Radbound University

IN·DEI·NOMINE·FELICITER

HONOURS THESIS

Properties of the Lattice $\mathcal{O}(\Sigma_A)$ Concerning Intuitionistic Quantum Logic

Evert-Jan M. Hekkelman
supervised by
Prof. Dr. N.P. LANDSMAN

Abstract

While traditionally quantum logic is regarded as a non-distributive type of logic, intuitionistic logic may suit the philosophy of quantum mechanics better. *Quantum toposophy* is the application of topos theory to quantum mechanics, an approach used by Landsman, Heunen and Spitters [8] resulting in an allocation of a topological space Σ_A to each C^*-algebra A, of which the topology $\mathcal{O}(\Sigma_A)$ forms a frame. This thesis explores some properties of that lattice. An attempt is made to calculate the Esakia dual of $\mathcal{O}(\Sigma_A)$, resulting in two candidates being written off while no conclusion is reached. It is proven that $A \mapsto \mathcal{O}(\Sigma_A)$ is a functor for the categories **CCStar** and $\mathbf{CStar}_{\hookrightarrow}$, of commutative C^*-algebras and C^*-algebras with injective morphisms respectively. Furthermore, it is proven that Σ_A is a sober space if and only if A is finite-dimensional, and for approximately-finite, commutative scattered C^*-algebras (such that $A = \overline{\bigcup D_\alpha}$) it is proven that Σ_A is not the limit of the spaces Σ_{D_α}.

August 17, 2018

Preface

It is with great satisfaction that I present this thesis. Whereas normally a bachelor thesis in mathematics at the Radboud University is a project lasting a semester, I am very grateful that I have received the opportunity to work on my thesis for two semesters thanks to the Radboud Honours Academy. This is the main reason why this thesis has become as lengthy as it is. It is, however, not the only abnormal aspect about it.

When I was fourteen years old, Quinten Rutgers and I ended up in a class together at highschool. Ever since, we have not managed to get separated. After highschool, we both chose to study Mathematics and Physics at the Radboud University and ended up with an overlapping group of friends. The past few years at the Radboud University, we often chose the same courses to follow and we even *both* opted to join the Radboud Honours Academy. The first year of the honours programme consisted of a group project which we did not do together, whereas the second year was supposed to be an opportunity to work individually on a thesis for a longer time than normal. That said, we managed to both choose the same supervisor and general research area for our theses independent of each other.

Thankfully, we became good friends in the past years, and as such it was decided that we would work together on the same subject as a thesis. Therefore, all results found in this thesis have been shared work. That is why it would also have made sense to make a single, joint thesis out of this project, however, we decided to both write our own thesis to incorporate our own views on the matter. Still, we have decided to share some chapters between our theses and to omit others. That means that a few chapters in this thesis are shared and appear in both Quinten's and my thesis, some of the chapters in this thesis will not appear at all in Quinten's, and vice versa. However, the chapters I omit are not solely Quinten's work, and the same goes for the other way around as well.

Working on this project was a fun journey, not in the least place due to the guidance by our supervisor, Klaas Landsman. He was always, and still is, ready to answer our every question. The weekly meetings with Klaas, Quinten and me always were pleasant and useful. I could not have wished for a better supervisor!

Lastly, it would be a crime not to thank Bert Lindenhovius for all he did for Quinten and me. As a former PhD student of Klaas Landsman working as a postdoctoral researcher at Tulane University, New Orleans, we had the opportunity of visiting him for two weeks in May 2018 at Tulane University. Not only was he very welcoming and friendly, he had made the full two weeks we were there free to work with us on this project. For that we also have to thank prof. dr. Michael Mislove, who was very welcoming too. The ideas of looking at AW^*-algebras and scattered C^*-algebras both came from Bert, and every time we were stuck on a proof he was willing to take the time to help us out. Bert even went so far as to sacrifice Memorial Day, one of the (very) few vacation days one gets when working in the USA, to help us out. As if that was not enough, Bert showed us around the city in the weekends and invited us to a great movie night with another postdoctoral researcher. All in all we had the time of our lives in New Orleans thanks to Bert, and for that we are very grateful.

Contents

Introduction

Ever since its inception at the beginning of the 20^{th} century, quantum mechanics has been one of the most troublesome theories in physics. Its postulates and rules seem incredibly unintuitive, to the point that many famous physicists have stated quantum mechanics cannot be *understood*:

> "I think I can safely say that nobody understands quantum mechanics." - Richard Feynman [4]

and

> "Quantum mechanics, that mysterious, confusing discipline, which none of us really understands but which we know how to use. It works perfectly, as far as we can tell, in describing physical reality, but it is a 'counter-intuitive discipline', as social scientists would say. Quantum mechanics is not a theory, but rather a framework, within which we believe any correct theory must fit." - Murray Gell-Mann [6]

are two of many examples. One of the reasons that this spectacularly successful theory remains an enigma is that the very *logic* underlying the theory is different from the logic we encounter in our macroscopic life. The first to coin the term 'quantum logic' were Garret Birkhoff and John von Neumann in their joint paper published in 1936. Their approach remains the dominant way to describe the logic behind quantum mechanics, yet today a rival is in development. This thesis will follow the break in tradition started by Christopher Isham and Jeremy Butterfield in 1999.

The starting point of all approaches to quantum logic is classical logic. However, then some axiom of classical logic will have to be dropped in order to make quantum mechanics 'fit'. The two branches of quantum mechanics disagree on which axiom has to bite the dust. To illustrate: say we perform measurements on a particle in a one-dimensional system, and conclude its position has to be in a certain region, for example between 0 and 2 (in some arbitrary units). Its momentum has been measured to be p, with certain uncertainty Δp. Then, classically, one could say the particle *either* has momentum $p \pm \frac{1}{2}\Delta p$ and position between 0 and 1, *or* momentum $p \pm \frac{1}{2}\Delta p$ and position between 1 and 2. This is not true in general in quantum mechanics, however! A most celebrated result of quantum mechanics is the uncertainty principle - which prevents the uncertainty in momentum, multiplied by the uncertainty in position, to be lower than some value. It could be that the statement 'the particle has momentum $p \pm \frac{1}{2}\Delta p$ and position between 0 and 1' presumes a higher simultaneous certainty in position and momentum than allowed by the uncertainty principle, while 'the particle has momentum $p \pm \frac{1}{2}\Delta p$ and position between 0 and 2' does not. Thus somewhere we have used an axiom of classical logic that is invalid in the world of quantum mechanics.

Birkhoff and Neumann would have analysed the situation as follows: if we take

- p = 'the particle has momentum $p \pm \frac{1}{2}\Delta p$',
- q = 'the particle has position between 0 and 1',
- r = 'the particle has position between 1 and 2',

then $p \wedge (q \vee r) \neq (p \vee q) \wedge (p \vee r)$. Therefore, in quantum logic there cannot be a distributive law.

This approach seems fair, but there is a different explanation. Landsman, Heunen and Spitters argued that it is the law of excluded middle that should be dropped. To explain that in this context, here the law is also used yet in a more subtle way. The crux lies in saying that the statement 'the particle has position between 0 and 2' is equal to stating 'q or r' with q and r defined as before. The law of the excluded middle, used often in classical logic, states that for any proposition p, the statement 'p is true or p is false' is always true itself. 'Removing' this law from logic does not mean, however, that propositions can be anything else but true or false, a fact proven by Valery Glivenko [7]. Say we start with the proposition 'the particle particle has position between 0 and 2', and we then make the statement 'the position has position between 0 and 1'. If that is false, then its position *must* be between 1 and 2. Therefore, we can view q and r as each other's negation, and to say that the original proposition is equal to saying 'q or r' we need to use the law of excluded middle. Therefore, instead of dropping distributivity, we could alternatively drop the law of the excluded middle to solve this dilemma.

Both approaches seem not without merit, however in my opinion the last approach roots out a more fundamental problem. The issue behind the dilemma posed above is that a proposition concerning some observable can be ripped apart in smaller propositions which all individually presume a higher certainty than the original proposition. This missing uncertainty has been transferred into the logical 'or' connection. By removing the distributive law, the first approach merely makes this procedure 'useless'. The second approach prohibits it altogether, which makes more sense. The latter propositions with lower uncertainty really reflect a different physical situation from what has been measured, especially since uncertainty plays a major role in quantum mechanics.

That said, there is no physical proof speaking for or against either approach, and both seem valid ways to describe quantum logic. My preference clearly lies with the second approach. The law of the excluded middle is a logical law not in line with the spirit of quantum mechanics, seen also in the case of Schrödinger's cat which is neither alive nor dead until measured. The distributive law on the other hand does not seem intrinsically unphysical. The logic that results is a subset of intuitionistic logic, and we will mean this kind of logic in the rest of this thesis when talking about quantum logic. More literature on this approach to quantum logic can be found in [11] [9] [2] [20].

A brief overview of the chapters in this thesis:

- In **chapter 1**, the foundations will be laid for the concept of frames and some properties of those;
- In **chapter 2**, the partially ordered sets $\mathcal{C}(A)$ and $\mathcal{P}(A)$ are introduced for C^*-algebras A;
- In **chapter 3**, Gelfand duality will be explained. This is mainly written by Quinten Rutgers, except for section 3.3;
- **In chapter 4**, Stone, Priestley and Esakia Dualities are discussed. This entire chapter is work by Patrick J. Morandi [14], with only slight changes;
- In **chapter 5**, the lattices $\mathcal{O}(\Sigma_A)$ and $Q(A)$ are introduced for C^*-algebras. These lattices are the focus of this thesis;
- In **chapter 6**, an attempt is made to calculate the Esakia dual of $\mathcal{O}(\Sigma_A)$. While two candidates can be written off, no conclusion is reached;
- In **chapter 7**, the lattices $\mathcal{O}(\Sigma_A)$ and $Q(A)$ are viewed from the perspective of AW^*-algebras. The definition of $Q(A)$ is then expanded to this category of AW^*-algebras. The idea of this expansion was Bert Lindenhovius's;
- In **chapter 8**, the functoriality of the mappings $A \mapsto \mathcal{O}(\Sigma_A)$ and $A \mapsto Q(A)$ is proven;
- In **chapter 9**, it is proven that the space Σ_A is sober if and only if A is finite-dimensional;

- In **chapter 10**, $\mathcal{O}(\Sigma)$ is considered for approximately finite, scattered C^*-algebras. This was also Bert Lindenhovius's idea.
- In **chapter 11**, the main results are summarised with some concluding remarks.

Finally, some chapters have been added as an appendix. These chapters are optional to read, but are considered as preliminary knowledge in the main body. They may be summarised as follows:

- In **appendix A**, elementary order theory is handled;
- In **appendix B**, some category theory is discussed. This appendix has been written by Quinten Rutgers;
- In **appendix C**, elementary functional analysis is explained.

Chapter 1

Lattices, Frames and Locales

There are a few kinds of lattices that will be important to us. One type that is used throughout classical logic is called a Boolean algebra.

Definition 1.0.1. A **Boolean algebra** L is a bounded distributive lattice in which for any a in L there exists a unique b in L that satisfies $a \wedge b = 0$, $a \vee b = 1$. This b is then called the **complement** of a, denoted by $b = \neg a$. A lattice morphism between Boolean algebras for which $f(\neg a) = \neg f(a)$ is called a **Boolean morphism**.

One should note that to define a Boolean algebra, either the relation $\leq$ or the operations $\wedge, \vee$ should be specified. That is, by giving defining the relation $\leq$ the operations $\wedge$ and $\vee$ can be recovered as these are respectively the largest lower bound and the highest upper bound of two elements. Likewise, one can recover the partial order $\leq$ from the operation $\wedge$ by defining $a \leq b$ if and only if $a \wedge b = a$.

In classical logic, the elements of a Boolean algebra would be interpreted as (equivalence classes of) propositions ordered by implication, i.e. if $x \leq y$ if the proposition x implies y. The operations $\wedge$, $\vee$ and $\neg$ are then interpreted as the logical 'and', 'or', and negation. The law of excluded middle then always holds in this kind of lattice, since $\neg\neg x = x$ in any Boolean algebra. For intuitionistic purposes a different lattice structure is needed, which is called a Heyting algebra.

Definition 1.0.2. A **Heyting algebra** H is a bounded, distributive lattice with an operation $\rightarrow: H \times H \rightarrow H$, called the *implication*, such that $c \wedge a \leq b$ is equivalent to $c \leq a \rightarrow b$. Equivalently, $\rightarrow$ is an operation such that:

- $a \rightarrow a = 1$,
- $a \wedge (a \rightarrow b) = a \wedge b$,
- $b \wedge (a \rightarrow b) = b$,
- $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.

A lattice morphism between Heyting algebras for which $f(a \rightarrow b) = f(a) \rightarrow f(b)$ is called a **Heyting morphism**.

Intuitionistic mathematicians likewise interpret Heyting algebras as (equivalence classes of) propositions ordered by implication. This time, however, the negation which can be introduced by $\neg x = x \rightarrow 0$, does not satisfy the law of excluded middle. Note that the negation in Heyting algebras is a derived operation, whereas in Boolean algebras it is part of the definition.

Since the identity map $f(a) = a$ from any Boolean algebra to itself is a Boolean morphism, and the composite $g \circ f$ of any two morphisms f and g is a Boolean morphism, Boolean algebras form a category. The same goes for Heyting algebras. These categories are denoted by **BA** and **HA**, respectively.

One fact that will be important to us is that the open sets $\mathcal{O}(X)$ of any topological space X form a bounded, distributive lattice, where join and meet correspond with the union and intersection of sets. In such a lattice we have even more structure, however, as any union of open sets $\bigcup_{i\in I} U_i$ is again an open set. Furthermore, these lattices satisfy the infinite distributive law:

$$V \cap \left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I}(V \cap U_i).$$

This is the motivation for the following definition.

Definition 1.0.3. A **frame** is a bounded lattice L in which any subset $S \subseteq L$ has a supremum $\bigvee S \in L$ that satisfies the *infinite distributive law*:

$$a \wedge \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \wedge b_i).$$

A **frame morphism** $\phi : L \to K$ is a lattice morphism between two frames which preserves all infinite suprema. This forms a category, **Frm**. [15]

A potentially confusing aspect of this definition is that frames are in fact complete lattices. As the infimum of any subset $S \subseteq L$ is merely the supremum of all lower bounds, a frame must be complete. Another important aspect of frames is that all frames are also Heyting algebras, by defining the implication as follows:

$$a \to b = \bigvee \{c \mid c \wedge a \leq b\}.$$

Likewise, any complete Heyting algebra is also be a frame [15].

Finally, the category of locales will play a role.

Definition 1.0.4. The category **Loc** of locales is the opposite category of **Frm**.

These categories are mainly relevant in the subject *pointless topology*, which will not be reviewed in-depth here. Some concepts originating from this line of research will come up, however.

Definition 1.0.5. In a topological space X, a set $S \subseteq X$ is called **meet-irreducible** if, for all open sets $U, V \subseteq X$, S satisfies:

$$\text{if } U \cap V \subseteq S, \text{ then either } U \subseteq S \text{ or } V \subseteq S.$$

S is called **join-irreducible** if for all closed sets $F, G \subseteq X$, S satisfies:

$$\text{if } S \subseteq F \cup G, \text{ then either } S \subseteq F \text{ or } S \subseteq G.$$

From this it follows that:

An *open* set $S \subseteq X$ is called **meet-irreducible** if for all *open* sets $U, V \subseteq X$, S satisfies:

$$\text{if } U \cap V = S, \text{ then either } U = S \text{ or } V = S.$$

A *closed* set $S \subseteq$ is called **join-irreducible** if for all *closed* sets $F, G \subseteq X$, S satisfies:

$$\text{if } F \cup G = S, \text{ then either } S = F \text{ or } S = G.$$

In fact, any set of the form $X\backslash\overline{\{x\}}$ is meet-irreducible, and any set of the form $\overline{\{x\}}$ is join-irreducible. That leads us to the following definition:

Definition 1.0.6. A topological space X is said to be **sober** if the only open meet-irreducible sets are of the form $X\backslash\overline{\{x\}}$, or, equivalently, if the only closed join-irreducible sets are of the form $\overline{\{x\}}$.

The motivation for these definitions is that it can be helpful to determine which frames or locales are (isomorphic to) $\mathcal{O}(X)$, i.e. the lattice of a topology. These frames and locales are called *spatial*. Sobriety is precisely a condition that forces the lattice of open subsets of X to determine X up to homeomorphism, which can be used to establish an equivalence between the category **Sob** of sober spaces, and the category **SLoc** of spatial locales [15].

There is more to be said about locales, however.

Definition 1.0.7. A map $f : L \to M$ between locales is called **localic**, if it preserves all infima, and the corresponding left adjoint $f^* : M \to L$ preserves finite meets. These are exactly the morphisms in the category **Loc**.

Definition 1.0.8. A subset $S \subseteq L$ of a locale L is a **sublocale** if both:

1. S is closed under all meets,
2. for every $s \in S$ and $x \in L$, $x \to s \in S$.

The second condition may seem strange, but this is needed to make sublocales behave 'nicely'.

Proposition 1.0.9. *Let L be a locale. A subset $S \subseteq L$ is a sublocale if and only if it is a locale in the induced order and the embedding map $j :\to L$ is a localic map. When this is the case, the meets and Heyting operation in S coincide with those in L. In general, the joins differ.*

The proof of this proposition is omitted, but can be found in the book 'Frames and Locales' by J. Picado and A. Pultr [15]. Sublocales can also be characterized via another route.

Definition 1.0.10. A **nucleus** in a locale L is a mapping $\nu : L \to L$ such that for any $a, b \in L$:

1. $a \leq \nu(a)$,
2. $a \leq b$ implies $\nu(a) \leq \nu(b)$,
3. $\nu(\nu(a)) = \nu(a)$,
4. $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

Proposition 1.0.11. *For a sublocale $S \subseteq L$, the map*

$$\begin{aligned} \nu_S &: L \to L; \\ &a \mapsto \bigwedge \{s \in S \mid a \leq s\}, \end{aligned}$$

is a nucleus. Likewise, for a nucleus $\nu : L \to L$, the set $\nu(L)$ is a sublocale.

Again, the proof is omitted, but it can be found in the same book as above [15]. Finally, we briefly touch upon the concept of the closure of a sublocale.

Definition 1.0.12. Let $S \subseteq L$ be a sublocale. Then S is a **closed** sublocale of L if $S = \uparrow U$, for some $U \in L$.

As with many concepts concerning locales, this definition may seem unintuitive. There exists a way, however, to make this definition coincide with the standard topological definition of a closed set. The gist of it is that any element in a locale can generate a sublocale in L. When L is spatial these sublocales are also spatial, which turns out to recover the open sets in the topological space X completely. In that context, the definition above coincides with taking the complement of such an open set. With the notion of closed sets comes the notion of taking a closure of a set:

Definition 1.0.13. The **closure** of a sublocale $S \subseteq L$ is defined by $\overline{S} = \uparrow(\bigwedge S)$. Furthermore, S is **dense** in L if $\overline{S} = L$.

Proposition 1.0.14. *A sublocale $S \subseteq L$ is dense if and only if the corresponding nucleus $\nu_S : L \to L$ satisfies $\nu_S(0) = 0$.*

Proof. The proof is elementary. S is dense if $L = \uparrow(\bigwedge S)$, which happens if and only if $0 = \bigwedge S$. By the correspondence from proposition 1.0.11, this is the case only if and only if $\nu_S(0) = 0$. □

Chapter 2

Lattices in C^*-Algebras

Definition 2.0.1. Let A be a C^*-algebra. Then $\mathcal{C}(A)$ is defined as the partially ordered set of commutative C^*-subalgebras of A, ordered by inclusion.

Much can be said about this concept, as indicated by the fact that an entire PhD-thesis has been written solely about $\mathcal{C}(A)$ by Bert Lindenhovius [12]. Some interesting results are that $A \mapsto \mathcal{C}(A)$ is a functor **CStar** $\to$ **DCPO**, where the category **DCPO** is a subcategory of the category **Poset** of partially ordered sets consisting of directed-complete partially ordered sets (DCPOs). A DCPO is a poset in wich every directed subset has a supremum. A few other interesting facts concerning $\mathcal{C}(A)$ are the following:

Proposition 2.0.2. *Let A be a C^*-algebra. Then the following statements are equivalent:*

1. *A is commutative;*
2. *$\mathcal{C}(A)$ has a greatest element;*
3. *$\mathcal{C}(A)$ is bounded;*
4. *$\mathcal{C}(A)$ is a complete lattice.*

Theorem 2.0.3. *Let A be a C^*-algebra. Then:*

1. *The C^*-subalgebra $\mathbb{C} \cdot \mathbb{1}_A$ is the least element of $\mathcal{C}(A)$;*
2. *The infimum of a non-empty subset $S \subseteq \mathcal{C}(A)$ is given by $\bigcap S$;*
3. *If a subset $\mathcal{D} \subseteq \mathcal{C}(A)$ is directed, then its supremum $\bigvee \mathcal{D}$ exists and is given by $\bigvee \mathcal{D} = \overline{\bigcup \mathcal{D}}$; in case A is finite-dimensional, then $\bigvee \mathcal{D} = \bigcup \mathcal{D}$;*
4. *For each $C \in \mathcal{C}(A)$, there is an $M \in max(\mathcal{C}(A))$ such that $C \subseteq M$. In particular, $max(\mathcal{C}(A))$ is non-empty, and its elements are exactly the maximal commutative C^*-subalgebras of A.*

Both proofs are omitted. $\mathcal{C}(A)$ is not the only poset made from a C^*-algebra that will be relevant to us.

Definition 2.0.4. Let A be a C^*-algebra. An element $e \in A$ is called a **projection** if $e = e^* = e^2$. The set of all projections in A is denoted by $\mathcal{P}(A)$.

Proposition 2.0.5. *Let A be a C^*-algebra. Define a relation on the projections in A by $e \leq f$ if and only if $e = ef$. This relation defines a partial order on $\mathcal{P}(A)$.*

Proof. We check all requirements in the definition of a partial order A.0.1. Take $e, f, g \in \mathcal{P}(A)$. Then:

- $e = e^2$ and therefore $e \leq e$;
- If $e = ef$ and $f = fe$, then $e = e^* = (ef)^* = f^*e^* = fe = f$;
- If $e = ef$ and $f = fg$, then $e = ef = e(fg) = (ef)g = eg$.

□

This partially ordered set has more structure, however.

Proposition 2.0.6. *Let A be a unital, commutative C^*-algebra. Then $\mathcal{P}(A)$ forms a Boolean algebra.*

Proof. For projections $e, f \in \mathcal{P}(A)$, define the relations $\vee, \wedge$ as follows:

$$\begin{aligned} p \vee q &= p + q - pq, \\ p \wedge q &= pq. \end{aligned}$$

These are indeed projections:

$$\begin{aligned} (p+q-pq)^2 &= p^2 + pq - p^2q + qp + q^2 - pq^2 - p^2q - pq^2 + p^2q^2 = p + q - pq, \\ (p+q-pq)^* &= p^* + q^* - (pq)^* = p + q - pq, \\ (pq)^2 &= p^2q^2 = pq, \\ (pq)^* &= q^*p^* = pq. \end{aligned}$$

Note that it is vital that A is a commutative C^*-algebra. Let us now show that these indeed define the meet and join in this lattice:

$$\begin{aligned} p(p+q-pq) &= p^2 + pq - p^2q = p, \\ p(pq) &= pq, \end{aligned}$$

and therefore $p \wedge q \leq p \leq p \vee q$ for any $q \in \mathcal{P}(A)$. Furthermore, suppose that for two projections $e, f \in \mathcal{P}(A)$: $p \wedge q \leq e \leq p, q \leq f \leq p \vee q$. Then:

$$\begin{aligned} e(pq) &= pq, \\ ep &= e, \\ eq &= e, \\ \Rightarrow pq &= e(pq) = eq = e, \\ f(p+q-pq) &= f, \\ fp &= p, \\ fq &= q, \\ \Rightarrow f &= f(p+q-pq) = fp + fq - (fp)q = p + q - pq. \end{aligned}$$

And thus $e = p \wedge q$, $f = p \vee q$.

Now take a projection $e \in \mathcal{P}(A)$. Then $\mathbb{1} - e$ also is a projection:

$$\begin{aligned} (\mathbb{1} - e)^2 &= \mathbb{1} - 2e + e^2 = \mathbb{1} - e, \\ (\mathbb{1} - e)^* &= \mathbb{1}^* - e^* = \mathbb{1} - e. \end{aligned}$$

This is in fact the complement of e:

$$\begin{aligned} e \vee (\mathbb{1} - e) &= e + \mathbb{1} - e - e(\mathbb{1} - e) = \mathbb{1} - e + e^2 = \mathbb{1}, \\ e \wedge (\mathbb{1} - e) &= e(\mathbb{1} - e) = e - e^2 = 0. \end{aligned}$$

Therefore, $\mathcal{P}(A)$ is a Boolean algebra. □

Proposition 2.0.7. *Let A be a finite-dimensional, commutative unital C^*-algebra. Then $\mathcal{P}(A)$ is a finite lattice, and therefore complete.*

Proof. For finiti-dimensional, commutative unital C^*-algebras, $A \cong \mathbb{C}^n$ by theorem C.0.11. Let $e = (e_1, ..., e_n) \in A$ be a projection. Then since $e^2 = e$, $e_i^2 = e_i$ for each $i = 1, ...n$. Therefore, $e_i = 0$ or $e_i = 1$ for each $i = 1, ...n$. Each of these combinations defines a projection, and therefore $\mathcal{P}(A)$ has 2^n elements.

Furthermore, every subset $S \subseteq \mathcal{P}(A)$ is finite, and therefore

$$\bigvee S = s_1 \vee ...s_m$$

exists, where $s_1, ...s_m$ are the elements of S. As such, $\mathcal{P}(A)$ is a complete lattice. □

$\mathcal{P}(A)$ is also a complete lattice for commutative Boolean AW^*-*algebras*, which will be explained in section 7.

Chapter 3

Gelfand Duality

In this section we will explore Gelfand duality, which gives a characterization of commutative C*-algebras in terms of topological spaces. We start with the following.

3.1 Main Theorem

Lemma 3.1.1. *For X a compact Hausdorff space the space of continuous functions $X \to \mathbb{C}$ denoted by $C(X)$ is a C*-algebra if we define addition and scalar multiplication pointwise and furthermore*

1. $||f|| = \sup_{x \in X} |f(x)|$
2. $f^*(x) = \overline{f(x)}$

It turns out that all unital commutative C*-algebras are *-isomorphic to $C(X)$ for some compact Hausdorff space X. There are many possible realizations of this space but the easiest definition is in terms of characters, also called multiplicative functionals.

Definition 3.1.2. A **character** of a C*-algebra A is a *-homomorphism $\phi : A \to \mathbb{C}$ from A to the C*-algebra of complex numbers. We denote the set of characters of A by $\Sigma(A)$. It is called the **Gelfand spectrum** of A.

Lemma 3.1.3. *Let $\phi : A \to \mathbb{C}$ be a character. Then ϕ is bounded with $||\phi|| = 1$.*

This space is notably a subspace of the *dual space* of A (see appendix C). A natural topology on this dual space A^* is the **weak-* topology**. This topology is defined by the following:

Definition 3.1.4. If $(\phi_\alpha)_{\alpha \in I}$ is a net in A^* and ϕ is in A^* for some C^*-algebra A, then we have $\phi_\alpha \to \phi$ **weak-*** in A^* if and only if $\phi_\alpha(a) \to \phi(a)$ for every $a \in A$. This is why the weak-* topology is also sometimes called the **topology of pointwise convergence**.

As a subspace of A^*, $\Sigma(A)$ inherits this **weak-* topology**. This is equivalent with the initial topology with respect to the mapping

$$\begin{aligned} \hat{a} : \Sigma(A) &\to \mathbb{C}; \\ \phi &\mapsto \phi(a), \end{aligned}$$

where $a \in A$.

The following is a special case of the **Banach-Alaoglu Theorem** in functional analysis [13]:

Proposition 3.1.5. *Let A^* be the dual space of some C^*-algebra A. The norm-closed unit ball, $\{\phi \in A^* \mid ||\phi|| \leq 1\}$, is compact in the weak-* topology.*

Another useful proposition from functional analysis is the following:

Proposition 3.1.6. *Let A^* be the dual space of some C^*-algebra A. Then A^* is Hausdorff.*

Proofs of both propositions can be found in [13]. With these in hand, we can prove that $\Sigma(A)$ is indeed a compact Hausdorff space.

Lemma 3.1.7. *Let A be a unital commutative C^*-algebra. Then $\Sigma(A)$ is a compact Hausdorff space in the weak-* topology.*

Proof. The space $\Sigma(A)$ is a subspace of a Hausdorff space by proposition 3.1.6, and as such is Hausdorff itself. For the compactness, it is sufficient to prove $\Sigma(A)$ is a closed in the weak-* topology, as then proposition 3.1.5 will guarantee that $\Sigma(A)$ is compact. Let $(\phi_\alpha)_{\alpha \in I}$ be a net in $\Sigma(A)$, converging to a $\phi \in A^*$, that is, $\phi_\alpha \to \phi$ weak-*. To show that $\phi \in \Sigma(A)$ we have to show it is multiplicative. Let a, b be in A. Since $\phi_\alpha \to \phi$ weak-*, we have that $\phi_\alpha(a) \to \phi(a)$, $\phi_\alpha(b) \to \phi(b)$, and $\phi_\alpha(ab) \to \phi(ab)$. On the one hand,

$$\phi_\alpha(ab) = \phi_\alpha(a)\phi_\alpha(b) \to \phi(a)\phi(b),$$

while on the other hand $\phi_\alpha(ab) \to \phi(ab)$. In a Hausdorff space, a net converges to at most one point and thus $\phi(ab) = \phi(a)\phi(b)$. Since $1 = \phi_\alpha(\mathbb{1}_A) \to \phi(\mathbb{1}_A)$, ϕ is indeed a *-morphism, and therefore an element of $\Sigma(A)$. □

We now have a map $\gamma_A : A \to C(\Sigma(A))$ defined by

$$a \mapsto \hat{a},$$
$$\hat{a}(\phi) = \phi(a).$$

This is called the **Gelfand representation** of A.

Theorem 3.1.8 (Gelfand duality)**.** *Let A be a unital commutative C^*-algebra. Then for every $a \in A$, $\gamma_A(a) = \hat{a}$ is a continuous map. Furthermore, the Gelfand representation is a *-isomorphism between A and $C(\Sigma(A))$.*

The proof is omitted, and can be found in [13].

3.2 Functoriality of Gelfand duality

In order to make Gelfand duality a true duality of categories we need to consider morphisms. We will state the following for the record.

Definition 3.2.1. The category of unital commutative C*-algebra **CCStar** has

1. Unital commutative C*-algebras as objects
2. *-homomorphisms as morphisms

Definition 3.2.2. The category of compact Hausdorff spaces **CptHaus** has

1. Compact Hausdorff spaces as objects
2. Continuous maps as morphisms

Lemma 3.2.3. *Gelfand spectrum Σ is a contravariant functor* **CCStar** $\to$ **CptHaus**.

Proof. We have already established that for a unital commutative C*-algebra A, $\Sigma(A)$ is a compact Hausdorff space. Now consider a *-homomorphism $f : A \to B$. We obtain a map $\Sigma(f) : \Sigma(B) \to \Sigma(A)$, $\Sigma(f)(\phi) = \phi \circ f$. We claim that this map is continuous. Take some net $(\phi_\alpha)_{\alpha \in I}$ converging weak-* to ϕ, and an element $a \in A$. Then:

$$\left(\Sigma(f)(\phi_\alpha)\right)(a) = \phi_\alpha(f(a)) \to \phi(f(a)) = \phi \circ f(a),$$

and therefore $\Sigma(f)(\phi_\alpha) \to \Sigma(f)(\phi)$ weak-* and thus $\Sigma(f)$ is continuous. The functoriality is easy. $\Sigma(\mathrm{id}_A)(\phi) = \phi \circ \mathrm{id}_A = \phi$ so $\Sigma(\mathrm{id}_A) = \mathrm{id}_{\Sigma(A)}$. If $f : A \to B$ and $g : B \to C$, then $\Sigma(f)(\Sigma(g)(\phi)) = (\phi \circ g) \circ f = \phi \circ (g \circ f) = \Sigma(g \circ f)(\phi)$. □

Lemma 3.2.4. *$C(\bullet)$ is a contravariant functor* **CptHaus** $\to$ **CCStar**.

Proof. We know that $C(X)$ is a unital commutative C*-algebra. Let $\phi : X \to Y$ be a continous map. We obtain $C(\phi) : C(Y) \to C(X)$ by $C(\phi)(f) = f \circ \phi$. This is a *-homomorphism because all operations are defined pointwise. The proof of functoriality is exactly the same. □

Lemma 3.2.5. *Let X be a compact Hausdorff space. Then the map $X \to \Sigma(C(X))$ given by $x \mapsto \mathrm{ev}_x$ is a homeomorphism.*

Theorem 3.2.6. *There is a duality of categories* **CCStar** $\to$ **CptHaus**.

Proof. We need to prove that $C \circ \Sigma \cong \mathrm{id}_{\mathbf{CCStar}}$ and $\Sigma \circ C \cong \mathrm{id}_{\mathbf{CptHaus}}$. For the first take a *-homomorphism $f : A \to B$ and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow\wr & & \downarrow\wr \\ C(\Sigma(A)) & \xrightarrow{C(\Sigma(f))} & C(\Sigma(B)) \end{array}$$

Let $a \in A$. We then have

$$C(\Sigma(f))(\hat{a})(\phi) = (\hat{a} \circ \Sigma(f))(\phi) = \hat{a}(\phi \circ f) = (\phi \circ f)(a) = \phi(f(a)) = \widehat{f(a)}(\phi)$$

so that $C(\Sigma(f))(\hat{a}) = \widehat{f(a)}$. Now take a continuous map $\phi : X \to Y$ and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad \phi \quad} & Y \\ \downarrow\wr & & \downarrow\wr \\ \Sigma(C(X)) & \xrightarrow{\Sigma(C(\phi))} & \Sigma(C(Y)) \end{array}$$

Let $x \in X$. We then have

$$\Sigma(C(\phi))(\mathrm{ev}_x)(f) = (\mathrm{ev}_x \circ C(\phi))(f) = \mathrm{ev}_x(C(\phi)(f)) = \mathrm{ev}_x(f \circ \phi) = f(\phi(x)) = \mathrm{ev}_{\phi(x)}(f)$$

so that $\Sigma(C(\phi))(\mathrm{ev}_x) = \mathrm{ev}_{\phi(x)}$. □

3.3 Restriction to Projections

Recall that the set of projections in a C^*-algebra is denoted by $\mathcal{P}(A) \subseteq A$, and that for any projection $e \in \mathcal{P}(A)$ we have that $e^2 = e$. Now take any character $\phi : A \to \mathbb{C}$. It is then clear that $\phi(e)^2 = \phi(e)$ holds as well, and therefore $\phi(e) = 0$ or $\phi(e) = 1$. With this, a nice isomorphism can be constructed.

Proposition 3.3.1. *Take a commutative C^*-algebra A, and consider the mapping:*

$$\begin{aligned} \beta_A : \mathcal{P}(A) &\to \mathrm{Cl}(\Sigma(A)); \\ e &\mapsto (\gamma_A(e))^{-1}\{1\} = \{\phi \in \Sigma(A) \mid \phi(e) = 1\}. \end{aligned}$$

Then β_A is an isomorphism of partially ordered sets.

Proof. We prove this in a few steps:

- β_A is well defined: since $\{\phi \in \Sigma(A) \mid \phi(e) = 1\} = \Sigma(A) \setminus \{\phi \in \Sigma(A) \mid \phi(e) = 0\}$, $\beta_A(e)$ is both an open and a closed set in the weak-* topology.

- β_A is injective: the functionals in $\Sigma(A)$ separate the elements of A [13], which implies that for every two projections $e \neq f \in \mathcal{P}(A)$ there exists a $\phi \in \Sigma(A)$ such that $\phi(e) = 1$, $\phi(f) = 0$ or $\phi(e) = 0$, $\phi(f) = 1$. This implies that $\beta_A(e) \neq \beta_A(f)$ for each $e \neq f$.

- β_A is surjective: take some clopen set $U \in \mathrm{Cl}(\Sigma(A))$. Then consider the functional $\chi_U : \Sigma(A) \to \mathbb{C}$, defined as the characteristic function of the set U. By Gelfand duality, this can be seen as an element of A: $\gamma_A^{-1}(\chi_U)$. We claim that $\gamma_A^{-1}(\chi_U)$ is a projection and that $\beta_A(\gamma_A^{-1}(\chi_U)) = U$. It is clear that $\chi_U^* : \Sigma_A \to \mathbb{C}$ and $\chi_U^2 : \Sigma_A \to \mathbb{C}$ are both equal to χ_U, and therefore via Gelfand duality $\gamma_A^{-1}(\chi_U)$ is indeed a projection. Lastly,

$$\beta_A(\gamma_A^{-1}(\chi_U)) = \left\{\phi \in \Sigma(A) \mid \phi(\gamma_A^{-1}(\chi_U)) = 1\right\} = \{\phi \in \Sigma(A) \mid \chi_U(\phi) = 1\} = U.$$

- β_A is strict order preserving: take projections $e \leq f \in \mathcal{P}(A)$. Then $ef = e$, and therefore also $\phi(e)\phi(f) = \phi(ef) = \phi(e)$ for each $\phi \in \Sigma(A)$. Therefore, if $\phi(e) = 1$ this implies that $\phi(f) = 1$ and thus $\beta_A(e) \subseteq \beta_A(f)$. Furthermore, take $e, f \in \mathcal{P}(A)$ such that $\beta_A(e) \subseteq \beta_A(f)$. Then $\phi(e) = 1$ implies that $\phi(f) = 1$, and therefore $\phi(ef) = \phi(e)\phi(f) = \phi(e)$ holds for every $\phi \in \Sigma(A)$. Since the characters $\Sigma(A)$ separate all elements, it follows that $e = ef$ and thus $e \leq f$.

As a bijective, strictly order preserving map, β_A is an isomorphism of partially ordered sets. □

By this proposition, it follows that β_A is also an isomorphism of Boolean algebras, as it gives an isomorphism between two partially ordered sets and which can therefore not be different Boolean algebras.

Chapter 4

Stone, Priestley and Esakia Dualities

There exist many duality theorems regarding distributive lattices, of which three will be discussed. Priestley duality is the most general case, Stone and Esakia duality turn out to be special cases of the former. That said, Stone duality is the simplest theorem to prove and will be handled first. As indicated in the introduction, this chapter is mainly work by Patrick J. Morandi [14], with only slight changes.

4.1 Stone Duality

Stone duality concerns Boolean algebras and Stone spaces, which are defined as follows:

Definition 4.1.1. A **Stone space** X is a topological space that is compact and Hausdorff, and in which all the clopen subsets in X form a basis of the topology. (This means any open set U can be written as a union of clopen subsets of X.)

Recall that the category of Boolean algebras is denoted by **BA**, in which the morphisms are lattice homomorphisms that preserve complementation, i.e. $f(\neg x) = \neg f(x)$ which we will call Boolean homomorphisms. The category of Stone spaces is denoted by **Stone**, and its morphisms are continuous maps.

To prove the duality between these categories, some steps are needed. Firstly, a construction is needed of the partially ordered space $\mathcal{PF}(L)$ for a bounded distributive lattice. This is defined as the set of prime filters in L ordered by inclusion, with a topology generated by the sets $\phi(a) = \{F \in \mathcal{PF}(L) \mid a \in F\}$ and their complements.

If we denote the opens in this topology on $\mathcal{PF}(L)$ by $\mathcal{O}(\mathcal{PF}(L))$, then this ϕ can be seen as a lattice homomorphism.

Lemma 4.1.2. *The map*

$$\begin{aligned} \phi &: L \to \mathcal{O}(\mathcal{PF}(L)); \\ & a \mapsto \{F \in \mathcal{PF}(L) \mid x \in F\}, \end{aligned}$$

is an injective lattice homomorphism.

Proof. First, note that $\phi(0) = \emptyset$, and $\phi(1) = \mathcal{PF}(L)$. Next, $\phi(a) \cup \phi(b) \subseteq \phi(a \vee b)$ and $\phi(a \wedge b) \subseteq \phi(a) \cap \phi(b)$ are both clear since filters are upsets. The inclusion $\phi(a) \cap \phi(b) \subseteq \phi(a \wedge b)$ is true since filters are closed under finite meets. Finally, if $P \in \phi(a \vee b)$, then $a \vee b \in P$ and since P is a prime filter, either a or b is in P. Thus $P \in \phi(a) \cup \phi(b)$.

Now for the injectivity, suppose that $a \neq b$. Then we may assume without loss of generality that $a \not\leq b$, so $a \wedge \neg b \neq 0$. By lemma A.0.13 (which can be found in appendix A) there exists a prime filter P containing $a \wedge \neg b$. Therefore, a and $\neg b$ are in P. This means P cannot be an element of $\phi(b)$, yet it is an element of $\phi(a)$. Therefore, $\phi(a) \neq \phi(b)$. $\square$

With this lemma we can now also conclude that the collection of sets $\phi(a) \cap \phi(b)^c$ forms a basis of the topology on $\mathcal{PF}(L)$.

Lemma 4.1.3. *If L is a bounded distributive lattice, then $\mathcal{PF}(L)$ is a Stone space.*

Proof. Since the topology is generated by clopen sets by definition, one property is already satisfied. What needs to be checked is if $\mathcal{PF}(L)$ is Hausdorff and compact.

- $\mathcal{PF}(L)$ is Hausdorff: Let $F \neq G$ be filters in L. Suppose without loss of generality $F \not\subseteq G$. Take an $a \in F - G$, then $F \in \phi(a)$, but $G \in (\phi(a))^c$. Therefore, these are disjoint open sets that separate F and G.

- $\mathcal{PF}(L)$ is compact: It suffices to prove that any cover of $\mathcal{PF}(L)$ by opens in the defined basis has a finite subcover. The reason is that if an arbitrary open covering $\mathcal{U}$ is chosen, for any $U \in \mathcal{U}$ and $x \in U$ a basic open $V_{x,U}$ can be found such that $x \in V_{x,U} \subseteq U$. Now the opens $V_{x,U}$ define a new covering $\mathcal{V}$. Since this is a covering by basic opens, a finite subcover $V_{x_1,U_1}, ..., V_{x_n,U_n}$ can be found. Since $V_{x_i,U_i} \subseteq U_i$ for every i, $U_1, ..., U_n$ is then a finite subcover of $\mathcal{U}$.

  Now suppose that $\mathcal{PF}(L) = \bigcup \phi(x_i) \cup \bigcup \phi(y_j)^c$. It follows that

  $$\bigcap \phi(y_j) = \left(\bigcup \phi(y_j)^c\right)^c \subseteq \bigcup \phi(x_i).$$

  Let I be the ideal generated by the x_i, and F the filter generated by the y_i. If $F \cap I = \emptyset$, then lemma A.0.13 can be used to find a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. Since F is the filter generated by the y_i, $y_i \in F \subseteq P$ for every y_i. Therefore, $P \in \bigcap \phi(y_j) \subseteq \bigcup \phi(x_i)$, and hence there exists some x_i for which $P \in \phi(x_i)$. This means that $x_i \in P$ for some x_i, and therefore $P \cap I \neq \emptyset$. This gives a contradiction, so that $F \cap I \neq \emptyset$.

  This means there are $x_1, ..., x_n$ and $y_1, ..., y_m$ for which $y_1 \wedge ... \wedge y_m \leq x_1 \vee ... \vee x_n$. This gives

  $$\phi(y_1) \cap ... \cap \phi(y_m) \subseteq \phi(x_1) \cup ... \cup \phi(x_n),$$

  and therefore

  $$\mathcal{PF}(L) = \phi(x_1) \cup ... \cup \phi(x_n) \cup \phi(y_1)^c \cup ... \cup \phi(y_m)^c.$$

$\square$

It will come in handy to simplify the basis of the topology on $\mathcal{PF}(L)$ in the case that L is a Boolean algebra.

Lemma 4.1.4. *If B is a Boolean algebra, then $\phi : B \to \mathcal{O}(\mathcal{PF}(B))$ is an injective Boolean homomorphism.*

Proof. Due to lemma 4.1.2 we already know that ϕ is an injective lattice homomorphism. Since B is a Boolean algebra, any prime filter F is also a maximal filter: given some element a which is not in F, we know that $a \vee \neg a = 1$, and 1 is in F. Since F is prime, it follows that $\neg a$ is in F. Note that a and $\neg a$ can never both be in a prime filter together, since then it would follow that $a \wedge \neg a = 0$ is in the filter, and thus the prime filter would be B itself, which cannot be. Hence, for any prime filter F, a is in F if and only if $\neg a$ is not in F. Thus $\phi(a)^c = \phi(\neg a)$. $\square$

With this lemma, the basis for the topology on $\mathcal{PF}(B)$ can simply be written as the collection $\phi(a)$, for a in B.

It is now also possible to define the functors from **BA** to **Stone** and vice-versa. For the functor from **BA** to **Stone**, we send a Boolean algebra B to $\mathcal{PF}(\mathcal{B})$. This mapping is well defined thanks to the lemma above. For the morphisms, given a Boolean homomorphism $f : B \to C$, we define $\mathcal{PF}(f) : \mathcal{PF}(C) \to \mathcal{PF}(B)$ by sending a filter Q to $f^{-1}(Q)$. This is again a prime filter. Defined this way, the functor $\mathcal{PF} : \mathbf{BA} \to \mathbf{Stone}$ is contravariant. To show that $\mathcal{PF}(f)$ is continuous, note that

$$\begin{aligned}\mathcal{PF}(f)^{-1}(U_b) &= \left\{Q \in \mathcal{PF}(C) \mid b \in f^{-1}(Q)\right\} \\ &= \{Q \in \mathcal{PF}(C) \mid f(b) \in Q\} \\ &= \phi(f(b)),\end{aligned}$$

which means that $\mathcal{PF}(f)$ is indeed a continuous map. It is easily checked that $\mathcal{PF}$ defines a functor.

For the other way around, let X be a Stone space. Then $\mathcal{CP}(X)$, defined as the set of all clopens in X, is a Boolean algebra ordered by inclusion. This gives the functor the other way around, $\mathcal{CP} : \mathbf{Stone} \to \mathbf{BA}$. For the morphisms, if $f : X \to Y$ is a continuous map, define $\mathcal{CP}(f) : \mathcal{CP}(Y) \to \mathcal{CP}(X)$ by sending an open U to $f^{-1}(U)$. Since f is continuous, $f^{-1}(U)$ is again clopen. It is again easily verified that this is a Boolean homomorphism, and that $\mathcal{CP}$ is a contravariant functor.

To show that the functors $\mathcal{PF}$ and $\mathcal{CP}$ provide a duality between **BA** and **Stone**, we first show that $B \cong \mathcal{CP}(\mathcal{PF}(B))$, and then that $X \cong \mathcal{PF}(\mathcal{CP}(X))$.

Lemma 4.1.5. *Let B be a Boolean algebra. Then the map $F_B : B \to \mathcal{CP}(\mathcal{PF}(B))$, defined by $F_B(b) = \phi(b)$, is an isomorphism of Boolean algebras.*

Proof. $\mathcal{CP}(\mathcal{PF}(B))$ is a Boolean algebra, and necessarily a Boolean subalgebra of the power set of $\mathcal{PF}(B)$ by construction. F_B is a well-defined map and an injective Boolean homomorphism due to lemma 4.1.4. It still needs to be shown to be surjective. Let C be a clopen subset of $\mathcal{PF}(B)$. Then C is open, so $C = \bigcup \phi(x_i)$ for some collection x_i in B. Since C is also a closed subset of a compact space, it is compact itself. Therefore, since ϕ is a Boolean homomorphism, $C = \bigcup_{i=1}^{n} \phi(x_i) = \phi(a)$, where $a = \bigvee_{i=1}^{n} x_i$. □

Lemma 4.1.6. *Let X be a Stone space. Then the map*

$$\begin{aligned}G_X : X &\to \mathcal{PF}(\mathcal{CP}(X)); \\ x &\mapsto \{U \in \mathcal{CP}(X) \mid x \in U\},\end{aligned}$$

is a homeomorphism.

Proof. First it needs to be checked if $G_X(x)$ is indeed a prime filter in $\mathcal{CP}(X)$. It is already clear that $G_X(x)$ is an upset in $\mathcal{CP}(X)$. Now if U, V are in $G_X(x)$, then x is in both U and V, and so $U \cap V$ is in $G_X(x)$. If x is in $U \cup V$, then clearly x is in U or in V. Therefore, U or V is in $G_X(x)$, and thus $G_X(x)$ is a prime filter.

Now we check if $G_X(x)$ is continuous. Let $U \in \mathcal{CP}(X)$, and consider the basic clopen set $V = \{P \in \mathcal{PF}(\mathcal{CP}(X)) \mid U \in P\}$. Then

$$\begin{aligned}G_X^{-1}(V) &= \{x \in X \mid G_X(x) \in V\} \\ &= \{x \in X \mid U \in G_X(x)\} \\ &= \{x \in X \mid x \in U\} \\ &= U,\end{aligned}$$

and therefore G_X is continuous.

Next, we note that $\{z\} = \bigcap G_X(z)$ for any $z \in X$. This follows because X is Hausdorff, and the basis of the topology on X consists of clopens. Therefore, if $G_X(x) = G_X(y)$, then $x = y$, and hence G_X is injective. It is also surjective; if P is a prime filter in $\mathcal{CP}(X)$, consider $\bigcap P$. This is a collection of closed subsets of the compact set X, which implies it has the finite intersection property [17]. Furthermore, for any finite collection $F_1, ..., F_N \in P$, their intersection $F_1 \cap ... \cap F_N$ is also in P and therefore non-empty (as $\emptyset \notin P$). Thus $\bigcap P$ is non-empty. If $\bigcap P$ contains distinct points x and y, then there is a clopen set U with $x \in U$ and $y \in U^c$. Moreover, either U is in P, or U^c is in P. Without loss of generality, assume that U is in P. Then y cannot be in $\bigcap P$. Thus $\bigcap P = \{x\}$ for some $x \in X$ and so $P \subseteq G_X(x)$. However, P and $G_X(x)$ are both prime filters and therefore maximal filters in $\mathcal{CP}(X)$, and hence $P = G_X(x)$. We have now shown that G_X is a bijective continuous map. To show that it is a homeomorphism, note that G_X is a map between Stone spaces, which are compact and Hausdorff by definition. Therefore, if $A \subseteq X$ is closed, it is then also compact. The set $G_X(A)$ then also has to be compact, and as a compact subset of a Hausdorff space it is closed. Thus G_X is a closed map, and we can conclude it is a homeomorphism. □

Theorem 4.1.7. *The functors $\mathcal{PF}$ and $\mathcal{CP}$ give a co-equivalence between the categories* **BA** *and* **Stone**.

Proof. To show that $\mathcal{PF}$ and $\mathcal{CP}$ yield an equivalence of categories, we define a natural isomorphism $F : \mathrm{id}_{\mathbf{BA}} \to \mathcal{CP} \circ \mathcal{PF}$. For a Boolean algebra B, define $F_B : B \to \mathcal{CP}(\mathcal{PF}(B))$ like before, $F_B(b) = \phi(b)$. As we have seen in Lemma 4.1.5, F_B is an isomorphism of Boolean algebras. Now it can be seen that F is a natural transformation, since if $f : A \to B$ is a Boolean homomorphism, the diagram

$$\begin{array}{ccc} A & \xrightarrow{F_A} & \mathcal{CP}(\mathcal{PF}(A)) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{CP}(\mathcal{PF}(f))} \\ B & \xrightarrow{F_B} & \mathcal{CP}(\mathcal{PF}(B)) \end{array}$$

commutes, since if $a \in A$, then

$$\begin{aligned} \mathcal{CP}(\mathcal{PF}(f))(F_A(a)) &= \mathcal{CP}(\mathcal{PF}(f))(\phi(a)) \\ &= \mathcal{PF}(f)^{-1}(\phi(a)) \\ &= \big\{Q \in \mathcal{PF}(\mathcal{B}) \mid f^{-1}(Q) \in \phi(a)\big\} \\ &= \big\{Q \in \mathcal{PF}(B) \mid a \in f^{-1}(Q)\big\} \\ &= \{Q \in \mathcal{PF}(B) \mid f(a) \in Q\} \\ &= F_B(f(a)). \end{aligned}$$

Next, define $G : \mathrm{id}_{\mathbf{Stone}} \to \mathcal{PF} \circ \mathcal{CP}$ for a Stone space X like before, by $G_X(x) = \{U \in \mathcal{CP}(X) \mid x \in U\}$. From lemma 4.1.6 we already know that G_X is a homeomorphism. Moreover, G is a natural transformation, since if $g : X \to Y$ is continuous, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{G_X} & \mathcal{PF}(\mathcal{CP}(X)) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{PF}(\mathcal{CP}((g))} \\ Y & \xrightarrow{G_Y} & \mathcal{PF}(\mathcal{CP}(Y)) \end{array}$$

is commutative. This can be seen since if $x \in X$, then

$$G_Y(g(x)) = \{V \in \mathcal{CP}(Y) \mid g(x) \in V\},$$

and therefore

$$\begin{aligned}
\mathcal{PF}(\mathcal{CP}(g))(G_X(x)) &= \mathcal{PF}(\mathcal{CP}(g))(\{U \in \mathcal{CP}(X) \mid x \in U\}) \\
&= \mathcal{CP}(g)^{-1}(\{U \in \mathcal{CP}(X) \mid x \in U\}) \\
&= \left\{V \in \mathcal{CP}(Y) \mid x \in g^{-1}(V)\right\} \\
&= \{V \in \mathcal{CP}(Y) \mid g(x) \in V\} \\
&= G_Y(g(x)).
\end{aligned}$$

□

4.2 Priestley Duality

Now we extend Stone duality to the case of bounded distributive lattices. This is often called Priestley duality. If L is a bounded distributive lattice, $\mathcal{CP}(\mathcal{PF}(L))$ is a Boolean algebra. Therefore, we need to determine how to recover L from $\mathcal{PF}(L)$. If a is in L and P is in $\phi(a)$, then for any prime filter Q with $P \subseteq Q$, Q is also in $\phi(a)$. Inclusion is of course a partial order on $\mathcal{PF}(L)$, and so we see that $\phi(a)$ is a clopen upper set of $\mathcal{PF}(L)$ for any a in L.

Definition 4.2.1. $(X, \leq)$ is called a **Priestley space** if it is a Stone space with a partial order satisfying the *Priestley separation axiom*: for all x and y in X with $x \not\leq y$, there is a clopen upset U with $x \in U$ and $y \notin U$.

The category **Pries** consists of Priestley spaces where the maps are continuous and order preserving. We will show that this category is dually equivalent to the category of bounded distributive lattices, **BDL**, where the maps are lattice homomorphisms.

Lemma 4.2.2. *If L is a bounded distributive lattice, then $(\mathcal{PF}(L), \subseteq)$ is a Priestley space.*

Proof. By lemma 4.1.3 we know that $\mathcal{PF}(L)$ is a Stone space. For the Priestley separation axiom, let P and Q be prime filters in L with $P \not\subseteq Q$. Then there exists an a in L which is in P but not in Q. Therefore, P is in $\phi(a)$ but Q is not. Thus we have found a clopen upset separating P and Q in the required way. □

Now we recover L from $\mathcal{PF}(L)$:

Lemma 4.2.3. *The clopen upsets of $\mathcal{PF}(L)$ are precisely the sets $\phi(a)$, for a in L.*

Proof. As noted, $\phi(a)$ is a clopen upset. Conversely, let U be a clopen upset of $\mathcal{PF}(L)$. For each P in U and Q in U^c, we have $P \not\subseteq Q$, since U is an upset. Thus there must be some a_{PQ} in L which is in P but not in Q. Therefore, P is in $\phi(a_{PQ})$, and Q in $\phi(a_{PQ})^c$. Now U^c is covered by the various $\phi(a_{PQ})^c$. Since $\mathcal{PF}(L)$ is compact and U^c is closed, U^c is also compact. Thus for some fixed P in U we have:

$$U^c \subseteq \bigcup_{i=1}^{n} \phi(a_{PQ_i})^c = \phi(a_P)^c,$$

where $a_P = a_{PQ_1} \wedge ... \wedge a_{PQ_n}$. Consequently, P is an element of $\phi(a_P) \subseteq U$. This gives us an open cover of U by these $\phi(a_P)$, for P in U. Once again, U is a closed subset of a compact space and thus compact. This gives the finite cover

$$U \subseteq \bigcup_{i=1}^{m} \phi(a_{P_i}) = \phi(a),$$

where $a = a_{P_1} \vee ... \vee a_{P_m}$. Since all $\phi(a_{P_i})$ were contained in U, we may conclude that $U = \phi(a)$. □

If $(X, \leq)$ is a Priestley space, we denote the clopen upsets of $(X, \leq)$ by $\mathcal{CU}(X, \leq)$. Then we can define one contravariant functor by $\mathcal{PF} : \mathbf{BDL} \to \mathbf{Pries}$ which turns a lattice homomorphism $f : L \to M$ into a Priestley homomorphism $\mathcal{PF}(f) : \mathcal{PF}(M) \to \mathcal{PF}(L)$, defined by $\mathcal{PF}(f)(Q) = f^{-1}(Q)$. For the other way around, we have $\mathcal{CU} : \mathbf{Pries} \to \mathbf{BDL}$ which turns a Priestley homomorphism $g : X \to Y$ into a lattice homomorphism $\mathcal{CU}(g) : \mathcal{CU}(Y) \to \mathcal{CU}(X)$, defined by $\mathcal{CU}(g)(V) = g^{-1}(V)$. It is elementary to verify that these are well-defined functors.

Lemma 4.2.4. *If L is a distributive lattice, then the map $F_L : L \to \mathcal{CU}(\mathcal{PF}(L), \subseteq)$, defined by $F_L(a) = \phi(a)$, is a lattice isomorphism.*

Proof. We can use lemma 4.1.2 once more to conclude that F_L is an injective lattice homomorphism. The previous lemma shows that it is also surjective. □

Lemma 4.2.5. *If $(X, \leq)$ is a Priestley space, then*

$$\begin{aligned} G_X : (X, \leq) &\to \mathcal{PF}(\mathcal{CU}(X, \leq)) \\ x &\mapsto \{U \in \mathcal{CU}(X, \leq) \mid x \in U\} \end{aligned}$$

is an isomorphism of Priestley spaces.

Proof. The proof that $G_X(x)$ is indeed a prime filter is the same as in lemma 4.1.6. To see that G_X is order preserving, take $x \leq y$ and $U \in G_X(x)$. Then $x \in U$, and since U is an upset, $y \in U$. Thus $U \in G_X(y)$. G_X is also continuous: let V be a clopen upset in $(X, \leq)$, and consider the basic clopen set $\phi(U) = \{P \in \mathcal{PF}(\mathcal{CU}(X, \leq)) \mid U \in P\}$. Then

$$\begin{aligned} G_X^{-1}(\phi(U)) &= \{x \in X \mid G_X(x) \in \phi(U)\} \\ &= \{x \in X \mid U \in G_X(x)\} \\ &= \{x \in X \mid x \in U\} \\ &= U, \end{aligned}$$

so G_X is continuous, and G_X is indeed a valid Priestley homomorphism.

The Priestley separation axiom shows that if $z \in X$, then any point in X not above z can be separated from z by a clopen upset. Therefore, $\uparrow z = \bigcap G_X(z)$. From this, it is clear that if $G_X(x) \subseteq G_X(y)$ then $x \leq y$, so G_X is strictly order preserving and hence also injective. Moreover, we note that G_X is a closed map, since its domain is compact and its codomain is Hausdorff. To finish the proof, we only need to prove that G_X is surjective (since a continuous map that is bijective and closed is also a homeomorphism). Now note that $G_X(X)$ is closed in $\mathcal{PF}(\mathcal{CU}(X, \leq))$. If G_X is not surjective, there is some prime filter P in $\mathcal{CU}(X, \leq)$ not contained in $G_X(X)$. Therefore, there must be some basic open set $V = \phi(U_1) \cup \phi(U_2)^c$ containing P but disjoint from $G_X(X)$, for some U_1, U_2 in $\mathcal{CU}(X, \leq)$. Now, $\emptyset = G_X^{-1}(V) = G_X^{-1}(\phi(U_1)) \cap G_X^{-1}(\phi(U_2))^c$. We have already seen above that $G_X^{-1}(\phi(U)) = U$. Therefore, $\emptyset = U_1 \cap U_2^c$, implying that $U_1 \subseteq U_2$. But then $V = \phi(U_1) \cap \phi(U_2)^c = \emptyset$. This contradiction shows that G_X is surjective. □

With these lemmas we are in the position to prove Priestley duality in full categorical glory!

Theorem 4.2.6. *The functors $\mathcal{CU}$ and $\mathcal{PF}$ give a co-equivalence of categories between* **BDL** *and* **Pries**.

Proof. Define the natural transformation $F : \mathrm{id}_{\mathbf{BDL}} \to \mathcal{CU} \circ \mathcal{PF}$ where, for a bounded distributive lattice L, the map $F_L : L \to \mathcal{CU}(\mathcal{PF}(L))$ is defined by $F_L(a) = \phi(a)$. Then F_L is a lattice isomorphism as we have seen before in lemma 4.2.4. To see that the diagram

$$\begin{array}{ccc} L & \xrightarrow{F_L} & \mathcal{CU}(\mathcal{PF}(L)) \\ f\big\downarrow & & \big\downarrow \mathcal{CU}(\mathcal{PF}(f)) \\ M & \xrightarrow{F_M} & \mathcal{CU}(\mathcal{PF}(M)) \end{array}$$

commutes, let $l \in L$. Then

$$\begin{aligned} \mathcal{CU}(\mathcal{PF}(f))(F_L(a)) &= \mathcal{CU}(\mathcal{PF}(f))(\phi(a)) \\ &= \mathcal{PF}(f)^{-1}(\phi(a)) \\ &= \left\{Q \in \mathcal{PF}(\mathcal{M}) \mid f^{-1}(Q) \in \phi(a)\right\} \\ &= \left\{Q \in \mathcal{PF}(M) \mid a \in f^{-1}(Q)\right\} \\ &= \{Q \in \mathcal{PF}(M) \mid f(a) \in Q\} \\ &= F_M(f(a)). \end{aligned}$$

Next, for a Priestley space $(X, \leq)$, define $G : \mathrm{id}_{\mathbf{Pries}} \to \mathcal{PF} \circ \mathcal{CU}$ as before by $G_X(x) = \{U \in \mathcal{CU}(X, \leq) \mid x \in U\}$. From lemma 4.2.5 we already know that G_X is a Priestley isomorphism. Moreover, G is a natural transformation, since if $g : (X, \leq) \to (Y, \preceq)$ is continuous, then the diagram

$$\begin{array}{ccc} (X, \leq) & \xrightarrow{G_X} & \mathcal{PF}(\mathcal{CU}(X, \leq)) \\ g\big\downarrow & & \big\downarrow \mathcal{PF}(\mathcal{CU}((g)) \\ (Y, \preceq) & \xrightarrow{G_Y} & \mathcal{PF}(\mathcal{CU}(Y, \preceq)) \end{array}$$

commutes. Indeed, if $x \in X$, then

$$G_Y(g(x)) = \{V \in \mathcal{CU}(Y) \mid g(x) \in V\},$$

hence

$$\begin{aligned} \mathcal{PF}(\mathcal{CU}(g))(G_X(x)) &= \mathcal{PF}(\mathcal{CU}(g))(\{U \in \mathcal{CU}(X) \mid x \in U\}) \\ &= \mathcal{CU}(g)^{-1}(\{U \in \mathcal{CU}(X) \mid x \in U\}) \\ &= \left\{V \in \mathcal{CU}(Y) \mid x \in g^{-1}(V)\right\} \\ &= \{V \in \mathcal{CU}(Y) \mid g(x) \in V\} \\ &= G_Y(g(x)). \end{aligned}$$

Therefore, F and G yield a co-equivalence between **BDL** and **Pries**. □

4.3 Esakia Duality

In this section we specialize Priestley duality from bounded distributive lattices to the category **HA** of Heyting algebras. If we wish to restrict Priestley duality to this category, we need to determine which Priestley spaces are duals of Heyting algebras, and which morphisms of such spaces are dual to Heyting morphisms.

Definition 4.3.1. Let $g : (X, \leq) \to (Y, \preceq)$ be a morphism of posets. We say that g is a ***p*-morphism** if for every $x \in X$ and $y \in Y$ with $g(x) \preceq z$ there is an $x' \in X$ with $x \leq x'$ and $g(x') = z$.

Definition 4.3.2. An **Esakia space** is a Priestley space $(X, \leq)$ such that if U is clopen, then so is $\downarrow U$.

We denote the category of Esakia spaces by **Esa**, where the morphisms are continuous p-morphisms. In this section we see that Priestley duality restricts to a duality between **HA** and **Esa**. We start with some preliminary lemmas.

Lemma 4.3.3. *Let $(X, \leq)$ be a Priestley space.*

- *The relation $\leq$ is closed, i.e. the set $R = \{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$.*
- *If C is closed in X, then so are $\uparrow C$ and $\downarrow C$.*

Proof. Let $(x, y) \in (X \times X) \backslash R$, i.e. $x \not\leq y$. Then there is a clopen upset U with $x \in U$ and $y \in U^c$. Since U is an upset and so U^c a downset, we see that $(U \times U^c) \cap R = \emptyset$. Therefore, $U \times U^c$ is an open neighbourhood of (x, y) disjoint from R. Thus R is closed in $X \times X$.

To prove the second statement, we note that $\uparrow C = \pi_2((C \times X) \cap R)$. Since X is compact, the projection maps are closed, and hence $\uparrow C$ is closed. Similarly, $\downarrow C = \pi_1((X \times C) \cap R)$ means $\downarrow C$ is closed. □

Lemma 4.3.4. *Let H be a Heyting algebra. If $a, b \in H$, then*

$$\downarrow (\phi(a) \cap \phi(b)^c) = \phi(a \to b)^c.$$

Proof. Let $a, b \in H$. Since $a \wedge (a \to b) \leq b$ by definition of the Heyting implication, we have $\phi(a) \cap \phi(a \to b) \subseteq \phi(b)$, so $\phi(a) \cap \phi(b)^c \subseteq \phi(a \to b)^c$. Because $\phi(a \to b)^c$ is a downset, $\downarrow (\phi(a) \cap \phi(b)^c) \subseteq \phi(a \to b)^c$. For the reverse inclusion, let $P \in \phi(a \to b)^c$. Then P is a prime filter with $a \to b \notin P$. We wish to find a prime filter Q with $P \cup \{a\} \subseteq Q$ and $b \notin Q$. Note that if $a \to b$ and a are both in Q, then b has to be in Q, since Q is an upset and $a \wedge (a \to b) \leq b$. Conversely, if a and b are in Q, then $a \wedge b$ has to be in Q. Since $a \wedge b \leq b$, we have $a \wedge b \leq a \to b$. Again, since Q is an upset, this implies that $a \to b$ is in Q. Thus if a is in Q, then $a \to b$ is in Q if and only if b is in Q. This shows that it is enough to make sure that $a \to b \notin Q$.

By lemma A.0.13, such a prime filter Q exists if the filter F generated by $P \cup \{a\}$ does not contain $a \to b$. If F contains $a \to b$, then there must be an $x \in P$ with $a \wedge x \leq a \to b$. By definition of $a \to b$, we have $(a \wedge b) \wedge a \leq b$. But then $a \wedge x \leq b$, forcing $x \leq a \to b$. This is a contradiction, since $x \in P$ and $a \to b \notin P$. Thus we have a prime filter Q with $P \subseteq Q$, $a \in Q$ and $b \notin Q$. Therefore, $P \in \downarrow (\phi(a) \cap \phi(b)^c)$. □

Lemma 4.3.5. *Let $f : (X, \leq) \to (Y, \preceq)$ be a poset morphism. Then the following conditions are equivalent:*

1. *f is a p-morphism,*
2. *$f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$ for every subset $A \subseteq Y$,*
3. *$f^{-1}(\downarrow y) = \downarrow f^{-1}(\{y\})$ for every $y \in Y$.*

Proof. (1) ⇒ (2): Suppose that f is a p-morphism and $A \subseteq Y$. Since $f^{-1}(A) \subseteq f^{-1}(\downarrow A)$ and the latter is a downset itself, we have $\downarrow f^{-1}(A) \subseteq f^{-1}(\downarrow A)$. For the reverse inclusion, let $x \in f^{-1}(\downarrow A)$. Then $f(x) \preceq a$ for some $a \in A$. Since f is a p-morphism, $a = f(x')$ for some $x' \in X$ with $x \leq x'$. Then $x' \in f^{-1}(A)$, so $x \in \downarrow f^{-1}(A)$. Thus $f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Suppose that $x \in X$ and $y \in Y$ with $f(x) \preceq y$. Then $f(x) \in \downarrow y$, so $x \in f^{-1}(\downarrow y) = f^{-1}(\{y\})$. Therefore, $x \leq z$ for some $z \in f^{-1}(\{y\})$, which proves that $f(z) = y$. Hence, f is a p-morphism. □

We now consider the functor $\mathcal{PF} : \mathbf{HA} \to \mathbf{Pries}$ defined by the restriction of $\mathcal{PF} : \mathbf{BDL} \to \mathbf{Pries}$.

Lemma 4.3.6. *If H is a Heyting algebra, then $(\mathcal{PF}(H), \subseteq)$ is an Esakia space.*

Proof. We already know that $(\mathcal{PF}(H), \subseteq)$ is a Priestley space. Let U be a clopen set in $\mathcal{PF}(H)$. Then $U = \bigcup_{i=1}^{n} \phi(a_i) \cap \phi(b_i)^c$ for some a_i, b_i in H. By lemma 4.3.4, we have

$$\downarrow U = \bigcup_{i=1}^{n} \downarrow(\phi(a_i) \cap \phi(b_i)^c) = \bigcup_{i=1}^{n} \phi(a_i \to b_i)^c,$$

a clopen set. Therefore, $(\mathcal{PF}(H), \subseteq)$ is an Esakia space. □

Lemma 4.3.7. *Let $f : H \to H'$ be a Heyting morphism. Then $\mathcal{PF}(f) : \mathcal{PF}(H') \to \mathcal{PF}(H)$ is a p-morphism.*

Proof. Let $Q \in \mathcal{PF}(H')$ and $P \in \mathcal{PF}(H)$ with $f^{-1}(Q) \subseteq P$. For notational convenience, we write $\mathcal{PF}(f) = g$. Let C be a clopen set in $\mathcal{PF}(H)$ containing P. Then C is a finite union of sets of the form $\phi(a) \cap \phi(b)^c$ with $a \in P$ and $b \notin P$. We have

$$\begin{aligned} g^{-1}(\downarrow(\phi(a) \cap \phi(b)^c)) &= g^{-1}(\phi(a \to b)^c) \\ &= g^{-1}(\phi(a \to b))^c \\ &= \phi(f(a \to b))^c \\ &= \phi(f(a) \to f(b))^c \\ &= \downarrow(\phi(f(a)) \cap \phi(f(b))^c). \end{aligned}$$

By considering finite unions, we then see that $g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen set. Since $g(Q) \subseteq P$, we see that $Q \in g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen C containing P. Thus $\uparrow Q \cap g^{-1}(C) \neq \emptyset$. Since the set of clopens containing P is closed under finite intersections, compactness implies that $\bigcap(\uparrow Q \cap g^{-1}(C)) \neq \emptyset$, where the intersection is over all clopens C containing P. This yields $\uparrow Q \cap \bigcap g^{-1}(C) \neq \emptyset$, so $\uparrow Q \cap g^{-1}(\{P\}) \neq \emptyset$. Therefore, there is some Q' with $Q \subseteq Q'$ and $g(Q') = P$. This proves that $g = \mathcal{PF}(f)$ is a p-morphism. □

The previous two lemmas show that $\mathcal{PF}$ is a functor from **HA** to **Esa**. We now consider the functor $\mathcal{CU} : \mathbf{Pries} \to \mathbf{BDL}$ restricted to **Esa**.

Lemma 4.3.8. *Let $(X, \leq)$ be an Esakia space. Then $\mathcal{CU}(X, \leq)$ is a Heyting algebra, where implication is defined by $U \to V = (\downarrow(U \cap V^c))^c$.*

Proof. We already know that $\mathcal{CU}(X, \leq)$ is a bounded distributive lattice. Now let U and V be clopen upsets. Then $U \cap V^c$ is clopen and since $(X, \leq)$ is an Esakia space, $\downarrow(U \cap V^c)$ is clopen. Then $(\downarrow(U \cap V^c))^c$ is a clopen upset, so we define

$$U \to V = (\downarrow(U \cap V^c))^c.$$

To see that this is a Heyting implication, we need to check that for any clopen upset W, we have $U \cap W \subseteq V$ if and only if $W \subseteq U \to V$.

Since $U \to V \subseteq (U \cap V^c)^c$, we have

$$U \cap (U \to V) \subseteq U \cap (U \cap V^c)^c = U \cap (U^c \cup V) = V.$$

Therefore, if $W \subseteq U \to V$, then

$$U \cap W \subseteq U \cap (U \to V) \subseteq V.$$

Suppose that $U \cap W \subseteq V$. Then $U \cap V^c \subseteq W^c$. Since W^c is a downset, we obtain $\downarrow(U \cap V^c) \subseteq W^c$. Thus $W \subseteq (\downarrow(U \cap V^c))^c = U \to V$. □

Lemma 4.3.9. *Let $g : (X, \leq) \to (Y, \preceq)$ be a morphism of Esakia spaces. Then the map*

$$\begin{aligned} \mathcal{CU}(g) : \mathcal{CU}(Y, \preceq) &\to \mathcal{CU}(X, \leq); \\ U &\mapsto g^{-1}(U), \end{aligned}$$

is a Heyting morphism.

Proof. We know that $\mathcal{CU}(g)$ is a lattice homomorphism, so we only need to show that it preserves implication. Let U, V be clopen upsets of Y. Since

$$g^{-1}(U) \cap g^{-1}(U \to V) = g^{-1}(U \cap (U \to V)) \subseteq g^{-1}(V),$$

we see that $g^{-1}(U \to V) \subseteq g^{-1}(U) \to g^{-1}(V)$. For the reverse inclusion, suppose that $x \notin g^{-1}(U \to V)$. Since $U \to V = (\downarrow (U \cap V^c))^c$, we have $x \in g^{-1}(\downarrow (U \cap V^c))$, so $g(x) \in \downarrow (U \cap V^c)$. Therefore, there is a $y \in U \cap V^c$ with $g(x) \preceq y$. Since g is a p-morphism, there is a $z \in X$ with $x \leq z$ and $y = g(z)$. Then $z \in g^{-1}(U \cap V^c) = g^{-1}(U) \cap g^{-1}(V)^c$. Thus $x \in \downarrow (g^{-1}(U) \cap g^{-1}(V)^c)$, and so $x \notin g^{-1}(U) \to g^{-1}(V)$. This proves the reverse inclusion. Therefore,

$$g^{-1}(U \to V) = g^{-1}(U) \to g^{-1}(V),$$

so $\mathcal{CU}(g)$ is a Heyting morphism. □

We have shown that $\mathcal{CU}$ is a functor from **Esa** to **HA**. To prove that these categories are dual to each other, we have little work left to do.

Lemma 4.3.10. *Let H be a Heyting algebra. Then the map*

$$\begin{aligned} F_H : H &\to \mathcal{CU}(\mathcal{PF}(H)); \\ a &\mapsto \phi(a), \end{aligned}$$

is a Heyting isomorphism.

Proof. We have seen in lemma 4.2.4 that F_H is an isomorphism of bounded distributive lattices. Therefore, we only need to check if F_H preserves implication. Let $a, b \in H$. Then by lemmas 4.3.4 and 4.3.8,

$$F_H(a \to b) = \phi(a \to b) = (\downarrow (\phi(a) \cap \phi(b)^c))^c = \phi(a) \to \phi(b).$$

Thus F_H is an isomorphism of Heyting algebras. □

Lemma 4.3.11. *Let $(X, \leq)$ be an Esakia space. Then*

$$\begin{aligned} G_X : (X, \leq) &\to \mathcal{PF}(\mathcal{CU}(X, \leq)); \\ x &\mapsto \{U \in \mathcal{CU}(X, \leq) \mid x \in U\}, \end{aligned}$$

is an Esakia isomorphism.

Proof. We have seen in lemma 4.2.5 that G_X is an isomorphism of Priestley spaces. Since G_X and G_X^{-1} are then in particular poset isomorphisms, they are both p-morphisms. Thus G_X is an Esakia isomorphism. □

Theorem 4.3.12. *The functors $\mathcal{CU}$ and $\mathcal{CU}$ give a co-equivalence of categories between* **HA** *and* **Esa**.

Proof. We have natural transformations $F : \mathrm{id}_{\mathbf{HA}} \to \mathcal{CU} \circ \mathcal{PF}$ and $G : \mathrm{id}_{\mathbf{Esa}} \to \mathcal{PF} \circ \mathcal{CU}$ defined in the two lemmas above. It follows from the same arguments as in theorem 4.2.6, along with the last two lemmas, that they are natural transformations. □

Chapter 5

The Internal Gelfand Spectrum

Now that all preliminary theorems have been touched upon, it is time to delve into the core of what this thesis is about. Via topos theory, which is a part of category theory, a recipe for constructing a frame has been cooked up corresponding to a C^*-algebra, which represents the logic of the quantum system associated to this C^*-algebra. Here, it will not be explained how this came to be or how this exactly represents the logic of the system; more can be read about this in [11]. What will be relevant to us is the following recipe:

Definition 5.0.1. Let A be a C^*-algebra. Then the internal Gelfand spectrum of A can be captured in the following topological space

$$\Sigma_A = \bigsqcup_{C \in \mathcal{C}(A)} \Sigma(C),$$

i.e. the disjoint union over all Gelfand spectra $\Sigma(C)$, where C is a commutative subalgebra of A. For the precise link between the internal Gelfand spectrum and Σ_A, see [11].

Now the topology on this space is defined by its opens $\mathcal{U} \subseteq \Sigma_A$, which are of the form

$$\mathcal{U} = \bigsqcup_{C \in \mathcal{C}(A)} \mathcal{U}_C;$$
$$\mathcal{U}_C = \mathcal{U} \cap \Sigma(C),$$

where the following two conditions are satisfied for each $C \in \mathcal{C}(A)$:

1. $\mathcal{U}_C \in \mathcal{O}(\Sigma(C))$,
2. For all $D \supseteq C$, if $\lambda \in \mathcal{U}_C$ and $\lambda' \in \Sigma(D)$ such that $\lambda'|_C = \lambda$, then $\lambda' \in \mathcal{U}_D$.

This definition of the topology makes it hard to grasp what exactly is happening here. To expand upon these conditions of an open set: the whole space can be seen as all the Gelfand spectra stacked on top of each other. The opens should then be a stack of opens, but if some λ is an element of a certain layer $\mathcal{U}_C$, then every possible extension of the character $\lambda : C \to \mathbb{C}$ to a continuous function $\lambda' : D \supseteq C \to \mathbb{C}$ should also be an element of the layer $\mathcal{U}_D$. This way, the opens resemble upsets in some manner.

This topology, $\mathcal{O}(\Sigma_A)$, seen as a frame, is what we were looking for. Note that it is indeed a frame, since it comes from the topology of some space. An important simplification can be made when the C^*-algebra A is finite dimensional. Let us introduce a new frame, $Q(A)$:

$$Q(A) = \{S : \mathcal{C}(A) \to \mathcal{P}(A) \mid S(C) \in \mathcal{P}(C), S(C) \leq S(D) \text{ if } C \subseteq D\}.$$

Here some explanation is needed as to why this is a frame. As stated before, the projections $\mathcal{P}(A)$ are a partially ordered set, and therefore we can define a pointwise partial order on $Q(A)$ by $S \leq T$

if and only if $S(C) \leq T(C)$ for all $C \in \mathcal{C}(A)$. Also its frame operations are defined pointwise, viz.

$$(S \wedge T)(C) = S(C) \wedge T(C),$$
$$(S \vee T)(C) = S(C) \vee T(C).$$

These operations are well defined since for every $C \in \mathcal{C}(A)$ we have $S(C) \in \mathcal{P}(C)$, and $\mathcal{P}(C)$ is a complete Boolean algebra when C is commutative and finite dimensional, as we assume. From here it can easily be seen that arbitrary joins can also be defined pointwise. In fact, when A is a finite dimensional C^*-algebra, $Q(A)$ turns out to be isomorphic to $\mathcal{O}(\Sigma_A)$.

Lemma 5.0.2. *Let A be a finite-dimensional C^*-algebra. Then $Q(A) \cong \mathcal{O}(\Sigma_A)$.*

Proof. Let us start by defining two frame morphisms:

$$\begin{aligned} \Phi : \mathcal{O}(\Sigma_A) &\to Q(A); \\ \mathcal{U} &\mapsto S_{\mathcal{U}}; \\ S_{\mathcal{U}}(C) &= \beta_C^{-1}(\mathcal{U}_C), \end{aligned}$$

where $\beta_C : \mathcal{P}(C) \to \mathrm{Cl}(\Sigma(C))$ are the lattice isomorphisms from lemma 3.3.1. Note that all commutative C^*-subalgebras are finite dimensional by assumption, and are therefore isomorphic to $\mathbb{C}^n$. Furthermore, $\Sigma(\mathbb{C}^n)$ is the discrete space with n elements, and thus all the $\mathcal{U}_C$ are indeed clopen. For the other way around:

$$\begin{aligned} \Psi : Q(A) &\to \mathcal{O}(\Sigma_A); \\ S &\mapsto \mathcal{U}_S = \bigsqcup_{C \in \mathcal{C}(A)} \beta_C(S(C)). \end{aligned}$$

Let us first show that these maps are indeed well defined, which is more difficult than may be expected.

- First, let us start with Φ. We have to check if $S_{\mathcal{U}}$ is indeed an element of $Q(A)$. $S_{\mathcal{U}}(C) = \beta_C^{-1}(\mathcal{U}_C)$ is a projection of C by definition of β_C. Now take $C, D \in \mathcal{C}(A)$, with $C \subseteq D$. We have to check if $S_{\mathcal{U}}(C) \leq S_{\mathcal{U}}(D)$. To do that we need a short intermezzo involving some category theory.

  From the inclusion mapping $C \overset{j}{\hookrightarrow} D$ we obtain a continuous surjection $\Sigma(D) \xrightarrow{\Sigma(j)} \Sigma(C)$ by Gelfand duality. Consider the following diagram:

$$\begin{array}{ccc} C & \xrightarrow{\quad j \quad} & D \\ \gamma_C \downarrow & & \downarrow \gamma_D \\ C(\Sigma(C)) & \xrightarrow{C(\Sigma(j))} & C(\Sigma(D)) \end{array}$$

  Because Gelfand duality gives a natural transformation, this diagram commutes. Therefore, for projections $p \in C$ we have

$$\gamma_D(j(p)) = C(\Sigma(j)) \circ \gamma_C(p) = \gamma_C(p) \circ \Sigma(j).$$

  Now if we look at the following diagram:

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{\quad j \quad} & \mathcal{P}(D) \\ \beta_C \downarrow & & \downarrow \beta_D \\ \mathcal{O}(\Sigma(C)) & \xrightarrow{\Sigma(j)^{-1}} & \mathcal{O}(\Sigma(D)), \end{array}$$

we can conclude that this commutes as well, since

$$\begin{aligned}\beta_D(j(p)) &= (\gamma_D(j(p)))^{-1}\{1\}\\ &= (\gamma_C(p)\circ\Sigma(j))^{-1}\{1\}\\ &= \Sigma(j)^{-1}\left(\gamma_C(p)^{-1}\{1\}\right)\\ &= \Sigma(j)^{-1}\left(\beta_C(p)\right).\end{aligned}$$

Hence $\beta_D\circ j = \Sigma(j)^{-1}\circ\beta_C$. Moreover, using the notation above we can simplify the conditions for open sets in Σ_A. Originally, an open $\mathcal{U}$ in Σ_A was defined as the disjoint union of open sets $\mathcal{U}_C$ for which for all $D\supseteq C$, if $\lambda\in\mathcal{U}_C$ and $\lambda'\in\Sigma(D)$ such that $\lambda'|_C=\lambda$, then $\lambda'\in\mathcal{U}_D$. Now $\lambda'|_C=\lambda$ iff $\lambda'(j(a))=\lambda(a)$ for all $a\in C$, which can be written as $(\Sigma(j)(\lambda'))(a)=\lambda(a)$ for all $a\in C$. The definition dictates that if this is the case and $\lambda\in\mathcal{U}_C$, then $\lambda'\in\mathcal{U}_D$. That is, if $\lambda'\in\Sigma(j)^{-1}(\mathcal{U}_C)$, then $\lambda'\in\mathcal{U}_D$. Therefore, the condition boils down to $\Sigma(j)^{-1}(\mathcal{U}_C)\subseteq\mathcal{U}_D$.

To come back to our goal, we have to check if $S_{\mathcal{U}}(C)\leq S_{\mathcal{U}}(D)$. By definition, we then have to check if $\beta_C^{-1}(\mathcal{U}_C)\leq\beta_D^{-1}(\mathcal{U}_D)$. We know that $\beta_D\circ j=\Sigma(j)^{-1}\circ\beta_C$, and since $\Sigma(j)$ is surjective, we conclude that $\beta_C=\Sigma(j)\circ\beta_D\circ j$, since

$$\begin{aligned}\beta_C^{-1}(\mathcal{U}_C) &= (\Sigma(j)\circ\beta_D\circ j)^{-1}(\mathcal{U}_C)\\ &= j^{-1}\left(\beta_D^{-1}\left(\Sigma(j)^{-1}(\mathcal{U}_C)\right)\right).\end{aligned}$$

Because we are comparing the projections in the ambient lattice $\mathcal{P}(A)$, we can ignore this j^{-1}. By doing this we are slightly abusing notation, because to make this rigorous we would have to introduce all inclusions $C\xrightarrow{i_C}A$. To continue, we have

$$\begin{aligned}\beta_C^{-1}(\mathcal{U}_C) &= \beta_D^{-1}\left(\Sigma(j)^{-1}(\mathcal{U}_C)\right)\\ &\leq \beta_D^{-1}(\mathcal{U}_D),\end{aligned}$$

since β_D is a lattice isomorphism and therefore so is its inverse, and $\Sigma(j)^{-1}(\mathcal{U}_C)\subseteq\mathcal{U}_D$. Thus Φ is indeed well defined.

- Now for Ψ. What we have to check is if $\mathcal{U}_S=\bigsqcup_{C\in\mathcal{C}(A)}\beta_C(S(C))$ is an open set in $\mathcal{O}(\Sigma_A)$. First of all, it is clear that $\mathcal{U}_C=\beta_C(S(C))$ is open in $\Sigma(C)$ for every $C\in\mathcal{C}(A)$. Now by the simplification made above, we have to check if $\Sigma(j)^{-1}(\mathcal{U}_C)\subseteq\mathcal{U}_D$ for every $C\subseteq D$. Indeed,

$$\begin{aligned}\Sigma(j)^{-1}(\mathcal{U}_C) &= \Sigma(j)^{-1}(\beta_C(S(C)))\\ &= \beta_D(j(S(C)))\\ &\subseteq \beta_D(S(D))\\ &= \mathcal{U}_D.\end{aligned}$$

  Therefore, Ψ is well defined as well.

Next, let us show that these maps are frame morphisms. First,

$$\begin{aligned}\Phi(\emptyset)(C) &= \beta_C^{-1}(\emptyset)=0,\\ \Phi(\Sigma_A)(C) &= \beta_C^{-1}(\Sigma(C))=1,\end{aligned}$$

since β_C^{-1} is a lattice morphism. Thus $\Phi(\emptyset)=0$, and $\Phi(\Sigma_A)=1$. Next,

$$\begin{aligned}\Phi(\mathcal{U}\cap\mathcal{V})(C) &= S_{\mathcal{U}\cap\mathcal{V}}(C)\\ &= \beta_C^{-1}((\mathcal{U}\cap\mathcal{V})_C)\\ &= \beta_C^{-1}(\mathcal{U}_C)\wedge\beta_C^{-1}(\mathcal{V}_C)\\ &= \Phi(\mathcal{U})\wedge\Phi(\mathcal{V}),\end{aligned}$$

and again this argument works because β_C^{-1} is a lattice isomorphism. For the join the argument is analogous. Since β_C is a lattice isomorphism between finite lattices it also preservers arbitrary joins (because they must be finite), which concludes the proof that Φ is a frame morphism. Secondly,

$$\begin{aligned}\Psi(0) &= \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(0) = \emptyset,\\ \Psi(1) &= \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(1) = \Sigma_A.\\ \Psi(S\wedge T) &= \bigsqcup_{C\in\mathcal{C}(A)} \beta_C((S\wedge T)(C))\\ &= \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(S(C)\wedge T(C))\\ &= \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(S(C))\cap\beta_C(T(C))\\ &= \Psi(S)\cap\Psi(T).\end{aligned}$$

Here again, the fact that β_C is a lattice isomorphism between finite lattices in combination with an analogous argument to the above, makes it clear that Ψ is also a frame morphism.

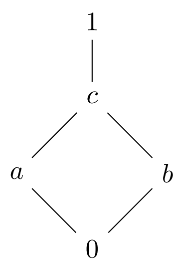
Now Ψ and Φ are frame morphisms, all that is left to show is that $\Psi\circ\Phi = \mathrm{id}_{\mathcal{O}(\Sigma_A)}$ and $\Phi\circ\Psi = \mathrm{id}_{Q(A)}$. This is indeed the case:

- $\Psi(\Phi(\mathcal{U})) = \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(\Phi(\mathcal{U})(C)) = \bigsqcup_{C\in\mathcal{C}(A)} \beta_C(\beta_C^{-1}(\mathcal{U}_C)) = \bigsqcup_{C\in\mathcal{C}(A)} \mathcal{U}_C = \mathcal{U}$;
- $\Phi(\Psi(S))(C) = \beta_C^{-1}(\Psi(S)_C) = \beta_C^{-1}(\beta_C(S(C))) = S(C)$.

Therefore, $Q(A)\cong\mathcal{O}(\Sigma_A)$ for finite-dimensional A. □

To finish this section, let us construct $Q(\mathbb{C}^2)$, as this space will serve in some (counter)examples further on.

Lemma 5.0.3. *The lattice $Q(\mathbb{C}^2)$ has the following Hasse diagram:*



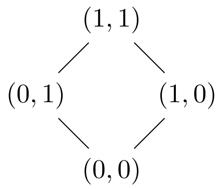
Proof. By definition,

$$Q(\mathbb{C}^2) = \left\{S : \mathcal{C}(\mathbb{C}^2)\to\mathcal{P}(\mathbb{C}^2) \mid S(C)\in\mathcal{P}(C), S(C)\le S(D) \text{ if } C\subseteq D\right\}.$$

Now in our case,

$$\begin{aligned}\mathcal{C}(\mathbb{C}^2) &= \left\{\mathbb{C}\cdot\mathbb{1}_{\mathbb{C}^2}, \mathbb{C}^2\right\},\\ \mathcal{P}(\mathbb{C}^2) &= \{(0,0),(0,1),(1,0),(1,1)\},\\ \mathcal{P}(\mathbb{C}\cdot\mathbb{1}_{\mathbb{C}^2}) &= \{(0,0),(1,1)\}.\end{aligned}$$

Let us construct the functions $S : \mathcal{C}(\mathbb{C}^2)\to\mathcal{P}(\mathbb{C}^2)$. The first condition, $S(C)\in\mathcal{P}(C)$, enforces $S(\mathbb{C}\cdot\mathbb{1}_{\mathbb{C}^2}) = (0,0)$ or $(1,1)$. If we choose $S(\mathbb{C}\cdot\mathbb{1}_{\mathbb{C}^2}) = (1,1)$, the second restriction, $S(C)\le S(D)$ if $C\subseteq D$, leaves only one option for $S(\mathbb{C}^2)$, namely $S(\mathbb{C}^2) = (1,1)$. This is clearly the top element of $Q(\mathbb{C}^2)$. If we had chosen $S(\mathbb{C}\cdot\mathbb{1}_{\mathbb{C}^2}) = (0,0)$, however, all options were still open for $S(\mathbb{C}^2)$. Since the four projections in $\mathcal{P}(\mathbb{C}^2)$ are ordered as follows:



we indeed reach the desired result. □

We will return to this situation in subsection 6.2.

Chapter 6

Esakia Dual of $\mathcal{O}(\Sigma_A)$

As seen in section 4.3, Esakia duality provides a way to uniquely represent a Heyting lattice as an Esakia space up to homeomorphism. This could then shed a new light on the properties of the lattice itself. Our main goal therefore was discovering what the Esakia dual of the lattice $\mathcal{O}(\Sigma_A)$ is exactly. In some sense this is of course already known, for the Esakia dual is simply the collection of prime ideals in the lattice $\mathcal{O}(\Sigma_A)$. This is hardly useful, however. To give an idea of what kind of information we want about the Esakia dual, let us look at classical logic. The relevant lattices there are Boolean, and therefore we can try to see what the Stone dual is of such a lattice B. In that case the Stone space $\mathcal{PF}(B)$ may actually be realized as $\mathcal{PF}(B) = \mathrm{Hom}_{Boolean}(B, \{0,1\})$, i.e. the set of all Boolean homomorphisms $f : B \to \{0,1\}$. This is interpreted as all 'valuations' of B, which are all valid ways to assign truth values to the elements (also called propositions) of B. Any proposition $a \in B$ can then be uniquely identified with the set of all valuations in which a is true. [16]

If we want our result to be interpretable in a similar manner, we ought to restrict ourselves to Heyting morphisms, as our logic is intuitionistic. Explicitly, what we are looking for is a lattice S_A for every C^*-algebra A such that the Esakia dual of $\mathcal{O}(\Sigma_A)$ is equal to $\mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_A), S_A)$. Now it seems reasonable from intuitionistic logic to surmise that the Esakia dual of $\mathcal{O}(\Sigma_A)$ will be equal to the space of Kripke models $\mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_A), \mathrm{Up}(\mathcal{C}(A))) = \mathrm{Mod}_{\mathcal{C}(A)}(\mathcal{O}(\Sigma_A))$ [11]. This does not turn out to be correct, however, as will become clear soon.

6.1 Back to Priestley

To determine the Esakia dual of some lattice, it can be helpful to go back to Priestley duality, since Esakia duality is a special case of this. Priestley duality, however, has the advantage of being *representable*, as indicated by the following theorem.

Theorem 6.1.1 (Representability of Priestley Duality)**.** *Let L be a bounded distributive lattice. Then the Priestley dual of this lattice L, $\mathcal{PF}(L)$, is equal to* $\mathrm{Hom}_{lattice}(L, \{0,1\})$*, ordered pointwise and with a topology generated by the sets*

$$U_a = \{f \mid f(a) = 1\}, V_a = \{f \mid f(a) = 0\}.$$

This theorem and more can be found in [1].

Proof. First, let us abbreviate $\{0,1\}$ by $\underline{2}$. Second, we associate a prime filter F in $\mathcal{PF}(L)$ with the function $\chi_F : L \to \underline{2}$ that sends every element of F to 1, and all other elements of L to 0. Evidently, $\chi_F(0_L) = 0$, en $\chi_F(1_L) = 1$. To show that this is a lattice morphism, we have to show that $\chi_F(x \wedge y) = \chi_F(x) \wedge \chi_F(y)$, and $\chi_F(x \vee y) = \chi_F(x) \vee \chi_F(y)$. Indeed,

$$
\begin{aligned}
\chi_F(x \vee y) = 1 &\iff x \wedge y \in F \\
&\iff x \in F \text{ or } y \in F && (F \text{ is a prime filter}) \\
&\iff \chi_F(x) = 1 \text{ or } \chi_F(y) = 1 \\
&\iff \chi_F(x) \vee \chi_F(y) = 1,
\end{aligned}
$$

$$
\begin{aligned}
\chi_F(x \wedge y) = 0 &\iff x \wedge y \in L \backslash F \\
&\iff x \in L \backslash F \text{ or } y \in L \backslash F && (L \backslash F \text{ is a prime ideal}) \\
&\iff \chi_F(x) = 0 \text{ or } \chi_F(y) = 0 \\
&\iff \chi_F(x) \wedge \chi_F(y) = 0.
\end{aligned}
$$

Therefore, χ_F is indeed a lattice morphism. Now for the other way around, let $f : L \to \{0, 1\}$ be a lattice homomorphism. Let us show that $\text{Ker}(f)$ is a prime ideal by checking all requirements of the definition A.0.11, which would make f an indicator function of a prime filter.

- $f(0) = 0$, and $f(1) = 1$, therefore $\text{Ker}(f)$ is not empty, but is a proper subset of L as well;
- $y \leq x \in L$, $x \in \text{Ker}(f)$. Then $f(y) = f(x \wedge y) = f(x) \wedge f(y) = 0 \wedge f(y) = 0$, so $y \in \text{Ker}(f)$;
- $x, y \in \text{Ker}(f)$. Then $f(x \vee y) = f(x) \vee f(y) = 0 \vee 0 = 0$, so $x \vee y \in \text{Ker}(f)$;
- If $f(x \wedge y) = 0$, then $f(x) \wedge f(y) = 0$, so $f(x) = 0$ or $f(y) = 0$. Therefore, $x \in \text{Ker}(f)$ of $y \in \text{Ker}(f)$.

This yields a bijection between the prime filters of L and the lattice morphisms from L to $\underline{2}$. All that is left is to show that the bijection is not only one of sets, but also of Priestley spaces.

- Since $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in L$, it is clear that $F \subseteq G$ if and only if $\chi_F \leq \chi_G$.
- Because the topology generated is by the sets U_a and V_a, it is again evident that this coincides with the topology on $\mathcal{PF}(L)$.

In this manner, the Priestley dual of L is identified in a natural way with $\text{Hom}_{lattice}(L, \underline{2})$. □

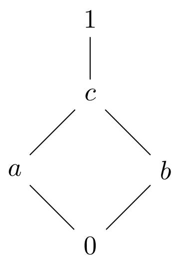
6.2 The Esakia Dual

For Esakia duality, no such theorem can exist. To be precise, there cannot be a set S such that the Esakia dual of every lattice is equal to $\text{Hom}_{Heyting}(L, S)$. An elegant proof of this fact can be found in [1]. The key argument is that for any linearly ordered Heyting algebra A there would have to be an injective function $h : A \to S$, forcing S to have an arbitrarily large cardinality.

However, since Esakia duality is a special case of Priestley duality, the theorem above holds for Heyting algebras as well. For a Heyting algebra L, the space $\text{Hom}_{lattice}(L, \underline{2})$ is then also its Esakia dual, since Esakia duality is merely the restriction of Priestley duality to Heyting algebras. The problem is that these *lattice* morphisms are not in general morphisms in the category of Heyting algebras. Therefore, the theorem cannot be interpreted in a meaningful way, as in classical logic.

It would be convenient if there would be some lattices for which all lattice homomorphisms turned out to be Heyting morphisms as well. However, it does not look like this is true for any lattice at all. We do not have a proof that there always exists a lattice homomorphism that is not a Heyting algebra, but such an example can already be found in the case of $\mathbb{C}^2$.

Recall that $Q(\mathbb{C}^2)$ has the following structure:



(6.1)

Now, consider the function $f : Q(\mathbb{C}^2) \to \underline{2}$ defined by

$$
\begin{aligned}
f(0) &= 0 \\
f(a) &= 0 \\
f(b) &= 0 \\
f(c) &= 0 \\
f(1) &= 1.
\end{aligned}
$$

It is easily verified that this is indeed a lattice homomorphism. However,

$$f(a \to b) = f(\bigvee \{c \mid a \wedge c \leq b\}) = f(b) = 0,$$

whereas

$$f(a) \to f(b) = 0 \to 0 = 1.$$

This means that the space $\mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_{\mathbb{C}^2}), \underline{2})$ cannot be the Esakia dual of $\mathcal{O}(\Sigma_{\mathbb{C}^2})$, and thus this puts a stop to this approach.

Another fact that may be important to note is that in the non-commutative case a more general result has been found by M. Caspers, C. Heunen, N. P. Landsman and B. Spitters, indicating that there exist *no* frame morphisms $f : Q(M_n(\mathbb{C})) \to \underline{2}$ [2] for $n > 2$ (this is the equivalence of the Kochen-Specker theorem in quantum theory), and thus certainly no complete Heyting morphisms either.

This approach, therefore, does not turn out to be useful. We still have our hypothesis, however, that the Esakia dual of $\mathcal{O}(\Sigma_A)$ might be equal to $\mathrm{Mod}_{\mathcal{C}(A)}(\mathcal{O}(\Sigma_A))$. Here the idea in finite dimension was that states ω on A would define canonical Kripke models for $Q(A)$ by

$$
\begin{aligned}
V_\omega : Q(A) &\to \mathrm{Up}(\mathcal{C}(A)); \\
S &\mapsto \{C \in \mathcal{C}(A) \mid \omega(S(C)) = 1\},
\end{aligned}
$$

see also [9]. $S \in Q(A)$ is then said to be *true* in a state ω provided $V_\omega(S) = \mathcal{C}(A)$, and *false* if $V_\omega(S) = \emptyset$.

What goes awry is that these functions fail to be lattice morphisms, let alone Heyting morphisms. Yet again, an example can already be found in $\mathbb{C}^2$. For the state ω on $\mathbb{C}^2$, take the function

$$
\begin{aligned}
\omega &: \mathbb{C}^2 \to \mathbb{C}; \\
(x, y) &\mapsto x.
\end{aligned}
$$

Consider the elements S and T of $Q(\mathbb{C}^2)$ defined by

$$S(C) = \begin{cases} (0,1) & C = \mathbb{C}^2, \\ (0,0) & C = \mathbb{C}, \end{cases}$$

$$T(C) = \begin{cases} (1,0) & C = \mathbb{C}^2, \\ (0,0) & C = \mathbb{C}, \end{cases}$$

which are exactly the elements a and b in the familiar diagram 6.1. Now

$$S \to T = \bigvee \left\{ R \in Q(\mathbb{C}^2) \mid R \wedge S \leq T \right\} = T,$$

as we have seen before.

$$V_\omega(S) = \left\{ C \in \mathcal{C}(\mathbb{C}^2) \mid \omega(S(C)) = 1 \right\} = \emptyset,$$
$$V_\omega(T) = \left\{ C \in \mathcal{C}(\mathbb{C}^2) \mid \omega(T(C)) = 1 \right\} = \mathbb{C}^2.$$

Therefore,

$$V_\omega(S \to T) = V_\omega(T) = \mathbb{C}^2,$$

whereas

$$V_\omega(S) \to V_\omega(T) = \emptyset \to \mathbb{C}^2 = \bigcup \left\{ U \in \mathrm{Up}(\mathcal{C}(\mathbb{C}^2)) \mid U \cap \emptyset \subseteq \mathbb{C}^2 \right\} = \mathcal{C}(\mathbb{C}^2).$$

Thus V_ω is indeed not a Heyting morphism.

Moreover, in the book 'Foundations of Quantum Theory' Nicolaas P. Landsman [11] it is proven that there are no lattice homomorphisms $f : Q(M_n(\mathbb{C})) \to \mathrm{Up}(\mathcal{C}(A))$ *either* for $n > 2$. The idea of the proof is that since there are no frame momorphisms $f : Q(M_n(\mathbb{C})) \to \underline{2}$ for $n > 2$, there can be no continuous cross-section of the bundle $\pi : \Sigma_A \to \mathcal{C}(A)$ (where the Alexandrov topology is considered on $\mathcal{C}(A)$). Such a cross-section $\sigma : \mathcal{C}(A) \to \Sigma_A$ is continuous if and only if $\sigma^{-1}(\mathcal{U})$ is an upper set in $\mathcal{C}(A)$ for every open set $\mathcal{U} \in \mathcal{O}(\Sigma_A)$. That this $\sigma^{-1} : \mathcal{O}(\Sigma_A) \to \mathrm{Up}(\mathcal{C}(A))$, if it would have existed, should have been a lattice morphism can easily be seen from the fact that taking unions and intersections always commutes with taking inverse images. Thus $\mathrm{Mod}_{\mathcal{C}(A)}(\mathcal{C}(A))$ cannot be the Esakia dual of $\mathcal{O}(\Sigma_A)$ either.

6.3 Conclusion

Our mission was to find a lattice S_A for every C^*-algebra A such that the Esakia dual of $\mathcal{O}(\Sigma_A)$ would be equal to $\mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_A), S_A)$. We only have a negative result as some candidates have been ruled out. Finding the right space has proven a daunting task, and it is not clear if there even exists such a space for every C^*-algebra. We will leave this problem open, and continue with various new results about $\mathcal{O}(\Sigma_A)$ we did manage to prove.

Chapter 7

Expanding $Q(A)$

In lemma 5.0.2 we have proven that for finite-dimensional C^*-algebras A a frame $Q(A)$ can be defined, which is isomorphic to the topology $\mathcal{O}(\Sigma_A)$. Now one can wonder what goes wrong when A becomes infinite-dimensional. The answer is that the lattice of projections $\mathcal{P}(C)$ for $C \in \mathcal{C}(A)$ does not have to be complete for infinite-dimensional C. That is, arbitrary joins do not have to exist.

This can be remedied however, by only considering infinite-dimensional C^*-algebras for which this projection lattice is also complete. For this purpose, let us introduce *AW^*-algebras*:

Definition 7.0.1. An **AW^*-algebra** A is a C^*-algebra for which $\mathcal{P}(A)$ is a complete lattice, and every maximal commutative subalgebra $M \in \mathcal{C}(A)$ is generated by its projections. Equivalently, an AW^*-algebra A is a C^*-algebra for which the Gelfand spectrum of every maximal commutative subalgebra $M \in \mathcal{C}(A)$, $\Sigma(M)$ is a Stonean space.

Definition 7.0.2. A topological space X is a **extremally disconnected** if for every open $U \subseteq X$, $\overline{U}$ is open as well. Then X is called **Stonean** if it is extremally disconnected, compact and Hausdorff.

Note that a Stonean space is necessarily a Stone space. From here on we will indicate the category of AW^*-algebras by **AWStar**, where the morphisms are normal *-homomorphisms, i.e. *-homomorphisms which preserve arbitrary suprema of orthogonal projections. Now for some examples of AW^*-algebras:

Lemma 7.0.3. *Let H be a Hilbert space. Then $B(H)$ is an AW^*-algebra.*

Lemma 7.0.4. *Let $A_1, ..., A_K$ be AW^*-algebras. Then $\bigoplus_{i=1}^{K} A_i$ is also an AW^*-algebra.*

The proofs are omitted, but can be found in [12]. These two results can be used to prove the following:

Lemma 7.0.5. *Let A be a finite-dimensional C^*-algebra. Then A is an AW^*-algebra.*

Proof. Since by lemma C.0.11 we know that $A \cong \bigoplus_{i=1}^{K} M_{N_i}(\mathbb{C})$, where $M_{N_i}(\mathbb{C}) = B(\mathbb{C}^{N_i})$. By the previous two lemmas, A is an AW^*-algebra. □

Finally, let us introduce the equivalent of $\mathcal{C}(A)$ for AW^*-algebras.

Definition 7.0.6. An **AW^*-subalgebra** of an AW^*-algebra A is a subset $C \subseteq A$ such that:

- C is an AW^*-algebra,
- if $\{p_i\}_{i \in I}$ is a collection of projections in B, then their join $\bigvee p_i$ calculated in A has to be an element of B.

The poset of commutative AW^*-subalgebras of some AW^*-algebra A is denoted by $\mathcal{A}(A)$.

We can now introduce our expanded $Q(A)$:

$$Q(A) = \{S : \mathcal{A}(A) \to \mathcal{P}(A) \mid S(C) \in \mathcal{P}(C), S(C) \leq S(D) \text{ if } C \subseteq D\}.$$

The reason $\mathcal{A}(A)$ has to be used instead of $\mathcal{C}(A)$ is that $\mathcal{P}(C)$ may not be a complete lattice for every C^*-subalgebra of A, but it certainly is for every AW^*-subalgebra. Considering this, $Q(A)$ is then a frame by the same arguments as before. No problems arise since $\mathcal{P}(C)$ is indeed a complete lattice for every $C \in \mathcal{A}(A)$.

One comment has to be made about this new object $Q(A)$. Another $Q(A)$ was introduced already for finite-dimensional C^*-algebras, and for these C^*-algebras it seems there are now two frames both named $Q(A)$. The confusion can be resolved by the following observation:

Lemma 7.0.7. *For finite-dimensional C^*-algebra A, $\mathcal{C}(A) = \mathcal{A}(A)$.*

Proof. Any AW^*-subalgebra of A is a C^*-algebra, and therefore also a C^*-subalgebra. For the other way around, take a C^*-subalgebra $C \subseteq A$. By lemma 7.0.5, this is an AW^*-algebra. By proposition 2.0.7, the projection lattice $\mathcal{P}(C)$ is complete. We can conclude that C is an AW^*-subalgebra of A, and therefore $\mathcal{C}(A) = \mathcal{A}(A)$. □

The downside of using $\mathcal{A}(A)$ instead of $\mathcal{C}(A)$ in the expanded definition of $Q(A)$ is that for infinite-dimensional A we cannot hope that $Q(A)$ will ever be isomorphic to $\mathcal{O}(\Sigma_A)$, since this last object is tied to $\mathcal{C}(A)$.

Instead, let us introduce a modified $\mathcal{O}(\Sigma_A)$:

Definition 7.0.8. Let A be a AW^*-algebra. Consider the set

$$\widetilde{\Sigma}_A = \bigsqcup_{C \in \mathcal{A}(A)} \Sigma(C),$$

i.e. the disjoint union over all Gelfand spectra $\Sigma(C)$, where C is a commutative AW^*-subalgebra of A.

Now the topology on this space is defined by its opens $\mathcal{U} \subseteq \widetilde{\Sigma}_A$, which are of the form

$$\mathcal{U} = \bigsqcup_{C \in \mathcal{A}(A)} \mathcal{U}_C;$$
$$\mathcal{U}_C = \mathcal{U} \cap \Sigma(C).$$

Then $\mathcal{U}$ is open if and only if the following two conditions are satisfied for each $C \in \mathcal{A}(A)$:

1. $\mathcal{U}_C \in \mathcal{O}(\Sigma(C))$,
2. For all $D \supseteq C$, if $\lambda \in \mathcal{U}_C$ and $\lambda' \in \Sigma(D)$ such that $\lambda'|_C = \lambda$, then $\lambda' \in \mathcal{U}_D$.

This is essentially the same definition as $\mathcal{O}(\Sigma_A)$, but every $\mathcal{C}(A)$ has been changed into $\mathcal{A}(A)$. It is the question now if this still turns out to be an isomorphism. For that we have to reconsider the lengthy calculations in lemma 5.0.2. While the frame morphism $\Psi : Q(A) \to \mathcal{O}(\Sigma_A)$ from that proof can be left unchanged, for $\Phi : \mathcal{O}(\Sigma_A) \to Q(A)$ something has to be done. The reason is that $\beta_C^{-1}(\mathcal{U}_C)$ is not well defined. Recall that β_C is an isomorphism of lattices between the projections of C and the clopen sets in $\Sigma(C)$. Since $\Sigma_A = \bigsqcup_{C \in \mathcal{C}(A)} \Sigma(C)$, for finite-dimensional A all $\Sigma(C)$ are finite, discrete spaces. Therefore, these $\mathcal{U}_C$ are already clopen. In general, however, this is not the case. Somehow, we have to make a clopen set from $\mathcal{U}_C$ in order to apply β_C^{-1}.

Thankfully, this can easily be done by the following lemma.

Lemma 7.0.9. *Let A be an AW^*-algebra. Then for every AW^*-subalgebra C of A, $\Sigma(C)$ is Stonean.*

Proof. Take such an AW^*-subalgebra C. By the definition of AW^*-algebras, every maximal commutative subalgebra $M \in \mathcal{C}(C)$ has the property that $\Sigma(M)$ is Stonean. Now, since C is itself a commutative AW^*-algebra, C is the maximal element of $\mathcal{C}(C)$. Thus we can conclude that $\Sigma(C)$ is Stonean. □

Due to this property we know that if we take the closure $\overline{\mathcal{U}_C}$, this is a clopen set in $\Sigma(C)$. We now have two candidates for frame morphisms. While promising, it turns out that one of the two is not a frame morphism.

Proposition 7.0.10. *$Q(A)$ is a dense sublocale of $\mathcal{O}(\widetilde{\Sigma}_A)$.*

Proof. Let us start by defining our two frame morphisms:

$$\begin{aligned}\Phi : \mathcal{O}(\widetilde{\Sigma}_A) &\to Q(A);\\ \mathcal{U} &\mapsto S_{\mathcal{U}};\\ S_{\mathcal{U}}(C) &= \beta_C^{-1}(\overline{\mathcal{U}_C}),\end{aligned}$$

$$\begin{aligned}\Psi : Q(A) &\to \mathcal{O}(\widetilde{\Sigma}_A);\\ S \mapsto \mathcal{U}_S &= \bigsqcup_{C\in\mathcal{A}(A)} \beta_C(S(C)).\end{aligned}$$

Let us first show that these are indeed well defined.

- For Φ, it is clear that $\beta_C^{-1}(\overline{\mathcal{U}_C})$ is an element of $\mathcal{P}(C)$. Now, if $C \subseteq D$, is then $S_{\mathcal{U}}(C) \leq S_{\mathcal{U}}(D)$? We know that $\Sigma(j)^{-1}(\mathcal{U}_C) \subseteq \mathcal{U}_D$, see also the proof of lemma 5.0.2. Therefore, $\overline{\Sigma(j)^{-1}(\mathcal{U}_C)} \subseteq \overline{\mathcal{U}_D}$ also holds. To continue, we will slightly abuse notation once more by omitting the inclusion j:

  $$\begin{aligned}\beta_C^{-1}(\overline{\mathcal{U}_C}) &= \beta_D^{-1}(\Sigma(j)^{-1}(\overline{\mathcal{U}_C}))\\ &\overset{*}{=} \beta_D^{-1}(\overline{\Sigma(j)^{-1}(\mathcal{U}_C)})\\ &\leq \beta_D^{-1}(\overline{\mathcal{U}_D}).\end{aligned}$$

  A step has been made here that will need its own (technical) proof:

  $$\Sigma(j)^{-1}(\overline{O}) = \overline{\Sigma(j)^{-1}(O)}$$

  for any open set O. This lemma will be proven below, see lemma 7.0.11. Other than that, we are done for now with Φ.

- Now consider Ψ. To prove that it is well defined we have to change little from the proof of last time. It is again clear that for all $C \in \mathcal{A}(A)$, $\beta_C(S(C))$ is an open set in $\Sigma(C)$. Now,

  $$\begin{aligned}\Sigma(j)^{-1}(\mathcal{U}_C) &= \Sigma(j)^{-1}(\beta_C(S(C)))\\ &= \beta_D(j(S(C)))\\ &\subseteq \beta_D(S(D))\\ &= \mathcal{U}_D.\end{aligned}$$

  Therefore, Ψ is also well defined.

The next part is determining whether or not Ψ or Φ are frame morphisms:

- Ψ turns out not to be a frame morphism, although it is a lattice morphism:

  $$\begin{aligned}\Psi(0) &= \bigsqcup_{C\in\mathcal{A}(A)} \beta_C(0) = \emptyset,\\ \Psi(1) &= \bigsqcup_{C\in\mathcal{A}(A)} \beta_C(1) = \widetilde{\Sigma}_A,\end{aligned}$$

$$\begin{aligned}\Psi(S \vee T) &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C((S \vee T)(C)) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S(C)) \cup \beta_C(T(C)) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S(C)) \cup \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(T(C)) \\ &= \Psi(S) \cup \Psi(T),\end{aligned}$$

$$\begin{aligned}\Psi(S \wedge T) &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C((S \wedge T)(C)) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S(C)) \cap \beta_C(T(C)) \\ &= \left(\bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S(C))\right) \cap \left(\bigsqcup_{C \in \mathcal{A}(A)} \beta_C(T(C))\right) \\ &= \Psi(S) \cap \Psi(T);\end{aligned}$$

Arbitrary joins are not preserved:

$$\begin{aligned}\Psi(\bigvee S_i) &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C((\bigvee S_i)(C)) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(\bigvee (S_i(C))) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \overline{\bigcup \beta_C(S_i(C))}, \\ \bigcup \Psi(S_i) &= \bigcup \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S_i(C)) \\ &= \bigsqcup_{C \in \mathcal{A}(A)} \bigcup \beta_C(S_i(C)).\end{aligned}$$

Where it goes wrong is that β_C is an isomorphism between the projections and the clopen sets of $\Sigma(C)$. This means that arbitrary joins can be "pulled through" β_C, however in the lattice of clopen sets, $\bigvee U_i = \overline{\bigcup U_i}$. The closure has to be taken since a union of infinitely many clopen sets in general does not result in a clopen set. Ψ is a function with co-domain $\mathcal{O}(\widetilde{\Sigma}_A)$ where an arbitrary join *is* just the union.

- Φ is a frame morphism, however:

$$\begin{aligned}\Phi(\emptyset)(C) &= \beta_C^{-1}(\emptyset) = 0, \\ \Phi(\Sigma_A)(C) &= \beta_C^{-1}(\Sigma(C)) = 1, \\ \Phi(\bigcup \mathcal{U}_i)(C) &= \beta_C^{-1}(\overline{\bigcup \mathcal{U}_{i,C}}) \\ &\overset{*}{=} \beta_C^{-1}(\overline{\bigcup \overline{\mathcal{U}_{i,C}}}) \\ &= \bigvee \beta_C^{-1}(\overline{\mathcal{U}_{i,C}}) \\ &= \bigvee \Phi(\mathcal{U})(C) \\ \Phi(\mathcal{U} \cap \mathcal{V})(C) &= \beta_C^{-1}(\overline{(\mathcal{U} \cap \mathcal{V})_C}) \\ &\overset{**}{=} \beta_C^{-1}(\overline{\mathcal{U}_C} \cap \overline{\mathcal{V}_C}) \\ &= \Phi(\mathcal{U})(C) \wedge \Phi(\mathcal{V})(C),\end{aligned}$$

where once again, some steps need some clarification. $\overline{\bigcup U_i} = \overline{\bigcup \overline{U_i}}$ holds in any topological space, which is proven in lemma 7.0.12 below.

Secondly, in general $\overline{U \cap V}$ is certainly not equal to $\overline{U} \cap \overline{V}$, however we are dealing with a Stonean space. That this identity does hold in Stonean spaces will be proven in lemma 7.0.13 below.

While Ψ is no frame morphism, Ψ and Φ are adjoints:

$$\begin{aligned}
\Phi(\mathcal{U}) \leq T &\Leftrightarrow S_{\mathcal{U}}(C) \leq T(C) && \forall C \in \mathcal{A}(C)\\
&\Leftrightarrow \beta_C^{-1}(\overline{\mathcal{U}_C}) \leq T(C) && \forall C \in \mathcal{A}(C)\\
&\Leftrightarrow \overline{\mathcal{U}_C} \subseteq \beta_C(T(C)) && \forall C \in \mathcal{A}(C)\\
&\overset{*}{\Leftrightarrow} \mathcal{U}_C \subseteq \beta_C(T(C)) && \forall C \in \mathcal{A}(C)\\
&\Leftrightarrow \mathcal{U}_C \subseteq \Psi(T)_C && \forall C \in \mathcal{A}(C)\\
&\Leftrightarrow \mathcal{U} \subseteq \Psi(T),
\end{aligned}$$

where the indicated step holds because $\beta_C(T(C))$ is always a closed set. Therefore, Φ is the left adjoint of Ψ. Furthermore:

$$\begin{aligned}
(\Phi \circ \Psi(S))(C) &= \Phi\left(\bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S(C))\right)(C)\\
&= \beta_C^{-1}\left(\overline{\beta_C(S(C))}\right)\\
&= \beta_C^{-1}\left(\beta_C(S(C))\right)\\
&= S(C),
\end{aligned}$$

thus $\Phi \circ \Psi = \mathrm{id}_{Q(A)}$. However:

$$\begin{aligned}
\Psi \circ \Phi(\mathcal{U}) &= \Psi(S_{\mathcal{U}})\\
&= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C(S_{\mathcal{U}}(C))\\
&= \bigsqcup_{C \in \mathcal{A}(A)} \beta_C\left(\beta_C^{-1}(\overline{\mathcal{U}_C})\right)\\
&= \bigsqcup_{C \in \mathcal{A}(A)} \overline{\mathcal{U}_C}\\
&\neq \bigsqcup_{C \in \mathcal{A}(A)} \mathcal{U}_C
\end{aligned}$$

While this last composition $\Psi \circ \Phi$ is not the identity on $\mathcal{O}(\widetilde{\Sigma}_A)$, it is a nucleus (definition 1.0.10):

$$\begin{aligned}
\bigsqcup_{C \in \mathcal{A}(A)} \mathcal{U}_C &\subseteq \bigsqcup_{C \in \mathcal{A}(A)} \overline{\mathcal{U}_C}\\
&= \Psi \circ \Phi(\mathcal{U})\\
\Psi \circ \Phi\left(\Psi \circ \Phi(\mathcal{U})\right) &= \bigsqcup_{C \in \mathcal{A}(A)} \overline{\overline{\mathcal{U}_C}}\\
&= \bigsqcup_{C \in \mathcal{A}(A)} \overline{\mathcal{U}_C}\\
&= \Psi \circ \Phi(\mathcal{U})
\end{aligned}$$

Therefore, by proposition 1.0.11, $Q(A)$ is a sublocale of $\mathcal{O}(\widetilde{\Sigma}_A)$. It is also clear that $\Psi \circ \Phi(\emptyset) = \emptyset$, and therefore $Q(A)$ is a dense sublocale of $\mathcal{O}(\widetilde{\Sigma}_A)$ by definition 1.0.13. □

We have skipped over three steps in the proof of the previous proposition, which can be found as promised in the form of three lemmas below.

Lemma 7.0.11. *Let $C(X)$ and $C(Y)$ be two commutative AW^*-algebras, (i.e. X and Y are Stonean), and let $f : X \to Y$ be such that $C(f) : C(Y) \to C(X)$ is a normal *-homomorphism. Then for any open $O \subseteq Y$, we have that $f^{-1}(\overline{O}) = \overline{f^{-1}(O)}$.*

Proof. Since $C(X)$ and $C(Y)$ are AW^*-algebras, X and Y are Stonean spaces. Stonean spaces are in particular Stone spaces, and thus we can write O as the union of clopen sets: $O = \bigcup U_i$. We know that these clopen sets U_i correspond to projections $p_i \in C(Y)$ by lemma 3.3.1, explicitly via $U_i = p_i^{-1}(1)$. Note that in this correspondence, $\bigvee p_i$ corresponds to $\overline{\bigcup U_i}$. Therefore:

$$\begin{aligned} C(f)(\bigvee p_i) &= (\bigvee p_i) \circ f \\ &= \bigvee (p_i \circ f) \\ &= \bigvee (C(f)(p_i)), \end{aligned}$$

Now we will use this to obtain our equality:

$$\begin{aligned} \left(C(f)(\bigvee p_i)\right)^{-1}(1) &= \left(\left(\bigvee p_i\right) \circ f\right)^{-1}(1) \\ &= f^{-1}\left(\left(\bigvee p_i\right)^{-1}(1)\right) \\ &= f^{-1}\left(\overline{\bigcup U_i}\right) \\ &= f^{-1}(\overline{O}). \end{aligned}$$

$$\begin{aligned} \left(\bigvee (C(f)(p_i))\right)^{-1}(1) &= \overline{\bigcup (C(f)(p_i))^{-1}(1)} \\ &= \overline{\bigcup (p_i \circ f)^{-1}(1)} \\ &= \overline{\bigcup (f^{-1} \circ p_i^{-1})(1)} \\ &= \overline{\bigcup f^{-1}(U_i)} \\ &= \overline{f^{-1}\left(\bigcup U_i\right)} \\ &= \overline{f^{-1}(O)}. \end{aligned}$$

Therefore, $f^{-1}(\overline{O}) = \overline{f^{-1}(O)}$ which is exactly what was claimed. □

To see how to use this lemma in the proof of proposition 7.0.10, simply take $\Sigma(j)$ for f. Then $C(\Sigma(j)) = j$ which is indeed a normal *-homomorphism. Now for the second lemma.

Lemma 7.0.12. *In a topological space X, $\overline{\bigcup U_i} = \overline{\bigcup \overline{U_i}}$.*

Proof. It is clear that $\overline{\bigcup U_i} \subseteq \overline{\bigcup \overline{U_i}}$. Now take some $x \in \overline{\bigcup \overline{U_i}}$. Then for all open neighbourhoods $x \in O$, $O \cap \overline{U_i} \neq \emptyset$ for some $i \in I$. Then consider such a $y \in O \cap \overline{U_i}$. Since $y \in \overline{U_i}$ any open neighbourhood $y \in V$ has to satisfy $V \cap U_i \neq \emptyset$. Because O is an open neighbourhood of y, we have that $O \cap U_i \neq \emptyset$. Therefore, $O \cap \bigcup U_i \neq \emptyset$ and thus $x \in \overline{\bigcup U_i}$. □

Lemma 7.0.13. *In a Stonean space X, $\overline{U \cap V} = \overline{U} \cap \overline{V}$ for any open sets U, V.*

Proof. $\overline{U \cap V} \subseteq \overline{U} \cap \overline{V}$ holds in any topological space. Now take some $x \notin \overline{U} \cap \overline{V}$. Then by definition there must be an open neighbourhood O of x for which $O \cap U \cap V = \emptyset$. Now we claim that for any two open sets in any topological space , $U \cap V = \emptyset$ if and only if $\overline{U} \cap V = \emptyset$:

$$\begin{aligned} U \cap V = \emptyset &\Leftrightarrow U \subseteq V^c \\ &\Leftrightarrow \overline{U} \subseteq V^c \\ &\Leftrightarrow \overline{U} \cap V = \emptyset. \end{aligned}$$

In the second step we can take the closure of U, because V^c is a closed set. Now recall that in a Stonean space, $\overline{U}$ is open as well! That means we can use this result twice:

$$\begin{aligned} O \cap U \cap V = \emptyset &\Leftrightarrow O \cap U \cap \overline{V} = \emptyset \\ &\Leftrightarrow O \cap \overline{U} \cap \overline{V} = \emptyset. \end{aligned}$$

Since $x \in O$, $x \notin \overline{U} \cap \overline{V}$. We can conclude that $x \in \overline{U \cap V}$ iff $x \in \overline{U} \cap \overline{V}$. □

Now the proof of proposition 7.0.10 is truly finished. What implications this proposition has, and of what purpose this expansion is in terms of logic remains to be seen.

Chapter 8

Functoriality

The next property we want to explore about $Q(A)$ and $\mathcal{O}(\Sigma_A)$ is their functoriality, that is to say, if the mappings $A \mapsto Q(A)$ and $A \mapsto \mathcal{O}(\Sigma_A)$ are functors. Let us dive straight in!

Proposition 8.0.1. *The mapping $A \mapsto Q(A)$ gives rise to covariant functors $Q : \mathbf{CAWStar} \to \mathbf{Frm}$ and $Q : \mathbf{AWStar}_{\hookrightarrow} \to \mathbf{Frm}$, where $\mathbf{CAWStar}$ denotes the category of commutative AW^*-algebras, and $\mathbf{AWStar}_{\hookrightarrow}$ is the category of AW^*-algebras with only the injective normal *-homomorphisms as morphisms.*

Proof. Let A and B be AW^*-algebras, with $\phi : A \to B$ a normal *-homomorphism. If A and B are commutative or if ϕ is injective, then the mapping

$$\begin{aligned} \mathcal{A}(\phi)_* : \mathcal{A}(B) &\to \mathcal{A}(A) \\ C &\mapsto \phi^{-1}(C), \end{aligned}$$

is well defined. This is in essence the bottleneck forcing us to only consider the categories mentioned. Now let us draw a diagram and see if it behaves nicely:

$$\begin{array}{ccccc} \mathcal{A}(A) & \xleftarrow{\mathcal{A}(\phi)_*} & \mathcal{A}(B) & \xleftarrow{\mathcal{A}(\psi)_*} & \mathcal{A}(C) \\ \downarrow S & & \downarrow S_\phi & & \downarrow (S_\phi)_\psi \stackrel{?}{=} S_{\psi\circ\phi} \\ \mathcal{P}(A) & \xrightarrow{\phi} & \mathcal{P}(B) & \xrightarrow{\psi} & \mathcal{P}(C) \end{array}$$

The gist of it is that we take AW^*-algebras A, B and C with normal *-homomorphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$. Then we consider an element $S \in Q(A)$ and transfer it to $Q(C)$ both directly and via B and see if both routes result in the same element in $Q(C)$. We define $S_\phi = \phi \circ S \circ \mathcal{C}(\phi)_*$. Let us check if this indeed well defined and yields an element of $Q(B)$:

$$S_\phi(D) = \phi\left(S\left(\phi^{-1}(D)\right)\right) \in \mathcal{P}(B)$$

holds if $\phi^{-1}(D) \in \mathcal{A}(A)$ and if $\phi(\mathcal{P}(A)) \subseteq \mathcal{P}(B)$. The first is the case when we restrict ourselves to the aforementioned categories. The last can be seen as follows:

$$\begin{aligned} \phi(e)^2 &= \phi(e^2) \\ &= \phi(e) \\ \phi(e)\phi(e)^* &= \phi(e)\phi(e^*) \\ &= \phi(ee^*) \\ &= \phi(e). \end{aligned}$$

Now to conclude that $S_\phi \in Q(B)$, take $D \subseteq E \in \mathcal{C}(B)$. Then:

$$\begin{aligned} D \subseteq E &\Rightarrow \phi^{-1}(D) \subseteq \phi^{-1}(E) \\ &\Rightarrow S(\phi^{-1}(D)) \leq S(\phi^{-1}(E)) \\ &\Rightarrow \phi(S(\phi^{-1}(D))) \leq \phi(S(\phi^{-1}(E))). \end{aligned}$$

ϕ is even a lattice morphism when restricted to $\mathcal{P}(A)$ [19]. It follows from the definition of the category of AW^*-algebras that ϕ also preserves arbitrary suprema. This can be used to show that $Q(\phi) : S \mapsto S_\phi$ is also a frame morphism:

$$\begin{aligned} Q(\phi)\left(\bigvee S_i\right)(D) &= \phi\left(\bigvee (S_i(\phi^{-1}(D))\right) \\ &= \bigvee \phi\left(S_i(\phi^{-1}(D)\right) Q(\phi)(S \wedge T)(D) \qquad = \phi((S \wedge T)(\phi^{-1}(D))) \\ &= \phi(S(\phi^{-1}(D)) \wedge T(\phi^{-1}(D))) \\ &= \phi(S(\phi^{-1}(D))) \wedge \phi(T(\phi^{-1}(D))). \end{aligned}$$

Lastly, let us check if $(S_\phi)_\psi = S_{\psi \circ \phi}$.

$$\begin{aligned} (S_\phi)_\psi(D) &= \psi(S_\phi(\psi^{-1}(D))) \\ &= \psi \circ \phi(S(\phi^{-1}(\psi^{-1}(D)))) \\ &= \psi \circ \phi(S((\psi \circ \phi)^{-1}(D))). \end{aligned}$$

□

Now for the functoriality of $\mathcal{O}(\Sigma_A)$. This we will do in a slightly different way, as we will prove that $A \mapsto \Sigma_A$ is a contravariant functor for certain categories. Since $\mathcal{O} : X \mapsto \mathcal{O}(X)$ is already a contravariant functor, this makes $A \mapsto \mathcal{O}(\Sigma_A)$ a covariant functor.

Proposition 8.0.2. *The mapping $A \mapsto \Sigma_A$ gives rise to contravariant functors Σ :* **CCstar** $\rightarrow$ **Top** *and Σ :* $\mathbf{CStar}_{\hookrightarrow}$ $\rightarrow$ **Top***, where* **CCstar** *denotes the category of commutative C^*-algebras, and* $\mathbf{CStar}_{\hookrightarrow}$ *the category of C^*-algebras with only the injective *-homomorphisms as morphisms.*

Proof. Consider two C^*-algebras, A and B, with a *-homomorphism $A \xrightarrow{\phi} B$. Then define:

$$\begin{aligned} \Sigma_\phi &: \Sigma_B \rightarrow \Sigma_A \\ &(C, \lambda) \mapsto (\phi^{-1}(C), \lambda \circ \phi). \end{aligned}$$

Once more, $\phi^{-1}(C)$ is only a commutative C^*-subalgebra of B if ϕ is injective or if A and B are commutative. Now let us see if Σ_ϕ is a well-defined, continuous function if one of these conditions hold. The first part is quickly verified, since $\lambda \circ \phi$ is indeed an element of $\Sigma(\phi^{-1}(C))$ whenever $\lambda \in \Sigma(C)$ as the composition of two continuous functions. Now for the continuity of Σ_ϕ:

$$\begin{aligned} \mathcal{V} = \Sigma_\phi^{-1}(\mathcal{U}) &= \left\{(C, \lambda) \in \Sigma_B \mid (\phi^{-1}(C), \lambda \circ \phi) \in \Sigma_A\right\} \\ \mathcal{V}_C &= \left\{\lambda \in \Sigma(C) \mid (\phi^{-1}(C), \lambda \circ \phi) \in \Sigma_A\right\} \\ &= \left\{\lambda \in \Sigma(C) \mid \lambda \circ \phi \in \mathcal{U}_{\phi^{-1}(C)}\right\} \\ &= \Sigma(\phi|_{\phi^{-1}(C)})^{-1}(\mathcal{U}_{\phi^{-1}(C)}), \end{aligned}$$

and therefore $\mathcal{V}_C$ is an open set in $\Sigma(C)$ for all $C \in \mathcal{C}(B)$. Now take commutative C^*-subalgebras $C \subseteq D \subseteq B$:

$$\begin{array}{ccc} C & \xrightarrow{\quad j \quad} & D \\ \uparrow{\scriptstyle \phi|_{\phi^{-1}(C)}} & & \uparrow{\scriptstyle \phi|_{\phi^{-1}(D)}} \\ \phi^{-1}(C) & \xrightarrow{\quad \phi_j \quad} & \phi^{-1}(D) \end{array}$$

where $\phi_j : \phi^{-1}(C) \to \phi^{-1}(D)$ is the inclusion mapping. Then it follows immediately that $j \circ \phi|_{\phi^{-1}(C)} = \phi|_{\phi^{-1}(D)} \circ \phi_j$, and thus:

$$\begin{aligned}
\Sigma(j)^{-1}(\mathcal{V}_C) &= \Sigma(j)^{-1}\left(\Sigma(\phi|_{\phi^{-1}(C)})^{-1}(\mathcal{U}_{\phi^{-1}(C)})\right) \\
&= \left(\Sigma(\phi|_{\phi^{-1}(C)}) \circ \Sigma(j)\right)^{-1}(\mathcal{U}_{\phi^{-1}(C)}) \\
&= \Sigma(j \circ \phi|_{\phi^{-1}(C)})^{-1}(\mathcal{U}_{\phi^{-1}(C)}) \\
&= \Sigma(\phi|_{\phi^{-1}(D)} \circ \phi_j)^{-1}(\mathcal{U}_{\phi^{-1}(C)} \\
&= \Sigma(\phi|_{\phi^{-1}(D)})^{-1}\left(\Sigma(\phi_j)^{-1}(\mathcal{U}_{\phi^{-1}(C)}\right) \\
&\subseteq \Sigma(\phi|_{\phi^{-1}(D)})^{-1}(\mathcal{U}_{\phi^{-1}(D)}) \\
&= \mathcal{V}_D.
\end{aligned}$$

From this calculation we can conclude that Σ_ϕ is indeed continuous. One last check remains: given $A \xrightarrow{\phi} B \xrightarrow{\psi} C$, does $\Sigma_{\psi\circ\phi} = \Sigma_\phi \circ \Sigma_\psi$ hold? It does:

$$\begin{aligned}
\Sigma_\phi \circ \Sigma_\psi(D, \lambda) &= \Sigma_\phi(\psi^{-1}(D), \lambda \circ \psi) \\
&= (\phi^{-1}(\psi^{-1}(D)), \lambda \circ \psi \circ \phi) \\
&= ((\psi \circ \phi)^{-1}(D), \lambda \circ (\psi \circ \phi)) \\
&= \Sigma_{\psi\circ\phi}(D).
\end{aligned}$$

□

Chapter 9

Sobriety

Another property worth exploring about Σ_A is if it is a sober space, in the sense of definition 1.0.6. It is known that if A is finite-dimensional, then Σ_A is a sober space [20]. In particular, this is proved by using the fact that if A is finite-dimensional, then $\mathcal{C}(A)$ has the so-called ascending chain condition: any infinite ascending chain $C_1 \subseteq C_2 \subseteq ...$ must eventually be constant (i.e. there is a number k such that for any $i \geq k : C_i = C_k$. If A is infinite-dimensional, then $\mathcal{C}(A)$ *does* always have infinite ascending chains [12]. From this we will prove that Σ_A is *not* a sober space if A is infinite-dimensional.

Definition 9.0.1. Let X be a compact Hausdorff space and let $K \subseteq X$ be closed. Then

$$C_K = \{f \in C(X) \mid f \text{ constant on } K\}$$

is a C^*-subalgebra of $C(X)$, called the *ideal subalgebra generated by* K.

The proof that this indeed defines a C^*-subalgebra can be found in Bert Lindenhovius's thesis [12]. The following lemma also originates from there:

Lemma 9.0.2. *Let X be a compact Hausdorff space and let $K, L \subseteq X$ be closed subsets. Then:*

1. *$C_K = C(X)$ if and only if K is empty or a singleton,*
2. *$K \subseteq L$ implies $C_L \subseteq C_K$,*
3. *$K \cap L \neq \emptyset$ implies $C_K \cap C_L = C_{K\cap L}$,*
4. *if $\#K \geq 2$, then $C_L \subseteq C_K$ implies $K \subseteq L$,*
5. *if $\#K \geq 2$, then $C_L = C_K$ implies $K = L$.*

The proof of this lemma is very technical and therefore omitted.

Proposition 9.0.3. *Let A be an infinite-dimensional C^*-algebra. Then $\mathcal{C}(A)$ has an infinite ascending chain.*

Proof. Take an infinite-dimensional $M \in \max(\mathcal{C}(A))$. Such a C^*-subalgebra must exist due to the assertion that A must contain a self-adjoint element with an infinite spectrum [10], from which a commutative infinite-dimensional C^*-subalgebra can be constructed via the *continuous functional calculus* [13]. That C^*-subalgebra must then be contained in a maximal $M \in \max(\mathcal{C}(A))$, which must be infinite-dimensional as well. Now $M = C(X)$ by Gelfand duality, with X a compact Hausdorff space. Then X cannot be finite, since that would result in a finite-dimensional M, which contradicts our assumption. X cannot have a discrete topology either, because then X

would not be compact. Therefore, choose some $x \in X$ for which $\{x\}$ is not open. Denote the collection of open neighbourhoods of x by $\mathcal{O}(x)$. Since X is Hausdorff:

$$\bigvee_{U \in \mathcal{O}(x)} C_{\overline{U}} = C_{\bigcap \overline{U}} = C_{\{x\}} = C(X) = M.$$

Therefore, we may construct an infinite chain consisting of C^*-subalgebras of the form $C_{\overline{U_i}}$ of which the supremum is M: a C^*-algebra which is *not* of that form. These are C^*-subalgebras of M and therefore of A as well. □

A remarkable result about closed meet-irreducible sets in Σ_A that will prove useful is the following:

Lemma 9.0.4. *Let $F \subseteq \Sigma_A$ be closed. Then F is join-irreducible if and only if the following two conditions are satisfied:*

1. *$\forall C \in \mathcal{C}(A)$: if $F_C \neq \emptyset$, then F_C is a singleton.*
2. *$\forall C_1, C_2 \in \mathcal{C}(A)$: if F_{C_1} and F_{C_2} are both nonempty, then there exists a $C_3 \in \mathcal{C}(A)$ such that $C_1, C_2 \subseteq C_3$ and F_{C_3} is nonempty.*

This lemmas originates from Sander Wolters's thesis [20], where its proof can also be found.

Theorem 9.0.5. *The space Σ_A is sober if and only if A is finite-dimensional.*

Proof. If A is finite-dimensional, then Σ_A is sober [20].

Let A be an infinite-dimensional C^*-algebra. Take an infinite-dimensional $M \in \max(\mathcal{C}(A))$ in combination with any $\lambda \in \Sigma(M)$, and consider the ascending chain

$$Q = \left\{C_{\overline{U_i}} \mid i \in I\right\} \subseteq \mathcal{C}(M)$$

constructed in proposition 9.0.3. Then construct the following set:

$$F = \bigsqcup_{C \in \mathcal{C}(A)} F_C;$$

$$F_C = \begin{cases} \{\lambda|_C\} & \text{C} \in \downarrow Q, \\ \emptyset & \text{C} \notin \downarrow Q. \end{cases}$$

Note that $\bigvee Q = M$ but $M \notin Q$ as seen in proposition 9.0.3, and thus $F_M = \emptyset$. By definition 5.0.1, any set $\bigsqcup_{C \in \mathcal{C}(A)} G_C$ is closed if and only if the following two conditions are satisfied. First, for every $C \in \mathcal{C}(A)$ the set G_C is closed in $\Sigma(C)$. Second, if $\lambda \in G_C$ and $D \subseteq C$, then $\lambda|_D \in F_D$. Since $\Sigma(C)$ is Hausdorff for any $C \in \mathcal{C}(A)$, singletons are indeed closed set and F satisfies the first condition. By construction, the second condition is also satisfied and therefore F is indeed a closed set.

By lemma 9.0.4, we have two conditions for determining whether F is join-irreducible. The first condition is indeed satisfied by construction. Since Q is an ascending chain, for any two $C, D \in \downarrow Q$ there exists some $E \in Q$, such that $C, D \subseteq E$. By construction F_E is nonempty, and thus we can conclude that F is a closed, join-irreducible set.

To show that F is not the closure of a singleton, we have to determine what sets of that form look like. Take any $(D, \mu) \in \Sigma_A$. By the aforementioned criteria, the smallest closed set we can construct containing (C, μ) is the following:

$$G = \bigsqcup_{C \in \mathcal{C}(A)} G_C;$$

$$G_C = \begin{cases} \{\mu|_C\} & \text{C} \in \downarrow D, \\ \emptyset & \text{C} \notin \downarrow D. \end{cases}$$

From this we may gather that the set F constructed above is actually $\overline{(M,\lambda)}\setminus\{(M,\lambda)\}$. Suppose that there would exist an element $(D,\mu)\in\Sigma_A$ such that $F=\overline{(D,\mu)}$. Then $(D,\mu)\in F$, i.e. $F_D=\{\mu\}$. For any nonempty F_C it must be that $C\subseteq D$. Therefore, D is an upper bound of the chain Q, and hence $M=\bigvee Q\subseteq D$. Since M is maximal, $D=M$. However, $F_M=\emptyset$ while $F_D=\{\mu\}$. This is a contradiction, and therefore F is a join-irreducible closed set that is not the closure of a singleton. $\square$

Chapter 10

Limit

In this last section we will look at Σ_A from an even different angle, and that is from the perspective of scattered approximately finite-dimensional C^*-algebras.

Definition 10.0.1. An **approximately finite-dimensional C^*-algebra** or **AF C^*-algebra** is a C^*-algebra A that is the direct limit of finite-dimensional C^*-algebras A_n, i.e. $A = \overline{\bigcup A_n}$.

Definition 10.0.2. Let A be a C^*-algebra. Then the following are equivalent:

1. A is a **scattered C^*-algebra**;
2. Each $C \in \mathcal{C}(A)$ is an AF C^*-algebra;
3. Each $M \in \max \mathcal{C}(A)$ is scattered, i.e. $M = C(X)$ for a **scattered space** X;
4. For each self-adjoint $a \in A$, the spectrum $\sigma(a)$ is countable;
5. $\mathcal{C}(A)$ is meet-continuous, i.e. for each $C \in \mathcal{C}(A)$ and directed set $\mathcal{D} \subseteq \mathcal{C}(A)$: $C \cap \bigvee \mathcal{D} = \bigvee_{D \in \mathcal{D}}(C \cap D)$;
6. For each $C \in \mathcal{C}(A)$: $C \ncong C([0,1])$.

   A topological space X is called scattered if each nonempty $S \subseteq X$ contains an isolated point.

It follows that any commutative scattered C^*-algebra is AF, and furthermore any scattered C^*-algebra that is a separable topological space with respect to the topology induced by the norm [12].

Proposition 10.0.3. *Any finite-dimensional C^*-algebra is scattered.*

Proof. Let A be finite-dimensional, then all its commutative C^*-subalgebras are finite-dimensional, so certainly AF C^*-algebras. It now follows directly from the definition that A is scattered. □

Proposition 10.0.4. *Let A be an AW^*-algebra that is also a scattered C^*-algebra. Then A is finite-dimensional.*

More on scattered C^*-algebras can be found in [12], where the proof which is omitted here can also be found.

Now the question we can ask is, when A is a scattered AF C^*-algebra such that $A = \bigvee \mathcal{D}$ for a directed set of finite-dimensional C^*-algebras $\mathcal{D} \subseteq \mathcal{C}(A)$, whether or not it is the case that $\Sigma_A = \lim_{D \in \mathcal{D}} \Sigma_D$.

Proposition 10.0.5. *Let A be an infinite-dimensional, commutative, scattered C^*-algebra such that $A = \bigvee \mathcal{D}$ for a directed set of finite-dimensional C^*-algebras*

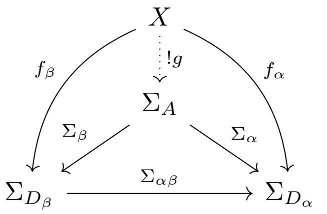
$$\mathcal{D} = \{D_\alpha \mid \alpha \in I\} \subseteq \mathcal{C}(A).$$

Then $\Sigma_A \neq \lim \Sigma_{D_\alpha}$.

Proof. Define the following mappings for $D_\alpha \subseteq D_\beta \in \mathcal{D}$:

$$\begin{aligned}
D_\alpha &\xhookrightarrow{j_{\alpha\beta}} D_\beta, \\
D_\alpha &\xhookrightarrow{j_{\alpha}} A, \\
\Sigma_{D_\beta} &\xrightarrow{\Sigma_{\alpha\beta}} \Sigma_{D_\alpha}, \\
\Sigma_{A} &\xrightarrow{\Sigma_{\alpha}} \Sigma_{D_\alpha},
\end{aligned}$$

and consider the following inverse limit:



with the mappings $f_\alpha, f_{\alpha\beta}$ such that:

$$\begin{aligned}
f_\alpha &= \Sigma_{\alpha\beta} \circ f_\beta, \\
\Sigma_\alpha &= \Sigma_{\alpha\beta} \circ \Sigma_\beta.
\end{aligned}$$

To prove the proposition, we will show that the unique $g : X \to \Sigma_A$ for which $\Sigma_\alpha \circ g = f_\alpha$ is not continuous. For every $x \in X$ we have to determine $g(x) = (C_x, \lambda_x)$. Since $\Sigma_\alpha \circ g = f_\alpha$:

$$\begin{aligned}
\pi_1(g(x)) &= C_x \\
&= C_x \cap A \\
&= C_x \cap \bigvee \mathcal{D} \\
&= \bigvee (C_x \cap D_\alpha) \\
&= \bigvee \pi_1(f_\alpha(x)) \\
&= \overline{\bigcup \pi_1(f_\alpha(x))}.
\end{aligned}$$

Now we have to define $\lambda_x \in \Sigma(C_x)$. Firstly, define

$$\begin{aligned}
\lambda'_x : &\bigcup C_x \cap D_\alpha \to \mathbb{C}; \\
&a \in C_x \cap D_\alpha \mapsto \pi_2(f_\alpha(x))(a).
\end{aligned}$$

This is necessary, since $\Sigma_\alpha \circ g = f_\alpha$. λ'_x is well defined, since if $a \in D_\alpha \subseteq D_\beta$:

$$\begin{aligned}
f_\alpha(x) &= \Sigma_{\alpha\beta}(f_\beta(x)) \\
\Rightarrow \lambda_x(a) &= \pi_2(f_\alpha(x))(a) \\
&= \pi_2(\Sigma_{\alpha\beta} \circ f_\beta(x))(a) \\
&= \pi_2(f_\beta(x))(j_{\alpha\beta}(a)).
\end{aligned}$$

Therefore, by the pasting lemma from topology and since $\mathcal{D}$ is a directed set, λ'_x is a continuous function. Therefore there exists a unique extension to a continuous function $\lambda_x : \overline{\bigcup C_x \cap D_\alpha} = C_x \to \mathbb{C}$. Thus we have defined $g(x)$ completely, which is now necessarily unique since we have made no choices anywhere.

Σ_A cannot have a weaker topology than the initial topology with respect to the projections Σ_A, as then these projections would not all be continuous by definition which is not the case by functoriality of Σ_A. On the other hand, if Σ_A has a stronger topology than the initial topology, one could take $X = \Sigma_A$ with the initial topology. Then it would necessarily follow that $g = \mathrm{id}_{\Sigma_A}$, which is then not continuous. In the case that Σ_A does indeed have the initial topology, sets of the form $\Sigma_\alpha^{-1}(U_\alpha)$ for some open $U_\alpha \subseteq \Sigma_{D_\alpha}$ form a basis for the topology. That is the case because sets of that form not only define a subbasis, but it is also the case that for any two sets $\Sigma_\alpha^{-1}(U_\alpha), \Sigma_\beta^{-1}(U_\beta)$, there exists some $\gamma \in I$ such that $D_\alpha, D_\beta \subseteq D_\gamma$ and thus:

$$\Sigma_\alpha^{-1}(U_\alpha) \cap \Sigma_\beta^{-1}(U_\beta) = \Sigma_\gamma^{-1}\left(\Sigma_{\alpha\gamma}^{-1}(U_\alpha) \cap \Sigma_{\beta\gamma}^{-1}(U_\beta)\right).$$

We also have that $g^{-1}\left(\Sigma_\alpha^{-1}(U_\alpha)\right) = f_\alpha^{-1}(U_\alpha)$, and so g is continuous. Therefore, g is continuous if and only if Σ_A has the initial topology.

Denote the space Σ_A equipped with the initial topology with respect to the projections Σ_α by $\widehat{\Sigma}_A$. By the reasons given above, we can say that $\widehat{\Sigma}_A = \lim \Sigma_{D_\alpha}$. However, in diagrams of the shape given, sobriety is a property preserved under limits. That means that $\widehat{\Sigma}_A$ is a sober space, while Σ_A is not as seen in theorem 9.0.5. Thus $\Sigma_A \neq \widehat{\Sigma}_A = \lim \Sigma_{D_\alpha}$.

Now let us prove that $\widehat{\Sigma}_A$ is a sober space. Assume that the index set I is ordered such that $\alpha \leq \beta$ if and only if $D_\alpha \subseteq D_\beta$.

$\widehat{\Sigma}_A$ is the limit of the sets Σ_{D_α}, and therefore:

$$\widehat{\Sigma}_A = \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} \Sigma_{D_\alpha} \mid \forall \alpha \leq \beta : \Sigma_{\alpha\beta}(x_\beta) = (x_\alpha) \right\}.$$

Now the functions Σ_α can be regarded as projections $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$, restricted to $\widehat{\Sigma}_A$.

Now take some closed join-irreducible set $K \subseteq \widehat{\Sigma}_A$. Then $\Sigma_\alpha(K) \subseteq \Sigma_{D_\alpha}$ is a closed set since Σ_α is a closed map. Furthermore, suppose that $\Sigma_\alpha(K)$ is not join-irreducible. Then there exist two non-empty closed sets $F, G \subseteq \Sigma_{D_\alpha}$ such that $F \cup G = \Sigma_\alpha(K)$. That means that $\Sigma_\alpha^{-1}(F) \cap K$ and $\Sigma_\alpha^{-1}(G) \cap K$ are two closed, non-empty subsets of $\widehat{\Sigma}_A$ for which:

$$\left(\Sigma_\alpha^{-1}(F) \cap K\right) \cup \left(\Sigma_\alpha^{-1}(G) \cap K\right) = K \cap \left(\Sigma_\alpha^{-1}(F \cup G)\right) = K \cap \Sigma_\alpha^{-1}(K) = K.$$

Since K is join-irreducible, either $K = \Sigma_\alpha^{-1}(F)$ or $K = \Sigma_\alpha^{-1}(G)$. Now, Σ_α is surjective, so we can conclude from this without loss of generality that $\Sigma_\alpha(K) = F$ and therefore $\Sigma_\alpha(K)$ is also join-irreducible. Because Σ_{D_α} is sober, $\Sigma_\alpha(K) = \overline{\{y_\alpha\}}$. This defines an element

$$y = (y_\alpha)_{\alpha \in I} \in \prod \Sigma_{D_\alpha}.$$

Now we have to show that $y \in \widehat{\Sigma}_A$ and $K = \overline{\{y\}}$. For that purpose:

$$\overline{\{y_\alpha\}} = \Sigma_\alpha(K) = \Sigma_{\alpha\beta}(\Sigma_\beta(K)) = \Sigma_{\alpha\beta}\left(\overline{\{y_\beta\}}\right) = \overline{\Sigma_{\alpha\beta}\left(\{y_\beta\}\right)}.$$

In the proof of theorem 9.0.5, we have seen that the closure of a singleton $(E, \lambda) \in \Sigma_{D_\alpha}$ is the following:

$$\overline{(E,\lambda)} = \bigsqcup_{C \in \mathcal{C}(A)} \overline{(E,\lambda)}_C;$$

$$\overline{(E,\lambda)}_C = \begin{cases} \{\mu|_C\} & \mathrm{C} \in \downarrow E, \\ \emptyset & \mathrm{C} \notin \downarrow E. \end{cases}$$

Now suppose that $\overline{(C,\lambda)} = \overline{(D,\mu)}$. Then:

$$C = \bigvee \left\{ E \in \mathcal{C}(A) \mid \overline{(C,\lambda)}_E \neq \emptyset \right\} = \bigvee \left\{ E \in \mathcal{C}(A) \mid \overline{(D,\mu)}_E \neq \emptyset \right\} = D,$$

from which we can conclude that:

$$\{\lambda\} = \overline{(C,\lambda)}_C = \overline{(D,\mu)}_D = \{\mu\}.$$

Therefore:

$$(C, \lambda) = (D, \mu).$$

What we may conclude from this is that

$$y_\alpha = \Sigma_{\alpha\beta}(y_\beta),$$

and so $y \in \widehat{\Sigma}_A$.

From the article [18] from A.H. Stone, it follows that $\widehat{\Sigma}_A$ is a compact space. $K \subseteq \widehat{\Sigma}_A$ is a closed subset of a compact space and therefore compact as well. Now since $\Sigma_\alpha(K) = \overline{\{y_\alpha\}}$, we may choose for every $\alpha \in I$ an element $x^\alpha \in K$ such that $f_\alpha(x^\alpha) = y_\alpha$. This results in a net $(x^\alpha)_{\alpha\in I} \subseteq \widehat{\Sigma}_\alpha$ which has a convergent subnet due to K being compact:

$$x^{\phi(j)} \xrightarrow{j} x,$$

where $\phi : J \to I$ is monotonous and final for a directed set J. Furthermore, $x \in K$. Then:

$$\begin{aligned}
\Sigma_\alpha(x) &= \Sigma_\alpha(\lim_j x^{\phi(j)}) \\
&= \lim_j \Sigma_\alpha(x^{\phi(j)}) \\
&= \lim_j \Sigma_{\alpha\phi(j)}\left(\Sigma_{\phi(j)}(x^{\phi(j)})\right) \\
&= \lim_j \Sigma_{\alpha\phi(j)}(y_{\phi(j)}) \\
&= \lim_j y_\alpha \\
&= y_\alpha.
\end{aligned}$$

Since the mappings Σ_α are projections from the product $\prod \Sigma_{D_\alpha}$ onto the sets Σ_{D_α}, we can conclude that $x = y$, and therefore $y \in K \Rightarrow \overline{\{y\}} \subseteq K$. For the other way around, take some $z \in K$ and some basic open neighbourhood $z \in O = \Sigma_\alpha^{-1}(U_\alpha)$. If we show that $y \in O$, then that means $z \in \overline{\{y\}}$ and thus $K = \overline{\{y\}}$. Now:

$$\begin{aligned}
&\Sigma_\alpha(z) \in U_\alpha, \\
&\Sigma_\alpha(z) \in \Sigma_\alpha(K) = \overline{\{y_\alpha\}}.
\end{aligned}$$

Therefore, $\Sigma_\alpha(y) = y_\alpha \in U_\alpha$. Thus we can conclude that $y \in \Sigma_\alpha^{-1}(U_\alpha) = O$. □

Chapter 11

Conlcusion

This thesis started out as a project to determine the Esakia dual of the lattice $\mathcal{O}(\Sigma_A)$. Whereas we have no answer to this problem except for the ruling out of the spaces $\mathrm{Mod}_{\mathcal{C}(A)}(\mathcal{O}(\Sigma_A))$ and $\mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_A), \underline{2})$, we have managed to reach some results.

- We have shown that the lattice $Q(A)$, which was only defined for finite-dimensional C^*-algebras, can also be defined for AW^*-algebras.
- We have shown that in that case, $Q(A)$ is a dense sublocale of $\mathcal{O}(\Sigma_A)$ and the mapping $A \mapsto Q(A)$ is a covariant functor for the categories **CAWStar** and $\mathbf{AWStar}_{\hookrightarrow}$.
- Likewise, we have shown that $A \mapsto \mathcal{O}(\Sigma_A)$ is a covariant functor for the categories of **CCStar** and $\mathbf{CStar}_{\hookrightarrow}$.
- We have proven that Σ_A is a sober space if and only if A is finite-dimensional.
- Finally, we have shown that if A is an infinite-dimensional, commutative, scattered C^*-algebra such that $A = \overline{\bigcup D_\alpha}$ for finite-dimensional D_α, that $\Sigma_A \neq \lim \Sigma_{D_\alpha}$.

All in all, while some properties have been proven, finding the Esakia dual of $\mathcal{O}(\Sigma_A)$ remains an open problem. We have not managed to find more probable candidates for lattices S_A such that the Esakia dual of $\mathcal{O}(\Sigma_A) = \mathrm{Hom}_{Heyting}(\mathcal{O}(\Sigma_A), S_A)$, and it may be a more difficult problem than we once expected. It is not even clear anymore if this object S_A even exists, and if there should exist an easy recipe to find it. Therefore, a different approach may be needed, which leaves opportunity for new research.

Appendices

Appendix A

Order Theory

Some basic order theory is relevant throughout the thesis, and for that reason it is included in this appendix.

Definition A.0.1. A **partially ordered** set (*poset*) $(P, \leq)$ is a set P with an order relation $\leq$ satisfying for all a, b, c in P:

- $a \leq a$,
- if $a \leq b$ and $b \leq a$, then $a = b$,
- if $a \leq b$ and $b \leq c$, then $a \leq c$.

In posets, the following notation is used:

Definition A.0.2. An **upper set** or **upset** U in a poset $(P, \leq)$ is a subset of P such that for any $x \in U$ and $y \geq x$ it follows that $y \in U$. Naturally, a **downset** is defined analogously. Some notation for $x \in P$, $C \subseteq P$:

- $\uparrow x = \{y \in P \mid x \leq y\}$
- $\uparrow C = \{y \in P \mid \exists c \in C : c \leq y\}$
- $\mathrm{Up}(P) = \{\uparrow C \mid C \subseteq P\}$

Similarly, $\downarrow x$, $\downarrow C$ and $\mathrm{Down}(P)$ are used. Note that $\mathrm{Up}(P)$ and $\mathrm{Down}(P)$ are subsets of the powerset of P.

Definition A.0.3. Let $(P, \leq)$ be a partially ordered set. Then $S \subseteq P$ is a **directed set** if any *finite* subset $A \subseteq S$ has an upper bound $s \in S$, i.e. for all $a \in A$: $a \leq s$.

Directed sets will play a marginal role in this thesis, but they are important in topology due to the notion of *nets*.

Definition A.0.4. Let $(I, \leq)$ be a directed set and X be a topological space. A function $f : I \to X$ is said to be a **net**. We often such a net in the form $(x_i)_{i \in I}$, which expresses the fact that the element i in I is mapped to the element x_i in X.

Definition A.0.5. A function $h : BA$ between directed sets A, B is **monotone** if $\beta_1 \leq \beta_2$ implies $h(\beta_1) \leq h(\beta_2)$, and **final** if for every α in A there exists a β in B such that $\alpha \leq h(\beta)$.

Definition A.0.6. If $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$ are nets from directed sets A and B respectively, then $(y_\beta)_{\beta \in B}$ is a **subnet** of $(x_\alpha)_{\alpha \in A}$ if there exists a *monotone, final* function $h : B \to A$ such that $y_\beta = x_{h(\beta)}$.

With these concepts some important propositions can be formulated. While most of these are not relevant for this thesis, the following proposition will prove useful:

Proposition A.0.7. *A topological space X is compact if and only if every net in X has a convergent subnet.*

For a proof, see [17].

A fundamental object for this thesis will be the lattice, which can be seen as a partially ordered set with more structure.

Definition A.0.8. A **lattice** L is a partially ordered set $(L, \leq)$ in which every two elements a, b in L have a unique *supremum* and *infimum*, also called *join* and *meet*, denoted by $a \vee b$ and $a \wedge b$ respectively. That is, for any a, b in L there exists an element $a \vee b$ such that if any c satisfies $a \leq c$, $b \leq c$ and $c \leq a \vee b$, then $c = a \vee b$. Likewise, for any c satisfying $c \leq a$, $c \leq b$ and $a \wedge b \leq c$, it should follow that $c = a \wedge b$.

A **bounded lattice** L is a lattice, with greatest and least elements 0 and 1 in L, which satisfy $0 \leq a \leq 1$ for every a in L. These are necessarily unique.

A **complete lattice** L is a lattice in which for every subset $A \subseteq L$ there exists a supremum and infimum, denoted by $\bigvee A$ and $\bigwedge A$ respectively.

A **distributive lattice** L is a lattice in which for all x, y and z in L $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A lattice is not necessarily complete, a simple example being an unbounded (thus also infinite) lattice. In that case the lattice in its entirety does not have an upper bound, hence the lattice is not complete. It is not the case, however, that every bounded lattice is complete.

Now that this construction has been given, a natural construction is that of the lattice homomorphism. It is a map that respects the particular structure of lattices:

Definition A.0.9. A **lattice homomorphism** from the lattice L to the lattice K is a map $f : L \to K$ such that for any a and b in L:

1. $f(a \vee_L b) = f(a) \vee_K f(b)$,
2. $f(a \wedge_L b) = f(a) \wedge_K f(b)$.

Where $\vee_L$ represents the join in the lattice L, and similar definitions of $\wedge_L$, $\vee K$ and $\wedge_K$.

Additionally, if L and K are both bounded lattices, a **bounded-lattice homomorphism** is a lattice homomorphism $f : L \to K$ with the additional requirements:

1. $f(0_L) = 0_K$,
2. $f(1_L) = 1_K$.

If f is a lattice homomorphism between two bounded lattices it will be assumed to be a bounded lattice homomorphism, unless specified otherwise.

The last statement is a *convention* we use, since there do exist lattice homomorphisms between bounded lattices L and K for which the image of 0_L is not 0_K, and likewise for 1_L.

The concept of *ideals* and *filters* in lattices is fundamental in order theory. They are defined as follows.

Definition A.0.10. An **ideal** I in a lattice L is a subset $I \subseteq L$ with the following properties for every x and y in L:

1. I is not empty,
2. If x is in I and $y \leq x$, then y is in I,
3. If x and y are in I, then $x \vee y$ is in I.

A **filter** $F \subseteq L$ satisfies above requirements dually, that is with every $\leq$ reversed and $\wedge$ and $\vee$ swapped. For any $x \in L$, the subset $\downarrow x$ is an ideal, whereas $\uparrow x$ is a filter. Trivially this is also the smallest ideal, respectively filter, that contains x. For a subset A of L, the filter **generated** by A is the smallest filter containing A. This filter exists and is constructed by $\bigcap\{F \subseteq L \mid A \subseteq F, F \text{ a filter}\}$ which is easily verified to be a filter. The ideal generated by A is constructed likewise.

Ideals and filters themselves are interesting concepts, but in many theorems special ideals and filters are considered, namely the prime ideals and filters:

Definition A.0.11. A **prime ideal** is an ideal $I \subseteq L$, whose complement $L \backslash I$ is a filter. Prime ideals can be characterised by the conditions:

- I is not equal to L
- For any x and y in L, $x \wedge y$ in I implies that x is in I or y is in I.

A **prime filter** is a filter $F \subseteq L$ for which, once again, the above holds dually. Note that the complement of a prime filter is then a prime ideal, and vice versa.

Definition A.0.12. A **maximal ideal** is an ideal $I \subseteq L$ that is not properly contained in any other ideal. A **maximal filter** is defined dually, and is often called an **ultrafilter**. Maximal ideals and filters are then prime as well.

A lemma which will prove useful concerns filters and ideals.

Lemma A.0.13. *Let F be a filter and I an ideal in a bounded distributive lattice L. If $F \cap I = \emptyset$, then there exists a prime filter P in L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. The proof needs Zorn's lemma. Let $\mathcal{F}$ be the set of all filters in L that contain F and are disjoint from I. Since F itself satisfies the condition, $F \in \mathcal{F}$. Let $\mathcal{C} = \{P_\alpha\}$ be a chain in $\mathcal{F}$. Since $\bigcup P_\alpha$ is a filter and $(\bigcup P_\alpha) \cap I = \emptyset$, $\mathcal{C}$ has an upper bound. Now it follows from Zorn's lemma that $\mathcal{F}$ has a maximal element, call it P. This is then a filter, but it needs to be proven that it is indeed prime. Suppose that $a \vee b \in P$. Consider the filters F_1 and F_2, generated by $P \cup \{a\}$ and $P \cup \{b\}$ respectively. If a and b are both not in P, then P is properly contained in F_1 and F_2, and therefore F_1 and F_2 are not in $\mathcal{F}$. This means that $F_1 \cap I$ is nonempty, and likewise for F_2. Let $x_i \in F_i \cap I$. Since the F_i are filters, there exist p_i in P such that $x_1 \geq p_1 \wedge a$ and $x_2 \geq p_2 \wedge b$. Therefore,

$$x_1 \vee x_2 \geq (p_1 \wedge a) \vee (p_2 \wedge b) \geq (p_1 \vee p_2) \wedge (p_1 \vee a) \wedge (p_2 \vee b) \wedge (a \vee b).$$

Since all four terms on the right hand side are in P, their meet must be in P as well. This in turn implies that $x_1 \vee x_2$ is in P, and thus in $P \cap I$. This gives a contradiction, and therefore either a or b must be in P, so P is a prime filter. □

This lemma and its proof both originate from [14].

Appendix B

Category Theory

This appendix on category theory is needed mainly for the categorical aspects that are covered in chapters 1, 4 and 8. We will cover the concepts of categories, functors, natural transformations and limits.

B.1 Categories

Definition B.1.1. A **category $\mathbf{C}$** consists of

1. A collection $\mathrm{Obj}(\mathbf{C})$ of **objects**.
2. For every $A, B \in \mathrm{Obj}(\mathbf{C})$ a collection $\mathrm{Hom}_{\mathbf{C}}(A, B)$ of **morphisms** between A and B.
3. For $A, B, C \in \mathrm{Obj}(\mathbf{C})$ a map

$$\circ : \mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C) \tag{B.1}$$

called **composition** (notation: $f \circ g := \circ(f, g)$)

such that:

1. Composition is associative.
2. For every $A \in \mathrm{Obj}(\mathbf{C})$ there is an **identity** morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ such that $f \circ \mathrm{id}_A = f$ if $f \in \mathrm{Hom}_{\mathbf{C}}(A, C)$ and $\mathrm{id}_A \circ g = g$ if $g \in \mathrm{Hom}_{\mathbf{C}}(C, A)$.

Remark 1. We usually write $A \in \mathbf{C}$ instead of $A \in \mathrm{Obj}(\mathbf{C})$. A morphism $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ is denoted by $f : A \to B$.

Example B.1.2.

1. The category **Sets** which has sets as objects and functions as morphisms. Composition is ordinary composition of functions.
2. The category **CStar** with C*-algebras as objects and *-homomorphisms as morphisms.
3. Any poset (or more generally any preorder) can be regarded as a category with an arrow $x \to y$ if and only if $x \leq y$. Transitivity of the order relation is then just composition of these arrows. Such a category is called a **posetal category**.
4. If $\mathbf{C}$ is a category, we can make a new category $\mathbf{C}^{op}$ called the **opposite category** of $\mathbf{C}$. $\mathbf{C}^{op}$ has the same objects as $\mathbf{C}$, but a morphism $X \to Y$ in $\mathbf{C}^{op}$ is a morphism $Y \to X$ in $\mathbf{C}$. The morphisms are 'reversed' in $\mathbf{C}^{op}$.

Remark 2. Most categories that we will work with are so-called **concrete** categories. This means that the objects are **sets**, possibly with extra structure, and that all morphisms are functions between these sets. Also we usually work with **locally small** categories in which each collection $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is actually a **set**.

B.2 Functors

Definition B.2.1. A **(covariant) functor** is a map $F : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ that associates to each $C \in \mathrm{Obj}(\mathbf{C})$ an object $F(C) \in \mathrm{Obj}(\mathbf{D})$ and to each morphism $f : A \to B$ in $\mathbf{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathbf{D}$, such that:

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathbf{C}$.
2. $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f, g for which the composite $f \circ g$ is defined.

Remark 3. We can also define **contravariant** functors. A contravariant functor assigs to a morphism $f : A \to B$ a morphism $F(f) : F(B) \to F(A)$ so the direction is reversed. In addition, condition no. 2 is replaced by $F(f \circ g) = F(g) \circ F(f)$.

Example B.2.2. The functor $\mathcal{P}$ that assigns to each set X its powerset $\mathcal{P}(X)$ can be made into a contravariant functor. To a function $f : X \to Y$ we associate the function $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$, $A \mapsto f^{-1}(A)$.

Example B.2.3. For any category $\mathbf{C}$ and object $X \in \mathbf{C}$ we have the functor $\mathrm{Hom}_{\mathbf{C}}(-, X)$ from $\mathbf{C}$ to **Sets**. To an object $Y \in \mathbf{C}$ it assigns the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ from Y to X. For any morphism $f : Y \to Z$ we get a morphism $f^* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \to \mathrm{Hom}_{\mathbf{C}}(Y, X)$ given by $g \mapsto g \circ f$.

B.3 Natural transformations

Definition B.3.1. A **natural transformation** $\alpha : F \to G$ between two covariant functors $F, G : \mathbf{C} \to \mathbf{D}$ consists of morphisms $\alpha_C : F(C) \to G(C)$ for each $C \in \mathbf{C}$ such that for each morphism $f : A \to B$ in $\mathbf{C}$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes. That is, $\alpha_B \circ F(f) = G(f) \circ \alpha_A$.

Example B.3.2. Let **Ring** be the category of rings. We can consider first the functor $\mathrm{GL}_n(-)$ that assigns to a ring R the ring $\mathrm{GL}_n(R)$ of n by n invertible matrices with coefficients in R. By applying a homomorphism $f : R \to S$ to the coefficients this becomes a functor. Next, we can also form the group of units R^* and any homomorphism $f : R \to S$ restricts to $f : R^* \to S^*$. The determinant assigns to an n by n invertible matrix A an element $\det(A) \in R^*$. This is an example of a natural transformation.

Definition B.3.3. A natural transformation $\alpha : F \to G$ is called a **natural isomorphism** when there is a natural transformation $\beta : G \to F$ such that $\alpha \circ \beta = \mathrm{id}_G$ and $\beta \circ \alpha = \mathrm{id}_F$.

Notation 1. We write $F \cong G$ if there is a natural isomorphism $F \to G$.

Remark 4. α is a natural isomorphism if and only if each component α_C is an isomorphism.

Example B.3.4. In the category **Sets** there is a natural isomorphism

$$\mathcal{P}(X) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \underline{2}), \tag{B.2}$$

given by associating a subset $A \subseteq X$ with its characteristic function χ_A. This is natural because for any function $f : X \to Y$ and subset $B \subseteq Y$ we have $\chi_{f^{-1}(B)} = \chi_B \circ f$.

Definition B.3.5. Two categories $\mathbf{C}, \mathbf{D}$ are said to be **equivalent** if there are (covariant) functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F \cong \mathrm{id}_{\mathbf{C}}$ and $F \circ G \cong \mathrm{id}_{\mathbf{D}}$. The functors F and G are then said to give an **equivalence of categories**.

Remark 5. If the functors F and G are contravariant the categories are said to be **dual** to each other and F and G give a **duality of categories**.

Natural transformations can be thought of as 'morphisms between functors'. This can be made precise using the concept of a **functor category**.

Definition B.3.6. Let $\mathbf{C}$, $\mathbf{D}$ be categories. Then the **functor category $\mathbf{D}^{\mathbf{C}}$** of functors from $\mathbf{C}$ to $\mathbf{D}$ has

- functors $\mathbf{C} \to \mathbf{D}$ as objects;
- The morphisms $F \to G$ are the natural transformations between F and G.

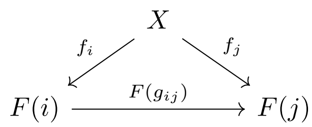
Remark 6. The identity natural transformation $\mathrm{id} : F \to F$, with components given by the identity $\mathrm{id}_{F(C)} : F(C) \to F(C)$ for $C \in \mathbf{C}$, serves as the identity morphism in this category. Also natural transformations can be composed componentwise, and this is associative.

B.4 Limits

Definition B.4.1. Let $\mathbf{J}$ be a category. A **diagram** of shape $\mathbf{J}$ in a category $\mathbf{C}$ is a functor $\mathbf{J} \to \mathbf{C}$.

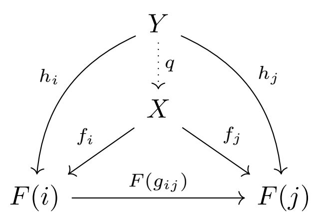
Remark 7. A diagram can be thought of as a collection of objects and morphisms that indexed by the category $\mathbf{J}$.

Definition B.4.2. A **cone** to a diagram $F : \mathbf{J} \to \mathbf{C}$ consists of an object $X \in \mathbf{C}$ and morphisms $f_i : X \to F(i)$ such that for every morphism $g_{ij} : i \to j$ in $\mathbf{J}$, $F(g_{ij}) \circ f_i = f_j$. This means that the following diagram



commutes for every morphism $g_{ij} : i \to j$.

Definition B.4.3. A **limit** of a diagram $F : \mathbf{J} \to \mathbf{C}$ is a **universal cone** to F. That is, a cone $f_i : X \to F(i)$ to F such that for every cone $h_i : Y \to F(i)$ there is a **unique** morphism $q : Y \to X$ with $h_i = f_i \circ q$ for all $i \in \mathbf{J}$.



Notation 2. A limit of a diagram $F : \mathbf{J} \to \mathbf{C}$ is denoted by $X = \lim_{\mathbf{J}} F$.

Proposition B.4.4. *Let $F : \mathbf{J} \to \mathbf{C}$ be a diagram with a limit $X = \lim_{\mathbf{J}} F$. Then X is unique up to unique isomorphism.*

Appendix C

Hilbert Spaces and C^*-algebras

Absolutely fundamental to all of modern physics is the concept of a Hilbert space. Any student in quantum mechanics, knowingly or unknowingly, uses its properties in many ways. In essence it is a generalization of Euclidian space, as it is a vector space with an inner product. However, whereas Euclidian space is a real vector space, Hilbert spaces are usually complex vector spaces. Any state of a particle or system in quantum mechanics is seen as an element of a Hilbert space.

Definition C.0.1. A **Hilbert space** H is a complete metric space that is also a normed vector space over the complex numbers with a map $\langle\ ,\ \rangle : H \times H \to \mathbb{C}$, called an inner product, satisfying for any x, y, z in H and any complex numbers a, b:

- $\langle x, x\rangle \geq 0$,
- $\|x\| = \sqrt{\langle x, x\rangle}$,
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
- $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$.

The norm induces the metric by $d(x, y) = \|x - y\|$. As noted, a Hilbert space must be complete with respect to this metric.

Some authors leave room for real Hilbert spaces, however in general only complex Hilbert spaces matter in physics. On that matter, observations inherently influence states in quantum mechanics which explains why observations are seen as operators on a Hilbert space. For more precise connections between observations and operators we refer to [11].

Definition C.0.2. A **linear operator** on a Hilbert space H is a function $T : H \to H$ that is linear, i.e.:

- $T(x + y) = T(x) + T(y)$,
- $T(ax) = aT(x)$,

for any x, y in H and complex number a. Furthermore, T is called **bounded** if there exists some real number $M \geq 0$ such that for every x in H: $\|T(x)\| \leq M\|x\|$. Note that if H is finite-dimensional, then all linear operators are bounded. The set of bounded linear operators on H is denoted by $B(H)$.

Lemma C.0.3. *For any linear operator $T : H \to H$, the following are equivalent:*

- *T is bounded,*
- *T is continuous,*
- *T is continuous at* 0.

Lemma C.0.4. *If H is a Hilbert space, $B(H)$ is a complete normed vector space.*

Proof. It is clear that $B(H)$ is a vector space, with addition and scalar multiplication defined pointwise. Since there exists a real number M such that $\|Tx\| \leq M\|x\|$ for every x in H, we can take the infimum of the set of numbers satisfying this condition: define

$$\|T\| = \inf\{M \in \mathbb{R}_{\geq 0} \mid \forall x \in H : \|Tx\| \leq M\|x\|\}.$$

For a proof that this is a norm and in fact equal to $\sup\{\|Tx\| \mid \|x\| = 1\}$, I refer to [13]. □

Definition C.0.5. A bounded linear operator $T \in B(H)$ is called **invertible** if there exists an $S \in B(H)$ such that the compositions $TS = ST = \mathbb{1}_H$, where ST and TS denote compositions.

To give a definition of a C^*-algebra, we need one last property of $B(H)$.

Definition C.0.6. For any bounded linear operator $T \in B(H)$, an adjoint of T is an operator T^* in $B(H)$ for which for every x, y in H: $\langle Tx, y\rangle = \langle x, T^*y\rangle$.

A proof of the fact that the adjoint of any operator T in $B(H)$ exists and is unique can be found in [13]. It then easily follows that $(T^*)^* = T$.

Now we define C^*-algebras, which are also crucial objects in the mathematics behind quantum mechanics.

Definition C.0.7. A **C^*-algebra** A is a linear subspace of $B(H)$ for a Hilbert space H, such that:

- for any T and S in A, their composition TS is also in A,
- A is a closed subset of $B(H)$ with respect to the topology induced by the operator norm,
- for any T in A, T^* is also in A.

Due to these properties, a C^*-algebra is automatically a complex algebra (which is not necessarily unital or commutative).

Note that $B(H)$ itself is then also an example of a C^*-algebra. Some important examples of C^*-algebras are, when $\mathbb{C}^n$ is taken as a Hilbert space $B(\mathbb{C}^n)$: the space of complex $n \times n$ matrices. $B(\mathbb{C}^n)$ will be denoted by $M_n(\mathbb{C})$ from now on. Secondly, the subspaces of these consisting of diagonal matrices: $D_n(\mathbb{C}) = \{z \cdot \mathbb{1}_n \mid z \in \mathbb{C}\} \cong \mathbb{C}^n$ gives another set of C^*-algebras which *are* commutative.

Another, more abstract, definition of a C^*-algebra is the following, which does not need an associated Hilbert space:

Definition C.0.8. A **C^*-algebra** A is a complex algebra equipped with a norm and a map $* : A \to A$ (notation: $*(a) = a^*$) such that A is complete with respect to its norm and satisfies the following for any a, b in A and complex number λ:

- $(a^*)^* = a$,
- $(a + b)^* = a^* + b^*$,
- $(ab)^* = b^*a^*$,
- $(\lambda a)^* = \overline{\lambda}a^*$,
- $\|aa^*\| = \|a\|\,\|a^*\|$.

Furthermore, A is a **commutative C^*-algebra** if $ab = ba$ holds as well.

The fact that these definitions are equivalent, in that for any abstractly defined C^*-algebra a Hilbert space H can be found such that A is a subset of $B(H)$ satisfying the conditions in the first definition, has been proven in 1943 by Israel Gelfand and Mark Naimark [5].

From here on out we will only consider *unital* C^*-algebras. Any C^*-algebra mentioned will be assumed to be unital. We should also specify the morphisms between C^*-algebras.

Definition C.0.9. Let $\phi : A \to B$ be a linear map between C^*-algebras A and B. Then ϕ is said to be a ***-homomorphism** if for each $a, b \in A$:

- $\phi(ab) = \phi(a)\phi(b)$,
- $\phi(a^*) = \phi(a)^*$,
- $\phi(\mathbb{1}_A) = \phi(\mathbb{1}_B)$.

A bijective *-homomorphism $\phi : A \to B$ is called a ***-isomorphism**, in which case ϕ has an inverse $\phi^{-1} : B \to B$ which itself a *-isomorphism. We call A and B ***-isomorphic**, abbreviated by $A \cong B$, if there exists a *-isomorphism between A and B.

We denote the category consisting of C^*-algebras with *-homomorphisms by **CStar**. Its subcategory of commutative C^*-algebras is denoted by **CCstar**.

Definition C.0.10. Let A be a C^*-algebra. If B is a *-closed subalgebra of A that is closed in the norm topology of A and that contains 1_A, then we say that B is a C^***-subalgebra** of A.

Theorem C.0.11. *Let A be a finite-dimensional C^*-algebra. Then there exist positive integers K and $N_1, ..., N_K$ such that*

$$A \cong \bigoplus_{i=1}^{K} M_{N_i}(\mathbb{C}).$$

Moreover, K is unique and $N_1, ..., N_K$ are unique, up to a permutation. In particular, if A is commutative as well, then

$$A \cong \mathbb{C}^n$$

for some positive integer n.

The proof of this theorem is omitted. It can be found in [3].

One concept concerning C^*-algebras is vital in many proofs and definitions in the field of operator algebras. While the concept comes up sparingly in this thesis, it is instructive to include.

Definition C.0.12. If $T \in B(H)$ for a Hilbert space H, the set of complex numbers λ for which $T - \lambda\mathbb{1}_H$ is not invertible is called the **spectrum** of T. This set is denoted by $\sigma(T)$.

Likewise, if $a \in A$ for a C^*-algebra A, the set of complex numbers λ for which $a - \lambda\mathbb{1}_A$ is not invertible is also called the **spectrum** of a. This set is denoted by $\sigma(a)$ as well.

Finally, a construct that is relevant for amongst others Gelfand duality is the *dual space* of a C^*-algebra.

Definition C.0.13. For a C^*-algebra A, the **dual space** A^* is defined as all *linear functionals* on A, i.e. the space $\{\Lambda : A \to \mathbb{C} \mid \forall\lambda, \mu \in \mathbb{C}, a, b \in A : \Lambda(\lambda a + \mu b) = \lambda\Lambda(a) + \mu\Lambda(b)\}$.

Bibliography

[1] Guram Bezhanishvili, Nick Bezhanishvili, David Gabelaia, and Alexander Kurz. Bitopological duality for distributive lattices and Heyting algebras. 20:359–393, 06 2010.

[2] Martijn Caspers, Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. Intuitionistic quantum logic of an n-level system. *Foundations of Physics*, 39(7):731–759, Jul 2009.

[3] K.R. Davidson. *C^*-algebras by Example.* Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs. American Mathematical Society, 1996.

[4] Richard Feynman. *The Character of Physical Law, chapter 6: "Probability and Uncertainty – the Quantum Mechanical View of Nature"*, page 129. Mit Press Ltd, 1989.

[5] Izrail Gel'fand and Mark Naimark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–217, 1943.

[6] Murray Gell-Mann. *The Nature of Matter, Wolfson College Lectures 1980, chapter: Questions for the Future*, pages 169–186. Clarendon, Oxford, 1981.

[7] Valery Glivenko. Sur la logique de M. Brouwer. *Acadmie Royale de Belgique, Bulletin de la classe des sciences*, 1928.

[8] Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. A topos for algebraic quantum theory. *Communications in Mathematical Physics*, 291(1):63–110, Oct 2009.

[9] Chris Heunen, Nicolaas P. Landsman, and Bas Spitters. Bohrification of operator algebras and quantum logic. *Synthese*, 186(3):719–752, Jun 2012.

[10] John Kadison, Richard; Ringrose. *Fundamentals of the Theory of Operator Algebras. Vol. I: Elementary Theory.* Academic Press, New York, 1983.

[11] Klaas Landsman. *Foundations of Quantum Theory: From Classical Concepts to Operator Algebras.* Springer International Publishing, Cham, 2017.

[12] Bert Lindenhovius. $\mathcal{C}(A)$. PhD thesis, Radboud University, Jul 2016.

[13] Barbara D. MacCluer. *Elementary Functional Analysis.* Springer New York, New York, NY, 2009.

[14] Patrick Morandi. Dualities in lattice theory. 08 2018.

[15] Jorge Picado and Aleš Pultr. *Frames and Locales: Topology without points.* Springer Basel, Basel, 2012.

[16] B. A. Davey; H. A. Priestley. *Introduction to Lattices and Order, chapter 11: "Representation: the General Case"*, pages 247–274. Cambridge University Press, Cambridge, UK, 1990.

[17] Volker Runde. *Set-Theoretic Topology*, pages 61–108. Springer New York, New York, NY, 2005.

[18] A.H. Stone. Inverse limits of compact spaces. *General Topology and its Applications*, 10(2):203 – 211, 1979.

[19] Sander Uijlen. Categorical Aspects of von Neumann Algebras and AW^*-algebras. Master's thesis, Radboud University, Aug 2013.

[20] Sander Wolters. *Quantum Toposophy.* PhD thesis, Radboud University, Oct 2013.