Commun. Math. Phys. 222, 97 – 116 (2001)

Communications in
Mathematical Physics

Operator Algebras and Poisson Manifolds Associated to Groupoids*

N. P. Landsman**

Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands. E-mail: npl@science.uva.nl

Received: 6 December 2000 / Accepted: 19 April 2001

Dedicated to Dai Evans at his 50th birthday

Abstract: It is well known that a measured groupoid G defines a von Neumann algebra $W^*(G)$, and that a Lie groupoid G canonically defines both a C^*-algebra $C^*(G)$ and a Poisson manifold $A^*(G)$. We construct suitable categories of measured groupoids, Lie groupoids, von Neumann algebras, C^*-algebras, and Poisson manifolds, with the feature that in each case Morita equivalence comes down to isomorphism of objects. Subsequently, we show that the maps $G \mapsto W^*(G)$, $G \mapsto C^*(G)$, and $G \mapsto A^*(G)$ are functorial between the categories in question. It follows that these maps preserve Morita equivalence.

1. Introduction

Kontsevich has introduced the idea of the "three worlds", viz. commutative, Lie, and associative algebras, relating these worlds to each other and to "formal" noncommutative geometry [17]. In the context of noncommutative geometry in the sense of Connes [4], and in particular of its relationship with quantum theory and quantization, three other worlds are relevant, namely von Neumann algebras, C^*-algebras, and Poisson manifolds. Groupoids provide access to each of these.

Firstly, measured groupoids G [29,38,13,10,2,33] define von Neumann algebras $W^*(G)$ in standard form [5,14,43,45,40]. Secondly, Lie groupoids G [27] canonically define C^*-algebras $C^*(G)$ [3,4]. Thirdly, one may canonically associate a Poisson manifold $A^*(G)$ with a Lie groupoid G [6,7,9].

For the most basic examples of these associations, first note that a set S defines two entirely different groupoids. The first has S as the total space G_1, and also as the base space G_0 of G. If S is an analytic measure space (X, μ), this leads to $W^*(X) \cong L^\infty(X, \mu)$, and if S is a manifold M one obtains $C^*(M) \cong C_0(M)$, and $A^*(M) \cong M$

* The results in this paper were first presented in Cardiff on 10. 10. 2000.

** Supported by a Fellowship from the Royal Netherlands Academy of Arts and Sciences (KNAW).

with zero Poisson bracket. The second is the pair groupoid of S, with $G_1 = S \times S$ and $G_0 = S$. In that case one has $W^*(X \times X) \cong \mathfrak{B}(L^2(X, \mu))$, $C^*(M \times M) \cong \mathfrak{K}(L^2(M))$, and $A^*(M \times M) \cong T^*(M)$.

If the groupoid is a group, one recovers the usual von Neumann algebra and C^*-algebra defined by a locally compact group. The Poisson manifold defined by a Lie group is just the dual of the Lie algebra, equipped with the Lie–Poisson structure. Group actions define the associated action groupoids [27], which in turn reproduce the group measure space construction of Murray and von Neumann, the notion of a transformation group C^*-algebra, and the class of semidirect Poisson structures, respectively (for the latter cf. [19]). For example, in the ergodic case all hyperfinite factors arise in this way.

Finally, the von Neumann algebras and C^*-algebras defined by foliations [2–4, 33] may be seen as special cases of the above constructions as well, where G is the holonomy groupoid of a smooth foliation. This class of examples formed a major motivation for the development of noncommutative geometry.

For fixed G, there are certain relationships between these constructions. Under appropriate technical conditions, both measured and Lie groupoids may be seen as special instances of locally compact groupoids with Haar system [40]; see [39] and [23, 21], respectively. The von Neumann algebra $W^*(G)$ is then simply the weak closure of $C^*(G)$ in its regular representation. The connection between $A^*(G)$ and $C^*(G)$ is deeper: $C^*(G)$ is a strict deformation quantization of $A^*(G)$[21–23]. This means, among other things, that there exists a continuous field of C^*-algebras over $[0, 1]$, whose fiber above 0 is the commutative C^*-algebra $C_0(A^*(G))$, all other fibers being $C^*(G)$. The C^*-algebra of continuous cross-sections of this continuous field turns out to be the C^*-algebra of the normal groupoid [15] defined by the embedding $G_0 \hookrightarrow G_1$ of the unit space of G into its total space (Connes's tangent groupoid [4] corresponds to the special case of a pair groupoid $G = M \times M$).

In the present paper, we examine and compare the properties of the associations $G \mapsto W^*(G)$, $G \mapsto C^*(G)$ and $G \mapsto A^*(G)$ as a function of G. Our main result is that each of these maps is functorial, though not with respect to the obvious arrows defining the pertinent categories. The categories that are involved have the desirable property that isomorphism of objects is the same as Morita equivalence (as previously defined by Rieffel for von Neumann algebras and C^*-algebras [42] and by Xu for Poisson manifolds [54]), so that functoriality implies that Morita equivalence is preserved.

Often involving different terminology, for von Neumann algebras many special cases of the latter property have been known for some time, starting with Mackey's ergodic imprimitivity theorem [29, 38], and including results in [10, 18, 39, 49]. For C^*-algebras and Poisson manifolds the preservation of Morita equivalence was already known in full generality; see [36] and [24], respectively. Special cases of our functoriality results include also [34, 35, 48, 47]. We surmise that the computations in [15], taking place in the category KK of separable C^*-algebras as objects and KK-groups as arrows, can be generalized to arbitrary Lie groupoids; they should then be related to our results as well.

The plan of this paper is as follows. In Sect. 2 we deal with measured groupoids and von Neumann algebras, in Sect. 3 we treat Lie groupoids and C^*-algebras, and in Sect. 4 we end with Lie groupoids and Poisson manifolds. Our main results are Theorems 1, 2, and 3.

The reader will notice that the category of measured groupoids and the category of Lie groupoids are defined in an apparently totally different way. The fact that these categories are actually closely related is explained in [26], to which we refer in general for motivation and for more details about the categories we use here. This includes the proof that, as already mentioned, in each case recognized notions of Morita equivalence turn out to coincide with isomorphism of objects in the pertinent category.

Notation. We use the notation

$$A \times_B^{f,g} C = \{(a, c) \in A \times C \mid f(a) = g(c)\}$$

for the fiber product of sets A and C with respect to maps $f : A \to B$ and $g : C \to B$. The total space of a groupoid G is denoted by G_1, and its base space by G_0. The source and target projections are called $s : G_1 \to G_0$ and $t : G_1 \to G_0$, and multiplication is a map $m : G_2 \to G_1$, with $G_2 = G_1 \times_{G_0}^{s,t} G_1$. The inversion is denoted by $I : G_1 \to G_1$. A functor $\Phi : G \to H$ decomposes into $\Phi_i : G_i \to H_i$, $i = 1, 2$, subject to the usual axioms.

2. Functoriality of $G \mapsto W^*(G)$

2.1. The category MG *of measured groupoids and functors.* The concept of a measured groupoid emerged from the work of Mackey on ergodic theory and group representations [29]. For the technical development of this concept see [38, 13, 10]. A different approach was initiated by Connes [2]. The connection between measured groupoids and locally compact groupoids is laid out in [40, 39].

Definition 1. *A Borel groupoid is a groupoid G for which G_1 is an analytic Borel space, I is a Borel map, $G_2 \subset G_1 \times G_1$ is a Borel subset, and multiplication m is a Borel map. It follows that G_0 is a Borel set in G_1, and that s and t are Borel maps.*

A left Haar system on a Borel groupoid is a family of measures $\{\nu^u\}_{u \in G_0}$, where ν^u is supported on the t-fiber $G^u = t^{-1}(u)$, which is left-invariant in that

$$\int d\nu^{s(x)}(y)\, f(xy) = \int d\nu^{t(x)}(y)\, f(y) \tag{2.1}$$

for all $x \in G_1$ and all positive Borel functions f on G_1 for which both sides are finite.

A measured groupoid is a Borel groupoid equipped with a Haar system as well as a Borel measure $\tilde{\nu}$ on G_0 with the property that the measure class of the measure ν on G_1, defined by

$$\nu = \int_{G_0} d\tilde{\nu}(u)\, \nu^u, \tag{2.2}$$

is invariant under I (in other words, $\nu^{-1} = I(\nu) \sim \nu$).

Recall that the push-forward of a measure under a Borel map is given by $t(\nu)(E) = \nu(t^{-1}(E))$ for Borel sets $E \subset G_0$.

This definition turns out to be best suited for categorical considerations. It differs from the one in [38, 13], which is stated in terms of measure classes. However, the measure class of ν defines a measured groupoid in the sense of [38, 13], and, conversely, the latter is also a measured groupoid according to Definition 1 provided one removes a suitable null set from G_0, as well as the corresponding arrows in G_1; cf. Thm. 3.7 in [13]. Similarly, Definition 1 leads to a locally compact groupoid with Haar system [40] after removal of such a set; see Thm. 4.1 in [39]. A measured groupoid according to Connes [2] satisfies Definition 1 as well, with $\tilde{\nu}$ constructed from the Haar system and a transverse measure [33]. See all these references for extensive information and examples.

The fact that a specific choice of a measure in its class is made in Definition 1 is balanced by the concept of a measured functor between measured groupoids, which is entirely concerned with measure classes rather than individual measures. Moreover, one merely uses the measure class of $\tilde{\nu}$.

The measure $\tilde{\nu}$ on G_0 induces a measure $\hat{\nu}$ on G_0/G, as the push-forward of $\tilde{\nu}$ under the canonical projection, and similarly for a measured groupoid H, for whose measures we will use the symbol λ instead of ν. We say that a functor Φ is Borel if both Φ_0 and Φ_1 are. If so, Φ_0 induces a Borel map $\hat{\Phi}_0 : G_0/G \to H_0/H$ in the obvious way.

Definition 2. *A measured functor $\Phi : G \to H$ between two measured groupoids is a Borel map that is algebraically a functor and satisfies $\hat{\Phi}_0(\hat{\nu}) \prec\prec \hat{\lambda}$.*

What we here call a measured functor is called a strict homomorphism in [38], and a homomorphism in [39]. Also, note that in [29,38,10] various more liberal definitions are used (in that one does not impose that Φ be a functor algebraically at all points), but it is shown in [39] that if one passes to natural isomorphism classes, this greater liberty gains little.

Definition 3. *The category* MG *has measured groupoids as objects, and isomorphism classes of measured functors as arrows. (Here a natural transformation $\nu : G_0 \to H_1$ between Borel functors from G to H is required to be a Borel map.) Composition is defined by $[\Psi] \circ [\Phi] = [\Psi \circ \Phi]$, and the unit arrow at a groupoid G is $1_G = [\mathrm{id}_G]$, where $\mathrm{id}_G : G \to G$ is the identity functor.*

2.2. The category W^* *of von Neumann algebras and correspondences.* Let $\mathfrak{M}, \mathfrak{N}$ be von Neumann algebras. Recall that an $\mathfrak{M}$-$\mathfrak{N}$ correspondence $\mathfrak{M} \rightarrowtail \mathcal{H} \leftarrowtail \mathfrak{N}$ is given by a Hilbert space $\mathcal{H}$ carrying commuting normal unital representations of $\mathfrak{M}$ and $\mathfrak{N}^{\mathrm{op}}$. See [4]. The notion of isomorphism of correspondences is the obvious one: one requires a unitary isomorphism between the Hilbert spaces in question that intertwines the left and right actions.

Given two matched correspondences $\mathfrak{M} \rightarrowtail \mathcal{H} \leftarrowtail \mathfrak{N}$ and $\mathfrak{N} \rightarrowtail \mathcal{K} \leftarrowtail \mathfrak{P}$, one may define an $\mathfrak{M}$-$\mathfrak{P}$ correspondence $\mathfrak{M} \rightarrowtail \mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K} \leftarrowtail \mathfrak{P}$, called the relative tensor product or "Connes fusion" of the given correspondences. This construction is a von Neumann algebraic version of the bimodule tensor product in pure algebra. Various definitions exist [4,44,51], which coincide up to isomorphism. This composition is associative up to isomorphism. A standard representation of a von Neumann algebra $\mathfrak{M}$ on $\mathcal{H} = L^2(\mathfrak{M})$, unique up to unitary equivalence, is best seen as an $\mathfrak{M}$-$\mathfrak{M}$ correspondence with special properties. One of these is that $L^2(\mathfrak{M})$ acts as a two-sided unit for $\boxtimes_{\mathfrak{M}}$, again merely up to isomorphism.

Definition 4. *The category* W^* *has von Neumann algebras as objects, and isomorphism classes of correspondences as arrows, composed by the relative tensor product, for which the standard forms $L^2(\mathfrak{M})$ are units.*

To detail, we here regard an isomorphism class $[\mathfrak{M} \rightarrowtail \mathcal{H} \leftarrowtail \mathfrak{N}]$ as an arrow from $\mathfrak{M}$ to $\mathfrak{N}$, so that the composition is

$$[\mathfrak{N} \rightarrowtail \mathcal{K} \leftarrowtail \mathfrak{P}] \circ [\mathfrak{M} \rightarrowtail \mathcal{H} \leftarrowtail \mathfrak{N}] = [\mathfrak{M} \rightarrowtail \mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K} \leftarrowtail \mathfrak{P}].$$

Using results in [42,44], it is easily seen that two von Neumann algebras are Morita equivalent iff they are isomorphic in W* [26], and this is true iff there is a correspondence in which the commutant of one is isomorphic to the opposite algebra of the other, or iff they are stably isomorphic.

2.3. The map $G \mapsto W^*(G)$ as a functor. It is well known that a measured groupoid defines a von Neumann algebra in standard form [5,14,45,43,40]. In this section, we extend the map $G \mapsto W^*(G)$ to a map from MG to W*, and establish its functoriality. The precise classes of Borel function f, g on G_1 for which the formulae below are well defined are spelled out in the above papers; for example, one may assume that $f, g \in II(G_1)$ as defined in [14].

Let G be a measured groupoid (cf. Definition 1). Convolution on G is defined by

$$f * g(x) = \int_{G_1} d\nu^{s(x)}(y)\, f(xy) g(y^{-1}), \tag{2.3}$$

and involution is

$$f^*(x) = \overline{f(x^{-1})}. \tag{2.4}$$

We here use the conventions in [40]; many authors include the modular homomorphism $\Delta : G_1 \to \mathbb{R}_+$ in (2.4), defined by $\Delta(x) = d\nu(x)/d\nu^{-1}(x)$. We write $L^2(G)$ for $L^2(G_1, \nu)$. For $\psi \in L^2(G)$ the formulae

$$\pi_L(f)\psi = (\Delta^{-1/2} f) * \psi; \tag{2.5}$$

$$\pi_R(f)\psi = \psi * f \tag{2.6}$$

define the left and right regular representations of $II(G_1)$; one then has $W^*(G) = \pi_L(II(G_1))''$, which is in standard form with respect to $J : L^2(G) \to L^2(G)$ defined by

$$J\psi(x) = \Delta(x)^{-1/2} \psi^*(x). \tag{2.7}$$

One then has $JW^*(G)J = W(G)' = \pi_R(II(G_1))''$.

We have now defined the alleged functor $G \mapsto W^*(G)$ on objects. To define it on arrows, let H be a second measured groupoid H (with Haar system λ), and let $\Phi : G \to H$ be a measured functor (cf. Definition 2). Define a Hilbert space

$$L^2(\Phi) = L^2\left(G_0 \times_{H_0}^{\Phi_0, t} H_1, \int_{G_0} d\tilde{\nu}(u)\, \lambda^{\Phi_0(u)}\right). \tag{2.8}$$

Compare (2.2). Also, define $\pi_\lambda : II(G_1) \to \mathfrak{B}(L^2(\Phi))$ and $\pi_\rho : II(H_1) \to \mathfrak{B}(L^2(\Phi))$ by

$$\pi_\lambda(f)\varphi(u,h) = \int_{G_1} d\nu^u(y)\, \Delta(y)^{-1/2} f(y) \varphi(s(y), \Phi_1(y^{-1})h); \tag{2.9}$$

$$\pi_\rho(g)\varphi(u,h) = \int_{H_1} d\lambda^{s(h)}(l)\, g(l^{-1}) \varphi(u, hl). \tag{2.10}$$

These expressions extend to $f \in W^*(G)$ and $g \in W^*(H)$ by continuity, and it is easily seen that one thus defines a correspondence $W^*(G) \rightarrowtail L^2(\Phi) \leftarrowtail W^*(H)$.

Theorem 1. *The map $W^* : \mathsf{MG} \to \mathsf{W}^*$, defined on objects by $W^*_0(G) = W^*(G)$ as above, and on arrows (i.e., natural isomorphism classes of measured functors $\Phi : G \to H$) by*

$$W^*_1([\Phi]) = [W^*(G) \rightarrowtail L^2(\Phi) \leftarrowtail W^*(H)],$$

is a functor.

Proof. For $H = G$ and $\Phi = \mathrm{id}$ one easily sees that $L^2(\mathrm{id}) \cong L^2(G)$, $\pi_\lambda \cong \pi_L$, and $\pi_\rho \cong \pi_R$ (the $\cong$ here standing for unitary equivalence). Hence one obtains the standard form

$$W^*_1(\mathrm{id}) = [W^*(G) \rightarrowtail L^2(G) \leftarrowtail W^*(G)].$$

Since the unit arrows in W^* are precisely the standard forms, this shows that W^* maps units into units.

We now need to show that, for a third measured groupoid K and a measured functor $\Psi : H \to K$, one has

$$W^*(G) \rightarrowtail L^2(\Phi) \boxtimes_{W^*(H)} L^2(\Psi) \leftarrowtail W^*(K) \tag{2.11}$$

$$\cong W^*(G) \rightarrowtail L^2(\Psi \circ \Phi) \leftarrowtail W^*(K). \tag{2.12}$$

Since $W^*(H) \rightarrowtail L^2(H)$ is in standard form, one can easily compute the relative tensor product by applying the general prescriptions in [44] to the case at hand. We use the notation in [44] and [14]. Thus $\mathfrak{A}_I \subset L^2(H)$ is the left Hilbert algebra associated to the above standard form. This defines a normal semi-finite faithful weight λ on $W^*(H)$ by $\lambda(f^* * f) = \|f\|^2_{L^2(H)}$ for $f \in \mathfrak{A}_I$, and $\lambda(f^* * f) = \infty$ otherwise. The space of λ-bounded vectors in $L^2(\Psi)$ is called $D(L^2(\Psi), \lambda)$. One defines a sesquilinear form on $L^2(\Phi) \otimes D(L^2(\Psi), \lambda)$ (algebraic tensor product over $\mathbb{C}$) by sesquilinear extension of

$$(\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2)_0 = (\varphi_1, \pi_\rho(\langle \psi_1, \psi_2 \rangle_\lambda)\varphi_2)_{L^2(\Phi)}, \tag{2.13}$$

where $\langle \psi_1, \psi_2 \rangle_\lambda \in W^*(H)$ in fact lies in $\mathfrak{A}_I$, and may be determined by its property

$$(f, \langle \psi_1, \psi_2 \rangle_\lambda)_{L^2(H)} = (\psi_1, \pi_\lambda(Jf)\psi_2)_{L^2(\Psi)}, \tag{2.14}$$

where $f \in \mathfrak{A}_I$ is arbitrary. The form $(\ ,\ )_0$ is positive semidefinite, and the completion of the quotient of $L^2(\Phi) \otimes D(L^2(\Psi), \lambda)$ by the null space of $(\ ,\ )_0$ in the induced norm is the Hilbert space $L^2(\Phi) \boxtimes_{W^*(H)} L^2(\Psi)$. The actions of $W^*(G)$ and $W^*(K)$ on $L^2(\Phi)$ and $D(L^2(\Psi), \lambda) \subset L^2(\Psi)$ (which is stable under $W^*(K)$), respectively, induce actions on $L^2(\Phi) \boxtimes_{W^*(H)} L^2(\Psi)$, defining this Hilbert space as a $W^*(G)$-$W^*(K)$ correspondence.

Denoting the Haar system on K by ρ, from (2.14) one easily finds

$$\langle \psi_1, \psi_2 \rangle_\lambda(h) = \int_{K_1} d\rho^{\Psi_0(s(h))}(k)\, \overline{\psi_1(s(h), k)}\psi_2(t(h), \Psi_1(h)k), \tag{2.15}$$

from which the form (2.13) may explicitly be computed. Now define

$$\tilde{U} : L^2(\Phi) \otimes D(L^2(\Psi), \lambda) \to L^2(\Psi \circ \Phi)$$

by linear extension of

$$\tilde{U}(\varphi \otimes \psi) : (u, k) \mapsto \int_{H_1} d\lambda^{\Phi_0(u)}(h)\, \varphi(u, h)\psi(s(h), \Psi_1(h^{-1})k). \tag{2.16}$$

Using (2.15) and (2.13), one finds that

$$(\tilde{U}(\varphi_1 \otimes \psi_1), \tilde{U}(\varphi_2 \otimes \psi_2))_{L^2(\Psi\circ\Phi)} = (\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2)_0. \tag{2.17}$$

Hence $\tilde{U}$ descends to an isometric map $U : L^2(\Phi) \boxtimes_{W^*(H)} L^2(\Psi) \to L^2(\Psi \circ \Phi)$. Using the fact that the underlying measure spaces are analytic, it is easily shown that the range of $\tilde{U}$ is dense, so that U is unitary. A simple computation finally shows that U intertwines the pertinent actions of $W^*(G)$ and $W^*(K)$. This proves (2.12). □

Since Morita equivalence for measured groupoids is isomorphism in MG, and Morita equivalence of von Neumann algebras is isomorphism in W^*, it follows that the map $G \mapsto W^*(G)$ preserves Morita equivalence.

3. Functoriality of $G \mapsto C^*(G)$

Most of the following constructions apply to locally compact groupoids with Haar system as well, but a key technical step in the proof of functoriality appears to be valid only in the smooth case; cf. the paragraph preceding (3.10). Another reason for our restriction to Lie groupoids is that the beautiful parallel with the classical case is only pertinent in the smooth case.

3.1. The category LG *of Lie groupoids and principal bibundles.* Lie groupoids [27] play a central role in differential geometry, once one starts looking for them. This applies, in particular, to foliation theory [3,4]. In addition, many physical systems can be modeled by Lie groupoids [21].

Definition 5. *A Lie groupoid is a groupoid for which G_1 and G_0 are manifolds, s and t are surjective submersions, and m and I are smooth.*

It follows that object inclusion is an immersion, that I is a diffeomorphism, that G_2 is a closed submanifold of $G_1 \times G_1$, and that for each $q \in G_0$ the fibers $s^{-1}(q)$ and $t^{-1}(q)$ are submanifolds of G_1. In this paper we include Hausdorffness in the definition of a manifold for simplicity, though the total space G_1 of the holonomy groupoid of a foliation usually fails to satisfy this condition. With more technical machinery, our results should extend to that case also.

The category LG, and the key concept of a principal bibundle occurring in its definition, arose in the work of Moerdijk [30], originally in the context of topos theory. Similar structures independently emerged in foliation theory [3,12,15]. The connection between these two points of entry was made by Mrčun [34,35], from which the following definitions are taken; for the basic underlying notion of a Lie groupoid action cf. [27].

Definition 6. *A G-H bibundle is a manifold M equipped with smooth maps $M \xrightarrow{\tau} G_0$ and $M \xrightarrow{\sigma} H_0$, a left G-action $(x, m) \mapsto xm$ from $G \times^{s,\tau}_{G_0} M$ to M, and a right H action $(m, h) \mapsto mh$ from $M \times^{t,\tau}_{H_0} H$ to M, such that $\tau(mh) = \tau(m)$, $\sigma(xm) = \sigma(m)$, and $(xm)h = x(mh)$ for all $(m, h) \in M \times H$ and $(x, m) \in G \times M$. We write $G \rightarrowtail M \leftarrowtail H$.*

Such a bibundle is called left principal when σ is a surjective submersion, the G action is free (in that $xm = m$ iff $x \in G_0$) and transitive along the fibers of σ. Equivalently, the map from $G_1 \times^{s,\tau}_{G_0} M \to M \times_{H_0} M$ given by $(x, m) \mapsto (xm, m)$ is a diffeomorphism.

A G-H bibundle M is called regular when it is left principal and the right H action is proper (in that the map $(m,h) \mapsto (m,mh)$ from $M \times_{H_0} H$ to $M \times M$ is proper).

Two G-H bibundles M, N are called isomorphic if there is a diffeomorphism $M \to N$ that intertwines the maps $M \to G_0$, $M \to H_0$ with the maps $N \to G_0$, $N \to H_0$, and in addition intertwines the G and H actions (the latter condition is well defined because of the former).

Note that the G action in a left principal G-H bibundle is automatically proper. In the topos literature a left principal bibundle is seen as a generalized map from H to G, whereas in the foliation literature it is regarded as the graph of a map between the leaf spaces of the foliations defining G and H.

Now suppose one has left principal bibundles $G \rightarrowtail M \leftarrowtail H$ and $H \rightarrowtail N \leftarrowtail K$. The fiber product $M \times_{H_0} N$ carries a right H action, given by $h : (m,n) \mapsto (mh, h^{-1}n)$ (defined as appropriate). We denote the orbit space by

$$M \circledast_H N = (M \times_H N)/H. \tag{3.1}$$

This is a manifold, and, indeed, a G-K bibundle under the obvious maps. The "tensor product" $\circledast$ is well defined on isomorphism classes. The canonical G-G bibundle G, defined by putting $M = H = G$, $\tau = t$, and $\sigma = s$ in the above definitions, with left and right actions given by multiplication in the groupoid, is a left and a right unit for the bibundle tensor product (3.1), up to isomorphism.

Definition 7. *The category* LG *has Lie groupoids as objects and isomorphism classes of regular (i.e., left principal and right proper) bibundles as arrows. The arrows are composed by (3.1), descending to isomorphism classes. The units 1_G in* G *are the isomorphism classes $[G \rightarrowtail G \leftarrowtail G]$ of the canonical bibundles.*

A number of definitions of Morita equivalence of Lie groupoids have appeared in the literature [12,36,30,53,34,35]; it can be shown that these are all equivalent, and that two Lie groupoids are Morita equivalent iff they are isomorphic objects in LG [34,26].

3.2. The category C^* *of C^*-algebras and Hilbert bibundles.* The definition of C^* is based on the concept of an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule, which is what Rieffel [42] called an Hermitian $\mathfrak{B}$-rigged $\mathfrak{A}$-module, with strict continuity of the $\mathfrak{A}$ action added. Thus an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule is a Hilbert C^* module $\mathcal{E}$ over $\mathfrak{B}$, along with a nondegenerate *-homomorphism of $\mathfrak{A}$ into $\mathcal{L}_{\mathfrak{B}}(\mathcal{E})$. We write $\mathfrak{A} \rightarrowtail \mathcal{E} \rightleftharpoons \mathfrak{B}$. Two $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodules $\mathcal{E}, \mathcal{F}$ are called isomorphic when there is a unitary $U \in \mathcal{L}_{\mathfrak{B}}(\mathcal{E}, \mathcal{F})$; cf. [20], p. 24.

The canonical bimodule $1_{\mathfrak{B}}$ over a C^*-algebra $\mathfrak{B}$ is defined by $\langle A, B\rangle_{\mathfrak{B}} = A^*B$, and the left and right actions are given by left and right multiplication, respectively. Rieffel's interior tensor product [42,20] maps an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule $\mathcal{E}$ and a $\mathfrak{B}$-$\mathfrak{C}$ Hilbert bimodule $\mathcal{F}$ into an $\mathfrak{A}$-$\mathfrak{C}$ Hilbert bimodule $\mathcal{E}\hat{\otimes}_{\mathfrak{B}}\mathcal{F}$. This operation is well defined on unitary isomorphism classes, and $1_{\mathfrak{B}}$ acts as a two-sided unit for $\hat{\otimes}_{\mathfrak{B}}$, up to isomorphism.

Definition 8. *The category* C^* *has C^*-algebras as objects, and isomorphism classes of Hilbert bimodules as arrows. The arrows are composed by Rieffel's interior tensor product, for which the canonical Hilbert bimodules $1_{\mathfrak{A}}$ are units.*

This category was introduced independently in [46], and, in the guise of a bicategory (where the arrows are Hilbert bimodules rather than isomorphism classes thereof), in [25]. It was shown in [46] that two C^*-algebras are Morita equivalent as defined by

Rieffel [42] iff they are isomorphic as objects in C^*; also see [26] for a detailed proof. The nondegeneracy condition in the definition of the arrows in C^* is essential for this result.

It should be noted that Thm. 2.2 in [1] implies that the category W^* of Definition 4 is isomorphic to the subcategory of C^* consisting of von Neumann algebras as objects and normal selfdual Hilbert bimodules as arrows.

3.3. The map $G \mapsto C^*(G)$ as a functor. We will now prove that the map $G \mapsto C^*(G)$ mentioned in the Introduction may be extended so as to associate Hilbert bimodules to regular bibundles, thus defining a functor from LG to C^*. Although it should be possible to use the geometric definition of $C^*(G)$ in terms of half-densities [4], as in our previous direct proof that $G \mapsto C^*(G)$ preserves Morita equivalence [24], we find it much easier to regard a Lie groupoid as a locally compact groupoid with smooth Haar system (cf. the Introduction).

Specifically, a Lie groupoid G has a left Haar system $\{\nu^q\}_{q\in G_0}$ such that ν^q is supported on $t^{-1}(q)$ and is equivalent to Lebesgue measure in each coordinate chart (recall that $t^{-1}(q)$ is a submanifold of G_1). Furthermore, for each $f \in C_c^\infty(G_1)$ the function $q \mapsto \int d\nu^q(x)\, f(x)$ on G_0 is smooth. This endows $C_c^\infty(G)$ with the structure of a *-algebra under the operations (2.3) and (2.4). The groupoid C^*-algebra $C^*(G)$ is a suitable completion of the *-algebra $C_c^\infty(G)$; see [40] for the analogous case of $C_c(G)$, or [4, 21] for the smooth case.

We have now defined the map $G \mapsto C^*(G)$ on objects. To define it on arrows, let $G \rightarrowtail M \leftarrowtail H$ be a regular bibundle (cf. Definition 6 for the notation that will be used throughout this chapter). A key fact is that a Haar system on G defines a family of measures $\{\mu^r\}_{r\in H_0}$ on M, where μ^r is supported on $\sigma^{-1}(r)$, on which it is equivalent to Lebesgue measure in each coordinate chart. Moreover, for each $f \in C_c^\infty(M)$ the function $r \mapsto \int d\mu^r(m)\, f(m)$ on H_0 is smooth, and the family is H-equivariant (in the sense of [41]) with respect to σ, the given H action on M, and the natural right H action on H_0. This means that for each $f \in C_c^\infty(M)$ one has

$$\int d\mu^{t(h)}(m)\, f(mh) = \int d\mu^{s(h)}(m)\, f(m). \tag{3.2}$$

Namely, for fixed $r \in H_0$ this system is defined by choosing $m_0 \in \sigma^{-1}(r)$, and putting

$$\int d\mu^r(m)\, f(m) = \int d\nu^{\tau(m_0)}(x) f(x^{-1}m_0). \tag{3.3}$$

Using (2.1), one verifies that this is independent of the choice of m_0 (despite the fact that $\tau(m_0)$ is not constant on $\sigma^{-1}(r)$). This definition is evidently possible because in a regular bibundle the G action is principal over σ.

The following lemma is similar to Thm. 2.8 in [36], and also appeared in [48] for the locally compact case (this paper was drawn to our attention after the circulation of an earlier draft of this paper as an e-print); our assumptions are weaker, since we do not have an equivalence bibundle but merely a regular one. However, what is really used in [36] is precisely our regularity properties.

Lemma 1. *Let $G \rightarrowtail M \leftarrowtail H$ be a regular bibundle. The formulae*

$$\langle \varphi, \psi \rangle : h \mapsto \int d\mu^{t(h)}\, \overline{\varphi(m)}\psi(mh); \tag{3.4}$$

$$f \cdot \varphi : m \mapsto \int d\nu^{\tau(m)}\,(x) f(x)\varphi(x^{-1}m); \tag{3.5}$$

$$\varphi \cdot g : m \mapsto \int d\lambda^{\sigma(m)}\, g(h^{-1})\varphi(mh), \tag{3.6}$$

where $\varphi, \psi \in C_c^\infty(M)$, $f \in C_c^\infty(G)$, and $g \in C_c^\infty(H)$, define functions in $C_c^\infty(H)$, $C_c^\infty(M)$, and $C_c^\infty(M)$, respectively. This equips $C_c^\infty(M)$ with the structure of a pre Hilbert C^*-module over $C_c^\infty(H)$ (seen as a dense subalgebra of $C^*(H)$), on which $C_c^\infty(G)$ (seen as a dense subalgebra of $C^*(G)$) acts nondegenerately by adjointable operators. This structure may be completed to a $C^*(G)$-$C^*(H)$ Hilbert bimodule, which we call $\mathcal{E}(M)$.

Proof. It should now be obvious why the right H action on a regular bibundle has to be proper, since this guarantees $C_c^\infty(H)$-valuedness of the inner product (otherwise, one could land in $C^\infty(H)$).

The necessary algebraic properties may be checked by elementary computations. The property $\langle \varphi, \psi \rangle^* = \langle \psi, \varphi \rangle$ follows from (3.2), the property $\langle \varphi, \psi \cdot g \rangle = \langle \varphi, \psi \rangle * g$ is an identity, the properties $\langle \varphi, f \cdot \psi \rangle = \langle f^* \cdot \varphi, \psi \rangle$ and $(f_1 * f_2) \cdot \varphi = f_1 \cdot (f_2 \cdot \varphi)$ require (3.3) and (2.1), and finally $\varphi \cdot (g_1 * g_2) = (\varphi \cdot g_1) \cdot g_2$ follows from (2.1) for λ.

The proof of positivity of $\langle\, ,\, \rangle$ is the same as in [36]; it follows from Prop. 2.10 in [36] and the argument of P. Green (see the remark following Lemma 2 in [11]). This also proves the nondegeneracy of the action of $C_c^\infty(G)$ (and hence of the ensuing action of $C^*(G)$).

We cannot use the entire argument in [36] to the effect that everything can be completed, since in [36] one has a $C_c^\infty(G)$-valued inner product as well. However, it is quite trivial to proceed, since by the above results $C_c^\infty(M)$ is a pre Hilbert C^*-module over $C_c^\infty(H)$, which can be completed to a Hilbert C^*-module $\mathcal{E}(M)$ over $C^*(H)$ in the standard way (cf. Ch. 1 in [20] or Cor. IV.2.1.4 in [21]). One then copies the proof in [36] of the property

$$\langle f \cdot \varphi, f \cdot \varphi \rangle \leq \|f\|^2 \langle \varphi, \varphi \rangle, \tag{3.7}$$

where the norm is in $C^*(G)$, to complete the argument. □

Theorem 2. *The map $C^* : \mathsf{LG} \mapsto \mathsf{C}^*$, defined on objects by $C_0^*(G) = C^*(G)$, and on arrows by*

$$C_1^*([G \rightarrowtail M \leftarrowtail H]) = [C^*(G) \rightarrowtail \mathcal{E}(M) \rightleftharpoons C^*(H)],$$

is a functor.

Proof. We begin with the unit arrows. We claim that the construction in Lemma 1 maps the canonical bibundle $G \rightarrowtail G \leftarrowtail G$ into the canonical Hilbert bimodule $C^*(G) \rightarrowtail C^*(G) \rightleftharpoons C^*(G)$. It is easy to check from (3.4)–(3.6) that $\langle \varphi, \psi \rangle = \varphi^* * \psi$, $f \cdot \varphi = f * \varphi$, and $\varphi \cdot g = \varphi * g$. These properties pass to the completions by continuity. Hence C^* preserves units.

Now let $H \rightarrowtail N \leftarrowtail K$ be a second regular bibundle, so that one may form the bibundle tensor product $M \circledast_H N$ (cf. (3.1)) and its associated $C^*(G)$-$C^*(K)$ Hilbert bimodule $\mathcal{E}(M \circledast_H N)$. To compare this with the $C^*(G)$-$C^*(K)$ Hilbert bimodule $\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)$, we define a map $\tilde{U} : C_c^\infty(M) \otimes_{\mathbb{C}} C_c^\infty(N) \to C_c^\infty(M \circledast_H N)$ by

$$\tilde{U}(\varphi \otimes_{\mathbb{C}} \psi) : [m, n]_H \mapsto \int d\lambda^{\sigma(m)}(h)\, \varphi(mh)\psi(h^{-1}n). \tag{3.8}$$

Note that the right-hand side is well defined on $[m, n]_H$ rather than (m, n) because of the invariance property (2.1) for H. This map was introduced by Mrčun [34] for smooth étale groupoids; we have merely replaced the counting measure by a general Haar system.

We now show that the map $\tilde{U}$ leaves the kernel of the canonical projection

$$C_c^\infty(M) \otimes_{\mathbb{C}} C_c^\infty(N) \to \mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)$$

stable, that $\tilde{U}$ has dense range, and that accordingly the corresponding quotient map U, extended by continuity, defines an isomorphism

$$\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N) \simeq \mathcal{E}(M \circledast_H N) \tag{3.9}$$

as $C^*(G)$-$C^*(K)$ Hilbert bimodules.

A lengthy but straightforward computation shows that

$$\langle \tilde{U}(\varphi_1 \otimes_{\mathbb{C}} \psi_1, \tilde{U}(\varphi_2 \otimes_{\mathbb{C}} \psi_2)\rangle_{C^*(K)}^{\mathcal{E}(M \circledast_H N)},$$

is equal to

$$\langle \psi_1, \langle \varphi_1, \varphi_2\rangle_{C^*(H)}^{\mathcal{E}(M)} \cdot \psi_2\rangle_{C^*(K)}^{\mathcal{E}(N)},$$

which by definition is equal to

$$\langle \varphi_1 \otimes_{C^*(H)} \psi_1, \varphi_2 \otimes_{C^*(H)} \psi_2\rangle_{C^*(K)}^{\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)}.$$

Here $\varphi \otimes_{C^*(H)} \psi$ is the image of $\varphi \otimes_{\mathbb{C}} \psi$ in $\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)$. In view of the definitions of the various Hilbert C^*-modules over $C^*(K)$ involved, this computation implies that $\tilde{U}$ quotients and extends to an isometry U from $\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)$ to $\mathcal{E}(M \circledast_H N)$.

Moreover, using the fact that M and N are manifolds, it is easily seen that $\tilde{U}$ has a dense range in $C_c^\infty(M \circledast_H N)$ with respect to the inductive limit topology, so that it certainly has a dense range for the topology induced on $C_c^\infty(M \circledast_H N)$ by the norm on $\mathcal{E}(M \circledast_H N)$ as a Hilbert C^*-module over $C^*(K)$ (since the latter topology is finer than the former). Since $C_c^\infty(M \circledast_H N)$ is itself dense in $\mathcal{E}(M \circledast_H N)$ in the latter topology, it follows that $\tilde{U}$ has dense range when seen as a map taking values in $\mathcal{E}(M \circledast_H N)$. Hence U is an isometric isomorphism between $\mathcal{E}(M)\hat{\otimes}_{C^*(H)}\mathcal{E}(N)$ and $\mathcal{E}(M \circledast_H N)$ as Banach spaces. Note that the first claim in this paragraph is not obvious in the general locally compact case; this is one of the reasons why we have restricted ourselves to Lie groupoids in this chapter.

Another elementary computation shows that

$$\tilde{U}(\varphi \otimes_{\mathbb{C}} (\psi \cdot g)) = \tilde{U}(\varphi \otimes_{\mathbb{C}} \psi) \cdot g \tag{3.10}$$

for $\varphi \in C_c^\infty(M)$, $\psi \in C_c^\infty(N)$, and $g \in C_c^\infty(H)$. This implies that

$$U(\varphi \otimes_{C^*(H)} (\psi \cdot g)) = U(\varphi \otimes_{C^*(H)} \psi) \cdot g \tag{3.11}$$

for all $\varphi \in \mathcal{E}(M)$, $\psi \in \mathcal{E}(N)$, and $g \in C^*(H)$. The reason for this is that a continuous $\mathfrak{B}_0$-linear map between two pre Hilbert C^* modules over a dense subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ extends to a $\mathfrak{B}$-linear map between the completions; this easily follows from the bound $\|\psi B\| \leq \|B\| \, \|\psi\|$.

We conclude that U is a $C^*(K)$-linear isometric isomorphism, and hence by Thm. 3.5 in [20] it is actually unitary (in particular, it now follows that P is adjointable).

Finally, analogously to (3.10) one obtains the equality

$$\tilde{U}(f \cdot (\varphi \otimes_{\mathbb{C}} \psi) = f \cdot \tilde{U}(\varphi \otimes_{\mathbb{C}} \psi), \tag{3.12}$$

where $f \in C_c^\infty(G)$. This time, the passage of this property to the pertinent completions is achieved through (3.7), which leads to the bound $\|A\psi\| \leq \|A\| \, \|\psi\|$ for any adjointable operator on a (pre) Hilbert C^*-module. Thus U is $C^*(G)$-linear as well. This proves (3.9).

Hence C^* preserves composition of arrows, and Theorem 2 follows. □

Since Morita equivalence of Lie groupoids is isomorphism in LG, and Morita equivalence of C^*-algebras is isomorphism in C^*, we recover the known result that the map $G \mapsto C^*(G)$ preserves Morita equivalence [36,24].

4. Functoriality of $G \mapsto A^*(G)$

The category on which the map A^* is going to be defined is as follows.

Definition 9. *The category* LGc *has s-connected and s-simply connected Lie groupoids as objects, and isomorphism classes of left principal bibundles as arrows. The arrows and units are as in Definition 7.*

In contrast with Definition 7, the class of objects is more restricted; this will be necessary for our functor to preserve units. On the other hand, the bibundles need not be right proper.

4.1. The category Poisson *of Poisson manifolds and dual pairs.* The definition of a suitable category of Poisson manifolds [26] is based on the theory of symplectic groupoids (cf. [6,50] and refs. therein). The objects in Poisson are defined as follows.

Definition 10. *A Poisson manifold P is called integrable when there exists a symplectic groupoid $\Gamma(P)$ over P.*

This definition is due to [6]. Using Thms. 5.2, 5.3, and A1 in [28] and Prop. 3.3 in [31], it follows that if P is integrable, then there exists an s-connected and s-simply connected symplectic groupoid $\Gamma(P)$ over P, which is unique up to isomorphism [26].

The arrows in Poisson will be isomorphism classes of certain dual pairs. Given two Poisson manifolds P and Q, a dual pair $Q \leftarrow S \rightarrow P$ consists of a symplectic manifold S and Poisson maps $q : S \rightarrow Q$ and $p : S \rightarrow P^-$, such that $\{q^*f, p^*g\} = 0$ for all $f \in C^\infty(Q)$ and $g \in C^\infty(P)$ [52,16]. In a complete dual pair the maps p and q are complete; a Poisson map $J : S \rightarrow P$ is called complete when, for every $f \in C^\infty(P)$ with complete Hamiltonian flow, the Hamiltonian flow of J^*f on S is complete as well (that is, defined for all times). Two Q-P dual pairs $Q \stackrel{q_i}{\leftarrow} \tilde{S}_i \stackrel{p_i}{\rightarrow} P$, $i = 1, 2$, are isomorphic when there is a symplectomorphism $\varphi : \tilde{S}_1 \to \tilde{S}_2$ for which $q_2\varphi = q_1$ and $p_2\varphi = p_1$.

Based on results in [6,8,54], it can be shown that for integrable Poisson manifolds P and Q, with associated s-connected and s-simply connected symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$, there is a natural bijective correspondence between complete dual pairs $Q \leftarrow S \to P$ and symplectic bibundles $\Gamma(Q) \rightarrowtail S \leftarrowtail \Gamma(P)$. In particular, the canonical symplectic bibundle associated to the dual pair $P \xleftarrow{t} \Gamma(P) \xrightarrow{s} P$ is $\Gamma(P) \rightarrowtail \Gamma(P) \leftarrowtail \Gamma(P)$. Accordingly, we say that a dual pair is regular when it is complete and when the associated symplectic bibundle is left principal (it is not necessary to impose properness of the right $\Gamma(P)$ action).

Let R be a third integrable Poisson manifold, with associated s-connected and s-simply connected symplectic groupoid $\Gamma(R)$, and let $Q \leftarrow S_1 \to P$ and $P \leftarrow S_2 \to R$ be regular dual pairs. The embedding $S_1 \times_P S_2 \subset S_1 \times S_2$ is coisotropic [21]; we denote the corresponding symplectic quotient by $S_1 \circledcirc_P S_2$. This is the middle space of a regular dual pair $P \leftarrow S_1 \circledcirc_P S_2 \to R$, which we regard as the tensor product of the given dual pairs. An alternative way of defining this tensor product is to construct the groupoid tensor product $\Gamma(Q) \rightarrowtail S_1 \circledast_{\Gamma(P)} S_2 \leftarrowtail \Gamma(R)$ of the associated symplectic bibundles [53]. Thus we have

$$S_1 \circledcirc_P S_2 = S_1 \circledast_{\Gamma(P)} S_2 \tag{4.1}$$

as symplectic manifolds, as $\Gamma(Q)$-$\Gamma(R)$ symplectic bibundles, and as Q-R dual pairs. In any case, this tensor product is associative up to isomorphism, and the dual pair $P \xleftarrow{t} \Gamma(P) \xrightarrow{s} P$ is a two-sided unit for $\circledcirc_P$, up to isomorphism [26].

Definition 11. *The category* Poisson *has integrable Poisson manifolds as objects, and isomorphism classes of regular dual pairs as arrows. The arrows are composed by the tensor product* $\circledcirc$*, for which the dual pairs* $P \xleftarrow{t} \Gamma(P) \xrightarrow{s} P$ *are units. Here* $\Gamma(P)$ *is "the" s-connected and s-simply connected symplectic groupoid over* P.

The original reason for the introduction of this category was not so much the subsequent functoriality theorem, but rather the fact that two Poisson manifolds are Morita equivalent in the sense of Xu [54] iff they are isomorphic objects in Poisson [26]. In particular, a Poisson manifold is integrable iff it is Morita equivalent to itself. Moreover, we now have a classical analogue of the categories W^* and C^*.

4.2. The map $G \mapsto A^*(G)$ *as a functor.* A Lie groupoid G defines an associated "infinitesimal" object, its Lie algebroid $A(G)$ [37]; see [27,6,21] for reviews. The main point is that $A(G)$ is a vector bundle over G_0, endowed with an "anchor map" $\alpha : A(G) \to T(G_0)$ and a Lie algebra structure on its space of sections $C^\infty(G_0, A(G))$ that is compatible with the anchor map in a certain way.

It is of central importance to us that the dual vector bundle $A^*(G)$ is a Poisson manifold in a canonical way [6,7,9], which generalizes the well-known Lie–Poisson structure on the dual of a Lie algebra. We look at the passage $G \mapsto A^*(G)$ as a classical analogue of the map $G \mapsto C^*(G)$.

Another important construction is that of the cotangent bundle $T^*(G)$ of G. This is not merely a symplectic space (equipped, in our conventions [21,24], with minus the usual symplectic form on a cotangent bundle, so that we write $T^*(G)^-$ when this aspect is relevant), but a symplectic groupoid with $T^*(G)_1 = T^*(G_1)$ over $T^*(G)_0 = A^*(G)$

[6] (also see [50] for a review). For simplicity we will write $T^*(G)$ for $T^*(G_1)$, and denote the source and target projections of $T^*(G)$ by $\tilde{s}$ and $\tilde{t}$, respectively.

Lemma 2. *The s-connected and s-simply connected symplectic groupoid over $A^*(G)$ is $T^*(\tilde{G})$, where $\tilde{G}$ is the s-connected and s-simply connected Lie groupoid with Lie algebroid $A(G) = A(\tilde{G})$.*

Proof. The existence of $\tilde{G}$ is guaranteed by Prop. 3.3 in [31]. Since the Poisson structure on $A^*(G)$ is entirely determined by the Lie algebroid structure of $A(G)$, one has $A^*(G) = A^*(\tilde{G})$ as Poisson manifolds. It may be checked from its definition that $T^*(G)$ is s-connected and s-simply connected iff G is. □

In view of this lemma, we will henceforth assume that all Lie groupoids are s-connected and s-simply connected, and drop the tilde. Thus we have defined the map $A^* : \mathsf{LGc} \to \mathsf{Poisson}$ on objects.

In order to define this map on arrows, we recall a number of results from [24], which we here combine into a lemma.

Lemma 3. *Any bibundle $G \rightarrowtail M \leftarrowtail H$ (cf. Definition 6) defines a symplectic bimodule*

$$A^*(G) \xleftarrow{J_L^G} T^*(M)^- \xrightarrow{J_R^H} A^*(H), \tag{4.2}$$

with associated symplectic bibundle

$$T^*(G)^- \rightarrowtail T^*(M)^- \leftarrowtail T^*(H)^-. \tag{4.3}$$

The explicit form of the "momentum map" J_R^H is

$$\left\langle J_R^H(\theta_m), \frac{dh(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{\sigma(m)} = \left\langle \theta_m, \frac{dmh(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_m, \tag{4.4}$$

where $\theta_m \in T_m^*(M)$, $\sigma(m) = h(0)$, and $h(\lambda) \in t^{-1}(\sigma(m))$, so that $\dot{h}(0)$ lies in $A_{\sigma(m)}(H)$ and $J_R^H(\theta_m) \in A^*_{\sigma(m)}(H)$.

The associated right action of $T^*(H)$ on $T^*(M)$ is given by

$$\left\langle \theta_m \cdot (\alpha_h)^{-1}, \frac{dm(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{mh^{-1}} = \left\langle \theta_m, \frac{dm(\lambda)\tilde{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_m - \left\langle \alpha_h, \frac{d\tilde{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h, \tag{4.5}$$

where $m(0) = mh^{-1}$, and $\tilde{h}(\cdot)$ is a curve in H satisfying $\tilde{h}(0) = h$ and $\sigma(m(\lambda)) = t(\tilde{h}(\lambda))$. As explained in [24], Eq. (4.5) is independent of the choice of $\tilde{h}$ because of the compatibility condition $J_R^H(\theta_m) = \tilde{s}(\alpha_h)$ under which $\theta_m \cdot (\alpha_h)^{-1}$ is defined; cf. Definition 6. Explicitly, this condition reads $\sigma(m) = s(h)$, along with

$$\left\langle \theta_m, \frac{dm\chi(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_m = \left\langle \alpha_h, \frac{dh\chi(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h, \tag{4.6}$$

for all curves $\chi(\cdot) \in t^{-1}(s(h))$ subject to $\chi(0) = s(h)$. Note that these formulae for right actions are not given in [24], but they may be derived from those for left actions, together with the formula $\alpha^{-1} = -I^*(\alpha)$ for the inverse in $T^*(G)$ (where $I : G_1 \to G_1$ is the inverse in G) [6].

The explicit form of J_L will shortly be needed not for $G \rightarrowtail M \leftarrowtail H$, but for a second bibundle $H \rightarrowtail N \leftarrowtail K$; hence we state it for the latter. The momentum map $J_L^H : T^*(N) \to A^*(H)$, then, is given by [24]

$$\left\langle J_L^H(\eta_n), \frac{dh(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{\rho(n)} = -\left\langle \eta_n, \frac{dh(\lambda)^{-1}n}{d\lambda}_{|\lambda=0} \right\rangle_h, \tag{4.7}$$

where $\eta_n \in T_n^*(N)$, $\rho(n) = h(0)$, and $h(\lambda) \in t^{-1}(\rho(n))$; recall that $\rho : N \to H_0$ is the base map of the H action on N.

The associated left action of $T^*(H)$ on $T^*(N)$ is given by

$$\left\langle \alpha_h \cdot \eta_n, \frac{dn(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{hn} = \left\langle \eta_n, \frac{d\hat{h}(\lambda)^{-1}n(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_n + \left\langle \alpha_h, \frac{d\hat{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h, \tag{4.8}$$

where $n(0) = hn$, and $\hat{h}(\cdot)$ is a curve in H satisfying $\hat{h}(0) = h$ and $\rho(n(\lambda)) = t(\hat{h}(\lambda))$. The condition under which $\alpha_h \cdot \eta_n$ is defined is $J_L^H(\eta_n) = \tilde{s}(\alpha_h)$, which reads $\rho(n) = s(h)$, along with

$$-\left\langle \eta_n, \frac{d\chi(\lambda)^{-1}n}{d\lambda}_{|\lambda=0} \right\rangle_n = \left\langle \alpha_h, \frac{d\chi(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h, \tag{4.9}$$

for χ as specified after (4.6). This completes the exposition of Lemma 3.

Theorem 3. *The map* $A^* : \mathsf{LGc} \to \mathsf{Poisson}$, *defined on objects by* $A_0^*(G) = A^*(G)$ *and on arrows by*

$$A_1^*([G \rightarrowtail M \leftarrowtail H]) = [A^*(G) \leftarrow T^*(M)^- \to A^*(H)],$$

is a functor.

Proof. The object map A_0^* is well defined between the given categories by Lemma 2. Turning to the unit arrows, we note that the construction in Lemma 3 maps the canonical bibundle $G \rightarrowtail G \leftarrowtail G$ into the symplectic bimodule

$$A^*(G) \stackrel{\tilde{t}}{\leftarrow} T^*(G)^- \stackrel{\tilde{s}}{\to} A^*(G).$$

To see this, recall that $\tilde{s}$ and $\tilde{t}$ are the source and target maps of the symplectic groupoid $T^*(G)^-$. The lemma follows because, as already remarked in [24], $\tilde{s}$ and $\tilde{t}$ as defined in [6] coincide with the momentum mappings J_R^G and J_L^G defined by Lemma 3, applied to the canonical bibundle. It is here that the assumption of s-connectedness and s-simply connectedness is essential.

We now turn to the composition of arrows. Let $G \rightarrowtail M \leftarrowtail H$ and $H \rightarrowtail N \leftarrowtail K$ be regular bibundles, with associated symplectic bimodules $A^*(G) \leftarrow T^*(M)^- \to A^*(H)$ and $A^*(H) \leftarrow T^*(N)^- \to A^*(K)$, respectively (cf. Lemma 3). We will prove that the tensor product

$$A^*(G) \leftarrow T^*(M)^- \circledcirc_{A^*(H)} T^*(N)^- \to A^*(K) \tag{4.10}$$

of these symplectic bimodules is isomorphic to the symplectic bimodule

$$A^*(G) \leftarrow T^*(M \circledast_H N)^- \rightarrow A^*(K) \tag{4.11}$$

associated with the bibundle $G \rightarrowtail M \circledast_H N \leftarrowtail K$.

We omit all suffixes "$-$" (as in S^-), unless strictly necessary. By (4.1) and (3.1) we have

$$T^*(M) \circledcirc_{A^*(H)} T^*(N) = (T^*(M) *_{A^*(H)} T^*(N))/T^*(H) \tag{4.12}$$

as $A^*(G)$-$A^*(K)$ symplectic bimodules. By (3.1), one has

$$T^*(M \circledast_H N) = T^*((M *_{H_0} N)/H), \tag{4.13}$$

so we start by proving that

$$(T^*(M) *_{A^*(H)} T^*(N))/T^*(H) \simeq T^*((M *_{H_0} N)/H) \tag{4.14}$$

as symplectic manifolds. To do so, we first show that

$$T^*((M *_{H_0} N)/H) \simeq (T^*(M) *_{A^*(H)} T^*(N))/\sim \tag{4.15}$$

as manifolds, where $\sim$ is an equivalence relation defined as follows. For $(\theta_m, \eta_n) \in T^*(M) *_{A^*(H)} T^*(N)$ (i.e., $\sigma(m) = \rho(n)$ and $J_R^H(\theta_m) = J_L^H(\eta_n)$), $h \in s^{-1}(\sigma(m))$, and $(\theta'_{mh^{-1}}, \eta'_{hn}) \in T^*(M) *_{A^*(H)} T^*(N)$, we say that $(\theta'_{mh^{-1}}, \eta'_{hn}) \sim (\theta_m, \eta_n)$ iff for each pair of vectors $\dot{m}(0) \in T_{mh^{-1}}(M)$ and $\dot{n}(0) \in T_{hn}(N)$ such that

$$\sigma_*(\dot{m}(0)) = \rho_*(\dot{m}(0)), \tag{4.16}$$

there exists a curve $h(\cdot)$ in H with $h(0) = h$ and $t(h(\lambda)) = \sigma(m(\lambda)) = \rho(n(\lambda))$ (the latter equality may be imposed for convenience because of (4.16)), such that

$$\left\langle \theta'_{mh^{-1}}, \frac{dm(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{mh^{-1}} + \left\langle \eta'_{hn}, \frac{dn(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{mh^{-1}}$$
$$= \left\langle \theta_m, \frac{dm(\lambda)h(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_m + \left\langle \eta_n, \frac{dh^{-1}(\lambda)n(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_n . \tag{4.17}$$

To prove (4.15), note that for any (possibly singular) smooth foliation Φ of a manifold Q with smooth leaf space Q/Φ one has an isomorphism

$$C^\infty(Q/\Phi, T^*(Q/\Phi)) \simeq C^\infty(Q, T(\Phi)_0^0), \tag{4.18}$$

where the right-hand side consists of all 1-forms ω on Q that satisfy $i_\xi \omega = 0$ (forming $T(\Phi)^0 \subset T^*(Q)$) and $i_\xi d\omega = 0$ (defining $T(\Phi)_0$), for all $\xi \in C^\infty(Q, T(\Phi))$. This is well known for regular foliations (cf. [32]), and the proof is the same in the singular case (it merely depends on the smoothness of the leaf space). These conditions may be rewritten as $i_\xi \omega = \mathcal{L}_\xi \omega = 0$ (where $\mathcal{L}$ is the Lie derivative), or as $i_\xi \omega = 0$ for all vector fields ξ as above and $\varphi^* \omega = \omega$ for all diffeomorphisms φ of Q that are generated by such ξ. The isomorphism (4.18) is then given by $\alpha \leftrightarrow \pi^* \alpha$, where $\pi : Q \rightarrow Q/\Phi$ is the canonical projection. In addition, one has

$$C^\infty(Q, T(\Phi)_0^0) \simeq C^\infty(Q/\Phi, T(\Phi)^0/\sim), \tag{4.19}$$

where the equivalence relation $\sim$ on $T(\Phi)^0$ is defined by $\beta' \sim \beta$ iff $\beta' = \varphi^*\beta$ for some diffeomorphism φ as specified above. The isomorphism (4.19) associates a section $q \mapsto \beta(q)$ with a section $[q]_\Phi \mapsto [\beta(q)]_\sim$. Hence the ensuing isomorphism

$$C^\infty(Q/\Phi, T^*(Q/\Phi)) \simeq C^\infty(Q/\Phi, T(\Phi)^0/\sim) \tag{4.20}$$

is given by $\alpha \leftrightarrow [\pi^*\alpha]_\sim$.

We apply this to $Q = M *_{H_0} N$, where Φ is the foliation by the orbits of the diagonal H action. The condition of lying in $T(\Phi)^0$ then has $T^*(M) *_{A^*(H)} T^*(N)$ as its solution set, and the equivalence relation $\sim$ defined for Φ is precisely the one imposed by (4.17) and preceding text. This proves (4.15).

Next, we show that the equivalence relation $\sim$ on $T^*(M) *_{A^*(H)} T^*(N)$ coincides with $\backsim$, defined as follows. We say that $(\theta'_{mh^{-1}}, \eta'_{hn}) \backsim (\theta_m, \eta_n)$ iff there exists $\alpha_h \in T^*_h(H)$ satisfying

$$\tilde{s}(\alpha_h) = J^H_R(\theta_m) \tag{4.21}$$

(and therefore also $\tilde{s}(\alpha_h) = J^H_L(\eta_n)$), such that for *each* pair of vectors $\dot{m}(0) \in T_{mh^{-1}}(M)$ and $\dot{n}(0) \in T_{hn}(N)$ (not necessarily satisfying (4.16)), there exist curves $\hat{h}(\cdot)$ and $\tilde{h}(\cdot)$ in H subject to $\hat{h}(0) = \tilde{h}(0) = h$, $t(\tilde{h}(\lambda)) = \sigma(m(\lambda))$, $t(\hat{h}(\lambda)) = \rho(n(\lambda))$, for which one has

$$\begin{aligned} &\left\langle \theta'_{mh^{-1}}, \frac{dm(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{mh^{-1}} + \left\langle \eta'_{hn}, \frac{dn(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_{mh^{-1}} \\ &\quad = \left\langle \theta_m, \frac{dm(\lambda)\tilde{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_m + \left\langle \eta_n, \frac{d\hat{h}^{-1}(\lambda)n(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_n \\ &\quad + \left\langle \alpha_h, \frac{d\hat{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h - \left\langle \alpha_h, \frac{d\tilde{h}(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle_h . \end{aligned} \tag{4.22}$$

We stress that $\hat{h}$ and $\tilde{h}$ do, and α_h does not depend on the vectors $\dot{m}(0)$ and $\dot{n}(0)$. The full right-hand side of (4.22) is independent of the choice of $\hat{h}$ and $\tilde{h}$; cf. the comment following (4.6).

First, $\backsim$ implies $\sim$ (i.e., $A \backsim B \to A \sim B$), for if (4.16), and hence $\sigma(m(\lambda)) = \rho(n(\lambda))$, holds, one may choose $h = \tilde{h} = \hat{h}$, and the final line in (4.22) drops out, implying (4.17).

To prove the converse, we note that, since the bibundle $G \rightarrowtail M \leftarrowtail H$ is regular, the map $\sigma : M \to H_0$ is a surjective submersion, so that

$$T_m(M) \simeq T^\sigma_m(M) \oplus T_{\sigma(m)}(H_0).$$

Here $T^\sigma_m(M)$ is the kernel of $\sigma_* : T(M) \to T(H_0)$ at m. This induces the decomposition

$$T_m(M) \oplus T_n(N) \simeq T^{\sigma=\rho}_{(m,n)}(M \times N) \oplus T_{\sigma(m)}(H_0), \tag{4.23}$$

where $T^{\sigma=\rho}_{(m,n)}(M \times N)$ is the kernel of $\sigma_* - \rho_*$ at (m, n). Explicitly, the decomposition of a given vector according to (4.23) reads

$$(\xi_1, \xi_2, \zeta) = (\xi_1, \rho_*(\zeta), \zeta) + (0, \xi_2 - \rho_*(\zeta), 0),$$

where $\xi_1 \in T_m^\sigma(M)$, $\xi_2 \in T_{\sigma(m)}(H_0)$, and $\zeta \in T_n(N)$. Now, in order to verify (4.22) given (4.17), we examine the two possible cases allowed by (4.23). A dimension count shows that one can always choose α_h so as to satisfy (4.22) on $T_{\sigma(m)}(H_0)$. This is because in a Lie groupoid one has [27,21]

$$T_h(H) \simeq T_h^t(H) \oplus T_{t(h)}(H_0),$$

and the condition (4.21) constrains α_h only on $T_h^t(H)$, leaving its value on $T_{t(h)}(H_0)$ free. On the other hand, if (4.16) holds, so that $(\dot{m}(0), (\dot{n}(0))$ lies in $T_{(m,n)}^{\sigma=\rho}(M \times N)$, and we assume (4.17), then (4.22) is satisfied for any α_h, as one may choose $\tilde{h} = \hat{h} = h$.

Hence $\sim$ implies $\backsim$, and we have shown that these equivalence relations coincide. Comparing (4.22) with (4.5) and (4.8), and using (4.15), it is clear that (4.14) holds at the manifold level. But it is almost trivial that the identification we have made preserves the symplectic structure, so that (4.14) is valid for symplectic manifolds as well.

Finally, we need to verify that the symplectomorphism (4.14) is compatible with the $A^*(G)$-$A^*(K)$ symplectic bimodule structure that both sides have. This is, indeed, obvious from the explicit structure of the pertinent Poisson maps. For example, denoting the appropriate Poisson map from $T^*(M)^- \circledcirc_{A^*(H)} T^*(N)^-$ to $A^*(G)$ by $\hat{J}_L^G$, we have $\hat{J}_L^G([\theta_m, \eta_n]) = J_L^G(\theta_m)$, so that

$$\left\langle \hat{J}_L^G([\theta_m, \eta_n]), \frac{d\gamma(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle = -\left\langle \theta_m, \frac{d\gamma(\lambda)^{-1}m}{d\lambda}_{|\lambda=0} \right\rangle. \tag{4.24}$$

Here $[\theta_m, \eta_n]$ is the equivalence class of (θ_m, η_n) under either the $T^*(H)$ orbits or under the null foliation with respect to the inclusion $T^*(M)^- *_{A^*(H)} T^*(N)^- \hookrightarrow T^*(M)^- \times T^*(N)^-$; these coincide by (4.12).

On the other hand, $\tilde{J}_L^G : T^*(M \circledast_H N)^- \to A^*(G)$ is given by

$$\left\langle \tilde{J}_L^G(\Theta_{[m,n]_H}), \frac{d\gamma(\lambda)}{d\lambda}_{|\lambda=0} \right\rangle = -\left\langle \Theta_{[m,n]_H}, \frac{d[\gamma(\lambda)^{-1}m, n]_H}{d\lambda}_{|\lambda=0} \right\rangle. \tag{4.25}$$

It is trivial from the explicit form of the isomorphism (4.14) described above that (4.24) is duly transferred to (4.25).

This completes the proof of the isomorphism between (4.10) and (4.11), and therefore of Theorem 3. □

Since Morita equivalence of s-connected and s-simply connected Lie groupoids is isomorphism in LG', and Morita equivalence of Poisson manifolds is isomorphism in Poisson, we recover the known result [24] that the map $G \mapsto A^*(G)$ preserves Morita equivalence.

References

1. Baillet, M., Denizeau, Y., Havet, J.-F. : Indice d'une espérance conditionnelle. Compositio Math. **66**, 199–236 (1988)
2. Connes, A.: *Sur la théorie non commutative de l'intégration*. Lecture Notes in Math. **725**, 1979, pp. 19–143
3. Connes, A.: A survey of foliations and operator algebras. Proc. Sympos. Pure Math. **38**, 521–628 (1982)
4. Connes, A.: *Noncommutative Geometry*. San Diego: Academic Press, 1994
5. Connes, A., Takesaki, M.: The flow of weights on factors of type III. Tohoku Math. J. **29**, 473–575 (1977); Err. ibid. **30**, 653–655 (1978)

6. Coste, A., Dazord, P., Weinstein, A.: Groupoides symplectiques. Publ. Dépt. Math., Univ. C. Bernard, Lyon I **2A**, 1–62 (1987)
7. Courant, T.J.: Dirac manifolds. Trans. Am. Math. Soc. **319**, 631–661 (1990)
8. Dazord, P.: *Groupoïdes symplectiques et troisième théorème de Lie "non linéaire"*. Lecture Notes in Math. **1416**, Berlin–Heidelberg–New York: Springer, 1990, pp. 39–74
9. Dazord, P., Sondaz, D.: Variétés de Poisson – algébroïdes de Lie. Séminaire Sud-Rhodanien, Univ. Claude Bernard, Lyon, 1–66 (1988)
10. Feldman, J., Hahn, P., Moore, C.C.: Orbit structure and countable sections for actions of continuous groups. Adv. Math. **28**, 186–230 (1978)
11. Green, P.: The local structure of twisted covariance algebras. Acta Math. **140**, 191–250 (1978)
12. Haefliger, A.: Groupoïdes d'holonomie et classifiants. Astérisque **116**, 70–97 (1984)
13. Hahn, P.: Haar measure for measure groupoids. Trans. Am. Math. Soc. **242**, 1–33 (1978)
14. Hahn, P.: The regular representations of measure groupoids. Trans. Am. Math. Soc. **242**, 35–72 (1978)
15. Hilsum, M., Skandalis, G.: Morphismes K-orientés d'espaces de feuilles et fonctorialitéé en théorie de Kasparov (d'aprés une conjecture d'A. Connes). Ann. Sci. ÉÉcole Norm. Sup. (4) **20**, 325–390 (1987)
16. Karasev, M.V.: The Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fibre bundles of symplectic manifolds. I, II. Selecta Math. Soviet. **8**, 213–234, 235–258 (1989)
17. Kontsevich, M.: Formal (non)commutative symplectic geometry. In: *The Gelfand Mathematical Seminars, 1990–1992*, Boston: Birkhäuser, 1993, pp. 173–187
18. Krieger, W.: On ergodic flows and the isomorphism of factors. Math. Ann. **223**, 19–70 (1976)
19. Krishnaprasad, P.S., Marsden, J.E.: Hamiltonian structure and stability for rigid bodies with flexible attachments. Arch. Rat. Mech. An. **98**, 137–158 (1987)
20. Lance, E.C.: *Hilbert C^*-Modules. A Toolkit for Operator Algebraists*. Cambridge: Cambridge University Press, 1995
21. Landsman, N.P.: *Mathematical Topics Between Classical and Quantum Mechanics*. New York: Springer, 1998
22. Landsman, N.P.: Lie groupoid C^*-algebras and Weyl quantization. Commun. Math. Phys. **206**, 367–381 (1999); E-print math-ph/9807028
23. Landsman, N.P., Ramazan, B.: Quantization of Poisson algebras associated to Lie algebroids. In: Kaminker, J., Ramsay, A., Renault, J., Weinstein, A. (eds.), *Proc. Conf. on Groupoids in Physics, Analysis and Geometry*. Contemporary Mathematics, to appear; E-print math-ph/0001005
24. Landsman, N.P.: The Muhly–Renault–Williams theorem for Lie groupoids and its classical counterpart. Lett. Math. Phys. **54**, 43–59 (2001); E-print math-ph/0008005
25. Landsman, N.P.: Bicategories of operator algebras and Poisson manifolds. In: Longo, R. (ed.) *Mathematical Physics in Mathematics and Physics. Quantum and Operator Algebraic Aspects*. Fields Inst. Comm., to appear (2001); E-print math-ph/0008003
26. Landsman, N.P.: Quantized reduction as a tensor product. In: Landsman, N.P., Pflaum, M., Schlichenmaier, M. (eds.), *Quantization of Singular Symplectic Quotients*, Basel: Birkhäuser, 2001; E-print math-ph/0008004
27. Mackenzie, K.C.H.: *Lie Groupoids and Lie Algebroids in Differential Geometry*. Cambridge: Cambridge University Press, 1987
28. Mackenzie, K.C.H., Xu, P.: Integration of Lie bialgebroids. Topology **39**, 445–476 (2000)
29. Mackey, G.W.: Ergodic theory and virtual groups. Math. Ann. **166**, 187–207 (1966)
30. Moerdijk, I.: *Toposes and groupoids*. Lecture Notes in Math. **1348**, Berlin–Heidelberg–New–York: Springer, 1988, pp. 280–298
31. Moerdijk, I., Mrčun, J.: On integrability of infinitesimal actions. Eprint math.DG/0006042
32. Molino, P.: *Riemannian Foliations*. Basel: Birkhäuser, 1988
33. Moore, C.C., Schochet, C.: *Global Analysis on Foliated Spaces*. New York: Springer, 1988
34. Mrčun, J.: *Stability and Invariants of Hilsum–Skandalis Maps*. Ph.D. Thesis, Univ. of Utrecht (1996)
35. Mrčun, J.: Functoriality of the bimodule associated to a Hilsum–Skandalis map. K-Theory **18**, 235–253 (1999)
36. Muhly, P., Renault, J., Williams, D.: Equivalence and isomorphism for groupoid C^*-algebras. J. Operator Th. **17**, 3–22 (1987)
37. Pradines, J.: Théorie de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales. C. R. Acad. Sc. Paris **A263**, 907–910 (1966)
38. Ramsay, A.: Virtual groups and group actions. Adv. Math. **6**, 253–322 (1971)
39. Ramsay, A.: Topologies on measured groupoids. J. Funct. Anal. **47**, 314–343 (1982)
40. Renault, J.: *A Groupoid Approach to C^*-algebras*. Lecture Notes in Math. **793**. Berlin: Springer, 1980
41. Renault, J.: Représentation des produits croisés d'algèbres de groupoïdes. J. Operator Theory **18**, 67–97 (1987)
42. Rieffel, M.A.: Morita equivalence for C^*-algebras and W^*-algebras. J. Pure Appl. Alg. **5**, 51–96 (1974)

43. Samuélidès, M.: Mesures de Haar et W^*-couple d'un groupoïde mesuré. Bull. Soc. Math. France **106**, 261–278 (1978)
44. Sauvageot, J.-L.: Sur le produit tensoriel relatif d'espaces de Hilbert. J. Operator Theory **9**, 237–252 (1983)
45. Sauvageot, J.-L.: Image d'un homomorphisme et flot des poids d'une relation d'équivalence mesurée. Math. Scand. **42**, 71–100 (1978)
46. Schweizer, J.: Crossed products by equivalence bimodules. Univ. Tübingen preprint (1999)
47. Stachura, P.: Differential groupoids and C^*-algebras. E-print math.QA/9905097
48. Stadler, M.O., O'Uchi, M.: Correspondence of groupoid C^*-algebras. J. Operator Th. **42**, 103–119 (1999)
49. Sutherland, C.E., Takesaki, M.: Right inverse of the module of approximately finite-dimensional factors of type III and approximately finite ergodic principal measured groupoids. Fields Inst. Commun. **20**, 149–159 (1998)
50. Vaisman, I.: *Lectures on the Geometry of Poisson Manifolds*. Basel: Birkhäuser, 1994
51. Wassermann, A.: Operator algebras and conformal field theory. III. Fusion of positive energy representations of $\mathrm{LSU}(N)$ using bounded operators. Invent. Math. **133**, 467–538 (1998)
52. Weinstein, A.: The local structure of Poisson manifolds. J. Diff. Geom. **18**, 523–557 (1983); Err. ibid. **22**, 255 (1985)
53. Xu, P.: Morita equivalent symplectic groupoids. In: Dazord, P., Weinstein, A. (eds.), *Symplectic Geometry, Groupoids, and Integrable Systems,* New York: Springer, 1991, pp. 291–311
54. Xu, P.: Morita equivalence of Poisson manifolds. Commun. Math. Phys. **142**, 493–509 (1991)

Communicated by A. Connes