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Lie Groupoid C^*-Algebras and Weyl Quantization

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Abstract: A strict quantization of a Poisson manifold P on a subset $I \subseteq \mathbb{R}$ containing 0 as an accumulation point is defined as a continuous field of C^*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar\in I}$, with $\mathfrak{A}_0 = C_0(P)$, a dense subalgebra $\tilde{\mathfrak{A}}_0$ of $C_0(P)$ on which the Poisson bracket is defined, and a set of continuous cross-sections $\{\mathcal{Q}(f)\}_{f\in\tilde{\mathfrak{A}}_0}$ for which $\mathcal{Q}_0(f) = f$. Here $\mathcal{Q}_\hbar(f^*) = \mathcal{Q}_\hbar(f)^*$ for all $\hbar \in I$, whereas for $\hbar \to 0$ one requires that $i[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)]/\hbar \to \mathcal{Q}_\hbar(\{f, g\})$ in norm.

For any Lie groupoid G, the vector bundle $\mathfrak{G}^*$ dual to the associated Lie algebroid $\mathfrak{G}$ is canonically a Poisson manifold. Let $\mathfrak{A}_0 = C_0(\mathfrak{G}^*)$, and for $\hbar \neq 0$ let $\mathfrak{A}_\hbar = C^*(\mathsf{G})$ be the C^*-algebra of G. The family of C^*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar\in[0,1]}$ forms a continuous field, and we construct a dense subalgebra $\tilde{\mathfrak{A}}_0 \subset C_0(\mathfrak{G}^*)$ and an associated family $\{\mathcal{Q}^W_\hbar(f)\}$ of continuous cross-sections of this field, generalizing Weyl quantization, which define a strict quantization of $\mathfrak{G}^*$.

Many known strict quantizations are a special case of this procedure. On $P = T^*\mathbb{R}^n$ the maps $\mathcal{Q}^W_\hbar(f)$ reduce to standard Weyl quantization; for $P = T^*Q$, where Q is a Riemannian manifold, one recovers Connes' tangent groupoid as well as a recent generalization of Weyl's prescription. When G is the gauge groupoid of a principal bundle one is led to the Weyl quantization of a particle moving in an external Yang–Mills field. In case that G is a Lie group (with Lie algebra $\mathfrak{g}$) one recovers Rieffel's quantization of the Lie–Poisson structure on $\mathfrak{g}^*$. A transformation group C^*-algebra defined by a smooth action of a Lie group on a manifold Q turns out to be the quantization of the Poisson manifold $\mathfrak{g}^* \times Q$ defined by this action.

1. Introduction

The notion of quantization to be used in this paper is motivated by the desire to link the geometric theory of classical mechanics and reduction [18,32] with the C^*-algebraic

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formulation of quantum mechanics and induction [15], and also with non-commutative geometry [2]. Starting with Rieffel's fundamental paper [27], various C^*-algebraic definitions of quantization have been proposed [29,12,30,15,31]. Definition 2 below is closely related to these proposals, and is particularly useful in the context of the class of examples studied in this paper.

These examples come from the theory of Lie groupoids and their Lie algebroids (cf. Sect. 2). The idea that the C^*-algebra of a Lie groupoid is connected to the Poisson manifold defined by the associated Lie algebroid by (strict) quantization was conjectured in [12], and proved in special cases in [13,15]. The results of [28,29,23] also supported the claim. In this paper we prove the conjecture up to Dirac's condition (3); this is the content of Theorems 1 and 2. Following up on our work, Dirac's condition has finally been proved by Ramazan [25]. This leads to the Corollary at the end of Sect. 5, which is the main result of the paper.

Further to the examples considered in Sect. 6, it would be interesting to apply the point of view in this paper to the holonomy groupoid of a foliation [2], and to the Lie groupoid defined by a manifold with boundary [23,19]. Moreover, the approach to index theory via the tangent groupoid [2] and its recent generalization to arbitrary Lie groupoids [20] may now be seen from the perspective of "strict" quantization theory. This may be helpful also in understanding the connection between various other approaches to index theory which use (formal deformation) quantization [8,7].

The central notion in C^*-algebraic quantization theory is that of a continuous field of C^*-algebras [5]. For our purposes the following reformulation is useful [10].

Definition 1. *A continuous field of C^*-algebras $(\mathfrak{C}, \{\mathfrak{A}_x, \varphi_x\}_{x\in X})$ over a locally compact Hausdorff space X consists of a C^*-algebra $\mathfrak{C}$, a collection of C^*-algebras $\{\mathfrak{A}_x\}_{x\in X}$, and a set $\{\varphi_x : \mathfrak{C} \to \mathfrak{A}_x\}_{x\in X}$ of surjective *-homomorphisms, such that for all $A \in \mathfrak{C}$,*

1. *the function $x \to \|\varphi_x(A)\|$ is in $C_0(X)$;*
2. *one has $\|A\| = \sup_{x\in X} \|\varphi_x(A)\|$;*
3. *there is an element $fA \in \mathfrak{C}$ for any $f \in C_0(X)$ for which $\varphi_x(fA) = f(x)\varphi_x(A)$ for all $x \in X$.*

The continuous cross-sections of the field in the sense of [5] consist of those elements $\{A_x\}_{x\in X}$ of $\prod_{x\in X} \mathfrak{A}_x$ for which there is a (necessarily unique) $A \in \mathfrak{C}$ such that $A_x = \varphi_x(A)$ for all $x \in X$.

We refer to [18,32] for the theory of Poisson manifolds and Poisson algebras; the latter is the classical analogue of the self-adjoint part of a C^*-algebra [15].

Definition 2. *Let $I \subseteq \mathbb{R}$ contain 0 as an accumulation point. A strict quantization of a Poisson manifold P on I consists of*

1. *a continuous field of C^*-algebras $(\mathfrak{C}, \{\mathfrak{A}_\hbar, \varphi_\hbar\}_{\hbar\in I})$, with $\mathfrak{A}_0 = C_0(P)$;*
2. *a dense subspace $\tilde{\mathfrak{A}}_0 \subset C_0(P)$ on which the Poisson bracket is defined, and which is closed under pointwise multiplication and taking Poisson brackets (in other words, $\tilde{\mathfrak{A}}_0$ is a Poisson algebra);*
3. *a linear map $\mathcal{Q} : \tilde{\mathfrak{A}}_0 \to \mathfrak{C}$ which (with $\mathcal{Q}_\hbar(f) \equiv \varphi_\hbar(\mathcal{Q}(f))$) for all $f \in \tilde{\mathfrak{A}}_0$ and $\hbar \in I$ satisfies*

$$\mathcal{Q}_0(f) = f, \tag{1}$$

$$\mathcal{Q}_\hbar(f^*) = \mathcal{Q}_\hbar(f)^*, \tag{2}$$

and for all $f, g \in \tilde{\mathfrak{A}}_0$ *satisfies Dirac's condition*

$$\lim_{\hbar \to 0} \| \frac{i}{\hbar} [\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] - \mathcal{Q}_\hbar(\{f, g\}) \| = 0. \tag{3}$$

Elements of I are interpreted as possible values of Planck's constant $\hbar$, and $\mathfrak{A}_\hbar$ is the quantum algebra of observables of the theory at the given value of $\hbar \neq 0$. For real-valued f, the operator $\mathcal{Q}_\hbar(f)$ is the quantum observable associated to the classical observable f. This interpretation is possible because of condition (2) in Definition 2. In view of the comment after Definition 1, for fixed $f \in \tilde{\mathfrak{A}}_0$ each family $\{\mathcal{Q}_\hbar(f)\}_{\hbar \in I}$ is a continuous cross-section of the continuous field in question. In view of (1) this implies, in particular, that

$$\lim_{\hbar \to 0} \| \mathcal{Q}_\hbar(f)\mathcal{Q}_\hbar(g) - \mathcal{Q}_\hbar(fg) \| = 0. \tag{4}$$

This shows that strict quantization yields asymptotic morphisms in the sense of E-theory [2]; cf. [22]. See [15] for an extensive discussion of quantization theory from the above perspective, including an interpretation of the conditions (3) and (4).

2. Lie Groupoids and Lie Algebroids

Throughout this section, the reader is encouraged to occasionally skip to Sect. 6 to have a look at some examples of the objects defined.

We refer to [26,17,3,2,15,1] for the basic definitions on groupoids; here we merely establish our notation. Briefly, a groupoid is a category whose space of arrows G is a set (hence the space of objects Q is a set as well), and whose arrows are all invertible. The source and target projections are called $\tau_s : \mathsf{G} \to Q$ and $\tau_t : \mathsf{G} \to Q$, respectively.

The subset of $\mathsf{G} \times \mathsf{G}$ on which the groupoid multiplication (i.e., the composition of arrows) is defined is called G_2; hence $(\gamma_1, \gamma_2) \in \mathsf{G}_2$ iff $\tau_s(\gamma_1) = \tau_t(\gamma_2)$. The inversion $\gamma \to \gamma^{-1}$ defines the unit space $\mathsf{G}_0 = \{\gamma\gamma^{-1} | \gamma \in \mathsf{G}\}$, which is related to the base space Q by the "object inclusion map" $\iota : Q \hookrightarrow \mathsf{G}$; this is a bijection between Q and $\iota(Q) = \mathsf{G}_0$. The notation $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$ for a groupoid to some extent captures the situation.

A Lie groupoid is a groupoid $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$, where G and Q are manifolds (perhaps with boundary), the maps τ_s and τ_t are surjective submersions, and multiplication and inclusion are smooth [17,3,2,15,1]. Following [15], we now sharpen Def. I.2.2 in [26].

Definition 3. *A left Haar system on a Lie groupoid* $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$ *is a family* $\{\mu_q^t\}_{q \in Q}$ *of positive measures, where the measure* μ_q^t *is defined on* $\tau_t^{-1}(q)$*, such that*

1. *the family is invariant under left-translation in* G*;*
2. *each* μ_q^t *is locally Lebesgue (i.e., it is equivalent to the Lebesgue measure in every co-ordinate chart; note that each fiber* $\tau_t^{-1}(q)$ *is a manifold);*
3. *for each* $\mathsf{f} \in C_c^\infty(\mathsf{G})$ *the map* $q \mapsto \int_{\tau_t^{-1}(q)} d\mu_q^t(\gamma)\mathsf{f}(\gamma)$ *from* Q *to* $\mathbb{C}$ *is smooth.*

Here left-invariance means invariance under all maps L_γ, defined by

$$L_\gamma(\gamma') := \gamma\gamma' \tag{5}$$

whenever $(\gamma, \gamma') \in \mathsf{G}_2$. Note that L_γ maps $\tau_t^{-1}(\tau_s(\gamma))$ diffeomorphically to $\tau_t^{-1}(\tau_t(\gamma))$.

A Lie groupoid $\mathsf{G} \overset{\leftarrow}{\Rightarrow} Q$ has an associated Lie algebroid [17,3,15,1], which we denote by $\mathfrak{G} \overset{\rightarrow TQ}{\rightarrow_Q}$. This is a vector bundle over Q, which apart from the bundle projection $\tau : \mathfrak{G} \to Q$ is equipped with a vector bundle map $\tau_a : \mathfrak{G} \to TQ$ (called the anchor), as well as with a Lie bracket $[\ ,\ ]_{\mathfrak{G}}$ on the space $\Gamma(\mathfrak{G})$ of smooth sections of $\mathfrak{G}$, satisfying certain compatibility conditions.

For our purposes, the essential point in the construction of $\mathfrak{G} \overset{\rightarrow TQ}{\rightarrow_Q}$ from $\mathsf{G} \overset{\leftarrow}{\Rightarrow} Q$ lies in the fact that the vector bundle $\mathfrak{G}$ over Q is the normal bundle $N^{\iota}Q$ defined by the embedding $\iota : Q \hookrightarrow \mathsf{G}$; accordingly, the projection $\tau : N^{\iota}Q \to Q$ is given by τ_s or τ_t (these projections coincide on G_0). The tangent bundle of G at the unit space has a decomposition

$$T_{\iota(q)}\mathsf{G} = T_{\iota(q)}\mathsf{G}_0 \oplus T^t_{\iota(q)}\mathsf{G}, \tag{6}$$

where $T^t\mathsf{G} = \ker(T\tau_t)$ is a sub-bundle of $T\mathsf{G}$. Note that $T^t_\gamma\mathsf{G} = T_\gamma \tau_t^{-1}(\tau_t(\gamma))$. Hence $\mathfrak{G} \overset{\rightarrow TQ}{\rightarrow_Q}$ is isomorphic as a vector bundle to the restriction $\mathfrak{G}'$ of $T^t\mathsf{G}$ to G_0. Under this isomorphism the fiber $\mathfrak{G}_q$ above q is mapped to the vector space $T^t_{\iota(q)}\mathsf{G} = T_{\iota(q)}\tau_t^{-1}(q)$.

The following pleasant result was pointed out by Ramazan [25].

Proposition 1. *Every Lie groupoid possesses a left Haar system.*

Proof. A given strictly positive smooth density ρ on the vector bundle $\mathfrak{G}$ can be (uniquely) extended to a left-invariant density $\tilde{\rho}$ on the vector bundle $T^t\mathsf{G}$, which in turn yields a left Haar system by $\mu^t_q(f) = \int_{\tau_t^{-1}(q)} \tilde{\rho} f$. □

One may canonically associate a C^*-algebra $C^*(\mathsf{G})$ to a Lie groupoid $\mathsf{G} \overset{\leftarrow}{\Rightarrow} Q$ [2], and equally canonically associate a Poisson algebra $C^\infty(\mathfrak{G}^*)$ to its Lie algebroid $\mathfrak{G} \overset{\rightarrow TQ}{\rightarrow_Q}$ [4,3] (here $\mathfrak{G}^*$ is the dual vector bundle of $\mathfrak{G}$, with projection denoted by τ^*). From the point of view of quantization theory, these constructions go hand in hand [12, 13,15].

Although a left Haar system is not intrinsic, and an intrinsic definition of $C^*(\mathsf{G})$ may be given [2,15,25], it vastly simplifies the presentation of our results if we define this C^*-algebra relative to a particular choice of a left Haar system $\{\mu^t_q\}_{q\in Q}$. For $\mathsf{f}, \mathsf{g} \in C^\infty_c(\mathsf{G})$ the product $*$ in $C^*(\mathsf{G})$ is then given by the convolution [26]

$$\mathsf{f} * \mathsf{g}(\gamma) := \int_{\tau_t^{-1}(\tau_s(\gamma))} d\mu^t_{\tau_s(\gamma)}(\gamma_1)\, \mathsf{f}(\gamma\gamma_1)\mathsf{g}(\gamma_1^{-1}); \tag{7}$$

the involution is defined by

$$\mathsf{f}^*(\gamma) := \overline{\mathsf{f}(\gamma^{-1})}. \tag{8}$$

The groupoid C^*-algebra $C^*(\mathsf{G})$ is the completion of $C^\infty_c(\mathsf{G})$ in a suitable C^*-norm [2, 26,15].

On the classical side, the Poisson algebra $C^\infty(\mathfrak{G}^*)$ associated to a Lie algebroid $\mathfrak{G}$ [4,3,15] is most simply defined by listing special cases which uniquely determine the Poisson bracket. These are

$$\{f, g\} = 0; \tag{9}$$

$$\{\tilde{s}, f\} = -(\tau_a \circ s)f; \tag{10}$$

$$\{\tilde{s}_1, \tilde{s}_2\} = -\widetilde{[s_1, s_2]}_{\mathfrak{G}}. \tag{11}$$

Here $f, g \in C^\infty(Q)$ (regarded as functions on $\mathfrak{G}^*$ in the obvious way), and $\tilde{s} \in C^\infty(\mathfrak{G}^*)$ is defined by a section s of $\mathfrak{G}$ through $\tilde{s}(\theta) = \theta(s(\tau^*(\theta)))$, etc. See [3] for an intrinsic definition.

3. A Generalized Exponential Map

Throughout the remainder of the paper, $\mathfrak{G} \overset{\to TQ}{\to_Q}$ will be the Lie algebroid of a Lie groupoid $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$. In order to state and prove our main results we need to construct an exponential map $\mathrm{Exp}^W : \mathfrak{G} \to \mathsf{G}$, which generalizes the map Exp from a Lie algebra to an associated Lie group. The construction of such a map was outlined by Pradines [24], but in order to eventually satisfy the self-adjointness condition (2) on our quantization map we need a different construction [15]. As in [24], our exponential map depends on the choice of a connection on the vector bundle $\mathfrak{G}$. As before, the reader is referred to Sect. 6 for examples of the constructions below.

Lemma 1. *The vector bundles $T^t\mathsf{G}$ and $\tau_s^*\mathfrak{G}$ (over G) are isomorphic.*

Proof. The pull-back bundle $\tau_s^*\mathfrak{G}$ is a vector bundle over G with projection onto the second variable. The isomorphism is proved via the vector bundle isomorphism $\mathfrak{G} \simeq \mathfrak{G}'$; see Sect. 2. Recalling (5), one checks that $TL_{\gamma^{-1}} : T^t_\gamma\mathsf{G} \to T^t_{\gamma^{-1}\gamma}\mathsf{G}$ is the desired bundle isomorphism between $T^t\mathsf{G}$ and $\tau_s^*\mathfrak{G}'$. □

Let us now assume that $\mathfrak{G}$ has a covariant derivative (or, equivalently, a connection), with associated horizontal lift $\ell^{\mathfrak{G}}$. By Lemma 1 one then obtains a connection on $T^t\mathsf{G}$ (seen as a vector bundle over G, whose projection is borrowed from $T\mathsf{G}$) through pull-back. Going through the definitions, one finds that the associated horizontal lift ℓ of a tangent vector $X = \dot{\gamma} := d\gamma(t)/dt_{t=0}$ in $T_\gamma\mathsf{G}$ to $Y \in T^t_\gamma\mathsf{G}$ is

$$\ell_Y(\dot{\gamma}) = \frac{d}{dt}[L_{\gamma(t)*}\ell^{\mathfrak{G}}_{TL_{\gamma^{-1}}Y}(\tau_s(\gamma(t)))]_{t=0}, \tag{12}$$

which is an element of $T_Y(T^t\mathsf{G})$ (here $\ell^{\mathfrak{G}}(\dots)$ lifts a curve).

Since the bundle $T^t\mathsf{G} \to \mathsf{G}$ has a connection, one can define geodesic flow $X \to X(t)$ on $T^t\mathsf{G}$ in precisely the same way as on a tangent bundle with affine connection. That is, the flow $X(t)$ is the solution of

$$\dot{X}(t) = \ell_{X(t)}(X(t)), \tag{13}$$

with initial condition $X(0) = X$.

Definition 4. *Let the Lie algebroid $\mathfrak{G} \overset{\to TQ}{\to_Q}$ of a Lie groupoid $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$ be equipped with a connection. Relative to the latter, the left exponential map $\mathrm{Exp}^L : \mathfrak{G} \to \mathsf{G}$ is defined by*

$$\mathrm{Exp}^L(X) := \gamma_{X'}(1) = \tau_{T^t\mathsf{G}\to\mathsf{G}}(X'(1)), \tag{14}$$

whenever the geodesic flow $X'(t)$ on $T^t\mathsf{G}$ (defined by the connection on $T^t\mathsf{G}$ pulled back from the one on $\mathfrak{G}$) is defined at $t = 1$. Here $X' \in \mathfrak{G}' = T^t\mathsf{G} \restriction \mathsf{G}_0$ is the image of X under the isomorphism $\mathfrak{G}' \simeq \mathfrak{G}$.

Our goal, however, is to define a "symmetrized" version of Exp^L.

Lemma 2. *For all $X \in \mathfrak{G}$ for which $\mathrm{Exp}^L(X)$ is defined one has*

$$\tau_t(\mathrm{Exp}^L(X)) = \tau(X). \tag{15}$$

Here τ is the bundle projection of the Lie algebroid.

Proof. We write X for X' in (14). One has $\tau_t(\gamma_X(0)) = \tau(X)$ and

$$\frac{d}{dt}\tau_t(\gamma_X(t)) = T(\tau_t \circ \tau_{T^t\mathsf{G}\to\mathsf{G}})\ell_{X(t)}(X(t)) = T\tau_t X(t) = 0,$$

since $\ell_X(Y)$ covers Y, and $X(t) \in T^t\mathsf{G} = \ker(T\tau_t) \cap T\mathsf{G}$. □

We combine this with the obvious $\tau(\frac{1}{2}X) = \tau(-\frac{1}{2}X)$ to infer that

$$\tau_t(\mathrm{Exp}^L(\tfrac{1}{2}X)) = \tau_t(\mathrm{Exp}^L(-\tfrac{1}{2}X)) = \tau_s(\mathrm{Exp}^L(-\tfrac{1}{2}X)^{-1}).$$

Thus the (groupoid) multiplication in (16) below is well-defined.

Definition 5. *The Weyl exponential map $\mathrm{Exp}^W : \mathfrak{G} \to \mathsf{G}$ is defined by*

$$\mathrm{Exp}^W(X) := \mathrm{Exp}^L(-\tfrac{1}{2}X)^{-1}\mathrm{Exp}^L(\tfrac{1}{2}X). \tag{16}$$

The following result is closely related to the tubular neighbourhood theorem.

Proposition 2. *The maps Exp^L and Exp^W are diffeomorphisms from a neighbourhood $\mathcal{N}^\iota$ of $Q \subset \mathfrak{G}$ (as the zero section) to a neighbourhood $\mathcal{N}_\iota$ of $\iota(Q)$ in G, such that $\mathrm{Exp}^L(q) = \mathrm{Exp}^W(q) = \iota(q)$ for all $q \in Q$.*

Proof. The property $\mathrm{Exp}^L(q) = \iota(q)$ is immediate from Definition 4. The push-forward of Exp^L at q is $T\mathrm{Exp}^L : T_q\mathfrak{G} \to T_{\iota(q)}\mathsf{G}$. Now recall the decomposition (6). For X tangent to $Q \subset \mathfrak{G}$ one immediately sees that $T\mathrm{Exp}^L(X) = T\iota(X)$. For X tangent to the fiber $\tau^{-1}(q)$, which we identify with $T^t_{\iota(q)}\mathsf{G}$, one has $T\mathrm{Exp}^L(X) = X'$, as follows by the standard argument used to prove that $\exp_q$ in the theory of affine geodesics is a local diffeomorphism: for a curve $X(s) = sX$ in $T^t_{\iota(q)}\mathsf{G}$ one has $\mathrm{Exp}^L(X(s)) = \gamma_{X'(s)}(1) = \gamma_{X'}(s)$, so that $d/ds[\mathrm{Exp}^L(X(s))]_{s=0} = X'$.

Since $T\mathrm{Exp}^L$ is a bijection at q, the inverse function theorem implies that Exp^L is a local diffeomorphism. Since it maps Q pointwise to $\iota(Q)$, the local diffeomorphisms can be patched together to yield a diffeomorphism of the neighbourhoods stated in Proposition 2; we omit the details of this last step, since it is identical to the proof of the tubular neighbourhood theorem.

As for Exp^W, for $X \in T_qQ \subset T_q\mathfrak{G}$ we have $T\mathrm{Exp}^W(X) = T\iota(X)$. Also,

$$\frac{d}{ds}[\mathrm{Exp}^L(-\tfrac{1}{2}sX)^{-1}\mathrm{Exp}^L(\tfrac{1}{2}sX)]_{s=0} = -\tfrac{1}{2}TI(X') + \tfrac{1}{2}X',$$

where TI is the push-forward of the inversion I in G. The right-hand side lies in $\ker(T\tau_s + T\tau_t) \subset T\mathsf{G}$, and every element in this kernel is of the stated form. Similarly to (6), one may prove the decomposition

$$T_{\iota(q)}\mathsf{G} = T_{\iota(q)}\mathsf{G}_0 \oplus \ker(T\tau_s + T\tau_t)(\iota(q)). \tag{17}$$

It follows that $T\mathrm{Exp}^W$ is a bijection at q, and the second part of the theorem is derived as for Exp^L. □

4. The Normal Groupoid and Continuous Fields of C^*-Algebras

We now come to the first part of the proof of the conjecture that $C^*(\mathsf{G})$ is related to the Poisson manifold $\mathfrak{G}^*$ by a strict quantization.

Theorem 1. *Let* G *be a Lie groupoid, with associated Lie algebroid* $\mathfrak{G}$. *Take* $I = [0, 1]$ *and put* $\mathfrak{A}_0 = C_0(\mathfrak{G}^*)$, *where* $\mathfrak{G}^*$ *is the dual vector bundle of* $\mathfrak{G}$, *and* $\mathfrak{A}_\hbar = C^*(\mathsf{G})$ *for* $\hbar \in I\backslash\{0\}$.

There exists a C^**-algebra* $\mathfrak{C}$ *and a family of surjective* **-homomorphisms* $\{\varphi_\hbar : \mathfrak{C} \to \mathfrak{A}_\hbar\}_{\hbar\in I}$ *such that* $(\mathfrak{C}, \{\mathfrak{A}_\hbar, \varphi_\hbar\}_{\hbar\in I})$ *is a continuous field of* C^**-algebras.*

The proof uses the normal groupoid of Hilsum and Skandalis [9] (also cf. [33,15]), re-interpreted in terms of the Lie algebroid. We recall the definition; our construction of the smooth structure is different from the one in [9]. The essence is to regard the vector bundle $\mathfrak{G}$ as a Lie groupoid under addition in each fiber, and glue it to G so as to obtain a new Lie groupoid containing both G and $\mathfrak{G}$.

Definition 6. *Let* $\mathsf{G} \overset{\leftarrow}{\rightrightarrows} Q$ *be a Lie groupoid with associated Lie algebroid* $\mathfrak{G} \overset{\to}{\to}{}^{TQ}_{Q}$. *The normal groupoid* G_N *is a Lie groupoid with base* $[0, 1] \times Q$, *defined by the following structures:*

- *As a set,* $\mathsf{G}_N = \mathfrak{G} \cup \{(0, 1] \times \mathsf{G}\}$. *We write elements of* G_N *as pairs* $(\hbar, u)$, *where* $u \in \mathfrak{G}$ *for* $\hbar = 0$ *and* $u \in \mathsf{G}$ *for* $\hbar \neq 0$. *Thus* $\mathfrak{G}$ *is identified with* $\{0\} \times \mathfrak{G}$.
- *As a groupoid,* $\mathsf{G}_N = \{0 \times \mathfrak{G}\} \cup \{(0, 1] \times \mathsf{G}\}$. *Here* $\mathfrak{G}$ *is regarded as a Lie groupoid over* Q, *with* $\tau_s = \tau_t = \tau$ *and addition in the fibers as the groupoid multiplication. The groupoid operations in* $(0, 1] \times \mathsf{G}$ *are those in* G.
- *The smooth structure on* G_N, *making it a manifold with boundary, is as follows. To start, the open subset* $\mathcal{O}_1 := (0, 1] \times \mathsf{G} \subset \mathsf{G}_N$ *inherits the product manifold structure. Let* $Q \subset \mathcal{N}^\iota \subset \mathfrak{G}$ *and* $\iota(Q) \subset \mathcal{N}_\iota \subset \mathsf{G}$, *as in Theorem 2. Let* $\mathcal{O}$ *be the open subset of* $[0, 1] \times \mathfrak{G}$ *(equipped with the product manifold structure; this is a manifold with boundary, since* $[0, 1]$ *is), defined as* $\mathcal{O} := \{(\hbar, X) \mid \hbar X \in \mathcal{N}^\iota\}$. *Note that* $\{0\} \times \mathfrak{G} \subset \mathcal{O}$. *The map* $\rho : \mathcal{O} \to \mathsf{G}_N$ *is defined by*

$$\rho(0, X) := (0, X);$$
$$\rho(\hbar, X) := (\hbar, \mathrm{Exp}^W(\hbar X)). \tag{18}$$

Since $\mathrm{Exp}^W : \mathcal{N}^\iota \to \mathcal{N}_\iota$ *is a diffeomorphism (cf. Proposition 2) we see that* ρ *is a bijection from* $\mathcal{O}$ *to* $\mathcal{O}_2 := \{0 \times \mathfrak{G}\} \cup \{(0, 1] \times \mathcal{N}_\iota\}$. *This defines the smooth structure on* $\mathcal{O}_2$ *in terms of the smooth structure on* $\mathcal{O}$. *Since* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *cover* G_N, *this specifies the smooth structure on* G_N.

The fact that G_N is a Lie groupoid eventually follows from the corresponding property of G. The given chart is defined in terms of the Weyl exponential, which depends on the choice of a connection in $\mathfrak{G}$. However, one may verify that any (smooth) connection, or, indeed, any (Q-preserving) diffeomorphisms between $\mathcal{N}^\iota$ and $\mathcal{N}_\iota$ leads to an equivalent smooth structure on G_N. For example, we could have used Exp^L instead of Exp^W. Also, the smoothness of Exp^W makes the above manifold structure on G_N well defined, in that open subsets of $\mathcal{O}_1 \cap \mathcal{O}_2$ are assigned the same smooth structure.

Since G_N is a Lie groupoid, we can form the C^*-algebra $C^*(\mathsf{G}_N)$, which plays the role of $\mathfrak{C}$ in Theorem 1. To proceed, we need a result due to Lee [16].

Lemma 3. *Let $\mathfrak{C}$ be a C^*-algebra, and let $\psi : \text{Prim}(\mathfrak{C}) \to X$ be a continuous and open map from the primitive spectrum* Prim($\mathfrak{C}$) *(equipped with the Jacobson topology [5]) to a locally compact Hausdorff space X. Define $\mathfrak{I}_x := \cap\psi^{-1}(x)$; i.e., $A \in \mathfrak{I}_x$ iff $\pi_{\mathfrak{I}}(A) = 0$ for all $\mathfrak{I} \in \psi^{-1}(x)$ (here $\pi_{\mathfrak{I}}(\mathfrak{C})$ is the irreducible representation whose kernel is $\mathfrak{I}$). Note that $\mathfrak{I}_x$ is a (closed two-sided) ideal in $\mathfrak{C}$.*
Taking $\mathfrak{A}_x = \mathfrak{C}/\mathfrak{I}_x$ and $\varphi_x : \mathfrak{C} \to \mathfrak{A}_x$ to be the canonical projection, $(\mathfrak{C}, \{\mathfrak{A}_x, \varphi_x\}_{x\in X})$ is a continuous field of C^*-algebras.

For the proof cf. [6]. We apply this lemma with $\mathfrak{C} = C^*(\mathsf{G}_N)$ and $X = I = [0, 1]$. In order to verify the assumption in the lemma, we first note that $\mathfrak{I}_0 \simeq C_0((0, 1]) \otimes C^*(\mathsf{G})$, as follows from a glance at the topology of G_N. Hence $\text{Prim}(\mathfrak{I}_0) = (0, 1] \times \text{Prim}(C^*(\mathsf{G}))$, with the product topology. Furthermore, one has $C^*(\mathsf{G}_N)/\mathfrak{I}_0 \simeq C^*(\mathfrak{G}) \simeq C_0(\mathfrak{G}^*)$; the second isomorphism is established by the fiberwise Fourier transform (20) below (also cf. [9,2]). Hence $\text{Prim}(C^*(\mathsf{G}_N)/\mathfrak{I}_0) \simeq \mathfrak{G}^*$. Using this in Prop. 3.2.1 in [5], with $A = C^*_r(\mathsf{G}_N)$ and I the ideal $\mathfrak{I}_0$ generated by those $f \in C^\infty_c(\mathsf{G}_N)$ which vanish at $\hbar = 0$, yields the decomposition

$$\text{Prim}(C^*(\mathsf{G}_N)) \simeq \mathfrak{G}^* \cup \{(0, 1] \times \text{Prim}(C^*(\mathsf{G}))\}, \tag{19}$$

in which $\mathfrak{G}^*$ is closed. This does not provide the full topology on $\text{Prim}(C^*(\mathsf{G}_N))$, but it is sufficient to know that $\mathfrak{G}^*$ is not open. If it were, $(0, 1] \times \text{Prim}(C^*(\mathsf{G}))$ would be closed in $\text{Prim}(C^*(\mathsf{G}_N))$, and this possibility can safely be excluded by looking at the topology of G_N and the definition of the Jacobson topology.

Using (19), we can define a map $\psi : \text{Prim}(C^*(\mathsf{G}_N)) \to [0, 1]$ by $\psi(\mathfrak{I}) = 0$ for all $\mathfrak{I} \in \mathfrak{G}^*$ and $\psi(\hbar, \mathfrak{I}) = \hbar$ for $\hbar \neq 0$ and $\mathfrak{I} \in \text{Prim}(C^*(\mathsf{G}))$. It is clear from the preceding considerations that ψ is continuous and open. Using this in Lemma 3, one sees that $\mathfrak{I}_\hbar$ is the ideal in $C^*(\mathsf{G}_N)$ generated by those $f \in C^\infty_c(\mathsf{G}_P)$ which vanish at $\hbar$. Hence $\mathfrak{A}_0 \simeq C_0(\mathfrak{G}^*)$, as above, and $\mathfrak{A}_\hbar \simeq C^*(\mathsf{G})$ for $\hbar \neq 0$. Theorem 1 then follows from Lemma 3.

As pointed out to the author by G. Skandalis (private communication, June 1997), similar considerations lead to the following generalization of Theorem 1.

Let $\tilde{\mathsf{G}}$ be a Lie groupoid with base $\tilde{Q}$, and let p be a continuous and open map from $\tilde{Q}$ to some Hausdorff space X, which is $\tilde{\mathsf{G}}$-invariant in the sense that $p \circ \tau_s = p \circ \tau_t$. Define $\tilde{\mathsf{G}}_x := (p \circ \tau_s)^{-1}(x)$ (this is a sub-groupoid of $\tilde{\mathsf{G}}$ because of the $\tilde{\mathsf{G}}$-invariance of p), and $\mathfrak{A}^x := C^*(\tilde{\mathsf{G}}_x)$. Then the collection $(\{\mathfrak{A}^x\}_{x\in X}, C^*(\tilde{G}))$ is a continuous field of C^*-algebras at those points x where $C^*(\tilde{\mathsf{G}}_x) = C^*_r(\tilde{\mathsf{G}}_x)$. Here $f \in C^*(\tilde{G})$ is understood to define a section of the field $\{\mathfrak{A}^x\}_{x\in X}$ by $f(x) = f \restriction \tilde{\mathsf{G}}_x$.

We apply this to our situation by taking $\tilde{\mathsf{G}} = \mathsf{G}_N$ and $X = I$, hence $\tilde{Q} = I \times Q$, and p is just projection onto the first variable. Continuity away from $\hbar = 0$ follows from the triviality of the field for $\hbar \neq 0$ (whether or not $C^*_r(\mathsf{G}) = C^*(\mathsf{G})$). Continuity at $\hbar = 0$ follows by noticing that $C^*_r(\mathfrak{G}) = C^*(\mathfrak{G})$, both sides being equal to $C_0(\mathfrak{G}^*)$. In other words, from this point of view it is the amenability of $\mathfrak{G}$, regarded as a Lie groupoid, that lies behind Theorem 1.

5. Weyl Quantization on the Dual of a Lie Algebroid

Let $\mathfrak{G} \overset{\tau_Q}{\to} Q$ be a Lie algebroid, with bundle projection τ. We start by defining a fiberwise Fourier transform $\grave{f} \in C^\infty(\mathfrak{G})$ of suitable $f \in C^\infty(\mathfrak{G}^*)$. This transform depends on the choice of a family $\{\mu^L_q\}_{q\in Q}$ of Lebesgue measures, where μ^L_q is defined on the fiber

$\tau^{-1}(q)$. We will discuss the normalization of each μ_q^L in the proof of Theorem 2; for the moment we merely assume that the q-dependence is smooth in the obvious (weak) sense. For a function $\grave{f}$ on $\mathfrak{G}$ which is L^1 on each fiber we put

$$f(\theta) := \int_{\tau^{-1}(q)} d\mu_q^L(X)\, e^{-i\theta(X)}\, \grave{f}(X), \tag{20}$$

where $X \in \tau^{-1}(q)$. Each μ_q^L determines a Lebesgue measure μ_q^{L*} on the fiber $\tau^{-1}_{\mathfrak{G}^*\to Q}(q)$ of $\mathfrak{G}^*$ by fixing the normalization in requiring that the inverse to (20) is given by

$$\grave{f}(X) = \int_{\tau^{-1}_{\mathfrak{G}^*\to Q}(q)} d\mu_q^{L*}(\theta)\, e^{i\theta(X)} f(\theta). \tag{21}$$

Having constructed a Fourier transform, we define the class $C^\infty_{\text{PW}}(\mathfrak{G}^*)$ as consisting of those smooth functions on $\mathfrak{G}^*$ whose Fourier transform is in $C^\infty_c(\mathfrak{G})$; this generalizes the class of Paley-Wiener functions on $T^*\mathbb{R}^n \simeq \mathbb{C}^n$. We pick a function $\kappa \in C^\infty(\mathfrak{G}, \mathbb{R})$ with support in $\mathcal{N}^\iota$ (cf. Proposition 2), equaling unity in some smaller tubular neighbourhood of Q, as well as satisfying $\kappa(-X) = \kappa(X)$ for all $X \in \mathfrak{G}$.

Definition 7. *Let* G *be a Lie groupoid with Lie algebroid* $\mathfrak{G}$. *For* $\hbar \neq 0$, *the Weyl quantization of* $f \in C^\infty_{\text{PW}}(\mathfrak{G}^*)$ *is the element* $\mathcal{Q}^W_\hbar(f) \in C^\infty_c(\mathsf{G})$, *regarded as a dense subalgebra of* $C^*(\mathsf{G})$, *defined by* $\mathcal{Q}^W_\hbar(f)(\gamma) := 0$ *when* $\gamma \notin \mathcal{N}_\iota$, *and by*

$$\mathcal{Q}^W_\hbar(f)(\text{Exp}^W(X)) := \hbar^{-n}\kappa(X)\grave{f}(X/\hbar). \tag{22}$$

Here the Weyl exponential $\text{Exp}^W : \mathfrak{G} \to \mathsf{G}$ *is defined in (16), and the cutoff function* κ *is as specified above.*

This definition is possible by virtue of Proposition 2. By our choice of $C^\infty_{\text{PW}}(\mathfrak{G}^*)$, the operator $\mathcal{Q}^W_\hbar(f)$ is independent of κ for small enough $\hbar$ (depending on f).

Theorem 2. *Let* G *be a Lie groupoid with Lie algebroid* $\mathfrak{G} \overset{\to}{\to}{}^{TQ}_{Q}$, *and take* $\tilde{\mathfrak{A}}_0 = C^\infty_{\text{PW}}(\mathfrak{G}^*)$. *For each* $f \in \tilde{\mathfrak{A}}_0$ *operator* $\mathcal{Q}^W_\hbar(f)$ *of Definition 7 satisfies* $\mathcal{Q}^W_\hbar(f)^* = \mathcal{Q}^W_\hbar(f^*)$, *and the family* $\{\mathcal{Q}^W_\hbar(f)\}_{\hbar\in[0,1]}$, *with* $\mathcal{Q}^W_0(f) = f$, *is a continuous cross-section of the continuous field of* C^**-algebras of Theorem 1.*

Proof. Writing the Poisson bracket and the pointwise product in terms of the Fourier transform, one quickly establishes that $\tilde{\mathfrak{A}}_0$ is indeed a Poisson algebra.

It is immediate from (8) and (16) that for real-valued $f \in \tilde{\mathfrak{A}}_0$ the operator $\mathcal{Q}^W_\hbar(f)$ is self-adjoint in $C^*(\mathsf{G})$; this implies the first claim.

To prove the second claim, we pick a left Haar system $\{\mu^t_q\}_{q\in Q}$ on $\mathsf{G} \overset{\leftarrow}{\Rightarrow} Q$; see Proposition 1. The vector bundle $\mathfrak{G}$, regarded as a Lie groupoid under addition in each fiber (cf. Definition 6), has a left Haar system in any case, consisting of the family $\{\mu^L_q\}_{q\in Q}$ of Lebesgue measures on each fiber already used in the construction of the Fourier transform. Since we have a Lie groupoid, the Radon-Nikodym derivative $J_q(X) := d\mu^t_q(\text{Exp}^W(X))/d\mu^L_q(X)$ is well defined and strictly positive on $\mathcal{N}^\iota$ (since both measures are locally Lebesgue on spaces with the same dimension). We now fix the normalization of the μ_q^L by requiring that $\lim_{X\to 0} J_q(X) = 1$ for all q. This leads to a left Haar system for G_N, given by

$$\mu^t_{(0,q)} := \mu_q^L;$$
$$\mu^t_{(\hbar,q)} := \hbar^{-n}\mu_q^t, \tag{23}$$

where n is the dimension of the typical fiber of $\mathfrak{G}$. The factor $\hbar^{-n}$ is necessary in order to satisfy condition 3 in Definition 3 at $\hbar = 0$, as is easily verified using the manifold structure on G_N.

Thus the *-algebraic structure on $C_c^\infty(\mathsf{G}_N)$ defined by (7) and (8) with Definition 6 and (23) becomes

$$\mathsf{f} * \mathsf{g}(0, X) = \int_{\tau^{-1}\circ\tau(X)} d\mu^L_{\tau(X)}(Y)\, \mathsf{f}(0, X - Y)\mathsf{g}(0, Y); \tag{24}$$

$$\mathsf{f} * \mathsf{g}(\hbar, \gamma) = \hbar^{-n} \int_{\tau_t^{-1}(\tau_s(\gamma))} d\mu^t_{\tau_s(\gamma)}(\gamma_1)\, \mathsf{f}(\hbar, \gamma\gamma_1)\mathsf{g}(\hbar, \gamma_1^{-1}); \tag{25}$$

$$\mathsf{f}^*(0, X) = \overline{\mathsf{f}(0, -X)}; \tag{26}$$

$$\mathsf{f}^*(\hbar, \gamma) = \overline{\mathsf{f}(\hbar, \gamma^{-1})}. \tag{27}$$

One sees that, for given $f \in C^\infty_{\mathrm{PW}}(\mathfrak{G}^*)$, the function $\mathcal{Q}(f)$ on G_N defined by $\mathcal{Q}(f)(0, X) = \grave{f}(X)$, $\mathcal{Q}(f)(\hbar, \mathrm{Exp}^W(X)) = \kappa(X)\grave{f}(X/\hbar)$, and $\mathcal{Q}(f)(\hbar, \gamma) = 0$ for $\gamma \notin \mathcal{N}_\iota$, is smooth on G_N; cf. Definition 6. In other words, $\mathcal{Q}(f)$ is an element of $C^*(\mathsf{G}_N)$.

Recall that $\mathfrak{I}_\hbar$ is the ideal in $C^*(\mathsf{G}_N)$ generated by those functions in $C_c^\infty(\mathsf{G}_N)$ which vanish at $\hbar$. The canonical map $\mathsf{f} \to [\mathsf{f}]_\hbar$ from $C^*(\mathsf{G}_N)$ to $C_r^*(\mathsf{G}_N)/\mathfrak{I}_\hbar$ is given, for $\hbar \neq 0$, by $[\mathsf{f}]_\hbar(\cdot) = \mathsf{f}(\hbar, \cdot)$. However, in view of the factor $\hbar^{-n}$ in (25), this map is only a *-homomorphism from $C^*(\mathsf{G}_N)$ to $C^*(\mathsf{G})$ if we add a factor $\hbar^{-n}$ to the definition (7) of convolution on G. Since for $\hbar \neq 0$ we would like to identify $C^*(\mathsf{G}_N)/\mathfrak{I}_\hbar$ with $C^*(\mathsf{G})$, in which convolution is defined in the usual, $\hbar$-independent way, we should therefore define the maps $\varphi_\hbar$ of Theorem 1 by

$$\varphi_0(\mathsf{f}) : \theta \mapsto \acute{\mathsf{f}}(0, \theta);$$
$$\varphi_\hbar(\mathsf{f}) : \gamma \mapsto \hbar^{-n}\mathsf{f}(\hbar, \gamma) \quad (\hbar \neq 0). \tag{28}$$

Here $\varphi_0 : C^*(\mathsf{G}_N) \to C_0(\mathfrak{G}^*)$, and $\acute{\mathsf{f}}(0, \theta)$ and $\mathsf{f}(0, X)$ are related as $f(\theta)$ and $\grave{f}(X)$ are in (20). For $\hbar \neq 0$ one of course has $\varphi_\hbar : C^*(\mathsf{G}_N) \to C^*(\mathsf{G})$. These expressions are initially defined for $\mathsf{f} \in C_c^\infty(\mathsf{G}_N)$; since $\varphi_\hbar$ is contractive, they are subsequently extended to general $\mathsf{f} \in C^*(\mathsf{G}_N)$ by continuity.

This explains the factor $\hbar^{-n}$ in (22); the theorem then follows from the paragraph after (27). □

The important calculations of Ramazan [25] show that

$$\lim_{\hbar\to 0} \|\frac{i}{\hbar}[\mathcal{Q}^W_\hbar(f), \mathcal{Q}^W_\hbar(g)] - \mathcal{Q}^W_\hbar(\{f, g\})\| = 0 \tag{29}$$

for all $f, g \in \tilde{\mathfrak{A}}_0$; this is Dirac's condition (he in addition proves this to hold in formal deformation quantization).

Corollary 1. *Let* G *be a Lie groupoid, with associated*

- *Lie algebroid* $\mathfrak{G} \overset{\to TQ}{\to_Q}$;
- *Poisson manifold* $\mathfrak{G}^*$ *(the dual bundle to* $\mathfrak{G}$*, with Poisson structure (9)–(11));*
- *normal groupoid* G_N *(cf. Definition 6).*

In the context of Definition 2, the ingredients listed below yield a strict quantization of the Poisson manifold $P = \mathfrak{G}^*$:

1. *The continuous field of* C^**-algebras given by* $\mathfrak{C} = C^*(\mathsf{G}_N)$, $\mathfrak{A}_0 = C_0(\mathfrak{G}^*)$, $\mathfrak{A}_\hbar = C^*(\mathsf{G})$ *for* $\hbar \in I\backslash\{0\}$*, and* $\varphi_\hbar$ *as defined in (28); cf. Theorem 1.*
2. *The dense subspace* $\tilde{\mathfrak{A}}_0 = C^\infty_{PW}(\mathfrak{G}^*)$ *of fiberwise Paley–Wiener functions on* $\mathfrak{G}^*$ *(as defined below (21)).*
3. *The map* $\mathcal{Q} : C^\infty_{PW}(\mathfrak{G}^*) \to C^*(\mathsf{G}_N)$ *is defined by putting* $\mathcal{Q}_\hbar = \mathcal{Q}^W_\hbar$ *(as specified in Definition 7); this determines* $\mathcal{Q}$ *by Theorem 2 and the remark after Definition 1.*

6. Examples

In this section we illustrate the concepts introduced above, and show that a number of known strict quantizations are special cases of Corollary 1. Details of these examples will be omitted; see [17,3,15,1] for matters related to the Lie groupoids and Lie algebroids involved, and cf. [2,26,15,25] for the C^*-algebras that appear. The quantization maps are discussed in detail in [15].

It turns out that a number of examples are more naturally described by changing some signs, as follows. We denote $\mathfrak{G}^*$, seen as a Poisson manifold through (9)–(11), by $\mathfrak{G}^*_-$. Alternatively, we may insert plus signs on the right-hand sides of (10) and (11), defining the Poisson manifold $\mathfrak{G}^*_+$. The normal groupoid G_N may be equipped with a different manifold structure by replacing $\mathrm{Exp}^W(\hbar X)$ in (18) by $\mathrm{Exp}^W(-\hbar X)$; the original Definition 6 yields a manifold G^+_N, and the modified one defines G^-_N. (The original smooth structure is equivalent to the modified one by the diffeomorphism $(0, X) \mapsto (0, -X)$ and $(\hbar, \gamma) \mapsto (\hbar, \gamma)$.) In (22) we may replace $\grave{f}(X/\hbar)$ by $\grave{f}(-X/\hbar)$, defining a quantization map $\mathcal{Q}^W_\hbar(\cdot)_-$, differing from the original one $\mathcal{Q}^W_\hbar(\cdot)_+ = \mathcal{Q}^W_\hbar(\cdot)$.

Theorems 1 and 2, Eq. (29), as well as Corollary 1 remain valid if all signs are simultaneously changed in this way.

Example 1 (Weyl quantization on a manifold). The pair groupoid $Q \times Q \overset{\leftarrow}{\Rightarrow} Q$ on a set Q is defined by the operations $\tau_s(q_1, q_2) := q_2$, $\tau_t(q_1, q_2) := q_1$, $\iota(q) := (q, q)$, $(q_1, q_2) \cdot (q_2, q_3) := (q_1, q_3)$, and $(q_1, q_2)^{-1} := (q_2, q_1)$. This is a Lie groupoid when Q is a manifold. Any measure ν on Q which is locally Lebesgue defines a left Haar system. One has $C^*(Q \times Q) \simeq \mathfrak{B}_0(L^2(Q))$, the C^*-algebra of all compact operators on $L^2(Q, \nu)$.

The associated Lie algebroid is the tangent bundle TQ, with the usual bundle projection and Lie bracket, and the anchor is the identity. The Poisson bracket on T^*Q is the canonical one.

To define Exp^W one chooses an affine connection ∇ on TQ, with associated exponential map $\exp : TQ \to Q$. Then

$$\mathrm{Exp}^L(X) = (\tau(X), \exp_{\tau(X)}(X)); \tag{30}$$

$$\mathrm{Exp}^W(X) = (\exp_{\tau(X)}(-\tfrac{1}{2}X), \exp_{\tau(X)}(\tfrac{1}{2}X)), \tag{31}$$

where $X \in TQ$ and $\tau := \tau_{TQ\to Q}$.

On $Q = \mathbb{R}^n$ with flat metric and corresponding flat Riemannian connection this simplifies to $\mathrm{Exp}^W(v, q) = (q - \frac{1}{2}v, q + \frac{1}{2}v)$, where we have used canonical coordinates on $T\mathbb{R}^n$. The operator $\mathcal{Q}_\hbar^W(f)_-$ on $L^2(\mathbb{R}^n)$ defined by (22), where one may take $\kappa = 1$, with (21), is then given by

$$\mathcal{Q}_\hbar^W(f)_-\Psi(x) = \int_{T^*\mathbb{R}^n} \frac{d^n p d^n y}{(2\pi\hbar)^n}\, e^{ip(x-y)/\hbar} f(p, \tfrac{1}{2}(x+y))\Psi(y). \tag{32}$$

This is Weyl's original prescription. The associated continuous field of C^*-algebras is $\mathfrak{A}_0 = C_0(T^*\mathbb{R}^n)$ and $\mathfrak{A}_\hbar = \mathfrak{B}_0(L^2(\mathbb{R}^n))$ for $\hbar \neq 0$. The fact that this quantization map is strict, and in particular satisfies (3), was proved by Rieffel [29]; also cf. [15]. Replacing $I = [0, 1]$, as we have used so far in connection with Definition 2, by $I = \mathbb{R}$, the C^*-algebra $\mathfrak{C}$ in Definition 1 is $C^*(H_n)$, the group algebra of the simply connected Heisenberg group on $\mathbb{R}^n$ [6]. This is indeed the C^*-algebra of the tangent groupoid of $\mathbb{R}^n$ (see below).

When Q is an arbitrary manifold, the normal groupoid $(Q \times Q)_N$ is the tangent groupoid of Q [2]. If one takes the affine connection on TQ to be the Levi-Civita connection given by a Riemannian metric on Q, one recovers the extension of Weyl's prescription considered in [12,15]. One now has $\mathfrak{A}_0 = C_0(T^*Q)$ and $\mathfrak{A}_\hbar = \mathfrak{B}_0(L^2(Q))$ for $\hbar \neq 0$, and $\mathcal{Q}_\hbar^W$ duly satisfies (3); see [12,15], where references to alternative generalizations of Weyl's quantization prescriptions may be found.

Example 2 (Rieffel's quantization of the Lie–Poisson structure on a dual Lie algebra). A Lie group is a Lie groupoid with $Q = e$. A left-invariant Haar measure on G provides a left Haar system; the ensuing convolution algebra $C^*(G)$ is the usual group algebra. The Lie algebroid is the Lie algebra. The Poisson structure on $\mathfrak{g}^*_\pm$ is the well-known Lie–Poisson structure [18,15].

No connection is needed to define the exponential map, and one has

$$\mathrm{Exp}^L(X) = \mathrm{Exp}^W(X) = \mathrm{Exp}(X), \tag{33}$$

where $X \in \mathfrak{g}$ and $\mathrm{Exp} : \mathfrak{g} \to G$ is the usual exponential map. When G is exponential (in that Exp is a diffeomorphism), one may omit κ in (22). Taking the $+$ sign, the function $\mathcal{Q}_\hbar^W(f)_+ \in C^*(G)$ is then given by

$$\mathcal{Q}_\hbar^W(f)_+ : \mathrm{Exp}(X) \to \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\langle\theta, X\rangle/\hbar} f(\theta). \tag{34}$$

This is Rieffel's prescription [28], who proved strictness of the quantization for nilpotent groups. When G is compact one needs the cut-off function κ, obtaining another quantization already known to be strict before the present paper and [25] appeared; see [14] or [15].

Example 3 (Weyl quantization on a gauge groupoid). The gauge groupoid $\mathsf{P} \times_H \mathsf{P} \overset{\leftarrow}{\Rightarrow} Q$ of a smooth principal bundle P over a base Q with structure group H is defined by the projections $\tau_s([x, y]_H) = \tau(y)$ and $\tau_t([x, y]_H) = \tau(x)$, and the inclusion $\iota(\tau(x)) = [x, x]_H$. Accordingly, the multiplication $[x, y]_H \cdot [x', y']_H$ is defined when y and x' lie in the same fiber of P, in which case $[x', y']_H = [y, z]_H$ for some $z = y'h$, $h \in H$. Then $[x, y]_H \cdot [y, z]_H = [x, z]_H$. Finally, the inverse is $[x, y]_H^{-1} = [y, x]_H$. See [17].

An H-invariant measure μ on P which is locally Lebesgue produces a left Haar system. In general, each measurable section $s : Q \to \mathsf{P}$ determines an isomorphism $C^*(\mathsf{P} \times_H \mathsf{P}) \simeq \mathfrak{B}_0(L^2(Q)) \otimes C^*(H)$; this is a special case of Thm. 3.1 in [21] (also cf. [15], Thm. 3.7.1). When H is compact one has $C^*(\mathsf{P} \times_H \mathsf{P}) \simeq \mathfrak{B}_0(L^2(\mathsf{P}))^H$, where $L^2(\mathsf{P})$ is defined with respect to some H-invariant locally Lebesgue measure on P.

The associated Lie algebroid $(T\mathsf{P})/H \overset{\to}{\to}{}^{TQ}_{Q}$ is defined by the obvious projections (both inherited from the projection $\tau : \mathsf{P} \to Q$), the Lie bracket on $\Gamma((T\mathsf{P})/H)$ obtained by identifying this space with $\Gamma(T\mathsf{P})^H$, and borrowing the commutator from $\Gamma(T\mathsf{P})$; cf. [17]. The Poisson structure on $((T\mathsf{P})/H)^* = (T^*\mathsf{P})/H$ is given by the restriction of the canonical Poisson bracket on $C^\infty(T^*\mathsf{P})$ to $C^\infty(T^*\mathsf{P})^H$, under the isomorphism $C^\infty((T^*\mathsf{P})/H) \simeq C^\infty(T^*\mathsf{P})^H$.

One chooses an H-invariant affine connection on $T\mathsf{P}$, with exponential map exp : $T\mathsf{P} \to \mathsf{P}$. This induces a connection on $(T\mathsf{P})/H$, in terms of which

$$\mathrm{Exp}^L([X]_H) = [\tau(X), \exp_{\tau(X)}(X)]_H; \tag{35}$$

$$\mathrm{Exp}^W([X]_H) = [\exp_{\tau(X)}(-\tfrac{1}{2}X), \exp_{\tau(X)}(\tfrac{1}{2}X)]_H, \tag{36}$$

where $\tau = \tau_{T\mathsf{P}\to\mathsf{P}}$, and $[X]_H \in (T\mathsf{P})/H$ is the equivalence class of $X \in T\mathsf{P}$ under the H-action on $T\mathsf{P}$.

In the Riemannian case, for compact H the corresponding map $\mathcal{Q}_\hbar^W(\cdot)_-$ is simply the restriction of $\mathcal{Q}_\hbar^W(\cdot)_- : C^\infty_{\mathrm{PW}}(T^*\mathsf{P}) \to \mathfrak{B}_0(L^2(\mathsf{P}))$ as defined in Example 1 to $C^\infty_{\mathrm{PW}}(T^*\mathsf{P})^H$. Since $\mathcal{Q}_\hbar^W$ is invariant under isometries [15], the image of $C^\infty_{\mathrm{PW}}(T^*\mathsf{P})^H$ is contained in $\mathfrak{B}_0(L^2(\mathsf{P}))^H$. The ensuing quantization of $(T^*\mathsf{P})/H$ was already known to be strict; see [12,15]. Physically, this example describes the quantization of a nonabelian charged particle moving in a gravitational as well as a Yang–Mills field.

Example 4 (Transformation group C^*-algebras). Let a Lie group G act smoothly on a set Q. The transformation groupoid $G \times Q \overset{\leftarrow}{\Rightarrow} Q$ is defined by the operations $\tau_s(x, q) = x^{-1}q$ and $\tau_t(x, q) = q$, so that the product $(x, q) \cdot (y, q')$ is defined when $q' = x^{-1}q$. Then $(x, q) \cdot (y, x^{-1}q) = (xy, q)$. The inclusion is $\iota(q) = (e, q)$, and for the inverse one has $(x, q)^{-1} = (x^{-1}, x^{-1}q)$.

Each left-invariant Haar measure dx on G leads to a left Haar system. The corresponding groupoid C^*-algebra is the usual transformation group C^*-algebra $C^*(G, Q)$, cf. [26].

The Lie algebroid $\mathfrak{g} \times Q \overset{\to}{\to}{}^{TQ}_{Q}$ is a trivial bundle over Q, with anchor $\tau_a(X, q) = -\xi_X(q)$ (the fundamental vector field on Q defined by $X \in \mathfrak{g}$). Identifying sections of $\mathfrak{g} \times Q$ with $\mathfrak{g}$-valued functions $X(\cdot)$ on Q, the Lie bracket on $\Gamma(\mathfrak{g} \times Q)$ is

$$[X, Y]_{\mathfrak{g}\times Q}(q) = [X(q), Y(q)]_{\mathfrak{g}} + \xi_Y X(q) - \xi_X Y(q). \tag{37}$$

The associated Poisson bracket coincides with the semi-direct product bracket defined in [11].

The trivial connection on $\mathfrak{g} \times Q \to Q$ yields

$$\mathrm{Exp}^L(X, q) = (\mathrm{Exp}(X), q); \tag{38}$$

$$\mathrm{Exp}^W(X, q) = (\mathrm{Exp}(X), \mathrm{Exp}(\tfrac{1}{2}X)q). \tag{39}$$

The cutoff κ in (22) is independent of q, and coincides with the function appearing in Example 2. For small enough $\hbar$ a function $f \in C^\infty_{\mathrm{PW}}(\mathfrak{g}^* \times Q)$ is then quantized by

$$\mathcal{Q}^W_\hbar(f)_\pm : (\mathrm{Exp}(X), q) \to \int_{\mathfrak{g}^*} \frac{d^n\theta}{(2\pi\hbar)^n}\, e^{i\langle\theta, X\rangle/\hbar} f(\pm\theta, \mathrm{Exp}(-\tfrac{1}{2}X)q). \tag{40}$$

When $G = \mathbb{R}^n$ and Q has a G-invariant measure, the map $f \to \mathcal{Q}^W_\hbar(f)_\pm$ is equivalent to the deformation quantization considered by Rieffel [27], who already proved that it is strict (also cf. [15]).

Note added in proof. All results remain true when the groupoid C^*-algebras are replaced by reduced ones. This is clear both from the proof of Lemma 3 and from the argument at the end of Sect. 4 (which should be attributed to E. Blanchard).

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