(x) = \lim_{m \rightarrow \infty} \varphi(v_m)x$ . Hence  $u(x_j) = \lim_{m \rightarrow \infty} v_m(x_j)$  for each  $1 \leq j \leq n$ .

The claim is:  $u$  lies in the strong closure of  $A$ . To see this, let  $W$  be a strong neighbourhood of  $u$ . The goal is to show that  $W \cap A \neq \emptyset$ . Therefore, suppose that  $W - u$  is a strong neighbourhood of 0. This implies the existence of  $x_1 \dots x_n \in \mathcal{H}$  and  $\varepsilon > 0$  such that:

$$W - u = \{v \in B(\mathcal{H}) : \|v(x_j)\| < \varepsilon \forall j \in \{1, \dots, n\}\}$$

Therefore, there is a sequence  $(v_m)_{m \in \mathbb{N}}$  such that  $u(x_j) = \lim_{m \rightarrow \infty} v_m(x_j)$  for  $1 \leq j \leq n$ . This implies that for some  $N \in \mathbb{N}$ , one has  $v_N \in W$ . Therefore,  $W \cap A \neq \emptyset$ .  $\square$

Now we can state the Double Commutant Theorem, which gives us another characterization of von Neumann algebras, including  $id_{\mathcal{H}}$ . The proof is an immediate consequence of the previous lemma.

**Theorem 2.3.** *Let  $A$  be a  $*$ -algebra on a Hilbert space  $\mathcal{H}$  and suppose  $id_{\mathcal{H}} \in A$ . Then  $A$  is a von Neumann algebra on  $\mathcal{H} \iff A = A''$ .*

## 2. Von Neumann algebras

Observe, that in the definition of a von Neumann algebra we do not require that it is unital. But using our definition of a von Neumann algebra we see, by the next theorem, that a von Neumann always is unital, but the unit may not be the identity map of the underlying Hilbert space (however, some authors use the following definition of a von Neumann algebra:  $A$  is a von Neumann algebra on a Hilbert space  $\mathcal{H}$  if  $A$  is a  $*$ -algebra on  $\mathcal{H}$  such that  $A = A''$ . This implies that the identity map on  $\mathcal{H}$  is an element of  $A$ ).

**Theorem 2.4.** *If  $A$  is a non-zero von Neumann algebra, then it is unital.*

*Proof.* Suppose  $A$  acts on a Hilbert space  $\mathcal{H}$ . Then let  $(u_\lambda)_{\lambda \in \Lambda}$  be an approximate unit for  $A$ . By Vigier's Theorem,  $(u_\lambda)_{\lambda \in \Lambda}$  converges strongly to a self-adjoint operator  $e \in A$ , since  $A$  is strongly closed. If  $x \in \mathcal{H}$  and  $u \in A$ , then  $eu(x) = \lim_\lambda u_\lambda u(x) = u(x)$ , which means that  $eu = u$ . Hence  $e$  is a unit for  $A$ .  $\square$

**Proposition 2.1.** *Suppose  $A$  is a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . Then  $A$  is the strong closure of the linear span of its projections.*

With this proposition we can complete the theorem about irreducible representations (Theorem 1.23).

**Theorem 2.5.** *Let  $A$  be a non-zero  $C^*$ -algebra acting on a Hilbert space  $\mathcal{H}$ . Then:  $(\varphi, \mathcal{H})$  is irreducible  $\iff \varphi(A)' = \mathbb{C}\mathbf{1}$ .*

*Proof.* If  $p \in B(\mathcal{H})$  is a projection, then  $p \in A'$  if and only if  $p(\mathcal{H}) \subseteq \mathcal{H}$  is invariant under  $A$ . Since  $A'$  is a von Neumann algebra, it is the closed linear span of its projections. We assumed, that  $A$  acts irreducibly, hence  $A'$  has no projections except the trivial ones, hence  $A' = \mathbb{C}\mathbf{1}$ . The converse implication is clear.  $\square$

### 2.2.1. Topologies and Continuity on von Neumann Algebras

Theorem 2.1 of this section already gives a connection between the topologies we introduced in the beginning. The following theorem shows that weak continuity and strong continuity are equivalent if we talk about linear functionals on  $B(\mathcal{H})$  (for the proof look at [13]).

**Theorem 2.6.** *Let  $\mathcal{H}$  be a Hilbert space and  $\tau : B(\mathcal{H}) \rightarrow \mathbb{C}$  a linear functional. Then the following are equivalent:*

1.  $\tau$  is weakly continuous.
2.  $\tau$  is strongly continuous.

This theorem has a very useful application to convex subsets of  $B(\mathcal{H})$ . In particular, the notion of weak and strong closedness coincide on convex subsets. This is a consequence of this previous theorem.



**Corollary 2.1.** *Let  $\mathcal{H}$  be a Hilbert space and  $C$  a convex subset of  $B(\mathcal{H})$ . Then  $C$  is strongly closed  $\iff C$  is weakly closed.*

*Proof.* By the definition of the topologies, the  $\Leftarrow$ -direction is obvious. For the converse, assume that  $C$  is strongly closed and that  $u$  is an element of the weak closure of  $C$ . By definition of the closure we can construct a net  $(x_\lambda)_{\lambda \in \Lambda}$  such that  $x_\lambda \rightarrow u$  with respect to the weak topology. If we assume that  $\tau$  is a weakly continuous linear functional, then we conclude that  $\tau(x_\lambda) \rightarrow \tau(u)$ . Thus the previous theorem tells us that weakly continuous functionals are the same as the strongly continuous functionals on  $B(\mathcal{H})$ , hence  $u$  is an element of the strong closure of  $C$  since we have the following general result: if  $C$  is a convex set in a locally convex space  $X$ , then for any point  $x \in X$  one has  $x \in \overline{C}$  if and only if there exists a net  $(x_\lambda)_{\lambda \in \Lambda}$  in  $C$  such that  $(\tau(x_\lambda))_{\lambda \in \Lambda}$  converges to  $\tau(x)$  for all continuous linear functionals  $\tau$  on  $X$  (see [13]). Hence  $u \in C$ . Therefore,  $C$  is weakly closed.  $\square$

From this we obtain a second corollary, which gives another characterization of von Neumann algebras.

**Corollary 2.2.** *Let  $A$  be a  $*$ -algebra on a Hilbert space  $\mathcal{H}$ . Then  $A$  is a von Neumann algebra  $\iff A$  is weakly closed.*

The following two theorems are very important in the context of von Neumann algebras. We will give these theorems without proofs, since these are technical and difficult. For more information consult [4],[13],[19],[24]. The first theorem, is *Kaplansky's Density Theorem*. G.K. Pedersen says the following about this theorem: "*The density theorem is Kaplansky's great gift to mankind. It can be used every day, and twice on Sundays.*"

**Theorem 2.7.** *Let  $\mathcal{H}$  be a Hilbert space  $\mathcal{H}$  and  $A$  a  $C^*$ -subalgebra of  $B(\mathcal{H})$  with strong closure  $B$ . Then:*

- *The self-adjoint elements of  $A$  are strongly dense in the set of self-adjoint elements of  $B$ .*
- *The closed unit ball of the set of self-adjoint elements of  $A$  is strongly dense in the closed unit ball of the set of self-adjoint elements of  $B$ .*
- *The closed unit ball of  $A$  is strongly dense in the closed unit ball of  $B$ .*
- *If  $A$  contains the identity, then the unitaries of  $A$  are strongly dense in the unitaries of  $B$ .*

The second theorem we will discuss is *Sakai's Theorem*. Some books use the notion of  $W^*$ -algebras, which are characterized as  $C^*$ -algebras that are isomorphic to the dual space of a Banach space. Sakai shows that the notion of  $W^*$ -algebras and von Neumann algebras are the same.

## 2. Von Neumann algebras

**Theorem 2.8.** *Let  $A$  be a  $C^*$ -algebra. Then  $A$  is  $*$ -isomorphic with a von Neumann algebra  $\iff A$  is the dual of a Banach space.*

### 2.3. Abelian von Neumann Algebras

We saw that every commutative  $C^*$ -algebra is isometrically isomorphic to  $C_0(X)$ , for some locally compact Hausdorff space  $X$ . Hence we can see every  $C^*$ -algebra as a "non-commutative" function space. In this section we want to characterize commutative von Neumann algebras.

**Definition 2.2.** *Let  $A$  be a  $C^*$ -algebra acting on a Hilbert space  $\mathcal{H}$ . We say that  $y \in \mathcal{H}$  is a **separating vector** for  $A$  if  $u(y) = 0 \Rightarrow u = 0$  for all  $u \in A$ .*

The notion of a separating vector has a connection with cyclic vectors and non-degenerate actions. If  $x \in \mathcal{H}$  is cyclic, then it is separating for  $A'$ . If  $A$  acts non-degenerately on  $\mathcal{H}$  and if  $x$  is separating for  $A'$ , then it is cyclic for  $A$ .

**Lemma 2.2.** *Let  $A$  be an abelian von Neumann algebra acting non-degenerately on a separable Hilbert space  $\mathcal{H}$ . Then  $A$  has a separating vector.*

*Proof.* By Zorn's Lemma, we obtain a maximal set  $E$  of  $\mathcal{H}$ , consisting of unit vectors  $x$  such that the spaces  $\overline{Ax}$  ( $x \in E$ ) are pairwise orthogonal. If  $y \in \mathcal{H}$  is a unit vector such that  $y \perp \overline{Ax}$  for each  $x \in E$ , then  $\overline{Ay}$  is orthogonal to all  $\overline{Ax}$ . This contradicts the maximality of  $E$ , hence  $\mathcal{H}$  is the orthogonal sum of all spaces  $\overline{Ax}$  ( $x \in E$ ). By assumption  $\mathcal{H}$  is separable which implies that  $E$  is countable. Therefore,  $E = \{x_n : n \geq 1\}$ , where  $x_n \in \mathcal{H}$  and  $\|x_n\| = 1$  for each  $n \in \mathbb{N}$ . Define  $x := \sum_{n \in \mathbb{N}} 2^{-n} x_n$ . If  $u \in A$  and  $u(x) = 0$ , then  $u(x_n) = 0$  for each  $n \in \mathbb{N}$ , since  $(u(x_n))_{n \in \mathbb{N}}$  consists of pairwise orthogonal elements. If  $v \in A$ , then  $uv(x_n) = vu(x_n) = 0$ , hence  $u(\overline{Ax_n}) = 0$  for each  $n \in \mathbb{N}$ . It follows that  $u = 0$ , hence  $x$  is a separating vector for  $A$ .  $\square$

**Theorem 2.9.** *Let  $A$  be an abelian von Neumann algebra acting on a separable Hilbert space  $\mathcal{H}$ , that has a cyclic vector for  $A$ . Then there exist a second countable compact Hausdorff space  $\Omega$ , a positive measure  $\mu \in M(\Omega)$  (i.e. the Banach space of all regular complex Borel measures on  $\Omega$ ), and a unitary  $u : \mathcal{H} \rightarrow L^2(\Omega, \mu)$ , such that  $uAu^*$  is the von Neumann algebra of all multiplication operators  $M_\varphi$  on  $L^2(\Omega, \mu)$ , where  $\varphi \in L^\infty(\Omega, \mu)$ .*

Before we prove this theorem, we make some (crucial) observations.

**Lemma 2.3.** *Let  $\mathcal{H}$  be a separable Hilbert space. Then the closed unit ball of  $B(\mathcal{H})$  is metrizable and separable with respect to the strong topology.*

*Proof.* Since  $\mathcal{H}$  is separable, we can find a countable dense subset  $\{x_n \in \mathcal{H} : n \in \mathbb{N}\}$  in the unit ball of  $\mathcal{H}$ . Then define, for  $u$  and  $v$  in the closed unit ball of  $B(\mathcal{H})$ , the map  $d :$

$B(\mathcal{H}) \times B(\mathcal{H}) \rightarrow \mathbb{R}$  by  $d(u, v) = \sum_{n=1}^{\infty} 2^{-n} \|(u - v)(x_n)\|$ . This defines a metric on the closed unit ball of  $B(\mathcal{H})$ , which induces the strong topology.  $\square$

**Lemma 2.4.** *If  $A$  is an abelian and separable  $C^*$ -algebra, then the character space  $\Omega(A)$  is second countable.*

*Proof.* By assumption  $A$  is separable. Therefore, we can find a countable dense subset  $D := \{a_n \in A : n \in \mathbb{N}\}$  of  $A$ . Define  $\mathcal{T}$  to be the smallest topology on  $\Omega(A)$  for which  $\hat{a}_n$  is continuous for each  $n \in \mathbb{N}$ . Let  $B$  be the set of all elements  $a \in A$  such that  $\hat{a}$  is continuous with respect to  $\mathcal{T}$ . Then  $B$  is a  $C^*$ -subalgebra of  $A$  such that  $D \subset B$ . Therefore, we conclude that  $B = A$ . It follows that  $\mathcal{T}$  coincides with the weak\*-topology on  $\Omega(A)$ , since this is the smallest topology making all  $\hat{a}$  continuous for each  $a \in A$ . Now let  $E$  be a countable base for the topology on  $\mathbb{C}$ . Then all finite intersections of the sets  $\hat{a}_n^{-1}(U)$  ( $n \in \mathbb{N}, U \in E$ ) form a countable base for the topology of  $\Omega(A)$ . This proves the lemma.  $\square$

Now we are ready to prove the theorem.

*Proof.* Assume that  $x$  is a cyclic vector for  $A$ . We saw above, that the closed unit ball of  $B(\mathcal{H})$  is metrizable and separable with respect to the strong topology, since  $\mathcal{H}$  is separable by assumption. Therefore, this is also true for the unit ball of  $A$ . Thus there exists a separable  $C^*$ -subalgebra  $B$  of  $A$  that is strongly dense in  $A$ . Without loss of generality, we assume that  $id_{\mathcal{H}} \in B$ . Now let  $\varphi : B \rightarrow C(\Omega(B))$  be the Gelfand transform. By the lemma above  $\Omega(B)$  is second countable. Define  $\tau : C(\Omega(B)) \rightarrow \mathbb{C}$  to be the positive linear functional  $\tau(f) = \langle \varphi^{-1}(f)x, x \rangle$ . Now we can apply the Riesz-Markov Theorem. This gives us a positive measure  $\mu$  in the Banach space of all regular complex Borel measures on  $\Omega(B)$ , such that  $\tau(f) = \int f d\mu$  for each  $f \in C(\Omega(B))$ . Furthermore, we have an injective \*-homomorphism  $\psi : B \rightarrow B(L^2(\Omega(B), \mu))$  given by  $\psi(v) = M_{\varphi(v)}$  (multiplication operator). Let  $v \in B$ . Then

$$\int |\varphi(v)|^2 d\mu = \tau(|\varphi(v)|^2) = \langle \varphi^{-1}(\varphi(v)^* \varphi(v))(x), x \rangle = \|v(x)\|^2$$

Therefore, we obtain a map  $\mathcal{U}$  from  $Bx := \{v(x) : v \in B\}$  to  $C(\Omega(B))$ , which is in particular dense in  $L^2(\Omega(B), \mu)$ , by  $\mathcal{U}(v(x)) = \varphi(v)$ . Hence, this map is linear, well-defined and isometric. Since  $x \in A$  was cyclic,  $\overline{Ax} = \mathcal{H}$ . By construction  $B$  is strongly dense in  $A$ , which implies  $\overline{Bx} = \mathcal{H}$ . Therefore, we can extend  $\mathcal{U}$  to a surjective unitary  $\tilde{\mathcal{U}} : \mathcal{H} \rightarrow L^2(\Omega(B), \mu)$ . If  $v, w \in B$ , then  $\psi(v)\tilde{\mathcal{U}}w(x) = \varphi(vw) = \tilde{\mathcal{U}}vw(x)$ . Hence  $\psi(v)\tilde{\mathcal{U}} = \tilde{\mathcal{U}}v$  for each  $v \in B$ . This gives rise to a \*-isomorphism  $\alpha : B(\mathcal{H}) \rightarrow B(L^2(\Omega(B), \mu))$  by conjugation with  $\tilde{\mathcal{U}}$ , such that  $\alpha = \psi$  on  $B$ . Now let  $M$  be the von Neumann algebra of all multiplication operators on  $L^2(\Omega(B), \mu)$ . By construction,  $B$  is strongly dense in  $A$ . Therefore  $uAu^* = M$ , because  $\alpha$  is a homeomorphism with respect to the strong topologies.  $\square$

## 2. Von Neumann algebras

The following theorem gives a characterization of abelian von Neumann algebras on separable Hilbert spaces. For more information about the proof see [13].

**Theorem 2.10.** *Let  $A$  be an abelian von Neumann algebra on a separable Hilbert space  $\mathcal{H}$ . Then there exist a second countable compact Hausdorff space  $\Omega$  and a positive measure  $\mu \in M(\Omega)$  (the Banach space of all regular Borel complex measures on  $\Omega$ ) such that  $A$  is  $*$ -isomorphic to the  $C^*$ -algebra  $L^\infty(\Omega, \mu)$ .*

*Proof.* We may assume that  $id_{\mathcal{H}} \in A$ . By Lemma 2.2 there exists a separating vector  $x$  for  $A$ . If  $p$  is the projection of  $\mathcal{H}$  onto  $\overline{Ax}$ . Then  $p \in A'$ . If  $K$  is a closed vector subspace of  $\mathcal{H}$ , then we have a projection  $p$  of  $\mathcal{H}$  on  $K$  (see [18]). If  $u \in B(\mathcal{H})$ , let  $u_p = u_K$  be the compression of  $u$  to  $K$ . If  $A$  is a  $*$ -algebra on  $\mathcal{H}$  and  $p \in A'$ , set  $A_p = \{u_p : u \in A\}$  (for more details see [13]). Now the map  $\varphi : A \rightarrow A_p, u \mapsto u_p$  is a  $*$ -homomorphism onto  $A_p$ , and since  $x$  is separating for  $A$ , this map is injective and therefore a  $*$ -isomorphism. Clearly,  $\varphi$  is weakly continuous, so  $\varphi(A) = A_p$  is a von Neumann algebra (see [13]) on  $p(\mathcal{H})$ . Obviously,  $x$  is cyclic for  $A_p$ . Note also that  $id_{p(\mathcal{H})} \in A_p$ . Thus, to prove the theorem we have shown we may reduce to the case where  $A$  contains  $id_{\mathcal{H}}$  and has some cyclic vector  $x$ . The result now follows from Theorem 2.9.  $\square$

This theorem tells us that all abelian von Neumann algebras on separable Hilbert spaces are (isomorphic to) spaces of the form  $L^\infty(\Omega, \mu)$ . Therefore, we may see general von Neumann algebras as non-commutative measure spaces. As in the case of  $C^*$ -algebras, we can look at a von Neumann algebra generated by a single element.

**Definition 2.3.** *Let  $S$  be a bounded operator on a Hilbert space  $\mathcal{H}$ . The **von Neumann algebra generated by  $S$**  and the unit is denoted by  $W^*(S)$ . It is the set of bounded operators that commute with every operator commuting with  $S$  and is the smallest von Neumann algebra containing  $S$  and the identity map  $id_{\mathcal{H}}$ . If  $S$  is self-adjoint (or normal), then  $W^*(S)$  is a commutative von Neumann algebra.*

Observe that, if  $S$  is self-adjoint, then  $W^*(S)$  is a commutative  $C^*$ -algebra containing  $C^*(S)$ . Now assume that there exists a net  $(S_\lambda)_{\lambda \in \Lambda}$  in  $W^*(S)$  that strongly converges to some  $S_0 \in B(\mathcal{H})$ . Then  $S_0 \in W^*(S)$ , since  $\|(S_0 T - T S_0)x\| = \lim_{\lambda \in \Lambda} \|(S_\lambda T - T S_\lambda)x\| = 0$  for each  $x \in \mathcal{H}$  and for each operator  $T$  that commutes with  $S$ .

## 2.4. Weights on a von Neumann algebra

Weights on a von Neumann algebra generalize states. In particular, one obtains another version of the GNS construction. The following definition is also important in connection with von Neumann algebraic quantum groups.

**Definition 2.4.** Let  $A$  be a von Neumann algebra. A **weight** on  $A$  is a map  $\omega : A_+ \rightarrow [0, +\infty]$  such that:

$$\forall a, b \in A_+ : \omega(a + b) = \omega(a) + \omega(b)$$

$$\forall a \in A_+ \forall \lambda \in \mathbb{R}_0^+ : \omega(\lambda a) = \lambda \omega(a)$$

Here we use the convention that  $0 \cdot (+\infty) = 0$  and that  $\mu + \infty = \infty$  for each  $\mu \in \mathbb{R}_0^+$

From this definition we obtain that such a weight is an increasing function in the sense that for  $a, b \in A_+ : a \leq b$  implies  $\omega(a) \leq \omega(b)$ . We now introduce three important sets we need for the theory of weights:

$$\mathcal{F}_\omega := \{a \in A_+ : \omega(a) < \infty\}$$

$$\mathcal{N}_\omega := \{a \in A : \omega(a^*a) < \infty\}$$

$$\mathcal{M}_\omega := \mathcal{N}_\omega^* \mathcal{N}_\omega \subset \mathcal{N}_\omega^* \cap \mathcal{N}_\omega$$

One observes that  $\mathcal{N}_\omega$  is a left ideal in  $A$ . Moreover,  $\mathcal{M}_\omega$  is a  $*$ -subalgebra of  $M$  which is the linear span of  $\mathcal{F}_\omega$ . In particular, we can extend  $\omega$  uniquely to a positive linear form on  $\mathcal{M}_\omega$ , which we will denote by  $\tilde{\omega}$ . From this we obtain the following three important properties of a weight on a von Neumann algebra.

**Definition 2.5.** Let  $\omega$  be a weight on a von Neumann algebra  $A$ . We say that  $\omega$  is:

- **semi-finite**, if  $\mathcal{M}_\omega$  is weakly dense in  $A$ .
- **faithful**, if  $a \in A_+$  and  $\omega(a) = 0$  implies  $a = 0$ .
- **normal**, for any bounded increasing net  $(a_\lambda)_{\lambda \in \Lambda}$  in  $A_+$  one has  $\omega(\sup_\lambda a_\lambda) = \sup_\lambda \omega(a_\lambda)$ .
- **tracial** if  $\omega(a^*a) = \omega(aa^*)$  for each  $a \in A$ .

Here an example: Let  $\mathcal{H}$  be a separable Hilbert space with basis  $(e_n)_{n \in \mathbb{N}}$  and let  $A$  be the von Neumann algebra of all bounded operators on  $\mathcal{H}$ , i.e.  $A = B(\mathcal{H})$ . We then can define  $\omega(a) = \sum_{n \in \mathbb{N}} \langle ae_n, e_n \rangle$ . This defines a faithful, normal and tracial weight on  $A$ . Moreover,  $\mathcal{N}_\omega$  is the set of Hilbert-Schmidt operators and  $\mathcal{M}_\omega$  coincides with the trace-class operators.

Remember the GNS-construction from chapter 1. We will now discuss a related construction. Let  $\omega$  be a fixed weight on a von Neumann algebra  $A$ . Define the following set:

$$N_\omega := \{a \in A : \omega(a^*a) = 0\}$$

By construction,  $N_\omega$  is a left ideal of  $A$  such that  $N_\omega \subset \mathcal{N}_\omega$ . In particular, we can take the quotient  $\mathcal{N}_\omega/N_\omega$ . We denote the canonical quotient map from  $\mathcal{N}_\omega$  to the quotient by  $\pi_\omega$ , i.e.

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$\pi_\omega(a) = a + N_\omega$ . Have a look at the following expression:

$$\langle a + N_\omega, b + \omega \rangle := \omega(b^*a)$$

The same arguments we used in chapter 1, show that this is an inner product which is well defined on the quotient, hence we obtain a pre-Hilbert space. Its completion with respect to the norm induced by the inner product is denoted  $\mathcal{H}_\omega$ . Since by definition both  $\mathcal{N}_\omega$  and  $N_\omega$  are left-ideals, the quotient  $\mathcal{N}_\omega/N_\omega$  is a left  $A$ -module. Note that we can extend the action of  $A$  on  $\mathcal{N}_\omega/N_\omega$  to an action on  $\mathcal{H}_\omega$  by  $(ax)^*ax \leq \|a\|^2 x^*x$ . Hence we obtain a representation  $\varphi_\omega$  of  $A$  on  $\mathcal{H}_\omega$ .

**Proposition 2.2.** *Let  $\omega$  be a semi-finite normal weight on the unital von Neumann algebra  $A$ . Then  $(\varphi_\omega, \mathcal{H}_\omega)$  is a non-degenerate normal  $*$ -representation. Furthermore, if  $\omega$  is faithful, then so is  $\varphi_\omega$ .*

*Proof.* We first show that  $\varphi_\omega$  is a  $*$ -representation. For that let  $a \in A$  and  $x, y \in \mathcal{N}_\omega$ . Then:

$$\begin{aligned} \langle \varphi_\omega(a)(x + N_\omega), y + N_\omega \rangle &= \langle ax + N_\omega, y + N_\omega \rangle \\ &= \tilde{\omega}(y^*ax) = \tilde{\omega}((a^*y)^*x) \\ &= \langle x + N_\omega, \varphi_\omega(a^*)(y + N_\omega) \rangle \end{aligned}$$

Hence we have a  $*$ -representation. Since  $A$  is unital,  $\varphi_\omega$  becomes non-degenerate. Normality of  $\varphi_\omega$  follows from that of  $\omega$ .  $\square$

The triple  $(\varphi_\omega, \mathcal{H}_\omega, \pi_\omega)$  is called a *semi-cyclic representation* of  $A$ . If one is more interested in the theory of semi-cyclic representations, one can consider [25]. There is a lot more to say about weights, but this is not relevant for this thesis. We just remark that we want  $\varphi_\omega(A)$  to be a von Neumann algebra, in particular, we want that  $\varphi_\omega(A)'' = \varphi_\omega(A)$ , but for this we need normality of  $\omega$ . The problem is that an arbitrary von Neumann algebra in general does not have enough normal states to construct such a representation. The solution is that a von Neumann algebra does have enough normal weights. Observe that the assumption of a semi-finite normal faithful weight seems to be strong, but there is the following lemma.

**Lemma 2.5.** *Every von Neumann algebra admits a normal, faithful, semi-finite weight.*

The key of the proof is the following proposition, which gives us several equivalent conditions for a normal weight ([7]).

**Proposition 2.3.** *Let  $\omega$  be a weight on a von Neumann algebra  $A$ . The following are equivalent:*

1.  $\omega$  is completely additive, i.e.  $\omega(\sum_{i \in I} x_i) = \sum_{i \in I} \omega(x_i)$  for any set  $(x_i)_{i \in I}$  in  $A_+$  with  $\sum_{i \in I} x_i \in A_+$ .

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2.  $\omega$  is normal.

3. There is a set  $(\omega_\lambda)_{\lambda \in \Lambda}$  of positive, normal functionals on  $A$  such that  $\omega(a) = \sup_\lambda \omega_\lambda(a)$  for each  $a \in A$ .

4. There exists a set  $(\omega_\lambda)_{\lambda \in \Lambda}$  of positive, normal functionals on  $A$  such that  $\omega(a) = \sum_\lambda \omega_\lambda(a)$  for each  $a \in A$ .

Here we omit the proof since it is a paper on itself. For details and more information we refer to [4] and [7].





## 3. Unbounded Operators

In the first chapter we discussed  $C^*$ -algebras and showed that each  $C^*$ -algebra can be realized as a  $C^*$ -subalgebra of the bounded operators  $B(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$ . We are not going to discuss the theory of bounded operators, since we assume this to be known. If this is not the case, we refer to [18] as an introduction and to [17] for advanced theory.

### 3.1. Basic definitions

An *operator*  $T$  on a Hilbert space  $\mathcal{H}$  is a linear map  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$ , where the linear subspace  $\mathcal{D}(T) \subseteq \mathcal{H}$  is called the *domain* of  $T$ . We do not assume that  $T$  is bounded.

The *graph* of an operator  $T$  is defined by:

$$\mathcal{G}(T) := \{(x, Tx) \in \mathcal{H} \times \mathcal{H} : x \in \mathcal{D}(T)\}$$

We say that an operator  $S$  is an *extension* of an operator  $T$ , or, in symbols,  $T \subset S$ , if  $\mathcal{D}(T) \subset \mathcal{D}(S)$  and  $Sx = Tx$  for each  $x \in \mathcal{D}(T)$ . An operator is *closed* if  $\mathcal{G}(T)$  is a closed subspace of  $\mathcal{H} \times \mathcal{H}$  with respect to the norm coming from the inner product

$$\langle (a, b), (c, d) \rangle = \langle a, c \rangle + \langle b, d \rangle \tag{3.1.1}$$

which turns  $\mathcal{H} \times \mathcal{H}$  into a Hilbert space.

We know from the theory of bounded operators on a Hilbert space that each bounded operator  $T$  has an adjoint  $T^*$ . We want to have this notion also for the general setting of the operators we just introduced. Define  $\mathcal{D}(T^*)$  as the set of all  $y \in \mathcal{H}$  such that the map  $x \mapsto \langle Tx, y \rangle$  is continuous on  $\mathcal{D}(T)$ . Now suppose that  $y \in \mathcal{D}(T^*)$ . Then we can extend the map  $x \mapsto \langle Tx, y \rangle$  to a continuous linear functional on  $\mathcal{H}$  (by Hahn-Banach), so that by the Riesz-Fréchet Theorem there exists  $T^*y \in \mathcal{H}$  with  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for each  $x \in \mathcal{D}(T)$ . For uniqueness, suppose that  $z \in \mathcal{H}$  also satisfies  $\langle Tx, y \rangle = \langle x, z \rangle$ , then  $\langle x, T^*y \rangle = \langle x, z \rangle$  and therefore  $\langle x, T^*y - z \rangle = 0$  for each  $x \in \mathcal{D}(T)$ . To conclude that  $T^*y - z = 0$  and therefore that  $T^*y = z$ , we must require that  $\mathcal{D}(T)$  is dense in  $\mathcal{H}$ . Therefore, we may conclude that the operators that have an adjoint are the so called *densely defined* operators.

For the algebraic operations, we define in a natural way the following domains for the operators

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$T$  and  $S$  with the domains  $\mathcal{D}(T)$  and  $\mathcal{D}(S)$  respectively:

$$\mathcal{D}(S + T) = \mathcal{D}(T) \cap \mathcal{D}(S)$$

$$\mathcal{D}(ST) = \{x \in \mathcal{D}(T) : Tx \in \mathcal{D}(S)\}$$

Moreover:  $\mathcal{D}(\alpha T) = \mathcal{H}$  and  $\alpha T = 0$  for  $\alpha \in \mathbb{R}$  and  $\alpha = 0$ , and for  $\alpha \neq 0$  we set  $\mathcal{D}(\alpha T) = \mathcal{D}(T)$  with  $(\alpha T)(x) = \alpha(Tx)$  and  $x \in \mathcal{D}(T)$ .

The following theorem says how the adjoint operation acts on products.

**Theorem 3.1.** *Let  $S$  and  $T$  be densely defined operators on a Hilbert space  $\mathcal{H}$ . Then  $S \subset T$  implies  $T^* \subset S^*$ . Moreover,  $T^*S^* \subset (ST)^*$ . Furthermore: if  $S \in B(\mathcal{H})$ , then  $T^*S^* = (ST)^*$ .*

The proof of this is an exercise and is left to the reader.

Having covered the pertinent definition and properties we needed, we will now cover some special types of operators, namely the *symmetric* and the *self-adjoint* ones. As for the case of bounded operators, we will work out a spectral theorem for (possibly) unbounded self-adjoint operators.

#### 3.1.1. Symmetric and self-adjoint operators

We start with some definitions.

**Definition 3.1.** *An operator  $T$  on a Hilbert space  $\mathcal{H}$  is called **symmetric** if  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for each  $x, y \in \mathcal{D}(T)$ . This implies that a densely defined symmetric operator satisfies  $T \subset T^*$ . A symmetric operator  $T$  is called **maximally symmetric** if  $T \subset S$  implies  $S = T$  for  $S$  a symmetric operator. If  $T = T^*$ , we say that  $T$  is **self-adjoint**.*

First, we make some observations about these definitions. If  $T$  is bounded and  $\mathcal{D}(T) = \mathcal{H}$ , i.e.  $T \in B(\mathcal{H})$ , then symmetric means the same as self-adjoint. Notice that this does not hold in general. A second observation is: if an operator  $T$  is densely defined and  $\langle Tx, y \rangle = \langle x, Sy \rangle$  for each  $x \in \mathcal{D}(T)$  and for each  $y \in \mathcal{D}(S)$ , then  $S \subset T^*$ .

Let  $\mathcal{H} = L^2([0, 1])$  be the well-known Hilbert space of all equivalence classes of Lebesgue square integrable functions on the unit interval  $[0, 1]$ . Define the operator  $D$  for example on  $\mathcal{D}(D) = C^1([0, 1])$  by  $D(f) = f'$ , where  $C^1([0, 1])$  is the space of all continuously differentiable functions on  $[0, 1]$ . Notice that  $D$  is unbounded, since we can look at the sequence of functions  $(f_n)_{n \in \mathbb{N}}$  given by  $f_n(x) = x^n$ ; then  $\|f_n\|^2 = \frac{1}{2n+1}$  but  $\|D(f_n)\|^2 = \frac{n^2}{2n-1}$ , so that  $D$  is an unbounded operator.

Furthermore, define the operator  $M$  by  $M(f)(t) = tf(t)$  (which is a bounded operator).

We can perform the following calculation:

$$(DM-MD)(f)(t) = DM(f)(t) - MD(f)(t) = D(tf(t)) - M(f'(t)) = f(t) + tf'(t) - tf'(t) = f(t)$$

Therefore, we obtain that  $DM - MD$  is the identity operator on  $C^1([0, 1])$ . We know by construction that one of the operators is unbounded. The question is whether this can also happen if both operators are bounded. The following theorem about Banach algebras gives us the negative answer.

**Theorem 3.2.** *If  $A$  is a unital Banach algebra, then  $xy - yx \neq e$  for each  $x, y \in A$ .*

Notice that the theorem answer our above question, since  $B(\mathcal{H})$  is a Banach algebra. Moreover, this is an important result in connection with quantum mechanics, since we have the canonical commutation relation of Heisenberg  $[p, q] = -\frac{i\hbar}{2\pi}$ .

*Proof.* We prove by contradiction: Suppose there are  $x, y \in A$  with  $xy - yx = e$ . We will prove by induction that  $x^n y - yx^n = nx^{n-1}$  for each  $n \geq 1$ . The formula is true for  $n = 1$  by assumption. We have to show that the formula is also true for  $n + 1$ , to which end we make some computations:

$$x^{n+1}y - yx^{n+1} = x^n(xy - yx) + (x^n y - yx^n)x = x^n e + nx^{n-1}x = (n + 1)x^n$$

Therefore, we see by induction that the formula holds. Now we can make an estimate:

$$n \|x^{n-1}\| = \|x^n y - yx^n\| \leq 2 \|x^n\| \|y\| \leq 2 \|x^{n-1}\| \|x\| \|y\|$$

hence  $n \leq 2 \|x\| \|y\|$  for each  $n \geq 1$ , which cannot be true for fixed  $x$  and  $y$ .  $\square$

Now let us return to symmetric operators. Let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Then we can turn  $\mathcal{H} \times \mathcal{H}$ , too, into a Hilbert space, as we remarked in formula (3.1.1). Moreover, define  $V : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$  by  $V(a, b) = (-b, a)$ . With the help of this operator it is possible to obtain the graph of  $T^*$  from the graph of  $T$ . The following theorem makes this precise.

**Theorem 3.3.** *Suppose  $T$  is a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then*

$$\mathcal{G}(T^*) = [V(\mathcal{G}(T))]^\perp$$

where  $\perp$  means the orthogonal complement in  $\mathcal{H} \times \mathcal{H}$ .

*Proof.*  $(y, z) \in \mathcal{G}(T^*) \iff y \in \mathcal{D}(T^*), T^*y = z \iff \langle Tx, y \rangle = \langle x, z \rangle \forall x \in \mathcal{D}(T) \iff \langle (-Tx, x), (y, z) \rangle = 0 \forall x \in \mathcal{D}(T) \iff (y, z) \in [V(\mathcal{G}(T))]^\perp$ .  $\square$

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This theorem enables us to recover  $T^*$  (including its domain) from  $\mathcal{G}(T)$ . But this theorem has wider implications, since the orthogonal complement is always closed.

**Corollary 3.1.** *Let  $T$  be a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then  $T^*$  is a closed operator. In particular: self-adjoint operators are closed.*

A second corollary follows from the observation that  $(M^\perp)^\perp = M$ , if  $M \subset \mathcal{H}$  is a closed subspace in a Hilbert space, and from the fact that in every Hilbert space one has  $\mathcal{H} = M \oplus M^\perp$ , where once again  $M$  is closed subspace.

**Corollary 3.2.** *Suppose  $T$  is a densely defined closed operator on a Hilbert space  $\mathcal{H}$ . Then  $\mathcal{H} \times \mathcal{H} = V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*)$ .*

The next theorem gives us information about symmetric and self-adjoint operators.

**Theorem 3.4.** *Assume that  $T$  is a densely defined symmetric operator on a Hilbert space  $\mathcal{H}$ . Then:*

1. *If  $\mathcal{D}(T) = \mathcal{H}$ , then  $T$  is self-adjoint and  $T \in B(\mathcal{H})$  (Hellinger-Toeplitz Theorem).*
2. *If  $T$  is self-adjoint and injective, then the image under  $T$  is dense in  $\mathcal{H}$ ,  $T$  is invertible, and  $T^{-1}$  is self-adjoint.*
3. *If the image of  $T$  is dense in  $\mathcal{H}$ , then  $T$  is injective.*
4. *If  $T$  is surjective, then  $T$  is self-adjoint as well as invertible and  $T^{-1} \in B(\mathcal{H})$ .*

*Proof.* To prove the four statements we will use some theory we have already established.

1. Since the general assumption is that  $T$  is symmetric, we know that  $T \subset T^*$ . Now suppose  $\mathcal{D}(T) = \mathcal{H}$ . Then also  $T^* \subset T$ , hence  $T = T^*$  which means that  $T$  is self-adjoint. Now Corollary 3.1 gives us the information that  $T$  is closed and therefore it is continuous by the Closed Graph Theorem. Thus  $T \in B(\mathcal{H})$ .
2. We first prove that the image  $\mathcal{R}(T)$  of  $T$  is dense in  $\mathcal{H}$ . Therefore, suppose  $y \perp \mathcal{R}(T)$ . Then the map  $x \mapsto \langle Tx, y \rangle = 0$  is continuous for each  $x \in \mathcal{D}(T)$ . Thus  $y \in \mathcal{D}(T^*)$  by definition. Since  $T$  is symmetric, we obtain  $\langle Tx, y \rangle = \langle x, Ty \rangle = 0$  for each  $x \in \mathcal{D}(T)$ . This means that  $Ty = 0$ . But  $T$  was assumed to be injective, hence  $y = 0$ . Thus  $\mathcal{R}(T)$  is dense in  $\mathcal{H}$ . From this we may conclude that  $T^{-1}$  is densely defined, with  $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$ , and that  $(T^{-1})^*$  exists. The reader can easily verify that  $\mathcal{G}(T^{-1}) = V(\mathcal{G}(T))$  and  $V(\mathcal{G}(T^{-1})) = \mathcal{G}(-T)$ . Since  $T$  is assumed to be self-adjoint, we find that  $T$  is closed. Thus  $-T$  is closed and therefore  $T^{-1}$  is closed by the previous relations. Now we infer that  $\mathcal{H} \times \mathcal{H} = V(\mathcal{G}(T^{-1})) \oplus \mathcal{G}((T^{-1})^*)$  and hence  $\mathcal{H} \times \mathcal{H} = \mathcal{G}(T^{-1}) \oplus V(\mathcal{G}(T^{-1}))$ . We may conclude that  $\mathcal{G}((T^{-1})^*) = \mathcal{G}(T^{-1})$ , which means, in particular, that  $(T^{-1})^* = T^{-1}$ .

3. Assume that  $\mathcal{R}(T)$  is dense in  $\mathcal{H}$  and suppose  $Tx = 0$ . Then  $\langle x, Ty \rangle = \langle Tx, y \rangle = 0$  for each  $y \in \mathcal{D}(T)$ , hence  $y \perp \mathcal{R}(T)$ . Thus  $x = 0$ .
4. Assuming that  $T$  is surjective, we may conclude by statement (3) that  $T$  is injective and that  $\mathcal{D}(T^{-1}) = \mathcal{H}$ . Now let  $x, y \in \mathcal{H}$  be arbitrary. Then there are  $w, z \in \mathcal{D}(T)$  such that  $x = Tz$  and  $y = Tw$ . Thus  $\langle T^{-1}y, y \rangle = \langle z, Tw \rangle = \langle Tz, w \rangle = \langle x, T^{-1}y \rangle$ , whence  $T^{-1}$  is symmetric and therefore, by part (1) also  $T^{-1}$  is self-adjoint and bounded. Now part (2) gives us that  $T = (T^{-1})^{-1}$  is self-adjoint, too. This proves the theorem.  $\square$

At the end of this section we will discuss another important class of unbounded operators, namely the so-called *essentially self-adjoint* operators. Roughly speaking, one can say that essentially self-adjoint operators lie between symmetric and self-adjoint operators. This description becomes clearer if we study what it means for an operator to be *closable*.

**Definition 3.2.** *An operator  $R : \mathcal{D}(R) \rightarrow \mathcal{H}$  on a Hilbert space  $\mathcal{H}$  is called **closable** if there exists a closed operator  $U : \mathcal{D}(U) \rightarrow \mathcal{H}$  such that  $R \subset U$ . Every closable operator has a smallest closed extension, called the *closure*, which we denote by  $\overline{T}$ .*

Suppose we have an arbitrary operator  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  on a Hilbert space  $\mathcal{H}$ . One could expect that we can obtain the closure  $\overline{T}$  by taking the closure of the graph  $\mathcal{G}(T)$  of  $T$  in  $\mathcal{H} \times \mathcal{H}$ . The problem is that  $\overline{\mathcal{G}(T)}$  need not be the graph of an operator. But there is the following lemma.

**Lemma 3.1.** *Let  $\mathcal{H}$  be a Hilbert space and  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  an operator. If  $T$  is closable, then  $\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$ .*

*Proof.* Since  $T$  is closable, we may find a closed extension  $S : \mathcal{D}(S) \rightarrow \mathcal{H}$  of  $T$ . This means in particular that  $\overline{\mathcal{G}(T)} \subseteq \mathcal{G}(S)$ . Therefore, if  $(0, x) \in \overline{\mathcal{G}(T)}$ , then  $x = 0$ . Now we will define a new operator  $R : \mathcal{D}(R) \rightarrow \mathcal{H}$ . The domain will be

$$\mathcal{D}(R) := \left\{ x \in \mathcal{H} : \exists y \in \mathcal{H} : (x, y) \in \overline{\mathcal{G}(T)} \right\}$$

Since the vector we found is unique, we can define  $Rx = y$ . Observe that  $\mathcal{G}(R) = \overline{\mathcal{G}(T)}$ , hence  $R$  is a closed extension of  $T$ . But  $R \subset S$  which is an arbitrary closed extension. Therefore one has  $R = \overline{T}$ .  $\square$

The following theorem gives us a connection between closable operators and adjoints.

**Theorem 3.5.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a densely defined operator on a Hilbert space  $\mathcal{H}$ . Then:*

1.  $T$  is closeable  $\iff \mathcal{D}(T^*)$  is dense, in which case  $\overline{T} = T^{**}$ .
2. If  $T$  is closable, then  $(\overline{T})^* = T^*$ .

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*Proof.* The first part of the proof is a nice application of the theorems from the beginning of this chapter.

$\implies$ : Suppose  $\mathcal{D}(T^*)$  is not dense in  $\mathcal{H}$ . Then we can find  $x \in \mathcal{D}(T^*)^\perp$ . Then  $(x, 0) \in (\mathcal{G}(T^*))^\perp$ , hence  $V(\mathcal{G}(T^*))^\perp$  is not the graph of an operator. But  $\overline{\mathcal{G}(T)} = (V\mathcal{G}(T^*))^\perp$ , hence  $T$  is not closable.

$\impliedby$ : Observe that  $\mathcal{G}(T)$  is a linear subspace of  $\mathcal{H} \times \mathcal{H}$ , hence we have:

$$\overline{\mathcal{G}(T)} = \left(\mathcal{G}(T)^\perp\right)^\perp = \left(V^2\mathcal{G}(T)^\perp\right)^\perp = (V\mathcal{G}(T^*))^\perp$$

But we know that the adjoint is closed, since  $T$  was densely defined, hence  $\overline{\mathcal{G}(T)}$  is the graph of  $T^{**}$ .

Observe that the second part of the theorem follows from part one by  $T^* = \overline{T^*} = T^{***} = (\overline{T})^*$ , since we assume that  $T$  is closable.  $\square$

The following corollary is a direct application of the first part of the theorem.

**Corollary 3.3.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a densely defined symmetric operator on a Hilbert space  $\mathcal{H}$ . Then  $T$  is closable.*

Now we come to the class of essentially self-adjoint operators.

**Definition 3.3.** *A symmetric operator  $T$  is called essentially self-adjoint if its closure  $\overline{T}$  is self-adjoint.*

We want to develop a criterion that helps us to characterize essentially self-adjoint operators. To obtain this we first prove the so-called *basic criterion for self-adjointness*.

**Theorem 3.6.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a symmetric operator on a Hilbert space  $\mathcal{H}$ . Then the following three statements are equivalent:*

1.  $T$  is self-adjoint.
2.  $T$  is closed and  $\text{Ker}(T^* \pm i) = \{0\}$ .
3.  $\mathcal{R}(T \pm i) = \mathcal{H}$

*Proof.* (1)  $\implies$  (2) : Suppose  $T$  is self-adjoint and assume that there exists  $x \in \mathcal{D}(T^*) = \mathcal{D}(T)$  such that  $T^*x = ix$ . By self-adjointness we obtain  $Tx = ix$ . Moreover:

$$i \langle x, x \rangle = \langle ix, x \rangle = \langle Tx, x \rangle = \langle x, T^*x \rangle = \langle x, Tx \rangle = i \langle x, x \rangle$$

hence  $x = 0$ . In the same way we obtain that  $T^*x = -ix$  cannot have any non-trivial solution. The result of Corollary 3.1 gives us the complete formulation of statement (2).

(2)  $\implies$  (3) : By assumption  $T^*x = -ix$  has no non-trivial solutions. First, we show that

$\mathcal{R}(T - i)$  is dense in  $\mathcal{H}$ . Therefore, assume that this is not the case. Then take  $y \in \mathcal{R}(T - i)^\perp$ . Then  $\langle (T - i)x, y \rangle = 0$  for each  $x \in \mathcal{D}(T)$ , hence  $y \in \mathcal{D}(T^*)$  and  $(T - i)^*y = (T^* + i)y = 0$ . This contradicts the assumption that  $\text{Ker}(T^* + i) = \{0\}$ . Therefore,  $\overline{\mathcal{R}(T - i)} = \mathcal{H}$ . The next step is to show that  $\mathcal{R}(T - i)$  is closed. Notice that since  $T^* = T$  the following equation is true for each  $x \in \mathcal{D}(T)$ :

$$\|(T - i)x\|^2 = \|Tx\|^2 + \|x\|^2$$

Now let  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(T)$  such that  $(T - i)x_n \rightarrow y_0$ , then we can conclude from the previous equality that  $x_n \rightarrow x_0$  for some  $x_0 \in \mathcal{H}$ . Since we assume that  $T$  is closed, we obtain  $x_0 \in \mathcal{D}(T)$ . We observe that also  $(Tx_n)_{n \in \mathbb{N}}$  converges. Therefore,  $(T - i)x_0 = y_0$ . Thus  $\mathcal{R}(T - i)$  is closed, hence  $\mathcal{R}(T - i) = \mathcal{H}$ . The same arguments can be applied to prove  $\mathcal{R}(T + i) = \mathcal{H}$ .

(3)  $\implies$  (1) : By assumption  $T$  is symmetric and therefore  $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ . We will prove the other implication. Therefore, let  $x \in \mathcal{D}(T^*)$ . Since  $\mathcal{R}(T - i) = \mathcal{H}$ , we can find  $y \in \mathcal{D}(T)$  such that  $(T - i)y = (T^* - i)x$ . We recall that  $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ , hence  $x - y \in \mathcal{D}(T)$  and  $(T^* - i)(x - y) = 0$ . Since  $\mathcal{R}(T + i) = \mathcal{H}$  and  $\text{Ker}(T^* - i) = \{0\}$ , we obtain  $x = y$ . Therefore,  $\mathcal{D}(T^*) = \mathcal{D}(T)$ , which means that  $T$  is self-adjoint.  $\square$

This criterion has a direct consequence for our characterization of essential self-adjointness.

**Corollary 3.4.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a symmetric operator on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:*

- $T$  is essentially self-adjoint.
- $\text{Ker}(T^* \pm i) = \{0\}$ .
- $\overline{\mathcal{R}(T \pm i)} = \mathcal{H}$ .

## 3.2. Cayley Transform

We recall the map  $t \mapsto \frac{t-i}{t+i}$  from complex analysis (i.e the Cayley Transform), which gives us a bijective correspondence between the real line and the unit circle without 1. We will now extend this to a bijective correspondence between symmetric operators and isometries. Thus let  $T$  be self-adjoint operator (with  $\mathcal{D}(T)$  dense in  $\mathcal{H}$  and  $T = T^*$ ). Then we want to define  $U := (T - iI)(T + iI)^{-1}$ . For this, we can consult Theorem 3.6. From this we know that  $\mathcal{R}(T + iI) = \mathcal{H}$ . Moreover, we obtain that  $T + iI$  is injective. Thus we can define  $U : \mathcal{H} \rightarrow \mathcal{H}$  by  $Tx + ix \mapsto x \mapsto Tx - ix$ . This is a surjective isometry, which is therefore unitary. In particular,

$$U = (T - iI)(T + iI)^{-1} \tag{3.2.1}$$

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is well-defined. This operator  $U$  is called the *Cayley transform* of  $T$ . We can now make some observations about  $U$ : If  $x \in \mathcal{D}(T)$ , then  $(I - U)(Tx + ix) = Tx + ix - (Tx - ix) = 2ix$ . This means in particular that  $I - U$  is injective and that  $\mathcal{R}(I - U) = \mathcal{D}(T)$ . With this we find  $(I + U)(Tx + ix) = 2Tx$ , hence:

$$2Tx = (I + U)(Tx + ix) = (I + U)(I - U)^{-1}(2ix)$$

Therefore, we find  $Tx = i(I + U)(I - U)^{-1}x$  for each  $x \in \mathcal{D}(T)$ , or, in other words

$$T = i(I + U)(I - U)^{-1} \quad (3.2.2)$$

Before we state the main theorem about the Cayley transform, we need a lemma.

**Lemma 3.2.** *Let  $T$  be an operator on a Hilbert space  $\mathcal{H}$ , such that  $T$  is an isometry, i.e.  $\|Tx\| = \|x\|$  for each  $x \in \mathcal{D}(T)$ . Then the following holds:*

1. For all  $x, y \in \mathcal{D}(T)$  :  $\langle Tx, Ty \rangle = \langle x, y \rangle$ .
2. If  $\mathcal{R}(I - T)$  is dense in  $\mathcal{H}$ , then  $I - T$  is injective.
3. If one of the three spaces  $\mathcal{D}(T), \mathcal{R}(T)$  and  $\mathcal{G}(T)$  is closed, then so are the other two.

*Proof.* To prove part one of the lemma, we use the following statement (which is not hard to prove): For every Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot, \cdot \rangle$ , the following identity holds:

$$\langle x, y \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|x + e^{i\theta}y\|^2 e^{i\theta} d\theta$$

From this one can conclude (1) directly.

Now we prove the second statement: assume that  $x \in \mathcal{D}(T)$  and  $(I - U)x = 0$ , then  $Ux = x$ . Then, by a little computation:  $\langle x, (I - U)y \rangle = 0$  for each  $y \in \mathcal{D}(T)$ , hence  $x \perp \mathcal{R}(I - U)$ . This means that  $x = 0$ , since we assumed that  $\mathcal{R}(I - U)$  is dense in  $\mathcal{H}$ . The last part follows from the observation that we have the following equality:  $\|Ux - Uy\| = \|x - y\| = \frac{1}{\sqrt{2}} \|(x, Ux) - (y, Uy)\|$ .  $\square$

Now we are ready to formulate and prove the main theorem about the Cayley transform, due to von Neumann:

**Theorem 3.7.** *Let  $U$  be the Cayley transform (3.2.1) of a symmetric operator  $T$  on a Hilbert space  $\mathcal{H}$ . Then:*

1.  $U$  is closed  $\iff T$  is closed.
2.  $\mathcal{R}(I - U) = \mathcal{D}(T)$ ,  $I - U$  is injective, and  $T$  can be reconstructed from  $U$  by (3.2.2).
3.  $U$  is unitary  $\iff T$  is self-adjoint.



4. *Conversely: If  $V$  is an operator on  $\mathcal{H}$  that is an isometry and if  $I - V$  is injective, then  $V$  is the Cayley transform of a symmetric operator on  $\mathcal{H}$ .*

*Proof.* To prove this theorem we will use the theory we developed before:

(1) : Observe that, since  $\mathcal{D}(U) = \mathcal{R}(T + iI)$  by construction, the following equivalences hold:  
 $T$  is closed  $\iff \mathcal{R}(T + iI)$  is closed  $\iff \mathcal{D}(U)$  is closed  $\iff U$  is closed.

(2) : The second statement was already proven during our observations on  $I - U$  and  $I + U$  above.

(3) : Assume that  $T$  is self-adjoint. Then  $\mathcal{R}(I+T^2) = \mathcal{H}$ . But we have  $I+T^2 = (T+iI)(T-iI)$  and  $I + T^2 = (T - iI)(T + iI)$  (where all operators have domain  $\mathcal{D}(T^2)$ ). This implies that  $\mathcal{D}(U) = \mathcal{R}(T + iI) = \mathcal{H}$  and  $\mathcal{R}(U) = \mathcal{R}(T - iI) = \mathcal{H}$ . This means that  $U$  is a surjective isometry, and therefore it is unitary. For the converse, suppose  $U$  is unitary. Then  $(\mathcal{R}(I - U))^\perp = \text{Ker}(I - U) = \{0\}$  by part (2) of the theorem and hence by the normality of  $I - U$  we have that  $\mathcal{D}(T) = \mathcal{R}(I - U)$  is dense in  $\mathcal{H}$ . Therefore we can define  $T^*$  and we find  $T \subset T^*$ . Now we want to prove that  $T^* \subset T$  in order to prove that  $T = T^*$ . Thus suppose that  $y \in \mathcal{D}(T^*)$ . Then there exists  $y_0 \in \mathcal{D}(T)$  with  $(T^* - iI)y = (T + iI)y_0 = (T^* + iI)y_0$ , since  $\mathcal{R}(T + iI) = \mathcal{D}(U) = \mathcal{H}$ . Observe, that the last equality holds, since we already saw that  $T \subset T^*$ . Let  $y_1 = y - y_0$ . Then  $y_1 \in \mathcal{D}(T^*)$  and  $\langle (T - iI)x, y_1 \rangle = \langle x, (T^* + iI)y_1 \rangle = \langle x, 0 \rangle = 0$  for each  $x \in \mathcal{D}(T)$ . Therefore, we find  $y \perp \mathcal{R}(T + iI) = \mathcal{R}(U) = \mathcal{H}$ . Thus  $y_1 = 0$ , which implies that  $y = y_0 \in \mathcal{D}(T)$ . This proves that  $T^* \subset T$ , and since  $T \subset T^*$ , we obtain  $T^* = T$ .

(4) : Now assume that  $V$  is an isometric operator on  $\mathcal{H}$ , such that  $I - V$  is injective. Then this gives us a bijective correspondence  $z \leftrightarrow x$  between  $\mathcal{D}(V)$  and  $\mathcal{R}(I - V)$ , by  $x = z - Vz$ . Now define an operator  $S$  with domain  $\mathcal{D}(S) = \mathcal{R}(I - V)$ , given by  $Sx = i(z + Vz)$  if  $x = z - Vz$ . We want to show that  $S$  is symmetric. Let  $x, y \in \mathcal{D}(S)$ . Then there are  $z, u \in \mathcal{D}(V)$  with  $x = z - Vz$  and  $y = u - Vu$ . Moreover,  $V$  is assumed to be an isometry, hence by Lemma 3.1 we obtain:

$$\begin{aligned} \langle Sx, y \rangle &= i \langle z + Vz, u - Vu \rangle \\ &= i \langle Vz, u \rangle - i \langle z, Vu \rangle \\ &= \langle z - Vz, iu + iVu \rangle \\ &= \langle x, Sy \rangle \end{aligned}$$

This proves that  $S$  is symmetric. Since we also find that  $2iVz = Sx - ix$  and  $2iz = Sx + ix$  for  $z \in \mathcal{D}(V)$ , we may conclude that  $V(Sx + ix) = Sx - ix$  for  $x \in \mathcal{D}(S)$  and  $\mathcal{D}(V) = \mathcal{R}(S + iI)$ . Thus  $V$  is the Cayley transform of  $S$ .  $\square$

### 3. Unbounded Operators

#### 3.2.1. Spectral Theorem for Unbounded Operators

We recall one version of the spectral theorem for normal operators on a Hilbert space  $\mathcal{H}$  (for a proof see [13],[17]):

**Theorem 3.8.** *If  $T \in B(\mathcal{H})$  is a normal operator on a Hilbert space  $\mathcal{H}$ , then there exists a unique resolution of the identity  $E$  on the Borel subsets of  $\sigma(T)$  such that  $T = \int_{\sigma(T)} \lambda dE_\lambda$ .*

Here the notion of a resolution of the identity plays an important role. Since the version of the spectral theorem for unbounded operators that we want to state here also uses this construction, we will recall the definition.

**Definition 3.4.** *Let  $\mathcal{B}$  be the  $\sigma$ -algebra of a set  $\Omega$  and let  $\mathcal{H}$  be a Hilbert space. A **resolution of the identity** on  $\mathcal{B}$  is a map  $E : \mathcal{B} \rightarrow B(\mathcal{H})$  such that:*

- $E(\emptyset) = 0, E(\Omega) = \mathbf{1}$ ;
- $\forall A \in \mathcal{B}: E(A) = E(A)^* = E(A)^2$  (i.e. each  $E(A)$  is a projection);
- $\forall A, B \in \mathcal{B}: E(A \cap B) = E(A)E(B)$ ;
- $\forall A, B \in \mathcal{B}: A \cap B = \emptyset \implies E(A \cup B) = E(A) + E(B)$ ;
- For each  $x, y \in \mathcal{H}$  the function  $E_{x,y} : \mathcal{B} \rightarrow \mathbb{C}$  defined by  $E_{x,y}(A) = \langle E(A)x, y \rangle$  is a complex regular Borel measure on  $\mathcal{B}$ .

Now we make the connection with unbounded operators, unbounded measurable functions, and the Cayley transform. The following theorem is the main result in this set-up. The preparation and the proof may be found in [17].

**Theorem 3.9.** *Let  $T$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a unique resolution of the identity  $E$  on the Borel sets of  $\sigma(T)$  such that*

$$\langle Tx, y \rangle = \int_{-\infty}^{\infty} t dE_{x,y}(t)$$

for each  $x \in \mathcal{D}(T)$  and each  $y \in \mathcal{H}$ . Moreover, one has  $E(\sigma(T)) = I$ , i.e.

$$\langle x, y \rangle = \int_{-\infty}^{\infty} dE_{x,y}(t)$$

and

$$\langle f(T)x, y \rangle = \int_{-\infty}^{\infty} f(t) dE_{x,y}(t)$$

.

### 3.3. Strongly continuous one-parameter unitary groups

In this section we want to discuss strongly continuous one-parameter unitary groups. The main theorem of this section is Stone's Theorem (which plays an important role in quantum mechanics). We start with the definitions we need for this section.

**Definition 3.5.** Let  $\mathcal{H}$  be a Hilbert space and  $\{U_t\}_{t \in \mathbb{R}}$  a family of bounded operators. This collection is called a **one-parameter group of operators** if  $U_t U_s = U_{t+s}$  for each  $s, t \in \mathbb{R}$  and  $U_0 = id_{\mathcal{H}}$ . If  $U_t$  is unitary for all  $t \in \mathbb{R}$ , then we have a **one-parameter unitary group**.

We can also look at the property of continuity. If the map  $t \mapsto U_t$  is continuous in  $t_0 \in \mathbb{R}$  with respect to the weak topology or the strong topology (see subsection 2.1), then we say that  $\{U_t\}_{t \in \mathbb{R}}$  is *weakly continuous* or *strongly continuous* in  $t_0 \in \mathbb{R}$ , respectively. We call  $\{U_t\}_{t \in \mathbb{R}}$  *weakly continuous* or *strongly continuous* if  $t \mapsto U_t$  is weakly continuous or strongly continuous for all  $t \in \mathbb{R}$ , respectively. The following theorem tells us more about the connection of these properties in the case where we handle a one-parameter unitary group.

**Theorem 3.10.** Let  $\{U_t\}_{t \in \mathbb{R}}$  be a one-parameter unitary group on a Hilbert space  $\mathcal{H}$ . Then the following statements are equivalent:

1.  $\{U_t\}_{t \in \mathbb{R}}$  is weakly continuous at  $t = 0$ .
2.  $\langle x, U_t x \rangle \rightarrow \langle x, x \rangle$  as  $t \rightarrow 0$  for each  $x \in \mathcal{H}$ .
3.  $\{U_t\}_{t \in \mathbb{R}}$  is strongly continuous in  $t = 0$ .
4.  $\{U_t\}_{t \in \mathbb{R}}$  is strongly continuous.
5.  $\{U_t\}_{t \in \mathbb{R}}$  is weakly continuous.

The proof of this theorem is straightforward from the definitions of the topologies if one proves (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5)  $\Rightarrow$  (1) and is therefore left to the reader (see [11]). The following theorem, due to von Neumann, shows that the property of strong continuity is not very restrictive.

**Theorem 3.11.** Assume that  $\{U_t\}_{t \in \mathbb{R}}$  is a one-parameter unitary group on a separable Hilbert space  $\mathcal{H}$ . Then  $\{U_t\}_{t \in \mathbb{R}}$  is strongly continuous  $\iff$  the map  $t \mapsto \langle U_t x, y \rangle$  is Borel measurable for each  $x, y \in \mathcal{H}$ .

*Proof.*  $\implies$ : This implication is the easy one. If the group is continuous with respect to the strong topology, then  $t \mapsto \langle U_t x, y \rangle$  is measurable for each  $t \in \mathbb{R}$  and for all  $x, y \in \mathcal{H}$ , being continuous.

$\impliedby$ : For the converse we have to work a little bit more. Suppose that the maps are Borel measurable, hence Lebesgue measurable. Observe that by Cauchy-Schwarz and  $\|U_t\| = 1$  for

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each  $t \in \mathbb{R}$  one has  $|\langle U_t x, y \rangle| = \|x\| \|y\|$ , which implies that all maps  $t \mapsto \langle U_t x, y \rangle$  are bounded. Now define, for  $x \in \mathcal{H}$  and  $a \in \mathbb{R}$ , the linear functional

$$y \mapsto \int_0^a \langle U_t x, y \rangle dt$$

Here we can also apply Cauchy-Schwarz to conclude that this linear functional is bounded by  $|a| \|x\|$ . Now we can apply the Riesz-Fréchet Theorem to conclude the existence of  $x_a \in \mathcal{H}$  such that for each  $y \in \mathcal{H}$  one has

$$\langle x_a, y \rangle = \int_0^a \langle U_t x, y \rangle dt$$

Therefore,

$$\langle U_b x_a, y \rangle = \langle x_a, U_{-b} y \rangle = \int_0^a \langle U_t x, U_{-b} y \rangle dt = \int_0^a \langle U_{t+b} x, y \rangle dt = \int_b^{a+b} \langle U_t x, y \rangle dt$$

From this, we obtain  $|\langle U_b x_a, y \rangle - \langle x_a, y \rangle| \leq 2b \|x\| \|y\|$ , which implies that  $\langle U_b x_a, y \rangle \rightarrow \langle x_a, y \rangle$  if  $b \rightarrow 0$ . Therefore, by conjugation we obtain

$$\lim_{t \rightarrow 0} \langle y, U_t x_a \rangle = \langle y, x_a \rangle$$

We will be done if we can show that the span of the set  $\{x_a : x \in \mathcal{H}, a \in \mathbb{R}\}$  is dense in  $\mathcal{H}$ . Let  $z \in \{x_a : x \in \mathcal{H}, a \in \mathbb{R}\}^\perp$ . By assumption,  $\mathcal{H}$  is separable. Hence we can choose a countable orthonormal basis  $\{x^{(n)}\}_{n \in \mathbb{N}}$  for  $\mathcal{H}$ . Observe that for each  $n \in \mathbb{N}$  and  $a \in \mathbb{R}$ :

$$0 = \langle x_a^{(n)}, z \rangle = \int_0^a \langle U_t x^{(n)}, z \rangle dt$$

This implies that the map  $t \mapsto \langle U_t x^{(n)}, z \rangle$  is zero almost everywhere. Now define  $S_n \subset \mathbb{R}$  to be the set where the above map does not vanish and fix  $t_0 \in \mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} S_n$ . Notice that  $t_0$  exists, since  $\bigcup_{n \in \mathbb{N}} S_n$  is not equal to  $\mathbb{R}$  (because  $S_n$  has measure zero for each  $n \in \mathbb{N}$ ). Then  $\langle U_{t_0} x^{(n)}, z \rangle = 0$  for each  $n \in \mathbb{N}$ , implies  $z = 0$  since  $U_{t_0}$  is unitary and hence  $\{U_{t_0} x^{(n)}\}_{n \in \mathbb{N}}$  is a basis for  $\mathcal{H}$ . Since  $\{x_a : x \in \mathcal{H}, a \in \mathbb{R}\}^\perp = \{0\}$ , the span of  $\{x_a : x \in \mathcal{H}, a \in \mathbb{R}\}$  is dense.  $\square$

#### 3.3.1. Stone's Theorem

Now we want to formulate Stone's Theorem, which gives us a complete description of strongly continuous one-parameter unitary groups ([11],[20]).

**Theorem 3.12.** *Let  $\mathcal{H}$  be a Hilbert space. If  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$  is self-adjoint, then the operators  $U_t := e^{itA}$  (defined by Theorem 3.9 or by the bounded transform in the following section) form a strongly continuous one-parameter unitary group. Conversely, if  $\{U_t\}_{t \in \mathbb{R}}$  is a strongly*

### 3.3. Strongly continuous one-parameter unitary groups

continuous one-parameter unitary group, then there exists a unique (densely defined) self-adjoint operator  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$  such that  $U_t = e^{itA}$  for each  $t \in \mathbb{R}$ .

The second part of Stone's Theorem is highly non-trivial and is proven in [11],[15] and [20]. The unique self-adjoint operator mentioned in this theorem is called the *infinitesimal generator* of  $\{U_t\}_{t \in \mathbb{R}}$ . Observe that, in general, this generator  $A$  is unbounded. In particular  $A \in B(\mathcal{H})$  if and only if  $t \mapsto U_t$  is continuous in the norm topology (see [11]). In quantum mechanics  $A$  plays the role of the Hamiltonian of the system. Moreover one can solve differential equations if one knows the theory of one-parameter groups (see [20]).

In general it is very difficult to find the infinitesimal generator of a one-parameter group of unitaries. However, an simple example is the following (for more details see [20]): Let  $\mathcal{H} = L^2(\mathbb{R})$  and define  $U_t f(x) = f(x + t)$  for each  $t \in \mathbb{R}$ . Then  $\{U_t\}_{t \in \mathbb{R}}$  is a strongly one-parameter unitary group with infinitesimal generator  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ . Here  $\mathcal{D}(A) = H^1(\mathbb{R})$  and  $A(f) = \frac{1}{i} f'$ .

We said that it is difficult to say whether an operator is an infinitesimal generator or not. But in the case of a semigroup of contractions (see [11],[20]) on a Banach space, one has the Hille-Yosida Theorem ([17],[20]) which gives us a necessary and sufficient criterion for this.

**Theorem 3.13.** *An linear operator  $T$  on a Banach space  $X$  is an infinitesimal generator of a continuous one parameter contraction group  $\{Q_t\}_{t \geq 0} \iff T$  is densely defined, closed,  $(0, \infty) \subset \rho(T)$  and  $\|(T - \lambda I)^{-1}\| \leq \frac{1}{\lambda}$  for each  $\lambda > 0$ .*



## 4. The Bounded Transform

### 4.1. Basic properties

We just discussed the Cayley transform, which relates self-adjoint operators to unitary operators. We now want to relate (possibly unbounded) self-adjoint operators to self-adjoint bounded operators that are *pure contractions*. By definition: an operator  $T$  on a Hilbert space  $\mathcal{H}$  is a pure contraction if  $\|Tx\| < \|x\|$  for each  $x \in \mathcal{H} \setminus \{0\}$ . In particular, pure contractions are elements of  $B(\mathcal{H})$ . An equivalent definition is the following:  $\|T\| \leq 1$  and  $\pm 1 \notin \sigma_p(T)$ .

To see this assume that  $T$  is a self-adjoint operator such that  $\|T\| = 1$ . Then  $|\langle Tx, x \rangle| \leq \|T\| \|x\| = 1$  for each unit vector  $x \in \mathcal{H}$ . If  $|\langle Tx, x \rangle| = 1$  for some unit vector of  $\mathcal{H}$ , then

$$1 = |\langle Tx, x \rangle| = \|Tx\| \|x\| \leq \|T\| \|x\|^2 \leq 1$$

Hence  $|\langle Tx, x \rangle| = \|Tx\| \|x\|$  with  $x \neq 0$  forces  $Tx = \lambda x$  for some  $\lambda \in \mathbb{C}$  and some unit vector  $x \in \mathcal{H}$ . Therefore,  $|\lambda| = 1$ . So we can conclude that  $\pm 1 \notin \sigma_p(T)$  if and only if  $|\langle Tx, x \rangle| < 1$  for each unit vector  $x \in \mathcal{H}$ . Now assume that  $|\langle Tx, x \rangle| < 1$  for each unit vector  $x \in \mathcal{H}$ . Equivalently, assume  $|\langle Tx, x \rangle| < \|x\|^2$  for all  $x \neq 0$ . Let  $x, y \in H$  and choose  $\varphi \in \mathbb{R}$  such that  $e^{i\varphi} \langle Tx, y \rangle = |\langle Tx, y \rangle|$ . Then

$$\begin{aligned} \langle Tx, e^{-i\varphi} y \rangle &= \Re \frac{1}{4} \sum_{n=0}^3 i^n \langle T(x + i^n e^{-i\varphi} y), (x + i^n e^{-i\varphi} y) \rangle \\ &= \frac{1}{4} \langle T(x + e^{-i\varphi} y), x + e^{-i\varphi} y \rangle - \frac{1}{4} \langle T(x - e^{-i\varphi} y), x - e^{-i\varphi} y \rangle. \end{aligned}$$

If  $x = 0$  and  $y = 0$ , then

$$|\langle Tx, y \rangle| < \frac{1}{4} \|x + e^{-i\varphi} y\|^2 + \frac{1}{4} \|x - e^{-i\varphi} y\|^2 = \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2.$$

For  $x \neq 0$ , let  $y = Tx$  in the above to obtain:

$$\|Tx\|^2 < \frac{1}{2} \|x\|^2 + \frac{1}{2} \|Tx\|^2$$

and hence  $\|Tx\| < \|x\|$ . Conversely, if  $\|Tx\| < \|x\|$  for all  $x \neq 0$ , then  $|\langle Tx, x \rangle| < \|x\|^2$ .

#### 4. The Bounded Transform

We first observe that the map  $h : \mathbb{R} \rightarrow (-1, 1)$  given by

$$h(x) = \frac{x}{\sqrt{1+x^2}} \quad (4.1.1)$$

with inverse  $g : (-1, 1) \rightarrow \mathbb{R}$  given by

$$g(x) = \frac{x}{\sqrt{1-x^2}} \quad (4.1.2)$$

is a homeomorphism of  $\mathbb{R}$  to  $(-1, 1)$ . We will see that this gives rise to a bijective correspondence of (possibly unbounded) self-adjoint operators and self-adjoint pure contractions (in particular, one can extend this to a correspondence between closed operators and pure contractions, which preserves self-adjointness).

**Lemma 4.1.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then the domain  $\mathcal{D}(T^2)$  of  $T^2$  is dense in  $\mathcal{H}$  and  $T^2$  is self-adjoint on  $\mathcal{D}(T^2)$ . Moreover,  $I + T^2$  is a bijection from  $\mathcal{D}(T^2)$  to  $\mathcal{H}$ . The inverse of  $I + T^2$  is a bounded self-positive linear operator.*

*Proof.* By assumption  $T$  is self-adjoint and therefore densely defined and closed. The theorem of von Neumann (stated after this proof) gives us the result that  $\mathcal{D}(T^2)$  is dense and that  $T^2$  is self-adjoint on  $\mathcal{D}(T^2)$ . Now we want to show that the operator  $I + T^2$  is a bijection. We know by Corollary 3.2 that  $\mathcal{H} \times \mathcal{H} = \mathcal{G}(T^*) \oplus V(\mathcal{G}(T))$ . Now let  $y \in \mathcal{H}$  be arbitrary. Then we can find  $x, z \in \mathcal{D}(T) = \mathcal{D}(T^*)$  such that

$$\langle 0, y \rangle = \langle z, Tz \rangle + V \langle x, Tx \rangle = \langle z - Tx, Tz + x \rangle$$

Hence  $Tx = z$ , and

$$y = x + Tz = x + T^2x = (I + T^2)x$$

Therefore,  $I + T^2$  is surjective. Moreover, observe that

$$\|(I + T^2)x\|^2 = \langle x + T^2x, x + T^2x \rangle = \|x\|^2 + 2\|Tx\|^2 + \|T^2x\|^2 \geq \|x\|^2$$

for each  $x \in \mathcal{D}(T^2)$ , which means that  $I + T^2$  is injective. Now let  $y = (I + T^2)x$  for  $x \in \mathcal{D}(T^2)$ . Then  $x = (I + T^2)^{-1}y$  and

$$\|(I + T^2)^{-1}y\| = \|x\| \leq \|(I + T^2)x\| = \|y\|$$

Therefore,  $(I + T^2)^{-1}$  is bounded on  $\mathcal{H}$ . Moreover,

$$\langle (I + T^2)^{-1}y, y \rangle = \langle x, y \rangle = \|x\|^2 + \|Tx\|^2 \leq \|(I + T^2)x\|^2 = \|y\|^2$$

which shows that  $(I + T^2)^{-1}$  is symmetric, hence self-adjoint, since  $(I + T^2)^{-1}$  is bounded.



Furthermore, one obtains  $0 \leq (I + T^2)^{-1} \leq I$ .  $\square$

In the proof of the lemma we used a theorem of von Neumann ([8],[16],[20]). The general version of this theorem is the following.

**Theorem 4.1.** *Let  $\mathcal{H}$  be a Hilbert space and  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  a densely defined, closed operator. Then  $T^*T$  is a self-adjoint operator in  $\mathcal{H}$ , and  $\mathcal{D}(T^*T)$  is a core of  $T$ .*

Reclaim that  $\mathcal{D}(T^*T)$  is a core of  $T$  means the following:  $\mathcal{D}(T^*T) \subset \mathcal{D}(T)$  such that

$$\{(x, Tx) \in \mathcal{H} \times \mathcal{H} : x \in \mathcal{D}(T^*T)\}$$

is dense in  $\mathcal{G}(T)$ . This implies that  $\mathcal{D}(T^*T)$  is dense in  $\mathcal{D}(T)$ .

*Proof.* We know that  $\mathcal{H} \times \mathcal{H} = \mathcal{G}(T) \oplus V\mathcal{G}(T^*)$ . Therefore, let  $(x, y) \in \mathcal{H} \times \mathcal{H}$  and find  $z \in \mathcal{D}(T)$  and  $w \in \mathcal{D}(T^*)$  such that  $(x, y) = (z, Tz) + (-T^*w, w)$ . If  $y = 0$ , then  $x = z - T^*w$  and  $0 = Tz + w$ , hence  $Tz = -w \in \mathcal{D}(T^*)$  and  $x = (1 + T^*T)z$ . Since  $x \in \mathcal{H}$  was arbitrary, it follows that  $R := 1 + T^*T$  is surjective. Moreover,  $R^{-1}$  is symmetric such that  $\|R^{-1}\| \leq 1$ . Hence  $R^{-1}$  is symmetric and bounded. Therefore,  $R$  and  $T^*T$  are self-adjoint. This implies that  $T^*T$  is densely defined. Now we will also show that  $\mathcal{D}(T^*T)$  is a core for  $T$ . Therefore, let  $x \in \mathcal{D}(T)$  and  $y \in \mathcal{D}(T^*T)$  and assume  $(x, Tx) \perp (y, Ty)$ . Then

$$0 = \langle u, v \rangle + \langle Tu, Tv \rangle = \langle x, (1 + T^*T)y \rangle = \langle x, Ry \rangle$$

But  $R$  has range  $\mathcal{H}$  if the domain is  $\mathcal{D}(T^*T)$ . Hence  $x = 0$ . This proves the claim.  $\square$

**Proposition 4.1.** *Let  $T$  be a (possible unbounded) self-adjoint operator. Then  $T\sqrt{(I + T^2)^{-1}}$  is a pure contraction.*

*Proof.* By the previous lemma, the operator  $(I + T^2)^{-1}$  is a bounded positive operator. Hence we can take the square root. Injectivity of  $(I + T^2)^{-1}$  implies injectivity of  $\sqrt{(I + T^2)^{-1}}$ . By Theorem 3.4,  $\mathcal{R}(\sqrt{(I + T^2)^{-1}})$  is dense in  $\mathcal{H}$ . Observe that  $\sqrt{(I + T^2)^{-1}}$  maps  $\mathcal{R}(\sqrt{(I + T^2)^{-1}})$  to  $\mathcal{R}((I + T^2)^{-1}) = \mathcal{D}(T^2) \subseteq \mathcal{D}(T)$ . So we can define  $S := T\sqrt{(I + T^2)^{-1}}$  on  $\mathcal{R}(\sqrt{(I + T^2)^{-1}})$ . To show that  $S$  is bounded on the latter, let  $x \in \mathcal{H}$  be arbitrary:

$$\begin{aligned} \left\| S\sqrt{(I + T^2)^{-1}}x \right\|^2 &= \left\| T(I + T^2)^{-1}x \right\|^2 \\ &\leq \left\langle (I + T^2)^{-1}x, x \right\rangle \\ &= \left\| \sqrt{(I + T^2)^{-1}}x \right\|^2 \end{aligned}$$

#### 4. The Bounded Transform

So we may extend  $S$  to all of  $\mathcal{H}$  by continuity, with  $\|S\| \leq 1$ . In fact, this extension, which we also denote by  $S$ , is a pure contraction. Let  $x \in \mathcal{H} \setminus \{0\}$ . Then by injectivity of  $\sqrt{(I+T^2)^{-1}}$ :

$$\|Sx\|^2 = \langle x, S^2x \rangle = \|x\|^2 - \left\| \sqrt{(I+T^2)^{-1}}x \right\|^2 < \|x\|^2 \quad \square$$

As stated in the begin of this section, we also want to go from pure contractions to (possibly unbounded) self-adjoint operators. The following lemmas gives us the first piece of information we need for this.

**Lemma 4.2.** *If  $S$  is a self-adjoint pure contraction, then  $I - S^2$  is positive and  $\sqrt{I - S^2}$  is an injective operator with dense range.*

*Proof.* Supposing  $S$  is a self-adjoint pure contraction, we have  $\|Sx\| < \|x\|$  for each  $x \in \mathcal{H} \setminus \{0\}$ . Now we have

$$\langle (I - S^2)x, x \rangle = \langle x, x \rangle - \langle S^2x, x \rangle = \|x\|^2 - \|Sx\|^2$$

But since  $S$  is a pure contraction, we obtain  $\langle (I - S^2)x, x \rangle \geq 0$ , which means that  $I - S^2$  is positive. Thus we may conclude that the square root of the operator  $I - S^2$  exists. The following claim is that  $\sqrt{I - S^2}$  is injective. Suppose  $\sqrt{I - S^2}x = 0$ . Then

$$0 = \langle \sqrt{I - S^2}x, \sqrt{I - S^2}x \rangle = \langle (I - S^2)x, x \rangle = \|x\|^2 - \|Sx\|^2$$

Thus  $\|Sx\| = \|x\|$ , which implies that  $x = 0$ , since  $S$  is a pure contraction. Density of the range follows directly from the fact that  $T$  is self-adjoint and injective (cf. Theorem 3.4).  $\square$

By injectivity of  $\sqrt{I - S^2}$ , the inverse exists on  $\mathcal{R}(\sqrt{I - S^2})$ . Therefore, the expression  $T := S\sqrt{I - S^2}^{-1}$  is well defined on  $\mathcal{D}(T) = \mathcal{R}(\sqrt{I - S^2})$ . We will show that  $T$  is self-adjoint. For this, one observes that  $S\sqrt{I - S^2}^{-1}x = \sqrt{I - S^2}^{-1}Sx$  for each  $x \in \mathcal{D}(T)$  since  $S : \mathcal{D}(T) \rightarrow \mathcal{D}(T)$ . One can think that  $\sqrt{I - S^2}^{-1}Sx$  is defined for more  $x \in \mathcal{H}$  than  $S\sqrt{I - S^2}^{-1}x$ . But this is not the case. In particular one has  $x \in \mathcal{D}(T)$  if and only if  $Sx \in \mathcal{D}(T)$ . To see this, assume that  $Sx \in \mathcal{D}(T)$ . This means that there exists  $y \in \mathcal{H}$  such that  $Sx = \sqrt{I - S^2}y$ . This can be rewritten as

$$S^2x = S\sqrt{I - S^2}^{-1}y = \sqrt{I - S^2}^{-1}Sy$$

From this one obtains

$$(I - S^2)x = x - \sqrt{I - S^2}^{-1}Sy$$

and hence

$$\sqrt{I - S^2}^{-1} \left( \sqrt{I - S^2}^{-1}x + Sy \right) = x$$

which implies that  $x \in \mathcal{D}(T)$ . So we obtain  $x \in \mathcal{D}(T)$  if and only if  $Sx \in \mathcal{D}(T)$ . From this

we obtain that  $S\sqrt{I - S^2}^{-1}$  is symmetric since for  $x, y \in \mathcal{D}(T)$

$$\left\langle S\sqrt{I - S^2}^{-1}x, y \right\rangle = \left\langle \sqrt{I - S^2}^{-1}Sx, y \right\rangle = \left\langle Sx, \sqrt{I - S^2}^{-1}y \right\rangle = \left\langle x, S\sqrt{I - S^2}^{-1}y \right\rangle$$

hence  $\mathcal{G}(S\sqrt{I - S^2}^{-1}) \subseteq \mathcal{G}((S\sqrt{I - S^2}^{-1})^*)$ . For the other inclusion, let  $x \in \mathcal{D}((S\sqrt{I - S^2}^{-1})^*)$ . Then  $Sx \in \mathcal{D}(\sqrt{I - S^2}^{-1})$  and  $\sqrt{I - S^2}^{-1}Sx = (S\sqrt{I - S^2}^{-1})^*x$ . Since we saw that  $x \in \mathcal{D}(T)$  if and only if  $Sx \in \mathcal{D}(T)$ , we obtain

$$S\sqrt{I - S^2}^{-1}x = \sqrt{I - S^2}^{-1}Sx = (S\sqrt{I - S^2}^{-1})^*x$$

This yields the other inclusion. Hence,  $T$  is self-adjoint.

At this stage we can formulate our first result.

**Theorem 4.2.** *For a (possibly unbounded) self-adjoint operator  $T$ , define*

$$S := T\sqrt{I + T^2}^{-1} \tag{4.1.3}$$

*Then  $S$  is a pure contraction. Moreover,*

$$T = S\sqrt{I - S^2}^{-1} \tag{4.1.4}$$

*In particular, these formulate a bijective coorespondence between the class of (possibly unbounded) self-adjoint operators and the class of self-adjoint pure contractions.*

*Proof.* If  $T$  is determined by  $S$ , then  $I + T^2 = (I - S^2)^{-1}$ , so that  $\sqrt{I - S^2} = \sqrt{I + T^2}$ . This yields:

$$S = S\sqrt{I - S^2}^{-1}\sqrt{I - S^2} = T\sqrt{I - S^2} = T\sqrt{I + T^2}$$

In the same way, one obtains that if  $S$  is determined by  $T$ , then  $I - S^2 = (I + T^2)^{-1}$ , whence  $S$  can be reproduced by  $T$ . We still have to show that the domains match. For that we have to show that  $\mathcal{D}(T) = \mathcal{R}(\sqrt{(I + T^2)^{-1}})$  by the previous observation. The inclusion  $\mathcal{R}(\sqrt{(I + T^2)^{-1}}) \subseteq \mathcal{D}(T)$  was already mentioned in the proof of Proposition 4.1. For the other inclusion, let  $x \in \mathcal{D}(T)$ . Then  $x = \sqrt{(I + T^2)^{-1}}(ST + \sqrt{(I + T^2)^{-1}})x$ , where  $S = T\sqrt{(I - T^2)^{-1}}$ . To see this, let  $y \in \mathcal{H}$  be arbitrary. Then:

$$\left\langle y, \sqrt{(I + T^2)^{-1}}STx \right\rangle + \left\langle y, (I + T^2)^{-1}x \right\rangle = \langle y, x \rangle \quad \square$$

We want to understand this construction a little bit better. In particular, we are going to work out this for the bounded case.

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**Proposition 4.2.** *Suppose  $T \in B(\mathcal{H})$  is self-adjoint, and let  $S$  be its bounded transform. Then*

$$C^*(T) = C^*(S) \quad (4.1.5)$$

Moreover,  $\sigma(T) \subset \mathbb{R}$  and  $\sigma(S) \subset (-1, 1)$  are homeomorphic via the maps (4.1.1) and (4.1.2). In particular,

$$\begin{aligned} \sigma(T) &= \left\{ \lambda(1 - \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(S) \right\} \\ \sigma(S) &= \left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(T) \right\} \end{aligned}$$

*Proof.* The second part of the proposition follows from the *Spectral Mapping Theorem* from Functional Analysis (see [13],[17]). The same theorem gives also an isomorphism between  $C^*(T)$  and  $C^*(S)$ , but we claim that equality holds. For this, observe that the function given by formula (4.1.2) is continuous on  $\sigma(S)$ , such that  $T = g(S)$ , hence  $T \in C^*(S)$ , which gives us the inclusion  $C^*(T) \subseteq C^*(S)$ . Similary, we can apply (4.1.1) to obtain  $S = h(T)$ , hence we obtain  $C^*(S) \subseteq C^*(T)$ . This gives us the equality we want.  $\square$

Now we want to generalize the result to the case of real interest, where  $T$  is self-adjoint but possibly unbounded. The following theorem states how the spectra are related.

**Theorem 4.3.** *Let  $T$  be a possibly unbounded self-adjoint operator, with bounded transform  $S$ . Then we have the following relation between the two spectra:*

$$\sigma(T) = \left\{ \lambda(1 - \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(S) \cap (-1, 1) \right\} \quad (4.1.6)$$

$$\sigma(S) = \overline{\left\{ \lambda(1 + \lambda^2)^{-\frac{1}{2}} : \lambda \in \sigma(T) \right\}} \quad (4.1.7)$$

*Proof.* Observe that by Lemma 4.2 and formula (4.1.3) the operator  $\sqrt{1 - S^2}$  is a bijection from  $\mathcal{H}$  to  $\mathcal{R}(\sqrt{1 - S^2}) = \mathcal{D}(T)$ . Now let  $\lambda \in \mathbb{C} \setminus \sigma(T)$  (i.e.  $\lambda$  is an element of the resolvent  $\rho(T)$  of  $T$ ). This implies that  $T - \lambda I$  is invertible and therefore bijective. In particular,  $T - \lambda I$  is a bijection from  $\mathcal{D}(T)$  to  $\mathcal{H}$ . Thus by composition we have a bijection  $\mathcal{H} \rightarrow \mathcal{H}$ ; equivalently,  $(T - \lambda I)(\sqrt{I - S^2})$  is invertible. By our bounded transform, this is equivalent to the statement that  $S - \lambda\sqrt{I - S^2}$  is invertible. We thus come to the conclusion that  $\lambda \in \rho(T) \iff S - \lambda\sqrt{I - S^2}$  is a bijection. By negation of this statement we obtain  $\lambda \in \sigma(T) \iff S - \lambda\sqrt{I - S^2}$  is not invertible. Now recall the fact that  $\sigma(f) = \mathcal{R}(f)$  if  $f \in C(X)$  with  $X$  a compact Hausdorff space. Since  $S$  is bounded,  $\sigma(S)$  is compact (and Hausdorff). We now look at the function  $k_\lambda(x) = x - \lambda\sqrt{1 - x^2}$ , which is an element of  $C(\sigma(S))$ . In particular,  $k_\lambda$  is not invertible if and only if  $\lambda \in \mathcal{R}(k_0)$ . Now observe that  $k_\lambda(\pm 1) = \pm 1$  for

each  $\lambda$ . From this, (4.1.6) follows. The same argument yields that  $\mu \in \sigma(S) \cap (-1, 1)$  comes from  $\lambda \in \sigma(T)$ . But since  $\sigma(S)$  is compact and hence closed in  $[-1, 1]$  we obtain (4.1.7).  $\square$

## 4.2. Spectral Theorem

We already mentioned that  $\sigma(T)$  is compact if  $T$  is a bounded operator. In general, this is not the case if  $T$  is unbounded. It is necessary to look at the various function spaces we know, for example  $C(\sigma(T))$ ,  $C_b(\sigma(T))$  and  $C_0(\sigma(T))$ . Even though  $S$  is a bounded operator, so that  $\sigma(S)$  is compact, writing

$$\tilde{\sigma}(S) = \sigma(S) \cap (-1, 1) \tag{4.2.1}$$

it is also possible to look at  $C_0(\tilde{\sigma}(S))$ ,  $C_b(\tilde{\sigma}(S))$  or  $C(\tilde{\sigma}(S))$ . Notice that  $C(\sigma(S))$  consists of all  $u \in C_b(\tilde{\sigma}(S))$  for which  $\lim_{y \rightarrow \pm 1} u(y)$  exists, where this limit is 0 if and only if  $u \in C_0(\tilde{\sigma}(S))$ . Define the following two spaces:

$$C_0^*(S) := \{u(S) : u \in C_0(\tilde{\sigma}(S))\}$$

$$C_c^*(S) := \{u(S) : u \in C_c(\tilde{\sigma}(S))\}$$

as the relevant images under the continuous functional calculus for the the bounded operator  $S$ . This gives rise to the following inclusions:

$$C_c^*(S) \subset C_0^*(S) \subset C^*(S)$$

We first want to define  $f(T)$  for  $f \in C_0(\sigma)(T)$ . Therefore, recall that we have the map  $g : (-1, 1) \rightarrow \mathbb{R}$  from the start (see (4.1.2)), as well as the bounded transform  $S$  of  $T$ . Since we define  $\tilde{\sigma}(S)$  in (4.2.1), we obtain  $f \circ g : \tilde{\sigma}(S) \rightarrow \mathbb{C}$  such that  $f \circ g \in C_0(\tilde{\sigma}(S))$ . Therefore, we define

$$f(T) := (f \circ g)(S) \tag{4.2.2}$$

In what follows, we use the following lemma.

**Lemma 4.3.**  $C_c^*(S)\mathcal{H}$  is dense in  $\mathcal{H}$ , or in other words  $\overline{C_c^*(S)\mathcal{H}} = \mathcal{H}$ .

*Proof.* To prove this, we use the spectral theorem for self-adjoint operators (Theorem 3.6,[17],[20]). From this, we obtain  $S = \int_{\sigma(S)} \lambda dE_\lambda$  and  $f(S) = \int_{\sigma(S)} f(\lambda)dE_\lambda$  for all Borel functions  $f$  on  $\sigma(S)$ . Now we define a sequence of functions  $(g_n)_{n \in \mathbb{N}}$ , where  $g_n$  is the function that is zero on  $(-1, \frac{1}{n} - 1] \cup [1 - \frac{1}{n}, 1)$ , one on  $[\frac{2}{n} - 1, 1 - \frac{2}{n}]$  and linearly interpolated in between. Therefore, we get a sequence of compact supported functions, which tends, in the pointwise limit,

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to the identity  $\mathbf{1}$  on  $(-1, 1)$ . Now we can also restrict all  $g_n$  to  $\tilde{\sigma}(S)$ . Then  $g_n \rightarrow \mathbf{1}_{\tilde{\sigma}(S)}$ . Since  $g_n \leq \mathbf{1}$  for each  $n \in \mathbb{N}$ , we can use Lebesgue's Dominated Convergence Theorem to conclude:

$$\lim_{n \rightarrow \infty} g_n(S) = \lim_{n \rightarrow \infty} \int_{\sigma(S)} g_n(\lambda) dE(\lambda) = \int_{\sigma(S)} \lim_{n \rightarrow \infty} g_n(\lambda) dE(\lambda) = \mathbf{1}_{\tilde{\sigma}(S)} \quad \square$$

In the view of the inclusions above, we also have  $\overline{C_0^*(S)\mathcal{H}} = \mathcal{H}$ . Let  $h \in C_b(\sigma(T))$ . Then we want to define  $h(T)$  on  $\overline{C_0^*(S)\mathcal{H}}$  by

$$h(T)(f(T)x) := (hf(T))x \quad (4.2.3)$$

Since  $f \in C_0(\sigma(T))$  and  $h \in C_b(\sigma(T))$ , we may conclude that  $hf \in C_0(\sigma(S))$ , hence the right hand side is well-defined by the previous definitions. Since we know that  $\overline{C_0^*(S)\mathcal{H}} = \mathcal{H}$ , we want to extend  $h(T)$  to the whole Hilbert space  $\mathcal{H}$ , but this requires boundedness of  $h(T)$ . This is the following lemma.

**Lemma 4.4.** *The operator  $h(T)$  is bounded on  $C_0^*(S)\mathcal{H}$ , i.e.  $\|h(T)\| \leq \|h\|_\infty$ .*

*Proof.* To prove boundedness let  $\varepsilon > 0$  be arbitrary. Hence we can find a compact subset  $K \subset \mathbb{R}$  such that  $|h(x)f(x)| < \varepsilon$  for each  $x \notin K$ . During this proof let  $\tilde{h} = h \circ g$  (with  $g$  as in (4.1.2)). Furthermore let  $\chi_{X \setminus K}$  be the characteristic function on the complement of  $K$ . Now take  $h$  to be  $\chi_{X \setminus K} h f$ . Then, by the Borel functional calculus for bounded operators on  $\mathcal{H}$ , we obtain

$$\left\| (\widetilde{\chi_{X \setminus K} h f})(S)x \right\| \leq \left\| (\widetilde{\chi_{X \setminus K} h f})(S) \right\| \|x\| \leq \left\| (\widetilde{\chi_{X \setminus K} h f}) \right\|_\infty \|x\| < \varepsilon \|x\|$$

From that and the homomorphism property of the Borel functional calculus:

$$\begin{aligned} \|(hf)(T)x\| &= \left\| (\widetilde{hf})(S)x \right\| \\ &= \left\| (\widetilde{\chi_K hf})(S)x + (\widetilde{hf} - \widetilde{\chi_K hf})(S)x \right\| \\ &\leq \left\| (\widetilde{\chi_K hf})(S)x \right\| + \left\| (\widetilde{\chi_{X \setminus K} hf})(S)x \right\| \\ &= \left\| (\widetilde{\chi_K h})(S)\tilde{f}(S)x \right\| + \left\| (\widetilde{\chi_{X \setminus K} hf})(S)x \right\| \\ &< \left\| (\widetilde{\chi_K h}) \right\|_\infty \|f(T)x\| + \varepsilon \|x\| \\ &\leq \|h\|_\infty \|f(T)x\| + \varepsilon \|x\| \end{aligned}$$

For the last estimation we used that  $\left\| (\widetilde{\chi_K h}) \right\|_\infty \leq \|\tilde{h}\|_\infty = \|h\|_\infty$ . Observe that the last expression is independent of the compact set  $K$ . Therefore,  $\varepsilon$  may tend to zero. This yields boundedness and the inequality we wanted.  $\square$

Now we go one step further: we want to do the same for (possibly unbounded) functions in the space  $C(\sigma(T))$ . Thus let  $h \in C(\sigma(T))$  and define an operator  $h_0(T) : \mathcal{D}(h_0(T)) \rightarrow \mathcal{H}$  by

$$h_0(T)(f(T)x) = (hf)(T)x \quad (4.2.4)$$

where

$$\mathcal{D}(h_0(T)) := \text{Span} \{f(T)x : f \in C_c(\sigma(T)), x \in \mathcal{H}\} \quad (4.2.5)$$

We cannot repeat often enough that  $\sigma(T)$  is not compact in general, and therefore  $h$  is not bounded in general. But the above definition of  $h_0(T)$  is sound, since  $f \in C_c(\sigma(T))$  and therefore  $hf \in C_c(\sigma(T))$ . Hence the right-hand side is well-defined by the above formulae.

Now we will use Theorem 3.5 to show that  $h_0(T)$  is closable. Therefore, suppose that  $f(T)x, g(T)y \in \mathcal{D}(h_0(T))$ . Then we can make the following computation:

$$\langle g(T)y, h_0(T)(f(T)x) \rangle = \langle y, \bar{g}(T)(hf)(T)x \rangle = \langle y, (\bar{g}hf)(T)x \rangle \quad (4.2.6)$$

$$\langle (g\bar{h})(T)y, f(T)x \rangle = \langle y, \overline{(g\bar{h})}f(T)x \rangle = \langle y, (\bar{g}hf)(T)x \rangle \quad (4.2.7)$$

This implies that  $\mathcal{D}(h_0(T)) \subseteq \mathcal{D}(h_0(T)^*)$ . Moreover,  $\mathcal{D}(h_0(T)^*)$  is dense in  $\mathcal{H}$ , whence  $h_0(T)$  is closable by Theorem 3.5. Therefore, define

$$h_0(T)^*(g(T)x) := (g\bar{h})(T)x$$

Notice: if  $h$  is real-valued, then  $h_0(T)$  is a symmetric operator. Now we are ready to give our construction of  $h(T)$ : define  $h(T)$  to be the closure of  $h_0(T)$ , i.e.

$$h(T) := \overline{h_0(T)}$$

Theorem 3.5 now implies that  $\mathcal{D}(h(T))$  consists of all  $x \in \mathcal{H}$  for which there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(h_0(T))$  with limit  $x \in \mathcal{H}$ , with the property that the sequence  $(h_0(T)x_n)_{n \in \mathbb{N}}$  also converges in  $\mathcal{H}$ . Thus we obtain  $h(T)x = \lim_{n \rightarrow \infty} h_0(T)x_n$ . If now  $h = \bar{h}$ , then we want  $h_0(T)$  to be essentially self-adjoint in order to conclude that  $h(T)^* = h(T)$ . Hence we need the following theorem.

**Theorem 4.4.** *Let  $h \in C(\sigma(T))$  be real valued. Then  $h(T)$  is self-adjoint. More generally, one has  $h(T)^* = \bar{h}(T)$ .*

To prove this result we want to use the following auxiliary theorem of Nelson (see [15]).

**Theorem 4.5.** *Let  $\{Q(t)\}_{t \in \mathbb{R}}$  be a strongly continuous unitary group of operators on a Hilbert*

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space  $\mathcal{H}$ . Let  $K : \mathcal{D}(K) \rightarrow \mathcal{H}$  be densely defined and symmetric. Assume that  $\mathcal{D}(K)$  is invariant under  $\{Q(t)\}_{t \in \mathbb{R}}$ , i.e.  $Q(t) : \mathcal{D}(K) \rightarrow \mathcal{D}(K)$ , and assume that  $\{Q(t)\}_{t \in \mathbb{R}}$  is strongly differentiable on  $\mathcal{D}(K)$ , then  $i^{-1} \frac{dQ(t)}{dt}$  is essentially self-adjoint on  $\mathcal{D}(K)$  and its closure is the infinitesimal generator of  $\{Q(t)\}_{t \in \mathbb{R}}$ .

This theorem gives us the following: if we can prove that  $\frac{dQ(t)}{dt}x = iKQ(t)x$  for each  $x \in \mathcal{D}(K)$ , then we may infer that  $K$  is essentially self-adjoint. Having this we can start the proof of our theorem.

*Proof.* For the proof set  $c = h_0(T)$  for  $h \in C(\sigma(T))$  and define  $Q(t) = e^{ith(T)}$  for  $t \in \mathbb{R}$  and  $f \in C_c(\sigma(T))$  by  $Q(t)(f(T)x) = (e^{ith}f)(T)x$ . Then one can see that  $\{Q(t)\}_{t \in \mathbb{R}}$  defines a strongly continuous one-parameter group of unitaries. What we now want to show is that  $c = h_0(T)$  is essentially self-adjoint. This can be done with the aid of the previous theorem and the following estimate:

$$\begin{aligned} \left\| \frac{Q(t+s)f(T)x - Q(t)f(T)x}{s} - icQ(t)f(T)x \right\| &= \left\| \frac{e^{is\tilde{h}}\tilde{f}(S) - \tilde{f}(S)}{s} - i(\tilde{h}\tilde{f})(S) \right\| \\ &\leq \left\| \frac{e^{is\tilde{h}}\tilde{f} - \tilde{f}}{s} - i\tilde{h}\tilde{f} \right\|_{\infty} \end{aligned}$$

Here we use the fact that  $\{Q(t)\}_{t \in \mathbb{R}}$  is a one-parameter group of unitary operators, hence  $Q(t)$  preserves the norm for each  $t \in \mathbb{R}$ , which is the reason that we can omit it from the computation. The last inequality comes from the fact that  $S$  is a pure contraction, so that  $\|S\| \leq 1$ . Therefore, we can now take the limit to see the following:

$$\lim_{s \rightarrow 0} \left\| \frac{e^{is\tilde{h}}\tilde{f} - \tilde{f}}{s} - i\tilde{h}\tilde{f} \right\|_{\infty} \leq \lim_{s \rightarrow 0} \|f\|_{\infty} \cdot \left\| \frac{e^{is\tilde{h}} - 1}{s} - i\tilde{h} \right\|_{\infty} \leq \|f\|_{\infty} \cdot \|\tilde{h}\|_{\infty} \cdot \lim_{s \rightarrow 0} \sum_{n=2}^{\infty} \frac{s^n}{n!} = 0$$

Thus, we proved that  $c = h_0(T)$  satisfies the condition of the previous theorem, which implies that  $c = h_0(T)$  is essentially self-adjoint. Together with the observation before the theorem we can conclude that  $h(T)$  is self-adjoint, which means in particular that  $h(T)^* = h(T)$ .  $\square$

The idea to define  $h(T)$  in this way comes from multiplier algebras we introduced in section 1.6. If  $A$  is a commutative algebra,  $\mathcal{M}(A)$  only consists of maps  $\varphi : A \rightarrow A$  such that  $\varphi(ab) = a\varphi(b)$  for each  $a, b \in A$ . In the case that  $A$  is a  $C^*$ -algebra, an additional requirement is the boundedness of  $\varphi$ . If one omits this property, then we obtain the *unbounded multiplier algebra*. For this concept, let  $A$  be a commutative  $C^*$ -algebra. The unbounded multiplier algebra of  $A$ , denoted by  $\mathcal{UM}(A)$ , consists of closed maps  $\varphi : \mathcal{D}(\varphi) \rightarrow A$ , where  $\mathcal{D}(\varphi)$  is a dense ideal in  $A$  and  $\varphi(ab) = a\varphi(b)$  holds for each  $a \in A$  and  $b \in \mathcal{D}(\varphi)$ . As an important example we obtain

$$\mathcal{UM}(C_0(X)) \cong C(X)$$



Here we use the identification  $\varphi \leftrightarrow \varphi_f$ , where  $f \in C(X)$ ,  $\varphi_f(g) = fg$  and

$$\mathcal{D}(f) = \{g \in C_0(X) : fg \in C_0(X)\}$$

Moreover, this leads to the following inclusions:

$$C_c^*(S) \subset C_b^*(S) \subset C^*(S) \subset \mathcal{M}(C_0^*(S)) \subset \mathcal{M}(C_c^*(S))$$

Observe that  $\mathcal{M}(C_0^*(S))$  is a  $C^*$ -algebra contained in  $B(\mathcal{H})$ , whereas  $\mathcal{M}(C_c^*(S))$  consists also of unbounded operators. Conceptually, what is going on here is that we extend the homomorphism  $C_0(\sigma(T)) \rightarrow B(\mathcal{H})$  to the multiplier algebra  $\mathcal{M}(C_0(\sigma(T))) \cong C_b(\sigma(T))$ .

#### 4.2.1. Examples for special types of functions

We now have the theorem above and we want to have a look to some examples. Thus suppose that  $T$  is a (possibly unbounded) self-adjoint operator on a Hilbert space  $\mathcal{H}$  and let  $h = id_{\sigma(T)}$ , so that  $h : \sigma(T) \rightarrow \sigma(T)$  is defined by  $h(\lambda) = \lambda$ , hence  $h \in C(\sigma(T))$ . Recall that we first have to look at the operator  $h_0(T) : \mathcal{D}(h_0(T)) \rightarrow \mathcal{H}$  defined by

$$h_0(T)(f(T)x) = (hf)(T)x$$

for  $x \in \mathcal{H}$  and  $f \in C_c(\sigma(T))$ . In particular, we obtain

$$h_0(T)(f(T)x) = (hf)(T)x = Tf(T)x$$

Now let  $x \in \mathcal{D}(h(T))$ . Then there exists a sequence  $x_n \in \mathcal{D}(h_0(T))$  with  $x_n \rightarrow x$  and  $h_0(T)x_n \rightarrow h(T)x$ . Since we know what  $\mathcal{D}(h_0(T))$  looks like, we may say  $x_n = f_n(T)\psi_n$  for  $f_n \in C_c(\sigma(T))$  and  $\psi_n \in \mathcal{H}$  for each  $n \in \mathbb{N}$ . Then:

$$h(T)x = \lim_{n \rightarrow \infty} h_0(T)(f_n(T)\psi_n) = \lim_{n \rightarrow \infty} Tf_n(T)\psi_n = Tx$$

Therefore, we observe that the identity function does what it is supposed to do.

Another important example is the following: Define  $h(\lambda) = \frac{1}{\lambda - \mu}$  where  $\mu \notin \sigma(T)$  is fixed and  $\lambda \in \sigma(T)$ . Then  $h_0(T)(f(T)x) = (hf)(T)x$  for  $x \in \mathcal{H}$  and  $f \in C_c(\sigma(T))$ . Furthermore,

$$(hf)(T)x = \frac{f(T)}{T - \mu}x$$

Now assume that  $x \in \mathcal{D}(h(T))$ . Then there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(h_0(T))$  such that  $x_n \rightarrow x$  and  $h_0(T)x_n \rightarrow h(T)x$ . In particular,  $x_n = f_n(T)\psi_n$  for certain  $f_n \in C_c(\sigma(T))$  and

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$\psi_n \in \mathcal{H}$  for each  $n \in \mathbb{N}$ . Hence

$$h(T)x = \lim_{n \rightarrow \infty} h_0(T)(f_n(T)\psi_n) = \lim_{n \rightarrow \infty} \frac{f_n(T)}{T - \mu} x = \frac{1}{T - \mu} x$$

Hence by our construction we obtain the continuous functional calculus we were after.

#### 4.2.2. Bounded Borel functions

The goal of this section is to develop a functional calculus for *bounded Borel functions*. In particular, we assume the reader to be familiar with the results for the bounded Borel functional calculus for bounded operators. Here are some results for bounded self-adjoint operators (see [2]).

**Theorem 4.6.** *Let  $T$  be a bounded self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a unique map  $\varphi$  from the bounded Borel functions on  $\sigma(T)$  into the space of bounded operators on  $\mathcal{H}$ , in particular  $\varphi : \mathcal{B}_b(\sigma(T)) \rightarrow B(\mathcal{H})$ , given by  $f \mapsto f(T) := \varphi(f)$  such that:*

- $\varphi$  is a unital  $*$ -homomorphism
- $\varphi$  is norm-continuous, in particular  $\|\varphi(f)\| \leq \|f\|_\infty$
- $\varphi(\mathbf{1}) = T$
- If  $(f_n)_{n \in \mathbb{N}}$  is a bounded monotone sequence in  $\mathcal{B}_b(\sigma(T))$  converging pointwise to  $f$ , then  $\varphi(f_n)$  converges strongly to  $\varphi(f)$

The idea of the proof is to use the Riesz Representation Theorem to find a measure  $\mu_x$  for  $x \in \mathcal{H}$  on the Borel sets of  $\sigma(T)$  such that we can define  $f(T)$  implicitly by

$$\langle x, f(T)x \rangle := \int_{\sigma(T)} f(\lambda) \mu_x(\lambda)$$

We now want to extend this idea of the bounded Borel functional calculus to (possibly) unbounded operators on a Hilbert space  $\mathcal{H}$ . Therefore, suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded Borel function, e.g. a characteristic function  $\chi_A$ , for  $A \subset \mathbb{R}$  a Borel set. Let  $T$  be a (possibly unbounded) self-adjoint operator with bounded transform  $S$ , in particular  $g(S) = T$ . We want to give a meaning to  $f(T)$ . Therefore, we look at what we did for our continuous functional calculus. In particular, we look at formula (4.2.2) and therefore at the function  $f \circ g(S)$ . Since  $f$  is a bounded Borel function and  $g$  is a bounded continuous function, the composition is a bounded Borel function. Note that we can restrict  $g$  to  $\tilde{\sigma}(S)$ , hence  $g|_{\tilde{\sigma}(S)} : \tilde{\sigma}(S) \rightarrow \mathbb{R}$ . By composition with  $f$  we obtain  $f \circ g : \tilde{\sigma}(S) \rightarrow \mathbb{R}$ . Therefore, we can define

$$f(T) := f(g(S)) \tag{4.2.8}$$

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Notice that we can now apply all facts we know from the theorem above about the bounded functional calculus for bounded self-adjoint operators, since the bounded transform  $S$  is self-adjoint. Notice that the assignment  $f \mapsto f(T)$  gives us a map from  $\mathcal{B}_b(\mathbb{R})$  to  $B(\mathcal{H})$ , which supports the intuitive idea that we obtain something bounded if we apply a bounded function to an unbounded operator. We can now combine the previous theorem with our new definition to obtain the following theorem.

**Theorem 4.7.** *Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a (possibly) unbounded self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exists a map  $\psi : \mathcal{B}_b(\mathbb{R}) \rightarrow B(\mathcal{H})$  given by  $\psi(f) = f(T) := f(g(S))$  with the following properties:*

- $\psi$  is a unital  $*$ -homomorphism
- $\psi$  is norm-continuous, in particular  $\|\psi(f)\| \leq \|f \circ g\|_\infty$
- If  $(h_n)_{n \in \mathbb{N}}$  is a bounded sequence of functions converging pointwise to  $h(x)$ , then  $h_n(T)$  converges strongly to  $h(T)$  on  $\mathcal{D}(T)$

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The motivation of this section is the fact that unfortunately unbounded operators do not form an algebra under composition or even the Jordan product. But most physical applications use unbounded operators, i.e. the Laplacian or the momentum and position operators on  $L^2(\mathbb{R})$ . We want to make the idea that an unbounded operator is close to a set of bounded operators, precise. The idea comes from the fact that every self-adjoint operator has a family  $(E_\lambda)_{\lambda \in \Lambda}$  of spectral measures, which are bounded. Now one has the following characterization of elements in a von Neumann algebra.

**Proposition 4.3.** *Let  $A$  be a unital von Neumann algebra acting on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:*

- (1)  $a \in A$ .
- (2)  $ae = ea$  for each projection  $e \in A'$ .
- (3)  $uau^* = a$  for each unitary  $u \in A'$ .

*Proof.* Assume that  $a \in A$ . Then by the Double Commutant Theorem we obtain  $a \in A''$ , hence  $ua = au$  for each unitary  $u \in A'$ . Since  $u$  is unitary, we may say that  $uau^* = a$  for each unitary  $u \in A'$ . This proves the implication (1)  $\Rightarrow$  (3). If one uses the fact that every operator is the sum of four unitary operators, we directly obtain the implication (3)  $\Rightarrow$  (2). For the implication (2)  $\Rightarrow$  (1) we use the fact that a von Neumann algebra is the norm-closed linear span of its projections (see Proposition 2.1), hence  $a \in A''$  and again by the Double Commutant Theorem  $a \in A$ . This proves the proposition.  $\square$

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**Definition 4.1.** Let  $A$  be a von Neumann algebra. We say that a (possibly unbounded) operator  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  on a Hilbert space  $\mathcal{H}$  is **affiliated** to  $A$  if  $UTU^* = T$  for each unitary  $U$  in  $A'$ . In this case we write  $T\eta A$ .

**Lemma 4.5.** Let  $\mathcal{H}$  be a Hilbert space and  $T \in B(\mathcal{H})$ . Then  $T\eta A$  if and only if  $T \in A$ .

*Proof.* Suppose  $T \in A$ . Let  $U \in A'$  be an arbitrary unitary element. Then  $UT = TU$  for each  $T \in A$ . Since  $T$  is bounded and  $U$  is unitary this yields  $UTU^* = T$ , which means  $T\eta A$ . For the converse, suppose  $T\eta A$ . By definition we obtain  $UTU^* = T$  for each unitary  $U \in A'$ . Since  $T$  is bounded we have  $UT = TU$  for each unitary  $U \in A'$ . By the fact that each operator is a linear combination of at most four unitary elements (see below and [22]) we may conclude that  $T$  commutes with every operator in  $A'$ . By the Double Commutant Theorem we may conclude that  $T \in A$ .  $\square$

**Lemma 4.6.** Let  $T$  be a bounded operator on a Hilbert space  $\mathcal{H}$ . Then  $T$  is a linear combination of at most four unitary operators.

*Proof.* Observe that it suffices to show that every self-adjoint operator  $S \in B(\mathcal{H})$  is a linear combination of two unitary operators since we can write  $T$  as a linear combination of two self-adjoint operators, i.e.

$$T = \frac{1}{2}(T + T^*) + \frac{i}{2}(iT - iT^*)$$

Without loss of generality we may assume that  $\|S\| \leq 1$ . Then one observes that  $\frac{1}{2}(S \pm i\sqrt{I - S^2})$  is unitary (notice that  $I - S^2$  is positive since  $\|S\| \leq 1$ , hence the square root is well defined). Furthermore:

$$S = \frac{1}{2}(S + i\sqrt{I - S^2}) + \frac{1}{2}(S - i\sqrt{I - S^2})$$

Therefore, we may conclude that every operator is a linear combination of at most four linear operators.  $\square$

Also, one can easily verify that for a densely defined closable operator  $T$  one has  $T\eta A$  if and only if  $T^*\eta A$ . What we could do, is look at self-adjoint operators affiliated with  $A$ . Therefore suppose  $T\eta A$ .

**Proposition 4.4.** Let  $A$  be a von Neumann algebra and suppose  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  is a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . If  $T\eta A$ , then the bounded transform  $S$  of  $T$  is an element of  $A$ , i.e.  $S \in A$ .

*Proof.* To prove this result, we will use the previous lemma. Suppose that  $T\eta A$ . Then  $UTU^* = T$  for each unitary  $U$  in  $A'$ . The first step is that we show that  $(1 + T^2)^{-1}\eta A$ . To

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this end, let  $U$  be an arbitrary unitary in  $A'$ . Then:

$$\begin{aligned}
 U(1 + T^2)^{-1}U^* &= (U(1 + T^2)U^*)^{-1} \\
 &= ((U + UT^2)U^*)^{-1} \\
 &= (UU^* + UT^2U^*)^{-1} \\
 &= (1 + UTU^*UTU^*)^{-1} \\
 &= (1 + T^2)^{-1}
 \end{aligned}$$

We already showed in chapter 4 that this operator is everywhere defined, bounded, and positive. Therefore,  $(1 + T^2)^{-1}\eta A$  implies  $(1 + T^2)^{-1} \in A$ . Since  $A$  is a von Neumann algebra, we may conclude by positivity of the operator that  $\sqrt{(1 + T^2)^{-1}} \in A$ , hence  $\sqrt{(1 + T^2)^{-1}}\eta A$ . To show that  $S \in A$ , we can also show that  $S\eta A$ . Therefore, since by assumption  $T\eta A$ :

$$USU^* = U \left( T\sqrt{(1 + T^2)^{-1}} \right) U^* = (UTU^*) \left( U\sqrt{(1 + T^2)^{-1}}U^* \right) = T\sqrt{(1 + T^2)^{-1}}$$

Therefore,  $S\eta A$ , which implies by the previous lemma that  $S \in A$ .  $\square$

In addition, the converse for our construction is true:

**Proposition 4.5.** *Let  $A$  be a von Neumann algebra and suppose  $S$  is a self-adjoint pure contraction on a Hilbert space  $\mathcal{H}$  such that  $S \in A$ . Then the bounded transform  $T$  of  $S$  is affiliated with  $A$ , i.e.  $T\eta A$ .*

*Proof.* Observe that by our construction (Theorem 4.2)  $T = S\sqrt{1 - S^2}^{-1}$ . Our goal is to show that for any unitary  $U$  in  $A'$  we have:  $UTU^* = T$ . Therefore, let  $U$  be such a unitary. Then observe that  $SU = US$ , since  $S \in A$ . Now we have the following:

$$\begin{aligned}
 UTU^* &= US\sqrt{1 - S^2}^{-1}U^* \\
 &= SU\sqrt{1 - S^2}^{-1}U^* \\
 &= S \left( U\sqrt{1 - S^2}U^* \right)^{-1} \\
 &= S\sqrt{1 - S^2}^{-1}
 \end{aligned}$$

The last equality follows from the fact that  $\sqrt{1 - S^2}$  is everywhere defined and bounded. In particular, it is an element of  $A$  and hence also  $\sqrt{1 - S^2}\eta A$ . This proves the claim.  $\square$

Therefore, we can conclude with the following theorem.

**Theorem 4.8.** *Let  $A$  be a von Neumann algebra and suppose  $T$  is a self-adjoint operator on a Hilbert space  $\mathcal{H}$  with bounded transform  $S$ . Then  $T\eta A$  if and only if  $S \in A$ .*

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In chapter 2 (Definition 2.3) we introduced the von Neumann algebra  $W^*(S)$ , for  $S$  a bounded operator. This can be generalized as follows: We know that it means that two operators commute. Recall: if  $T$  is defined on  $\mathcal{D}(T) \subseteq \mathcal{H}$  for some Hilbert space  $\mathcal{H}$  and if  $S \in B(\mathcal{H})$ , then  $S$  and  $T$  commute if  $ST \subset TS$ . By  $\{T\}'$  we denote all bounded operators commuting with  $T$ . If  $T$  is closed, then  $\{T\}'$  is strongly closed. In this case, also  $S \in \{T\}'$  implies  $S^* \in \{T^*\}'$ . In particular,  $\{T\}' \cap \{T^*\}'$  is a unital, strongly closed  $*$ -subalgebra of  $B(\mathcal{H})$ , hence a von Neumann algebra. Therefore, we define

$$W^*(T) = \left( \{T\}' \cap \{T^*\}' \right)'$$

This extends the previous notion from chapter 2, in the sense that if  $T$  is bounded, then  $W^*(T)$  is the von Neumann algebra generated by  $T$ . In the special case where  $T$  is a (possibly) unbounded self-adjoint operator on  $\mathcal{H}$ ,  $T$  is also closed and we obtain

$$W^*(T) = \{T\}''$$

or in other words,  $W^*(T)$  contains all operators commuting with all operators that commute with  $T$ .

We recall that a (possibly) unbounded self-adjoint operator on a Hilbert space  $\mathcal{H}$  is by definition affiliated with a von Neumann algebra  $A$  ( $T\eta A$ ) if and only if  $UTU^* = T$  for each unitary operator in  $A'$ . The following proposition gives us two other equivalent characterizations.

**Proposition 4.6.** *Let  $A$  be a von Neumann algebra on the Hilbert space  $\mathcal{H}$  and let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a self-adjoint operator on  $\mathcal{H}$ . Then the following conditions are equivalent:*

- (i) *For all  $V \in A'$ :  $VT \subset TV$*
- (ii) *For all unitaries  $U \in A'$ :  $UT = TU$*
- (iii) *For all unitaries  $U \in A'$ :  $UTU^* = T$*

*Proof.* We start with the implication (i)  $\Rightarrow$  (ii). Therefore, assume that  $VT \subset TV$  for each  $V \in A'$ . This means that if  $x \in \mathcal{D}(T)$  and  $V \in A'$  is arbitrary, then  $Vx \in \mathcal{D}(T)$  and  $Tvx = VTx$ . In particular, if  $U$  is a unitary operator in  $A'$ , then  $Ux \in \mathcal{D}(T)$  and  $U^*x \in \mathcal{D}(T)$  and  $TUx = UTx$  and  $TU^*x = U^*Tx$ . Since  $U$  is unitary, hence bijective, it follows that  $TU = UT$ . The converse, hence the implication (ii)  $\rightarrow$  (i), follows from the fact that every operator is a linear combination of at most four unitary elements of  $A'$  (see [22]). The equivalence of (ii) and (iii) is trivial.  $\square$

Hence one also has  $T\eta A$  if and only if  $A' \subseteq \{T\}' \cap \{T^*\}'$  (which means  $A' \subseteq \{T\}'$  when  $T$  is self-adjoint). This leads to the following fact, which follows directly from the construction of the bounded transform and Theorem 4.8.

**Theorem 4.9.** *Let  $T$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ , with bounded transform  $S$ . Then  $V \in W^*(S)$  if and only if  $V \in W^*(T)$  for each  $V \in B(\mathcal{H})$ . In particular,  $W^*(T) = W^*(S)$ .*

## 4.4. Affiliated operators and integration theory

In this section we will need some integration theory. Since we are not going into the details of integration and measure theory, we will not prove the theorems we need. Good references are, for example, [14] and [17].

Let  $X$  be a locally compact,  $\sigma$ -compact Hausdorff space and  $f$  a Radon integral on  $X$  (here we use the convention from [14]: a Radon integral is a linear functional  $f : C_c(X) \rightarrow \mathbb{R}$  that is positive, i.e.  $f \geq 0$  implies  $f f \geq 0$ ). Notice that one can extend  $f$  to a bigger class of functions (see [14]). Furthermore, if  $X$  has such a Radon integral  $f$ , then  $X$  also has a measure  $\mu$  on it (by integration over characteristic functions). If we talk about a measure space in what follows, we mean the measure that comes from the Radon integral on it. Let  $\mathcal{L}(X)$ ,  $\mathcal{B}(X)$  and  $\mathcal{N}(X)$  be the spaces of measurable, Borel and null functions respectively. Then we obtain a  $*$ -algebra

$$L(X) := \mathcal{L}(X)/\mathcal{N}(X) = \mathcal{B}(X)/\mathcal{N}(X)$$

. This leads to the following definition.

**Definition 4.2.** *Let  $\mathcal{H}$  be a Hilbert space and  $X$  a locally compact,  $\sigma$ -compact Hausdorff space. We call a map  $\varphi$  from  $\mathcal{L}(X)$  into the space of (possibly) unbounded self-adjoint operators an **essential homomorphism** if  $\varphi(f + g) = \overline{\varphi(f) + \varphi(g)}$  and  $\varphi(fg) = \overline{\varphi(f)\varphi(g)}$ .*

On such spaces  $X$  with a Radon integral, one can define a multiplication operator  $M_f$  for each  $f \in \mathcal{L}(X)$  by  $M_f(g) = fg$  on  $\mathcal{D}(M_f) = \{g \in L^2(X) : fg \in L^2(X)\}$ . The following proposition gives us a link between normal operators on  $L^2(X)$  and multiplication operators.

**Proposition 4.7.** *Let  $X$  be a locally compact,  $\sigma$ -compact Hausdorff space with Radon integral  $f$ . The map  $f \mapsto M_f$  defines an essential  $*$ -isomorphism between  $L_{\mathbb{R}}(X)$  and the class of self-adjoint operators on  $L^2(X)$  affiliated with the von Neumann algebra  $A := \{M_f : f \in L^\infty(X)\}$ .*

*Proof.* Observe that the operator  $M_f$  is self-adjoint for each  $f \in \mathcal{L}_{\mathbb{R}}(X)$ . We want to show that the map  $f \mapsto M_f$  is an essential homomorphism. In particular, we have to show that  $\mathcal{G}(M_{f+g}) = \overline{\mathcal{G}(M_f + M_g)}$ . Therefore, let  $h \in \mathcal{D}(M_{f+g})$  and define for each  $n \in \mathbb{N}$ :

$$h_n := \frac{h}{1 + \frac{1}{n}(|f| + |g|)}$$

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Then  $h_n \in \mathcal{D}(M_f) \cap \mathcal{D}(M_g)$  for each  $n \in \mathbb{N}$ . By application of the *Monotone Convergence Theorem*, due to Lebesgue, one obtains  $h_n \rightarrow h$  and  $(M_f + M_g)h_n \rightarrow M_{f+g}h$ . This gives us  $\mathcal{G}(M_{f+g}) = \overline{\mathcal{G}(M_f + M_g)}$ . In the same way, one can show that  $\mathcal{G}(M_{fg}) = \overline{\mathcal{G}(M_f M_g)}$  holds. Moreover,  $M_f = 0$  if and only if  $f \in \mathcal{N}(X)$ . This gives us an essential isomorphism from  $L_{\mathbb{R}}(X)$  into a commutative  $*$ -algebra of self-adjoint operators on  $L^2(X)$  (by  $f \mapsto M_f$ ). Since  $M_f^* = M_{\bar{f}}$ , we conclude that this map is in particular an essential  $*$ -isomorphism. Recall that  $M_g M_f \subset M_f M_g$  for each  $f \in \mathcal{L}(X)$  and each  $g \in L^\infty(X)$ , hence each multiplication operator is affiliated with  $A'$ . One observes that  $A = A'$  (see [14]), which implies that we proved the result we wanted. Now assume that  $T$  is a self-adjoint operator on  $L^2(X)$  such that  $T\eta \in A$ . By the previous theorem, we obtain  $S \in A$  (where  $S$  is the bounded transform of  $T$ ). Hence, there exists a Borel function  $h$  on  $X$  such that  $|h(x)| = 1$  but  $h(x) \neq 1$  for almost all  $x \in X$  such that  $S = M_h$ . Now let  $g$  be the function defined by formula (4.1.2) and define  $f = g \circ h$ . Then:

$$M_f = S\sqrt{1 - S^2}^{-1} = T \quad \square$$

Now we will connect this notion to  $L_{\mathbb{R}}(\sigma(T))$  and with self-adjoint operators that are affiliated with  $W^*(T)$ .

**Theorem 4.10.** *Assume that  $T$  is a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$ , such that the bounded transform  $S$  admits a unit cyclic vector for  $C^*(S)$ . Then there exists a finite Radon integral on  $\sigma(T)$  and an isometry  $U : L^2(\sigma(T)) \rightarrow \mathcal{H}$  such that, if we define  $f(T) := UM_f U^*$  ( $f \in \mathcal{L}(\sigma(T))$ ) then the map  $f \mapsto f(T)$  is an essential  $*$ -isomorphism between  $L_{\mathbb{R}}(\sigma(T))$  and the  $*$ -algebra of self-adjoint operators in  $\mathcal{H}$  affiliated with  $W^*(T)$ . In particular,  $W^*(T) = A = \{UM_f U^* : f \in L^\infty(\sigma(T))\}$ .*

*Proof.* By assumption we obtain a unit cyclic vector for  $C^*(S)$ . If  $f \in C_0(\mathbb{R})$ , then  $f \circ g \in C_0(-1, 1)$  (for  $g$  as in formula (4.1.2)). Hence we can define

$$\langle f \circ g(S)x, x \rangle = \int f \, d\mu$$

for each  $f \in C_0(\sigma(T))$ . This is a finite Radon measure on  $\sigma(T)$ . Now we can start to construct our isometry  $U$ . For  $f \in C_0(\sigma(T))$  define

$$U(f) = (f \circ g(S))x$$

Extending by continuity gives the remaining part. By sending  $f$  to  $f(T)$  as defined in the theorem we obtain by the previous proposition, an essential  $*$ -isomorphism of  $L(\sigma(T))$  onto the  $*$ -algebra of self-adjoint operators affiliated with the von Neumann algebra

$$A = \{UM_f U^* : f \in L^\infty(\sigma(T))\}$$



. Now assume that  $f$  is a bounded Borel function on  $\mathbb{R}$ . Then  $f \circ g$  is a bounded Borel function on  $(-1, 1)$ . Since the bounded transform  $S$  of  $T$  is a self-adjoint pure contraction, 1 is not an eigenvalue of  $S$ , hence

$$A = \{u(S) : u \in B_b(\sigma(S))\} = W^*(S) = W^*(T) \quad \square$$

Before we deduce our next theorem, we will need a quite technical lemma due to Nussbaum (see [14]) about self-adjoint extensions.

**Lemma 4.7.** *If  $(\mathcal{H}_n)_{n \in \mathbb{N}}$  is a sequence of Hilbert spaces, and if for each  $n \in \mathbb{N}$ ,  $T_n$  is a self-adjoint operator on  $\mathcal{H}_n$ , then there exists a unique self-adjoint operator  $T$  on  $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$  such that  $T|_{\mathcal{D}(T_n)} = T_n$  for each  $n \in \mathbb{N}$ . Moreover,  $\mathcal{D}(T)$  consists of those vectors  $x = (x_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}$ , such that  $x_n \in \mathcal{D}(T_n)$  for each  $n \in \mathbb{N}$  and  $\sum_{n \in \mathbb{N}} \|T_n x_n\|^2 < \infty$ .*

*Proof.* Let  $\mathcal{D}_0$  denote the linear span in  $\mathcal{H}$  of the orthogonal subspaces  $\mathcal{D}(T_n)$  and let  $T_0$  be the operator in  $\mathcal{H}$  such that for each  $x = (x_n)_{n \in \mathbb{N}} \in \mathcal{D}_0$ :

$$T_0 \left( \sum_{n \in \mathbb{N}} x_n \right) = \sum_{n \in \mathbb{N}} T_n x_n$$

Furthermore, let  $T$  be the extension of  $T_0$  indicated in the lemma. In particular, for each  $x = (x_n)_{n \in \mathbb{N}} \in \mathcal{D}(T)$ :

$$T \left( \sum_{n \in \mathbb{N}} x_n \right) = \sum_{n \in \mathbb{N}} T_n x_n$$

Notice, that by construction  $\mathcal{D}(T_0)$  is dense in  $\mathcal{D}(T)$ . In particular,  $\mathcal{D}_0$  is a core for  $T$ , which is closed. Moreover,  $T$  and  $T_0$  are both symmetric. Now we want to show self-adjointness of  $T$ . Therefore, let  $x = (x_n)_{n \in \mathbb{N}} \in \mathcal{D}(T^*)$ . For each  $n \in \mathbb{N}$  and for each  $y_n \in \mathcal{D}(T_n)$ , the map  $y_n \mapsto \langle T y_n, x \rangle = \langle T_n y_n, x_n \rangle$  is bounded. Hence  $x_n \in \mathcal{D}(T_n^*) = \mathcal{D}(T_n)$ . Now let  $y \in \mathcal{D}_0$  be a finite sum, i.e.  $y = \sum_{n=1}^m y_n$ . Then

$$\langle T y, x \rangle = \sum_{n=1}^m \langle T_n y_n, x_n \rangle = \sum_{n=1}^m \langle y_n, T_n x_n \rangle$$

Since  $\mathcal{D}_0$  is dense in  $\mathcal{H}$ , this implies that  $\sum_{n \in \mathbb{N}} T_n x_n$  belongs to  $\mathcal{H}$ , which is equal to  $T^* x$ . In particular,  $\sum_{n \in \mathbb{N}} \|T_n x_n\|^2 < \infty$ . Hence  $x \in \mathcal{D}(T)$ , so  $T = T^*$ .  $\square$

The following theorem allows us to compute formally inside the class of self-adjoint operators affiliated with  $W^*(T)$ , without worrying about domains, closedness or self-adjointness

**Theorem 4.11.** *Let  $T$  be a self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then there exist a finite Radon measure on  $\sigma(T)$  and an essential  $*$ -isomorphism  $f \mapsto f(T)$  from  $L_{\mathbb{R}}(\sigma(T))$  onto the  $*$ -algebra of self-adjoint operators affiliated with  $W^*(T)$ .*

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*Proof.* Find an orthogonal sequence  $(p_n)_{n \in \mathbb{N}}$  of cyclic projections relative to  $W^*(T)$  such that  $\sum_{n \in \mathbb{N}} p_n$  converges strongly to the identity operator. If  $S$  is the bounded transform of  $T$ , then  $S p_n$  admits a cyclic vector for  $p_n(\mathcal{H})$ . Since  $p_n$  commutes with  $S$ ,  $T$  also commutes with  $p_n$ . Moreover, the bounded transform of  $T p_n$  is given by  $S p_n$ . By identifying  $\sigma(T p_n)$  as a closed subset of  $\sigma(T)$  for each  $n \in \mathbb{N}$  we obtain a Radon integral  $\int_n$  on  $\sigma(T)$  and an isometry  $U_n$  from  $L^2(\sigma(T))$  onto  $p_n(\mathcal{H})$  and an essential  $*$ -isomorphism  $f \mapsto U_n M_f U_n^*$  from  $L_{\mathbb{R}}(\sigma(T))$  onto the class of self-adjoint operators in  $p_n(\mathcal{H})$  affiliated with  $W^*(T p_n)$ . Now define  $\int := \sum_{n \in \mathbb{N}} 2^{-n} \int_n$ . This is a normalized Radon measure on  $\sigma(T)$ . Furthermore, for each  $f \in \mathcal{L}(\sigma(T))$  define  $f(T)$  as the self-adjoint extension of  $(U_n M_f U_n^*)$ , constructed in the lemma before. One has  $f(T) = 0$  if and only if  $f \in \mathcal{N}(\sigma(T))$  (with respect to  $\int$ ). Observe that now  $f \mapsto f(T)$  defines a  $*$ -isomorphism on  $L_{\mathbb{R}}(\sigma(T))$ . Now assume that  $K$  is a self-adjoint operator on  $\mathcal{H}$ , with bounded transform  $Q$ , affiliated with  $W^*(T)$ . Then  $Q \in W^*(S)$ . Hence there exists a unitary function  $u$  on  $\sigma(S)$  such that  $Q = u(S)$ . Let  $w = u \circ g$ . Then for each  $n \in \mathbb{N}$ :

$$Q p_n = u(S) p_n = u(g(T p_n)) = w(T p_n) = U_n M_w U_n^*$$

If one identifies  $Q p_n$  with  $g(K p_n)$  and if  $f = h \circ w$ , then  $Q p_n = U_n M_f U_n^*$  and  $K = f(S)$ .  $\square$

The next result yields continuity of the map  $f \mapsto f(T)$  from the previous theorem.

**Theorem 4.12.** *If  $T$  is a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$ , then there exists a finite Radon integral  $\int_{\sigma(T)} \bullet d\mu_x$  on  $\sigma(T)$  for each  $x \in \mathcal{H}$  such that for all positive bounded Borel functions on  $\sigma(T)$ :*

$$\int_{\sigma(T)} f d\mu_x = \langle f(T)x, x \rangle$$

*Proof.* Assume  $f \in C_0((-1, 1))$ , then  $k = f \circ g \in C_0(\mathbb{R})$ . Hence

$$\int_{\sigma(T)} k d\mu_x = \langle f(g(T))x, x \rangle$$

defines a Radon integral on  $\sigma(T)$ . In particular, one has  $\int_{\sigma(T)} 1 d\mu_x = \|x\|^2$ . Let  $x \in \mathcal{H}$  be arbitrary. Then we obtain an orthogonal decomposition  $x = \sum_{n \in \mathbb{N}} x_n$ , where  $x_n = p_n(x)$  with  $p_n$  as in the previous theorem. Then for  $f \in C_0(\mathbb{R})$ :

$$\int_{\sigma(T)} f d\mu_x = \langle f(T)x, x \rangle = \sum_{n \in \mathbb{N}} \langle f(T)x_n, x_n \rangle = \sum_{n \in \mathbb{N}} \int_{\sigma(T)} f d\mu_{x_n}$$

Hence  $\int_{\sigma(T)} \bullet d\mu_x = \sum_{n \in \mathbb{N}} \int_{\sigma(T)} \bullet d\mu_{x_n}$ . Therefore, the formula from the theorem holds by additivity. If  $f \in \mathcal{N}(\sigma(T))$  with respect to the integral from the previous theorem, then  $f(T) = 0$  if and only if  $f \in \mathcal{N}(\sigma(T))$ . Moreover,  $\int_x f = 0$  for every  $f \in \mathcal{N}(\sigma(T))$ , hence  $\int_{\sigma(T)} \bullet d\mu_x \ll \int$ , which means absolute continuous.  $\square$

## 4.5. The Joint Spectrum

We know that  $C^*(T)$  is a commutative  $C^*$ -algebra if  $T$  is a bounded self-adjoint operator on a Hilbert space  $\mathcal{H}$ . This can be generalized. In particular, let  $T_1, \dots, T_n$  be a finite sequence of bounded self-adjoint operators on  $\mathcal{H}$ . Look at the  $C^*$ -algebra generated by  $T_1, \dots, T_n$ , denoted by  $C^*(T_1, \dots, T_n)$ . If one assumes that  $[T_i, T_j] = 0$  if  $i \neq j$ , in other words if all operators commute,  $C^*(T_1, \dots, T_n)$  will be a commutative  $C^*$ -algebra. For this case we can talk about the so-called *joint spectrum* (see [20]).

**Definition 4.3.** Let  $T_1, \dots, T_n$  be a sequence of bounded self-adjoint operators on a Hilbert space  $\mathcal{H}$  such that  $T_i T_j = T_j T_i$  for each  $i, j$ . Then we define the **joint spectrum**  $\sigma(T_1, \dots, T_n) \subset \mathbb{R}^n$  to be the following set:

$$\sigma(T_1, \dots, T_n) := \{(\omega(T_1), \dots, \omega(T_n)) \in \sigma(T_1) \times \dots \times \sigma(T_n) : \omega \in \Omega(C^*(T_1, \dots, T_n))\}$$

Observe, that we obtain the original spectrum for the case  $n = 1$ .

Moreover,  $(\lambda_1, \dots, \lambda_n) \in \sigma(T_1, \dots, T_n)$  if and only if there exists a sequence  $(x_m)_{m \in \mathbb{N}}$  of unit vectors in  $\mathcal{H}$  such that  $\lim_{m \rightarrow \infty} \|(T_i - \lambda_i)x_m\| = 0$  for  $1 \leq i \leq n$ . To see this, we use the following lemma due to Weyl, which also covers the unbounded self-adjoint case.

**Lemma 4.8.** Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a (possibly) unbounded self-adjoint operator on a Hilbert space  $\mathcal{H}$ . Then  $\lambda \in \sigma(T)$  if and only if there exists a sequence of unit vectors  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(T)$  such that  $\|(T - \lambda)x_n\| \rightarrow 0$  if  $n \rightarrow \infty$ .

*Proof.* The first implication will be proved by contradiction. Therefore, assume that there exists such a sequence of unit vectors  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{D}(T)$  but suppose that  $\lambda \notin \sigma(T)$ . Then  $(T - \lambda)^{-1}$  exists and  $(T - \lambda)^{-1} \in B(\mathcal{H})$ . Taking an arbitrary sequence  $(y_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}$  with  $y_n \rightarrow 0$ , this would imply that  $(T - \lambda)^{-1}y_n \rightarrow 0$ . Hence, by taking  $y_n = (T - \lambda)x_n$  we observe that  $(T - \lambda)x_n \rightarrow 0$  implies  $x_n \rightarrow 0$ , which contradicts the assumption that  $\|x_n\| = 1$  for each  $n \in \mathbb{N}$ . Hence  $\lambda \in \sigma(T)$ .

For the converse, assume that  $\lambda \in \sigma(T)$ . This means that  $T - \lambda$  is not invertible. Therefore, we can find an unit vector  $x_n \in \mathcal{D}(T)$  such that  $\|(T - \lambda)x_n\| < \frac{1}{n}$  for each  $n \in \mathbb{N}$ . The existence of such unit vectors is an application of the so called *Closed Range Theorem* for closed (and hence also self-adjoint) operators which we stated after this proof. This gives us the sequence we wanted.  $\square$

**Theorem 4.13.** Let  $T : \mathcal{D}(T) \rightarrow \mathcal{H}$  be a closed operator on a Hilbert space  $\mathcal{H}$ . Then  $\overline{\mathcal{R}(T)} = \text{Ker}(T^*)^\perp$  and  $\mathcal{R}(T)^\perp = \text{Ker}(T^*)$ . In particular, one has  $\overline{\mathcal{R}(T)} = \mathcal{H}$  if and only if  $\text{Ker}(T^*) = \{0\}$ . Furthermore, we have the following:

- $\mathcal{R}(T)$  is closed if there exists  $\alpha > 0$  such that  $\|Tx\| \geq \alpha \|x\|$  for each  $x \in \mathcal{D}(T)$ .

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- $T$  is invertible if and only if there exists  $\alpha > 0$  such that  $\|Tx\| \geq \alpha \|x\|$  for each  $x \in \mathcal{D}(T)$  and  $\text{Ker}(T^*) = \{0\}$ .
- If  $T$  is self-adjoint, then  $T$  is invertible if and only if there exists  $\alpha > 0$  such that  $\|Tx\| \geq \alpha \|x\|$  for each  $x \in \mathcal{D}(T)$

*Proof.* The first equality follows from the fact that in general  $\overline{A} = A^{\perp\perp}$  for each subspace  $A \subseteq \mathcal{H}$  and the fact that  $\mathcal{R}(T)^\perp = \text{Ker}(T^*)$ . The second equality follows by closedness of the kernel, the equality  $\mathcal{R}(T^*)^\perp = \text{Ker}(T)$ , and the fact that  $A^{\perp\perp\perp} = A^\perp$  for each subspace  $A \subseteq \mathcal{H}$ . Now we are going to prove the three statements:

- Assume that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{R}(T)$  converging to  $x \in \mathcal{H}$ . Then  $x_n = Ty_n$  for some  $y_n \in \mathcal{D}(T)$ . Since one has  $\|y_m - y_n\| \leq \frac{1}{\alpha} \|x_m - x_n\|$  for each  $m, n \in \mathbb{N}$  we may conclude that  $(y_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. If now  $y_n \rightarrow y$ , then  $x_n \rightarrow Ty = x$  by closedness of the operator. Hence  $x \in \mathcal{R}(T)$ .
- Suppose  $T$  is invertible. Then  $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \|Tx\|$  for each  $x \in \mathcal{D}(T)$ . Since by assumption  $T$  is invertible,  $T^{-1} \in B(\mathcal{H})$ , hence we obtain the first property by taking  $\alpha = 1/\|T^{-1}\|$ . Moreover one obtains surjectivity directly from invertibility, hence  $\mathcal{R}(T) = \mathcal{H}$ . Therefore, by the first equality  $\text{Ker}(T^*) = \{0\}$ . For the converse: if both properties are given we obtain  $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$ , hence  $T$  is surjective, hence invertible.
- For self-adjoint operators one has  $\text{Ker}(T) = \text{Ker}(T^*)$ . Therefore, since there exists  $\alpha > 0$  such that  $\|Tx\| \geq \alpha \|x\|$  for each  $x \in \mathcal{D}(T)$ , one has  $\text{Ker}(T^*) = \{0\}$  and therefore  $\text{Ker}(T) = \{0\}$ .  $\square$

Our goal is now to generalize the joint spectrum to (possibly) unbounded operators on a Hilbert space  $\mathcal{H}$ . Therefore, assume that  $T_1, \dots, T_n$  are (possibly) unbounded self-adjoint operators on  $\mathcal{H}$ . We say that  $T_i$  and  $T_j$  *commute strongly* if their bounded transforms  $S_i$  and  $S_j$  commute. Hence we can do the following: let  $g^{(n)} := g \times \dots \times g$  (as in formula (4.1.2)) and define:

$$\sigma(T_1, \dots, T_n) := g^{(n)}(\tilde{\sigma}(S_1, \dots, S_n)) \quad (4.5.1)$$

We want to deduce that for  $\sigma(T_1, \dots, T_n)$  a relation similar to (4.1.6) holds. In particular, we have the following theorem.

**Theorem 4.14.** *Let  $T_1, \dots, T_n$  be (possibly) unbounded self-adjoint operators on a Hilbert space  $\mathcal{H}$  such that all  $T_i$  strongly commute with each other. Moreover, let  $S_1, \dots, S_n$  be the bounded transforms of  $T_1, \dots, T_n$ , respectively. Then one has:*

$$\sigma(T_1, \dots, T_n) = \left\{ (\lambda_1(1 - \lambda_1^2)^{-\frac{1}{2}}, \dots, \lambda_n(1 - \lambda_n^2)^{-\frac{1}{2}}) : (\lambda_1, \dots, \lambda_n) \in \sigma(S_1, \dots, S_n) \cap (-1, 1)^n \right\}$$

## 4.5. The Joint Spectrum

Moreover, one has  $(\lambda_1, \dots, \lambda_n) \in \sigma(T_1, \dots, T_n)$  if and only if there exists a sequence  $(x_m)_{m \in \mathbb{N}}$  of unit vectors in  $\mathcal{D}(T_i)$  such that  $\|(T_i - \lambda_i)x_m\| \rightarrow 0$  for  $m \rightarrow \infty$  for each  $1 \leq i \leq n$ .

*Proof.* The proof of the first part of the theorem is a componentwise application of the proof of Theorem 4.3, which can be done by (4.5.1). The second part of the theorem is a componentwise application of Lemma 4.5.  $\square$

The second part of this theorem gives us directly the link between our definition of the joint spectrum and the definition via the support of the spectral measures (see [20]). In particular, we see that these characterizations are equivalent. This means that the theorems one knows about the joint spectrum (see [20]) also go through for our definition of the joint spectrum. Here is such a theorem.

**Proposition 4.8.** *Let  $T_1, \dots, T_n$  be a finite sequence of (possibly) unbounded self-adjoint operators, such that all  $T_i$  strongly commute with each other. Assume that  $f : \sigma(T_1, \dots, T_n) \rightarrow \mathbb{C}$  is a continuous function. Then  $\sigma(f(T)) = \overline{f(\sigma(T))}$ .*



# A. Basic Functional Analysis

## A.1. Banach algebras

We first recall what it means to be a *Banach space*: A Banach space  $A$  is a vector space with a norm  $\|\cdot\|$ , such that  $A$  is complete with respect to this norm. Recall that  $\|\cdot\|$  is a norm if we have the following properties:

- $\forall a \in A \forall \lambda \in \mathbb{C} : \|\lambda a\| = |\lambda| \|a\|$
- $\forall a, b \in A : \|a + b\| \leq \|a\| + \|b\|$
- $\forall a \in A : \|a\| \geq 0$  and  $\|a\| = 0$  if and only if  $a = 0$

If, moreover  $A$  is an algebra with unit  $e$  such that  $\|ab\| \leq \|a\| \|b\|$  for each  $a, b \in A$  and  $\|e\| = 1$ , then  $A$  is called a Banach algebra. An involution on  $A$  is a map  $*$  :  $A \rightarrow A$  such that the following equalities hold:

- $\forall a, b \in A : (a + b)^* = a^* + b^*$
- $\forall a, b \in A : (ab)^* = b^* a^*$
- $\forall a \in A : (a^*)^* = a$
- $\forall a \in A \forall \lambda \in \mathbb{C} : (\lambda a)^* = \bar{\lambda} a^*$

If a Banach algebra  $A$  has such an involution, we call  $A$  a Banach  $*$ -algebra. In particular, a  $C^*$ -algebra  $A$  is a Banach  $*$ -algebra with the following norm property:  $\|a^* a\| = \|a\|^2$ , for each  $a \in A$ .

## A.2. Hilbert spaces

Assume that  $\mathcal{H}$  is a complex vector space. An inner product on  $\mathcal{H}$  is a function  $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  satisfying the following conditions:

- $\forall x \in \mathcal{H} : \langle x, x \rangle \geq 0$
- $\forall x \in \mathcal{H} : \langle x, x \rangle = 0 \iff x = 0$
- $\forall x, y, z \in \mathcal{H} \forall \alpha, \beta \in \mathbb{C} : \langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

## A. Basic Functional Analysis

- $\forall x, y \in \mathcal{H} : \langle x, y \rangle = \overline{\langle y, x \rangle}$

If  $\mathcal{H}$  has such an inner product, then  $\mathcal{H}$  is called an *inner product space*. Such an inner product induces a norm on  $\mathcal{H}$  by  $\|x\| := \sqrt{\langle x, x \rangle}$ . If  $\mathcal{H}$  is complete with respect to this induced norm, then  $\mathcal{H}$  is called a *Hilbert space*. A useful tool is the fact that a Hilbert space  $\mathcal{H}$  is separable if and only if  $\mathcal{H}$  has a countable orthonormal basis. If this is the case, let  $(e_n)_{n \in \mathbb{N}}$  be such a countable orthonormal basis, then the following holds:

- $\forall x \in \mathcal{H} : x = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle e_n$
- $\forall x \in \mathcal{H} : \|x\|^2 = \sum_{n \in \mathbb{N}} |\langle x, e_n \rangle|^2$
- $\forall x, y \in \mathcal{H} : \langle x, y \rangle = \sum_{n \in \mathbb{N}} \langle x, e_n \rangle \langle e_n, y \rangle$

## A.3. Spectral Theory

The following theorems hold already for Banach algebras and therefore we can also apply them to  $C^*$ -algebras and von Neumann algebras.

Let  $A$  be a Banach algebra. For  $a \in A$  we define the spectrum  $\sigma(a)$  to be the set of  $\lambda \in \mathbb{C}$  such that  $a - \lambda e$  is not invertible. The spectral radius is defined by

$$r(a) = \sup \{ |\lambda| : \lambda \in \sigma(a) \}$$

Geometrically, one can see  $r(a)$  as the radius of the smallest closed circular disc in  $\mathbb{C}$ , with center the origin, that contains  $\sigma(a)$ . The following properties hold for  $\sigma(a)$  and  $r(a)$ .

**Theorem A.1.** *If  $A$  is a Banach algebra and  $a \in A$ , then  $\sigma(a)$  is compact and non-empty. Moreover, the spectral radius formula holds:*

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|a^n\|^{\frac{1}{n}}$$

The following theorem characterizes Banach algebras, that are division algebras. In particular, if all non-zero elements are invertible, the Gelfand-Mazur Theorem states the following.

**Theorem A.2.** *If  $A$  is a Banach algebra in which every non-zero element is invertible, then  $A$  is isometrically isomorphic to  $\mathbb{C}$ .*

A very useful theorem in the theory of  $C^*$ -algebras is the so called *spectral mapping theorem*. This gives us information about the spectrum of  $f(a)$  where  $a \in A$  is (at least) normal and  $f \in C(\sigma(a))$ . In particular we obtain the following formulation.

**Theorem A.3.** *Let  $A$  be a  $C^*$ -algebra and  $a \in A$  be normal. Assume that  $f \in C(\sigma(a))$ . Then  $\sigma(f(a)) = f(\sigma(a))$ .*



## A.4. Fundamental Theorems

We will formulate the most important theorems of functional analysis we use in this thesis. The first theorem we will state is a version of the Hahn-Banach Theorem, which already is true for locally convex spaces, and two corollaries:

**Theorem A.4.** *Suppose  $M$  is a subspace of a real vector space  $X$ ,  $p : X \rightarrow \mathbb{R}$  satisfies  $p(x+y) \leq p(x)+p(y)$  and  $p(tx) = tp(x)$  for  $x, y \in X$  and  $t \in \mathbb{R}$  and  $f : M \rightarrow \mathbb{R}$  is linear such that  $f(x) \leq p(x)$  on  $M$ . Then there exists a linear map  $\tau : M \rightarrow \mathbb{R}$  such that  $\tau(x) = f(x)$  on  $M$  and  $-p(-x) \leq \tau(x) \leq p(x)$  for each  $x \in X$ .*

**Corollary A.1.** *Suppose  $M$  is a subspace of a vector space  $X$ ,  $p$  is a semi-norm on  $X$  and  $\tau$  is a linear functional on  $M$  such that  $|\tau(x)| \leq p(x)$  for each  $x \in M$ . Then  $\tau$  extends to a linear functional  $\varphi$  on  $X$  that satisfies  $|\varphi(x)| \leq p(x)$  for each  $x \in X$ .*

**Corollary A.2.** *If  $X$  is a normed space and  $x_0 \in X$ , there exists a continuous linear functional  $\tau$  on  $X$  such that  $\tau(x_0) = \|x_0\|$  and  $|\tau(x)| \leq \|x\|$  for each  $x \in X$ .*

Another important theorem in the context of locally convex spaces is the Krein-Milman Theorem.

**Theorem A.5.** *Let  $C$  be a non-empty convex compact set in a Hausdorff locally convex space  $X$ . Then the set  $E$  of extreme points of  $C$  is non-empty and  $C = \overline{\text{co}(E)}$ , where  $\text{co}(E)$  denotes the convex hull of  $E$ .*

The following theorem is the general formulation of the theorem of Banach-Alaoglu, which already holds for topological vector spaces.

**Theorem A.6.** *If  $V$  is a neighbourhood of 0 in a topological vector space  $X$ , then*

$$\{\varphi \in X^* : |\varphi(x)| \leq 1 \ \forall x \in V\}$$

*is weak\*-compact.*

The general statement of the Stone-Weierstrass Theorem is the following.

**Theorem A.7.** *Let  $X$  be a compact topological space. Let  $D$  be a linear subspace of  $C(X)$ , the space of continuous functions from  $X$  to  $\mathbb{C}$ . If  $D$  contains a unit and separates the points and if  $f \in D$  implies  $f^2 \in D$  or  $|f| \in D$ , then  $D$  is dense in  $C(X)$ , with respect to the sup-norm.*

A theorem we often used is the Riesz-Fréchet Theorem.

**Theorem A.8.** *Let  $\mathcal{H}$  be a Hilbert space and  $\varphi : \mathcal{H} \rightarrow \mathbb{C}$  a continuous linear functional on  $\mathcal{H}$ . Then there exists a unique  $y \in \mathcal{H}$  such that  $\varphi(x) = \langle x, y \rangle$  for each  $x \in \mathcal{H}$ . Moreover:  $\|\varphi\| = \|y\|$ .*

## A. Basic Functional Analysis

The last theorem is the well-known Lemma of Zorn (which is an equivalent formulation of the Axiom of Choice):

**Theorem A.9.** *Suppose that a partial ordered set  $P$  has the property that every totally ordered subset has an upper bound in  $P$ . Then  $P$  contains a maximal element.*

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