Review of *Linear Operators and their Spectra* by E. Brian Davies

(Cambridge studies in advanced mathematics 106, Cambridge University Press, Cambridge, 2007. 15 line figures, 274 exercises, 77 worked examples. xii + 451 pages, ISBN 0-521-86629-4. Hardcover, £42).

Brian Davies (King's College London) has had a long and distinguished career in the functional analysis/mathematical physics interface, including the authorship of classics like *Quantum theory of open systems* (1976), *One-parameter semigroups* (1980), *Heat kernels and spectral theory* (1989), and *Spectral theory and differential operators* (1995). He subsequently shocked at least his mathematical audience (and perhaps even his former self) by publishing *Science in the looking glass: What do scientists really know?* (2003), a critical survey of modern science, which includes an assault on Platonism as a philosophy of mathematics, and to some extent even on mathematics itself, portrayed as a subject with shaky foundations and questionable relevance to science. I considered this popular book a gem on appearance and feared the worst for the mathematical output of its author, but fortunately an uninterrupted sequence of research papers showed that his critical attitude towards mathematics has not stopped at least Davies himself in pursuing the subject. Subsequently, the book under review, which is a continuation of the technical series of books just listed and does not even mention *Science in the looking glass*, has removed all possible doubts about Davies' commitment to mathematics.

The only nod towards his previous book (which makes a case for constructive mathematics in the style of Bishop) is a dry comment in the Preface to the effect that "the present book has a slight philosophical bias towards explicit bounds and away from abstract existence theorems" (which in any case is a rather common stance among so-called 'hard analysts'), followed by a statement of preference for constructive rather than general proofs (which, however, is quite unusual for functional analysts). But in fact, this preference is by no means just "philosophical", as shown by the example of an "abstract existence theorem" Davies gives a few lines later, namely the nonemptiness of the spectrum of an operator. This is usually proved by *reductio ad absurdum*, which in particular means that the proof gives or suggests no procedure for actually finding explicit elements of the spectrum. Davies relates this to the phenomenon that the spectrum can be highly unstable under perturbation, a drawback that is not shared by the so-called pseudospectrum.

It is no accident that this notion already appears in the Preface, as its incorporation is one of the distinguishing features of the book. So what is the pseudospectrum of an operator - or rather, to begin with, what is the spectrum of an operator and which kind of operators are considered in this book? We know from linear algebra that a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ is injective if and only if it is surjective. Consequently, for any $z \in \mathbb{C}$, the map $A - z$ is invertible (i.e., injective and surjective) iff z is *not* an eigenvalue of a, i.e., if there exists no $f \in \mathbb{C}^n$ such that $Af = zf$. Hence the collection of eigenvalues of A may be characterized as the set of all $z \in \mathbb{C}$ for which $A - z$ is *not* invertible. It is the latter set that defines the *spectrum* $\mathrm{Spec}(A)$ also for bounded linear maps $A : \mathcal{B} \to \mathcal{B}$, where $\mathcal{B}$ is a Banach space (i.e. a complete normed vector space over $\mathbb{C}$); as has been known since the beginning of the 20th century, the possible definition of the spectrum as the collection of eigenvalues of A turns out to be too narrow for infinite-dimensional $\mathcal{B}$, where injectivity of a linear map is no longer equivalent to surjectivity. Perhaps surprisingly, this concept of a spectrum turned out to be the key to operator theory, on which “everything else is based” (p. 14).

One situation has been studied particularly well, from Hilbert to the present day, namely the case where $\mathcal{B}$ is a Hilbert space $\mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$ is self-adjoint (in that $(Af, g) = (f, Ag)$ for all $f, g \in \mathcal{H}$, in terms of the inner product $(\ ,\ )$ of $\mathcal{H}$). More generally, the notion of self-adjointness makes sense for unbounded operators $A : \mathcal{D}(A) \to \mathcal{H}$, where the domain $\mathcal{D}(A)$ of A is a dense linear subspace of $\mathcal{H}$, with essentially the same spectral theory as in the bounded case. Formalized by von Neumann (who brought ideas that had emerged from Hilbert and his school at Göttingen to maturity), the spectral theory of unbounded self-adjoint operators has been developed partly in response to quantum mechanics, where fundamental observables like position, momentum, and the Hamiltonian are of this form. More generally, there is a fertile interplay between boundary value problems for linear partial differential equations and unbounded self-adjoint operators on Hilbert space - it remains a miracle of the history of science how much of the theory of the Schrödinger equation of *quantum* mechanics (first written down in 1926) was already clear from the context of *classical* mathematical physics, as contained in the legendary monograph *Methoden der mathematischen Physik* by Courant and Hilbert from 1924. Today, one can find dozens of excellent books (and probably hundreds of books altogether) on the spectral theory of self-adjoint operators, including one by Davies himself (1995).

In contrast, the book under review is emphatically about the spectral theory of *non-self-adjoint* linear operators. What is meant by this is *a priori* not entirely clear, but the book pays attention to two different situations:

1. Non-self-adjoint operators on Hilbert space (where a comparison with the self-adjoint case is possible and often illuminating);

2. Operators on Banach space (where no such comparison can be made, so that the theory stands on its own).

Let us illustrate the first case with a little example (which is Problem 10.4.5 in the book). Most readers will know that the formula $(T_t f)(x) = f(x-t)$ describes a (strongly continuous) one-parameter unitary group on the Hilbert space $L^2(\mathbb{R})$, in the sense that the operator T_t is defined for each $t \in \mathbb{R}$, is unitary, and satisfies $T(s)T(t) = T(s+t)$ for all $s, t \in \mathbb{R}$ (as well as $\lim_{t\to 0} T(t)f = f$ for each f). By Stone's Theorem, such a unitary group is characterized by its generator H, which is defined and self-adjoint on the domain $\mathcal{D}(H)$ consisting of all f for which the limit $Hf = i \lim_{t\to 0} t^{-1}(T(t)f - f)$ exists. In the case at hand, this yields $H = -id/dx$, which is the momentum operator of quantum mechanics. Now let us slightly change this example, in replacing $L^2(\mathbb{R})$ by $L^2(\mathbb{R}^+)$. In that case, we need to redefine T_t by putting $(T_t f)(x) = f(x-t)$ for $0 \le t \le x$ and $(T_t f)(x) = 0$ for $0 \le x \le t$. This defines a semigroup (i.e. $t \in \mathbb{R}^+$) rather than a group, and the T_t are no longer unitary but merely isometric. Formally, the generator is still $-id/dx$, but it fails to be self-adjoint (and cannot even be extended to a self-adjoint operator): instead, it is just symmetric. Such a situation can be completely understood by comparison with the unitary case: self-adjoint generators produce unitary groups, but maximal symmetric generators merely give rise to semigroups of isometries.

Certain other problems in the theory of one-parameter (semi)groups of operators with non-self-adjoint generators relate to the Hilbert space setting, too; for example, one finds a complete characterization of generators of contractive one-parameter semigroups on Hilbert space, as well as of one-parameter groups for which $\|T_t\| \leq \exp(a|t|)$ for some $a \geq 0$ and all $t \in \mathbb{R}$. Further examples of case 1. above include Schrödinger operators with complex potentials, and the semi-classical analysis of pseudodifferential operators with complex symbols. Such topics receive extensive treatment in the book, and perhaps the only omission I noticed in this direction would be a discussion of incomplete motion in quantum mechanics as generated by non-self-adjoint Hamiltonians: note that in classical mechanics motion is not necessarily defined for all t (think of a particle falling off a table, or of the five-body problem with planets escaping to infinity in finite time), whereas in quantum theory self-adjoint Hamiltonians always generate complete motion by Stone's Theorem, as already mentioned. This obviously leaves a gap to be bridged, which quite splendidly would have fitted into Davies's book; a distinct lack of existing theory makes its absence understandable, though.

In any case, the thrust of the book lies in case 2. above, the spectral theory of general (linear) operators on Banach space. To set the stage, in the Preface the author laments that "Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to require a much wider range of knowledge, and accept that one cannot expect to have as high a rate of success when confronted with particular cases." Nonetheless, two general techniques appear to be widely applicable and could be said to dominate the book: *one-parameter semigroups* and *pseudospectra*. The former, already mentioned above in the Hilbert space setting, are an old specialty of the author, but he has much to add to his previous books on this subject (especially as far as the non-Hilbert-space setting is concerned). As he sees it, the theory of one-parameter semigroups on Banach spaces is a triangle, whose vertices are the semigroup $t \mapsto T_t$ itself, its generator $Z = -iH$ (as defined above), and the corresponding resolvent operators $(z - Z)^{-1}$ (defined for $z \notin \mathrm{Spec}(Z)$). Thus this triangle forms the basis of a fairly complete discussion, which largely supersedes earlier literature.

The principal innovative aspect of the book, however, lies in its coverage of pseudospectra and related notions. According to the online *Pseudospectra gateway* by Mark Embree and Nick Trefethen (see `http://web.comlab.ox.ac.uk/pseudospectra/`), pseudospectra have been independently invented at least five times between 1974 and 1990, motivated by the following observation. The linear equation $(A - \lambda)x = b$ for x has a unique solution whenever $\lambda \notin \mathrm{Spec}(A)$, obviously given by $x = (A - \lambda)^{-1}b$. Now if b is perturbed to b' with $\|b - b'\| \leq \varepsilon$ (or, alternatively, if b is only know with finite precision ε), then the corresponding solutions satisfy $\|x - x'\| \leq \varepsilon \, \|(A - \lambda)^{-1}\|$. For a self-adjoint operator A on a Hilbert space (more generally, for a normal operator), the number $\|(A - \lambda)^{-1}\|$ that evidently controls the stability of the solution turns out to be equal to the inverse of the distance between λ and $\mathrm{Spec}(A)$, so that, roughly speaking, it is small if λ is far from the spectrum and large if λ is almost an eigenvalue of A. For general operators A, however, $\|(A-\lambda)^{-1}\|$ can be large even if λ is far from $\mathrm{Spec}(A)$.

This motivates the introduction of the ε-pseudospectrum $\mathrm{Spec}_\varepsilon(A) \subseteq \mathbb{C}$ of A, defined for each $\varepsilon > 0$ as $\mathrm{Spec}_\varepsilon(A) = \{z \in \mathbb{C} \mid \|(A-z)^{-1}\| > \varepsilon^{-1}\}$ (in fact, Davies also includes $\mathrm{Spec}(A)$ into this set). The pseudospectrum of A then consists of the parametrized family $\{\mathrm{Spec}_\varepsilon(A)\}_{\varepsilon>0}$. Keeping in mind that for self-adjoint A on Hilbert space one has $\lambda \in \mathrm{Spec}(A)$ iff there exists a sequence (f_n) with $\|f_n\| = 1$ for all n such that $\lim_n \|(A-\lambda)f_n\| = 0$, it is interesting to know that $\lambda \in \mathrm{Spec}_\varepsilon(A)$ iff there exists a vector f with $\|f\| = 1$ such that $\|(A-\lambda)f\| < \varepsilon$.

Pseudospectra will be new to most functional analysts and mathematical physicists, and Davies provides extensive coverage with a good mix of theory and examples. The conclusion is that phenomena that formerly had been regarded as pathological behaviour of the spectrum of non-self-adjoint operators, such as the existence of approximate eigenvalues far from the spectrum or the instability of the spectrum under small perturbations, disappear if one replaces the spectrum by the pseudospectrum. In addition, Davies discusses other generalizations of the spectrum, like the hull and the numerical range of a bounded operator A on a Banach space $\mathcal{B}$. For a polynomial p, put $\mathrm{Hull}(p, A) = \{z \in \mathbb{C} \mid |p(z)| \leq \|p(A)\|\}$, and subsequently $\mathrm{Hull}_n(A) = \bigcap_{\deg(p)\leq n} \mathrm{Hull}(p, A)$. This is indeed a generalization of the spectrum, for if $\mathcal{B}$ has finite dimension n, one has $\mathrm{Spec}(A) = \mathrm{Hull}_n(A)$, whereas in general $\mathrm{Spec}(A) \subseteq \mathrm{Hull}_n(A)$ for all n.

Furthermore, if A acts on a Hilbert space $\mathcal{H}$, one may define its numerical range by $\mathrm{Num}(A) = \{\langle Af, f\rangle, f \in \mathcal{H}, \|f\| = 1\}$, as well as its closure $\overline{\mathrm{Num}}(A)$ in $\mathcal{H}$. Once, again, each polynomial p then leads to a set $\mathrm{Num}(p, A) = \{z \in \mathbb{C} \mid p(z) \in \overline{\mathrm{Num}}(p(A))\}$, with associated sets $\mathrm{Num}_n(A) = \bigcap_{\deg(p)\leq n} \mathrm{Num}(p, A)$. Then $\mathrm{Spec}(A) \subseteq \mathrm{Num}_n(A)$ for all n, and for self-adjoint A on Hilbert space one has $\mathrm{Spec}(A) = \mathrm{Num}_2(A)$. Amazingly, for general bounded operators on Hilbert space one has the equality $\mathrm{Num}_n(A) = \mathrm{Hull}_n(A)$ for all n (Burke–Greenbaum), and $\bigcap_n \mathrm{Hull}_n(A)$ (or $\bigcap_n \mathrm{Num}_n(A)$) turns out to be equal to the union of $\mathrm{Spec}(A)$ and all open subsets of $\mathbb{C}$ that are enclosed by $\mathrm{Spec}(A)$ (Nevanlinna).

One will look in vain for the notions of pseudospectrum, hull and numerical range in standard functional analysis texts, so Davies has done us a great service by explaining them through beautiful theorems and examples. More generally, his book is the first to offer a comprehensive survey of the spectral theory of non-self-adjoint operators, including both "classical" and "cutting edge" results, showing that this theory holds as much promise as the self-adjoint theory in both foundations and application. The scope of the book is truly enormous, and is only partly reflected by listing the chapter titles: Elementary operator theory, Function spaces, Fourier transforms and bases, Intermediate operator theory, Operators on Hilbert space, One-parameter semigroups, Special classes of semigroup (sic), Resolvents and generators, Quantitative bounds on operators, Quantitative bounds on semigroups, Perturbation theory, Markov chains and graphs, Positive semigroups, Non-self-adjoint Schrödinger operators.

My only criticisms would be that the organization of the book could have been better (for example, with a clearer chapterwise separation between Hilbert space and general Banach space results), and that historical or bibliographical notes are lacking. As the author states himself, the book is halfway between being a textbook and a monograph, which makes it difficult to say for whom the book is intended. I'd say that a firm background in Hilbert space theory and some feeling for basic Banach spaces such as $L^p(X)$ and $C(X)$ is necessary and sufficient to understand and appreciate this beautiful volume, which has no competitors.

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