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GROUPOID SYMMETRY AND CONSTRAINTS BRACKET OF GENERAL RELATIVITY REVISITED

M.SC. THESIS IN MATHEMATICAL PHYSICS

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Introduction

The constraints that arise from the ADM formulation of the Einstein equation are agreed to be both *crucially important* and *not fully understood*. This thesis is a result of our struggle for understanding the origin of their Poisson bracket relations that was proposed to *emerge* from a symmetry structure inherent to the initial value formulation by Weinstein, Blohmann and Fernandes in the paper "Groupoid symmetry and constraints of General Relativity" [2]. We find this idea, as well as the symmetry structure that they propose, very appealing. However, the analysis as presented there might not be fully satisfactory for the following reasons:

1. There exist *no* general framework for diffeological algebroids, which inevitably led to a bit sketchy character of some of the points of the analysis.
2. The diffeological structure that is put on the groupoid feels unnatural and does not make it crystal clear that the bracket structure thus derived is really *emergent* from the groupoid structure.
3. The resulting global structure of the algebroid, just like the general strategy of deriving it, seems to us to be more complex than necessary.

We aim to make some improvements at all the points mentioned above. The approach that we describe in details is sketched in the Appendix of [2], all the ideas that we present are basically already there. We make some changes on the level of the diffeology and develop a detailed analysis through the group of bisections. We aim to make the derivation conceptually and technically clearer and more convincing which, in our view, was accomplished this way.

The first part of the thesis aims to improve the situation described in the first point above. We introduce all the mathematical frameworks that we will use, beginning with a deep, yet elementary analysis of the theory of diffeological spaces from the *sheaf-theoretic point of view*. This perspective is not necessary for the future analysis but certainly helps to understand better the diffeological framework, especially *why* it is so useful and efficient. There are no new claims here but all the detailed and elementary proofs we invented ourselves. Such a self-contained and user-friendly exposition of the subject is nowhere to be found except here. At the end of this section, we briefly introduce the two different approaches to diffeological tangent spaces – the *internal* and *external* – and conclude that our analysis is *insensitive for this choice*.

Next we introduce the concept of a groupoid, Lie groupoid and Lie algebroid and discuss the generalisation to the diffeological setting, which is an *interesting open problem*. Since we are mainly interested in an application of this framework to the symmetry structure described in [2], and we discovered that the task of filling this gap exhaustively is not something we could do on the side, we only *filled the gaps relevant for our analysis*. The idea is, as suggested in the Appendix of [2], to define the diffeological algebroid through the group of bisections with the bracket being given by the Lie bracket there. An equivalence of this perspective to the standard one was recently proved for Lie groupoids on compact bases. It seems then a plausible path to take, although it is not clear under what assumptions such a construction makes sense and leads to something which we would like to call a *diffeological algebroid*. However, it *does* make sense for the case we are interested in – the algebroid we get is a vector bundle equipped with a bracket and an anchor, just like for the smooth case, with the only difference being that the base space is no longer a manifold but a diffeological space. We leave the general case unsolved.

The second part is an application of the partly-developed framework of diffeological algebroids to the groupoid structure described in [2]. After introducing the concepts and underlying their importance, we present the groupoid of [2] and take care of the second point above by putting a *different diffeological structure* on the groupoid, which we find simpler and more natural.

Next, by a detailed analysis of the group of bisections, we *confirm the choice* of g-gaussian extensions of vector fields defined on hypersurfaces as representing sections of the algebroid which, although crucial for establishing the bracket structure, we did not find to be done very clearly in the original approach [2]. We hope to have thus *improved the understanding* of the origin of this structure, certainly we clarified it for ourselves. We also provide a *new reasoning* pointing at the same representation via the Lie algebra structure of the group of bisections. On the way, we may have also provided an *interesting link* to the approach that Weinstein and Blohmann are sketching at the end of [3]. There they develop a general theory of Hamiltonian Lie algebroids, still motivated by the desire of understanding the dynamics of the Einstein's theory. The diffeological structure of our choice also *simplifies the global structure of the derived algebroid*, the approach we present is still involved, but in our view *more straightforward* than the original one, which was our third goal. The *triviality* of the algebroid bundle, mentioned but not proved in the original paper, is easily seen from the point of view of our analysis.

We close by summarising what has been done and sketching some interesting lines of further development of this approach. Among other perspectives, we describe the original motivation of Prof. Klaas Landsman for looking into these structures – namely, to see if it could be linked to a quantization framework based on integration of Lie algebroids that he developed in [12].

Part I
Mathematical framework

In our view, the main purpose of Mathematical Physics is to provide the *appropriate* mathematical frameworks for capturing theories of Physics in order to allow for better understanding of the theories themselves and, ultimately, the physical reality they describe. We think of a framework as *appropriate* iff it is capable to grasp all the physical notions that appear in the theory and allows for unambiguous and precise reasoning with them.

However, what we are going to do here does not fall exactly in the scope of this definition – we aim to provide an appropriate mathematical framework for providing a partial answer to a meta question *about* the structure of a physical theory, namely the theory of General Relativity. The question can be phrased as follows:

What is the origin of the bracket structure of the constraints of General Relativity?

The partial answer that we want to support and improve was given in [2] where the authors put forward a claim that an identical structure is given by the constant sections of an algebroid describing diffeomorphism invariant deformations of space-like hypersurfaces in Lorentzian manifolds, which can be derived from a groupoid that captures the global symmetry structure of the initial value formulation of the Einstein equation.

As we will see, a mathematical framework powerful enough to not only encode this symmetry structure, but also to support the analysis of the associated infinitesimal structure, i.e. *appropriate* for the task, is that of diffeological algebroids. In this part we aim to introduce all the necessary concepts and provide a sufficient technical and conceptual basis.

We begin by providing basics of the theory of diffeological spaces, which is one of the generalizations of the manifold framework. The advantages of diffeological spaces over manifolds are, among others, the simplicity of treatment of functional and quotient spaces – we will need both those features for the smooth description of our groupoid. Next we dive into the analysis of the structural properties of the theory of diffeological spaces, providing a detailed, self-contained and user-friendly description of the sheaf-theoretic perspective on this framework, which is not directly needed for the analysis but deepens the understanding of this tool.

In the second part we present the concept of a groupoid, Lie groupoid, and its associated Lie algebroid, and push it a bit in the direction of diffeological algebroids, establishing the ground for our future analysis. We discuss some perspectives and difficulties of developing a general notion of a diffeological Lie algebroid, which is central to our task and to the best of our knowledge has not been treated in the literature before.

1 Diffeological spaces

Diffeology, next to the theory of (very similar) Chen spaces, Frölicher spaces, Differential spaces of Sikorski etc., is an extension of differential geometry. It was introduced by Jean-Marie Souriau in 1979 and developed in a textbook [10] by Patric Iglesias-Zemmour. More recently, topology of diffeological spaces and different approaches to diffeological tangent spaces were developed and summarized in the papers [6] and [5] by J.D. Christensen, E. Wu and G. Sinnamon. As nicely put by Patrick [10]:

> With a minimal set of axioms, diffeology allows the geometer to deal simply but rigorously with objects which do not fall within the usual field of differential geometry: quotients of manifolds (even non-Hausdorff), spaces of functions, groups of diffeomorphisms, etc. The category of diffeological spaces is stable under standard set-theoretic operations, such as quotients, products, coproducts, subsets, limits, and colimits. With its right balance between rigor and simplicity, diffeology can be a good framework for many problems that appear in various areas of physics.

Sharing his belief in a huge potential for applications of this beautiful concept in different areas of mathematical physics, we aim to provide a relatively self-contained and accessible description by introducing all the notions from the ground up and providing detailed and elementary proofs that can not be found elsewhere.

First, we introduce the concept of a diffeological space as sets equipped with a specified set of smooth parametrizations (plots) subject to the three natural conditions (smooth compatibility, covering and sheaf conditions), describing along the way how manifolds fit into this framework. Next, we present basic constructions that allow to naturally equip products, sub-spaces, quotients, functional spaces etc. of diffeological spaces with their own, inherited diffeological structure.

Next, we switch to the the sheaf-theoretic perspective - after reflecting on the already discovered properties of the category $\{Diffeo\}$ of diffeological spaces and smooth maps between them, we carefully introduce the basic notions from the abstract sheaf theory. We then show how the category $\{Diffeo\}$ fits into this framework by proving its equivalence (Theorem 1.18) to the category $cSh_{Cov}\{Eucl\}$ of *concrete sheaves* on the *Euclidean site* modelled on the category $\{Eucl\}$ of open subsets of Euclidean spaces and smooth mappings. Having achieved that, we explore how the nice properties of the category of diffeological spaces can be understood from this perspective.

Finally, we describe different kinds of tangent structures that can be associated to diffeological spaces – again using the power of the Category Theory, we describe the concepts of an *internal* and *external* tangent spaces and discuss some of their properties. We also introduce the very natural notion of a diffeological *infinite jet bundle*. Even though it can be argued that Frölicher spaces or Differential spaces are better suited to deal with the tangent structures – there is less confusion about how these concepts should be generalized from the manifold framework – we find the richness of the theory of diffeological tangent spaces not only interesting but also potentially advantageous.

1.1 Basics

We will now introduce the basic notions and constructions from the theory of diffeological spaces.

1.1.1 Definitions

In this paragraph, we would like to introduce the notion of a diffeological space and diffeologically smooth map and explain how they can be understood as a generalization of a manifolds and smooth maps between them. Let us begin with the definition of a *diffeological space*:

Definition 1.1. *A parametrization of a set X is a map $\phi : U \to X$, where U is an open subset of a Euclidean space $\mathbb{R}^n$ of arbitrary dimension $n \in \mathbb{N}$.*

Definition 1.2. *A diffeological space, denoted (X, D_X) is a pair of sets, where X is the set we are concerned with and D_X is the set of parametrizations of X, called plots, subject to the following natural conditions:*

i) *D_X is closed under composition with smooth maps (smooth compatibility condition):*

$$\phi : U \to X,\ \{f : U' \to U\} \in C^\infty(U', U) \ \Rightarrow\ \{\phi \circ f : U' \to X\} \in \mathsf{D}_X,$$

ii) *compatible plots defined on an open cover of an open subset of a euclidean space can be (uniquely) glued together to give another plot (sheaf condition):*

$$\forall \big\{\{\phi_i : U_i \to X\} \in \mathsf{D}_X, U = \bigcup_i U_i\big\} :\ \phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}\ \exists! \{\phi : U \to X\} \in \mathsf{D}_X :\ \phi|_{U_i} = \phi_i\ \forall_i,$$

iii) *all constant maps are plots (covering condition):*

$$\phi : U \ni u \mapsto \phi(u) = x \in X\ \ \forall_{u \in U} \ \Rightarrow\ \phi \in \mathsf{D}_X,$$

We refer to the set D_X as a diffeology or a diffeological structure on X.

Definition 1.3. *On any set X we can put one of the "extreme" diffeologies: the one that considers all maps to X as plots, called the coarse diffeology, and the one for which the only plots are constant maps - the discrete diffeology.*

Very much like topology, diffeology can be generated:

Definition 1.4. *Let $\mathscr{F}$ denote some family of parametrizations of a set X. The diffeology generated by $\mathscr{F}$, denoted $\langle \mathscr{F} \rangle$, is the finest (smallest) diffeology on X containing $\mathscr{F}$:*

$$\langle \mathscr{F} \rangle := \bigcap_{\mathscr{F} \subset \mathsf{D}_X} \mathsf{D}_X,$$

where D_X runs over all the diffeologies on X.

Further, a diffeological structure does not require an underlying topology – it generates one:[1]

Definition 1.5. *The D-topology on a diffeological space (X, D_X) is the finest topology making all the plots smooth:*

$$A \in \mathscr{O}_D(X) \ \Leftrightarrow\ \phi^{-1}(A) \in \mathscr{O}(U)\ \ \forall \{\phi : U \to A \subset X\} \in \mathsf{D}_X,$$

where $\mathscr{O}_D(X)$ denotes the set of all subsets of X that are D_X-open.

[1] For the complete treatment of *D*-topologies we refer to [6].

Among other perspectives, the notion of a diffeological space can be regarded as generalization of that of a smooth manifold:

Lemma 1.1. *A manifolds becomes a diffeological space when plots are defined to be those parametrizations that are smooth (in the manifold sense).*

Proof. We will use small roman numbering to refer to the Definition 1.2 of a diffeological space. A map $f : \mathbb{R}^n \supset U \to M$ to the m-dimensional manifold M is smooth (in the manifold sense) iff for any local chart $\chi : M \supset A \to V \subset \mathbb{R}^m$, the map $\chi \circ f : U \to V$ is a smooth (subsets of) between Euclidean spaces. Constant parametrizations composed with charts give rise to constant maps, which are smooth – this assures (i); composition of smooth maps between open subsets of euclidean spaces (that arise *after* the composition with charts) is again smooth, which gives compatibility requirement (ii); finally, compatible family of smooth maps $f_i : U_i \to M$ defined on an open cover $\bigcup U_i = U$ gives rise to a family of smooth maps $\chi \circ f_i : U_i \to V$ and hence can be uniquely glued to a smooth map on the whole U and since this needs to work for *any* local chart, we get a smooth map $f : U \to M$ and hence (iii) also holds. □

Definition 1.6. *We call the diffeology described above the manifold diffeology.*

From now on, we consider all manifolds as diffeological spaces equipped with manifold diffeologies. Let us now have a look at the concept of a smooth map *between* diffeological spaces:

Definition 1.7. *We call a function between diffeological spaces $f : (X, \mathsf{D}_X) \to (Y, \mathsf{D}_Y)$ a smooth map, denoted $f \in C^\infty(X,Y)$, iff it takes plots to plots, i.e. it's composition with any plot on the source space gives a plot on the target space:*

$$\{f : X \to Y\} \in C^\infty(X,Y) \quad \Leftrightarrow \quad \{f \circ \phi : U \to Y\} \in \mathsf{D}_Y \;\; \forall \{\phi : U \to X\} \in \mathsf{D}_X.$$

Remark. Clearly, the identity map is always smooth.

Let us first acknowledge some nice properties of the smooth maps just defined:

Lemma 1.2. *Smooth maps between diffeological spaces compose.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\phi \in \mathsf{D}_X$. Since f is smooth, $\{f \circ \phi\} \in \mathsf{D}_Y$. Hence, since g is also smooth, we have $\{g \circ (f \circ \phi) = (g \circ f) \circ \phi\} \in \mathsf{D}_Z$ and since ϕ was arbitrary, we get that $g \circ f$ is smooth itself. □

Lemma 1.3. *Smooth maps are D-continuous.*

Proof. Let $f \in C^\infty(X,Y)$, $A \in \mathcal{O}_D(Y)$. Since f is smooth, for any $\phi \in \mathsf{D}_X$ we have $f \circ \phi \in \mathsf{D}_Y$. Further, since A is D_Y-open, for any $\phi \in \mathsf{D}_X$ we have that $(f \circ \phi)^{-1}(A) = \phi^{-1}(f^{-1}(A))$ is open, and hence, since ϕ was a plot on X, we get that $f^{-1}(A)$ is D_X-open. □

And back to the manifold example:

Lemma 1.4. *Smooth maps between manifolds are precisely smooth in the diffeological sense, when manifolds are considered diffeological spaces equipped with manifold diffeological structures.*

Proof. Let $f : M \to N$, where M and N are manifolds equipped with manifold diffeologies, and denote by χ and η the local charts defined on the neighbourhoods of $x \in M$ and $f(x) \in N$, respectively. Then, since χ and η were arbitrary, f is smooth as a map between manifolds iff $\chi^{-1} \circ f \circ \eta$ is smooth as a map between Euclidean spaces. Now take a plot on M, i.e. a map $\phi : U \to M$ such that $\chi \circ \phi$ is smooth (again as a map between Euclidean spaces). Moreover, let us assume that $f \circ \phi(U)$ fits into the domain of η. Then f is smooth as a map between diffeological spaces iff for each ϕ as above we have that $f \circ \phi$ is a plot on N, and hence $\eta \circ f \circ \phi$ is a smooth map between Euclidean spaces. But $\phi = \chi^{-1} \circ \chi \circ \phi$ and hence we can write $\eta \circ f \circ \phi = (\eta \circ f \circ \chi^{-1}) \circ (\chi \circ \phi)$. We now see that, that since

$(\chi \circ \phi)$ is smooth, $\eta \circ f \circ \phi$ is smooth iff $(\eta \circ f \circ \chi^{-1})$, i.e. the requirements of manifold-smoothness and diffeological smoothness are indeed equivalent. □

From now on, unless stated otherwise, we will use the term 'smooth map' for maps between diffeological spaces (including manifolds) that are smooth in the above sense. We also give $C^\infty(X,Y)$ the universal meaning of diffeologially smooth maps.

Remark. When we put on the open subsets of euclidean spaces the manifold diffeology, plots on a set X are precisely the smooth maps between diffeological spaces:

$$\phi : U \to X \in \mathsf{D}_X \quad \Leftrightarrow \quad \phi \in C^\infty(U,X),$$

and hence there is no ambiguity in referring to plots as *smooth parametrizations*.

Two diffeological spaces are indistinguishable as such iff they are *diffeomorphic* – just like for manifolds, we define:

Definition 1.8. *Two diffeological spaces (X,D_X) and (Y,D_Y) will be called diffeomorphic, denoted $(X,\mathsf{D}_X) \simeq (Y,\mathsf{D}_Y)$, iff there is smooth bijection with a smooth inverse between them. Such a map is called a diffeomorphism.*

Remark. Clearly, $(X,\mathsf{D}_X) \simeq (Y,\mathsf{D}_Y)$ iff $X \cong Y$ and $\mathsf{D}_X \cong Y,\mathsf{D}_Y$, where by "$\cong$" means bijective sets.

1.1.2 Basic constructions

In this paragraph we describe the basic constructions concerning diffeological spaces. In particular, we will explain how natural diffeological structures can be put on subsets, cartesian products, quotient and functional spaces, as well as on coproducts, fibered products and pushout sets. We will also prove the smoothness with respect to this diffeologies of the maps naturally arising in this contexts.

Definition 1.9. *The subspace diffeology, denoted $D_{A\subset X}$, is given on a subset $A \subset X$ of a diffeological space (X,D_X) by taking plots to be those parametrizations which composed with the inclusion $i : A \hookrightarrow X$ give plots on X:*

$$\{\phi : U \to A\} \in D_{A\subset X} \quad \Leftrightarrow \quad \{i \circ \phi : U \to X\} \in \mathsf{D}_X.$$

The subspace diffeology is simply the coarsest one for which the inclusion map is smooth. Naturally, we have:

Lemma 1.5. *The restriction of a smooth map to the subset is again smooth for the subspace diffeology.*

Proof. Take a smooth map $\{f : X \to Y\} \in C^\infty(X,Y)$ and a subset $i : A \hookrightarrow X$. The restricted map $f\rceil_A : A \to Y$ is smooth for the subspace diffeology iff for any plot $\phi \in \mathsf{D}_{A\subset X}$, we have $f\rceil_A \circ \phi \in \mathsf{D}_Y$. Now, since $f \in C^\infty(X,Y)$ and $i \circ \phi \in \mathsf{D}_X$, we have $\{f \circ (i \circ \phi) = (f \circ i) \circ \phi = f\rceil_A \circ \phi\} \in \mathsf{D}_Y$. □

Definition 1.10. *The product diffeology $D_{X\times Y}$ is given on a cartesian product $X \times Y$ of diffeological spaces (X,D_X) and (Y,D_Y) by taking plots to be those parametrizations which, when composed with projections, are plots on the factors:*

$$\phi : U \to X \times Y \in \mathsf{D}_{X\times Y} \quad \Leftrightarrow \quad \pi_X \circ \phi \in \mathsf{D}_X \;\; \& \;\; \pi_Y \circ \phi \in \mathsf{D}_Y,$$

where π_X and π_Y are the canonical projections.

We see that the product diffeology is in turn the finest one making the projections π_X and π_Y smooth, and that the definition can be easily generalized to arbitrary finite products. Somewhat similarly, we have a natural diffeological structure on a *disjoint union*:

Definition 1.11. *The* <u>*quotient diffeology*</u> *is a diffeology $D_{\pi(X)}$ given on an image $\pi(X)$ of a surjective function π defined on a diffeological space (X,D_X) by taking plots to be those parametrizations that are given by composition of π with a plot on X:*

$$\phi : U \to \pi(X) \in \mathsf{D}_{\pi(X)} \;\;\Leftrightarrow\;\; \phi = \pi \circ \psi,\;\; \psi \in \mathsf{D}_X.$$

This is the coarsest diffeology making π smooth. Given an equivalence relation on a diffeological space, the canonical projection $\pi : X \to X/\sim$ provides the diffeology on the quotient space, which the given name.

We can also easily define a diffeological structure on the space of smooth functions between diffeological spaces:

Definition 1.12. *The* <u>*functional diffeology*</u> *is a diffeology $D_{C^\infty(X,Y)}$ given on a set of smooth functions $C^\infty(X,Y)$ between two diffeological spaces (X,D_X) and (Y,D_Y) by taking plots to be those parametrizations for which the evaluation map is smooth:*

$$\phi : U \to C^\infty(X,Y) \in \mathsf{D}_{C^\infty(X,Y)} \;\;\Leftrightarrow\;\; ev_\phi : U \times X \to Y \in C^\infty(U \times X, Y),$$

where $ev_\phi : U \times X \ni (u,x) \mapsto \phi(u)(x) \in Y$ is the <u>*evaluation map*</u> *and we consider the manifold diffeology on U and the product diffeology on $U \times X$.*

As an illustration of the introduced notions, let us prove the following simple fact in details:

Lemma 1.6. *The composition of smooth maps is a smooth map between functional spaces.*

Proof. Consider $\{f : X \to Y\} \in C^\infty(X,Y)$ and $\{g : Y \to Z\} \in C^\infty(Y,Z)$ and equip the functional spaces with the functional diffeologies just defined. The composition of smooth functions is a map:

$$C^\infty(X,Y) \times C^\infty(Y,Z) \ni (f,g) \mapsto cmp(f,g) := g \circ f \in C^\infty(X,Z),$$

and we consider the product-functional diffeology on the domain and the functional diffeology on the codomain. It is smooth iff for any plot $\phi : U \ni u \mapsto (f_u, g_u) \in C^\infty(X,Y) \times C^\infty(Y,Z)$ the map:

$$cmp \circ \phi : U \ni u \mapsto g_u \circ f_u \in C^\infty(X,Z)$$

is a plot on $C^\infty(X,Z)$. Now, ϕ is a plot on a product space, i.e. we have that both maps:

$$\pi_1 \circ \phi : U \ni u \mapsto f_u \in C^\infty(X,Y) \;\;\&\;\; \pi_2 \circ \phi : U \ni u \mapsto g_u \in C^\infty(Y,Z)$$

are smooth, which because of the functional diffeology that we put on $C^\infty(X,Y)$ and $C^\infty(Y,Z)$ translates to the smoothness of the relevant evaluation maps:

$$ev_{\{\pi_1 \circ \phi\}} : U \times X \ni (u,x) \mapsto f_u(x) \in Y \;\;\&\;\; ev_{\{\pi_2 \circ \phi\}} : U \times Y \ni (u,y) \mapsto g_u(y) \in Z.$$

Hence, since the composition of smooth maps is again smooth (Lemma 1.2), we get that the map:

$$ev_{\{cmp \circ \phi\}} : U \times X \ni (u,x) \mapsto g_u(f_u(x)) \in Z$$

is smooth, which means precisely that $cmp \circ \phi$ is a plot on $C^\infty(X,Z)$. □

For the sake of completeness, let us also spell out how disjoint unions, fibered products and pushout sets are also naturally equipped with inherited diffeological structures:

Definition 1.13. *The coproduct diffeology $D_{X\sqcup Y}$ is given on a disjoint union $X \sqcup Y$ (coproduct) of diffeological spaces (X, D_X) and (Y, D_Y) by taking the plots to be those parametrizations that locally factor through a plot in either X or Y, i.e. for every connected component $U_i \subset U$ of an open subset of a Euclidean space we have:*

$$\{\phi : U \to X \sqcup Y\} \in D_{X\sqcup Y} \quad \Leftrightarrow \quad \{\exists_{\psi_i \in \mathsf{D}_X} :\ \phi\big|_{U_i} = i_X \circ \psi_i\} \ \vee\ \{\exists_{\eta_i \in \mathsf{D}_Y} :\ \phi\big|_{U_i} = i_Y \circ \eta_i\},$$

where $i_X : X \to X \sqcup Y \leftarrow Y : i_Y$ are the canonical inclusions.

This is the coarsest diffeology making the inclusions i_X and i_Y smooth, and we see that plots of coproduct diffeology on a disjoint union space are locally given along factors and that the definition above can also be easily generalized to arbitrary finite coproducts. Let us first recall the definitions of a fibered product and a pushout set:

Definition 1.14. *Given two functions with the same codomain, $f : X \to Z$ and $g : Y \to Z$, we define the fibered product of X and Y via:*

$$X \times_Z Y := \{(x,y) \in X \times Y : f(x) = g(y)\} \subseteq X \times Y.$$

A pushout set is a notion similar in some sense:[1]

Definition 1.15. *Given two functions with the same* domain*, $f : Z \to Z$ and $g : Z \to Y$, we define the pushout set to be:*

$$X \sqcup_Z Y := X \sqcup Y / \sim,$$

where "$\sim$" denotes the following equivalence relation: $x \sim y$ iff $\exists_{z\in Z}$ such that $f(z) = x$ and $g(z) = y$.

Combining the constructions already introduced, we can easily put diffeological structures on them:

Definition 1.16. *We equip a fibered product set $X \times_Z Y$ with the subspace-product diffeology, denoted $\mathsf{D}_{X\times_Z Y}$. This diffeological structure is referred to as the pullback diffeology.*

Definition 1.17. *We equip a pushout set with the coproduct-quotient diffeology, denoted $\mathsf{D}_{X\sqcup_Z Y}$, and refer to it as the pushout diffeology.*

[1] We will make this 'sense' precise in the next section, after introducing limits and colimits.

1.2 Sheaf-theoretic perspective

The category $\{Diffeo\}$ of diffeological spaces and smooth maps is equivalent to the category of concrete sheaves on the Euclidean site (Theorem 1.18). We find this perspective not only very appealing but also enlightening. The language of sheaf theory we are going to introduce, allows to see the concept of a diffeological space in a different light. Especially, the nice properties of $\{Diffeo\}$ that we have seen can be more deeply understood when we adapt this perspective. After carefully introducing general concepts, we aim to simplify things by making them explicit, providing detailed, elementary proofs and always looking for an optimal way of presentation. Our purpose is, except deepening our own understanding of this beautiful concept, to produce a relatively self-contained and comprehensible presentation of the theory of diffeological spaces with some didactic flavour. We refer to [1] for a slightly more complete treatment and to the relevant articles on *nlab* that we found very useful while writing this section.

1.2.1 Categorical preliminaries

While introducing basics of the Category Theory, we will also provide some important examples to be used later on and establish some notation. Let us start from the very beginning:

Definition 1.18. *A category $\mathscr{C} = \{\mathscr{C}_0, \mathscr{C}_1\}$ is a basic mathematical structure that consists of a set of objects $\mathscr{C}_0$ and a set*[1] *of arrows $\mathscr{C}_1$ connecting them, subject to the following conditions:*

i) arrows that meet can be composed (closedness of composition):

$$\forall f,g \in \mathscr{C}_1 : \; A \xrightarrow{f} B \xrightarrow{g} C \;\; \exists (g\circ f) \in \mathscr{C}_1 : \quad A \xrightarrow[f]{} B \xrightarrow[g]{} C \text{ with } A \xrightarrow{g\circ f} C \;\; \textit{commute},$$

ii) order of composition does not matter (associativity of composition):

$$f \circ (g \circ h) = (f \circ g) \circ h,$$

iii) for each object $C \in \mathscr{C}$ there is an identity arrow Id_C, pointing from C to itself, such that:

$$\forall \{ \; A \xrightarrow{f} C \; \} : \;\; Id_C \circ f = f, \quad \& \quad \forall \{ \; C \xrightarrow{g} B \; \} : \;\; g \circ Id_C = g.$$

Remark. We will use the terms arrow and morphism interchangeably. By a diagram in $\mathscr{C}$ we mean a drawing with objects of $\mathscr{C}$ represented (usually) by capital Latin letters and arrows representing morphisms between them, usually 'named' with the use of small Latin letters. We also introduce the notation $\mathscr{C}(A,B)$ for the collection of *all* morphisms in $\mathscr{C}_1$ from $A \in \mathscr{C}_0$ to $B \in \mathscr{C}_0$. In *Category Theory*, we consider two arrows $f, f' \in \mathscr{C}(A,B)$ to be *the same*, denoted $f = f'$, iff the diagram:

$$A \overset{f'}{\underset{f}{\rightrightarrows}} B$$

commutes, i.e the arrows f and f' can be interchanged in any diagram in $\mathscr{C}$ with no effect, and thus are indistinguishable in the realm of the category $\mathscr{C}$. Inherent in the category structure of $\mathscr{C}$ are two projections $s,t : \mathscr{C}_1 \to \mathscr{C}_0$ called the source and target projections, that take an arrow to it's starting or ending point, respectively:

$$\mathscr{C}_0 \ni A \xleftarrow{\;s\;} \{A \xrightarrow{f} B\} \xmapsto{\;t\;} B \in \mathscr{C}_0,$$

e.g. two arrows, f and g, can be composed to $(g \circ f)$ iff $t(f) = s(g)$.

[1] In general, neither $\mathscr{C}_0$ nor $\mathscr{C}_1$ are assumed to be sets, but merely proper classes. However, we do not need such a level of generality for our purposes. A category as we define it is usually called small.

Many interesting categories consist of objects that are sets, or sets with some additional structure,[1] and arrows that are just maps preserving this structure and forming a set that is closed with respect to the (associative) composition and contain the identity functions. Let us here give some examples of those that we will be referring to later on and introduce some notation:

- $\{Set\}$ denotes the category of *sets* and *functions*,
- $\{Vect\}$ denotes the category of *vector spaces* and *linear maps*,
- $\{Mfld\}$ denotes the category of *manifolds* and *smooth maps*,
- $\{Diffeo\}$ denotes the category of *diffeological spaces* and *smooth maps*.

As we have seen in the previous section, given a pair of diffeological spaces we can form a third one with the cartesian product or a disjoint union as the underlying set with a relevant, canonical diffeological structure, and that this construction generalizes to arbitrary finite products and coproducts. We say that the category $\{Diffeo\}$ has **finite products and coproducts**.

We have also seen that the set of smooth functions between any two diffeological spaces is also naturally equipped with a canonical diffeological structure – the functional diffeology. We say that $\{Diffeo\}$ is **closed**. The diffeological space $C^\infty(X,Y)$ is called the **exponential object** of (X,D_X) and (Y,D_Y). A category that is closed and has finite products, e.g. $\{Diffeo\}$, is called **cartesian closed**.

Last but not least, we have shown that a diffeological structure is naturally inherited from a diffeological space on *any* **quotient space**.

Notice also that there is a distinguished one-point diffeological space – indeed, there is only one way in which we can equip the one-point set with the diffeological structure, namely by declaring *all* of the functions from the open subsets of Euclidean spaces into $\{*\}$ to be plots. Notice, that in this degenerate case the *coarse* and *discrete* diffeologies coincide. This diffeological space has a property that for any other one (X,D_X) the is a *unique* function $!_X : X \to \{*\}$, which is trivially smooth since every parametrization into $\{*\}$ is a plot. Such an object in a category – with a unique arrow *from* any other object – is called a **terminal object**.

We also have another distinguished diffeological space with a similar property – the one modelled on the *empty set*. Indeed, since there are *no* maps into the empty set $\emptyset$, there is only one way we can make it a diffeological space, namely by declaring $D_\emptyset := \emptyset$, i.e. there are no plots on $\emptyset$. All the axioms are then trivially satisfied, and since there is always a *unique* function from $\emptyset$ to any diffeological space – the *empty function*, since there are no elements in $\emptyset$ – which is trivially smooth since there are no plots on $\emptyset$. Such an object in a category – with a unique arrow *to* any other object – is called an **initial object**.

The category $\{Mfld\}$ deals relatively well with products – we have a natural manifold structure on cartesian products of manifolds – but it fails to simply include the other derived spaces:

(a) The disjoint union of two manifolds $M_1 \sqcup M_2$ is naturally a manifold itself iff they have the same dimension: $dim(M_1) = dim(M_2)$.

(b) The theory of infinite-dimensional manifolds is rich and difficult, it's often far from obvious if a given functional space is a manifold or not (and in which sens this question should be conveniently considered) [citation].

(c) The quotient space M/G, where M is a manifold and G a Lie group is a manifold iff the action of G on M is free and proper [citation].

Thus we see that the category $\{Diffeo\}$ has many structural advantages over the category $\{Mfld\}$. Indeed - the simplicity in which it deals with functional spaces and quotients is precisely the reason for which it will be a convenient framework for our purpose.

[1] If such a category admits an object formed on the one-point set, we call it concrete.

As a next step towards the theory of sheaves we introduce the concept of a functor, which can be understood as a map between two categories respecting their structures. But before doing so, let us introduce a few more example of categories that will enrich our exposition and will be of future use:

Example 1.7. *A group $(G,\cdot)$ can be understood as a category $\mathscr{G}$, called a group category, with one object, i.e. $\mathscr{G}_0 = \{*\}$ and arrows from it to itself corresponding to each if the group elements. The identity arrow Id_* corresponds to the group identity $e \in G$, associativity and closure conditions being already present in the definition of a category. Notice also, that because of the group structure each arrow has a two-sided inverse. Groups are then one-point categories with all the arrows invertible.*

Example 1.8. *By $\{Eucl\}$ we will denote the Euclidean category with open subsets of Euclidean spaces as objects and smooth maps between them as arrows. It can be seen as a **full sub-category** of the category $\{Mfld\}$, meaning that its objects are manifolds and morphisms are precisely the smooth maps between them.*

Example 1.9. *Any topological space $(B,\mathscr{T})$ defines a category, denoted by $\mathscr{B}$, with open subsets of B as objects: $\mathscr{B}_0 = \mathscr{O}(B)$, and arrows given by inclusions: $U \to V \in \mathscr{B}_1$ iff $U \subseteq V$. Closedness for composition follows since the inclusion is transitive, associativity is obvious and we have identities since $U \subseteq U$ always.*

Definition 1.19. *Given two categories $\mathscr{C}$ and $\mathscr{D}$ a functor $F:\mathscr{C}\to\mathscr{D}$, is a pair of parallel maps:*

$$F = \{F_0 : \mathscr{C}_0 \to \mathscr{D}_0,\ F_1 : \mathscr{C}_1 \to \mathscr{D}_1\},$$

such that the image of any diagram in $\mathscr{C}$ is a diagram in $\mathscr{D}$, i.e. F:

i) respects the source and target projections: $s(F_1(f)) = F_0(s(f))$ & $t(F_1(f)) = F_0(t(f))$,

ii) is compatible with the composition: $F_1(g\circ f) = F_1(g)\circ F_1(f)$,

where by the abuse of notation we denote by s, t and "$\circ$" the source projection, target projection and composition, respectively, of both categories.

Remark. The first requirement means that for all $A,B \in \mathscr{C}_0$ and any $f \in \mathscr{C}(A,B)$ we have:

$$F:\ \{A \xrightarrow{f} B\} \mapsto \{F_0(A) \xrightarrow{F_1(f)} F_0(B)\}.$$

Remark. It follows that F respects the identity morphisms: $F_1(Id_C) = Id_{F_0(C)}$.

Example 1.10. *We always have the **identity functor**, which sends each object and arrow to itself.*

Example 1.11. *We can think of a functor that sends sets with some additional structure to simple sets and special maps preserving the given structure to themselves, e.g. $U:\{Vect\}\to\{Set\}$. This is an example of a **forgetful functor**, i.e. one that 'forgets' part of the structure.*

Example 1.12. *A group homomorphism $h:(G,\cdot)\to(H,*)$ is the same thing as a functor $\mathscr{G}\to\mathscr{H}$. Indeed, the requirement $h(g\cdot g') = h(g)*h(g')$ for $g,g'\in G$ can as well be understood as the second condition on the arrows that represent g and g' in $\mathscr{G}$, first one being trivial since both the categories have only one object. This is a perfect example of a functor that simply respects the categorical structures.*

Example 1.13. *We have a functor $\{Eucl\}\hookrightarrow\{Mfld\}$ that reinterprets open subsets of Euclidean spaces as special kind of manifolds. Similarly, we have a functor $\{Mfld\}\hookrightarrow\{Diffeo\}$, which understands manifolds as special diffeological spaces. This are examples of **inclusion functors**.*

A functor as defined above is sometimes called covariant to distinguish between another class of very natural objects:

Definition 1.20. *Given two categories $\mathscr{C}$ and $\mathscr{D}$ a contravariant functor $F:\mathscr{C}\to\mathscr{D}$, is a pair of parallel maps:*

$$F = \{F_0 : \mathscr{C}_0 \to \mathscr{D}_0,\ F_1 : \mathscr{C}_1 \to \mathscr{D}_1\},$$

such that the image of any diagram in $\mathscr{C}$ is a diagram in $\mathscr{D}$ with reversed arrows, i.e. F:

i) switches the source and target projections: $s(F_1(f)) = F_0(t(f))$ & $t(F_1(f)) = F_0(s(f))$,

ii) is compatible with the composition (of reversed arrows): $F_1(g \circ f) = F_1(f) \circ F_1(g)$.

Remark. The first requirement now means that for all $A, B \in \mathscr{C}_0$ and any $f \in \mathscr{C}(A,B)$ we have:

$$F: \ \{A \xrightarrow{\ f\ } B\} \ \mapsto \ \{F_0(A) \xleftarrow{F_1(f)} F_0(B)\}.$$

Remark. A contravariant functor $\mathscr{C} \to \mathscr{D}$ is the same thing as a covariant functor $\mathscr{C}^{op} \to \mathscr{D}$, where $\mathscr{C}^{op}$ denotes the category constructed from $\mathscr{C}$ by reversing the direction of all of the arrows, which is the same thing as switching s for t and vice versa.

1.2.2 Sheaves on sites

We are now ready to present the archetypal sheaf that will serve us an illustration and motivation for the more general concept that we are going to develop in this paragraph. Consider a *sheaf of continuous functions on a topological space*:

Example 1.14. *Given a topological space $(B, \mathscr{T})$ we define a contravariant functor $F : \mathscr{B} \to \{Set\}$, or equivalently a functor $\mathscr{C}^{op} \to \{Set\}$, called a sheaf of continuous functions on B, that assigns to any open subset $U \in \mathscr{O}(B)$ the set $F(U)$ of continuous real-valued functions defined on U, and the restriction map $\rceil_{V,U} : F(V) \ni f \mapsto f|_U \in F(U)$ to each of the arrows $U \to V$ in $\mathscr{B}$, i.e. inclusions $U \subseteq V$. We see, that reversing the arrows is the most natural thing to do in this kind of a situation. It can be pictured as a diagram:*

$$\begin{array}{ccc} U & \xrightarrow{\ \subseteq\ } & V \\ \downarrow F & & \downarrow F \\ F(U) & \xleftarrow{\ \rceil_{V,U}\ } & F(V) \end{array}$$

For the second requirement, let us take $U \subseteq V \subseteq W$ and $F \in F(W) = C^\infty(W, \mathbb{R})$. Composition of this chain of inclusions is simply the inclusion $U \subseteq W$, so F takes it to the restriction $\rceil_{W,U} : F(W) \to F(U)$. If we apply the functor to the inclusions separately and compose afterwards instead, we get a composition of restrictions $\rceil_{W,V} \circ \rceil_{V,U}$, which is the same thing since $(f|_V)|_U = f|_U$.

Let us now point out an important property of the *sheaf* defined above. Consider an open cover $U = \bigcup_i U_i$ of some $U \in \mathscr{O}(B)$ together with a family of continuous functions $f_i \in F(U_i)$ that are compatible on the intersections – such a collection will be called a *compatible family*:

Definition 1.21. *Given an open subset $U \in \mathscr{O}(B)$ with an open cover $U = \bigcup_i U_i$ we call a family of continuous functions $\mathscr{F} = \{f_i : U_i \to \mathbb{R} \mid i \in I\}$ a compatible family iff:*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}, \ \ \forall_{i,j}.$$

It is an elementary property of continuous functions, that such a compatible family can be *glued* together to give a continuous function on the whole U. We will refer to this as the **gluing property** and call such a *glued* function $f \in F(U)$ an *amalgamation* of the compatible family:

Definition 1.22. *Given an open subset $U \in \mathscr{O}(B)$ with an open cover $U = \bigcup_i U_i$ and a compatible family $\mathscr{F} = \{f_i : U_i \to \mathbb{R} \mid i \in I\}$ we call the function $f \in F(U)$ an amalgamation of $\mathscr{F}$ iff:*

$$f|_{U_i} = f_i \ \forall_i.$$

Moreover, the *gluing* has a unique outcome: given two functions $f, f' \in F(U)$, we know that if they agree on each of the members of the covering, they need to agree on the whole U. The elements of $F(U)$ are then defined *locally* with respect to the topology on B - this is called the **locality property**. Together, they give that the amalgamations as above not only exist but are *unique*:

For *every* covering $U = \bigcup_i U_i$ and a *compatible family* $\mathscr{F} = \{f_i \in F(U_i) \mid i \in I\}$ there is a *unique* amalgamation $f \in F(U)$ of $\mathscr{F}$.

We will refer to this as the **sheaf property**. It can be understood as follows: elements of $F(U)$ for the sheaf $F : \mathscr{B} \to \{Set\}$ on any object $U \in \mathscr{B}_0$ is fully determined by their compatible values on any covering of U.

The functor just described can be generalized in different directions. Firstly, we could consider functions to be only k-times differentiable or merely continuous, and so on. As a next step, we could give up on thinking of elements of $F(U)$ as functions at all and consider them as abstract elements of sets that are assigned to open subsets of B instead. However, we would probably like to put some extra conditions on such a generalized object to assure that the nice property described above still hold, in a more abstract sense. Indeed, this will be the defining property distinguishing *sheafs* from more general class of objects called *pre-sheafs*. Finally, we can also try to generalize this concept so that it makes sense for a broader class of categories than just those that come from topological spaces. This is the plan, so bear with us! More *concretely*, we are going to:

1) Define a *pre-sheaf* on a general category, forgetting about topological spaces and functions
2) Generalize the notion of an open covering to a general category by introducing *sites*[1]
3) Define sheaves in this general setting
4) Show how a diffeological space can be understood as a *concrete sheaf* on a site of $\{Eucl\}$
5) State and prove the equivalence of categories $\{Diffeo\}$ and the category of those sheaves

Definition 1.23. *A pre-sheaf F on a category $\mathscr{C}$ is a functor from the opposite category to $\{Set\}$, i.e.:*

$$F : \mathscr{C}^{op} \to \{Set\}.$$

Simple as that. Notice here, that we have a natural notion of sub-presheaf. Indeed, given a pre-sheaf $F : \mathscr{C}^{op} \to \{Set\}$ we can think of a sheaf H which sends objects of $\mathscr{C}$ to the *sub*sets of the sets that are assigned by F and respects the arrows:

Definition 1.24. *A sub-presheaf H of $F : \mathscr{C}^{op} \to \{Set\}$, denoted by $H \subseteq F$, is a pre-sheaf on $\mathscr{C}$ satisfying:*

$$H(C) \subseteq F(C) \;\; \forall_{C \in \mathscr{C}_o}, \qquad H(f) = F(f)\big|_{H(C)} \;\; \forall_{f \in \mathscr{C}(A,C)}.$$

Now, all we need to do in order to state the *sheaf property* in the more general context is to generalize the notions of an *open cover*, a *compatible family* and an *amalgamation*.

There are various ways to generalize a notion of an open cover to an arbitrary category, the *Grothendieck topology* being probably the most classical and widely used one. However, we find the notion of a *coverage* [16] advantageous for the following reasons:

1. it is simpler and easier to introduce – there is only one condition that we need to impose on the coverings and no additional notions,[2]
2. it is more natural as a generalization of the topological notion – taking the standard open covers constitute a *coverage* for $\mathscr{B}$, which is not the case for the *Grothendieck topology*,[3]

[1] We actually do not need the full generality as on out category the notion of an open cover is very natural and we feel that does not necessary need to be that strongly supported. Hence, the reader not interested in the general theory of sheaves (what a pity!) is welcome to skip this part and move directly to the Definition 1.29.

[2] Grothendieck distinguishes covering *sieves* and imposes three conditions in his definition. See also [14] and [11].

[3] *Sieves* are by definition downwards-closed while topological coverings very well might not be.

and yet it gives rise to a Grothendieck topology and moreover defines the same sheaves [1]. We will then use the concept of a *coverage* to define *sites*, and later *sheaves*.

Let us denote the collection of open covers of U by $Cov(U)$. In terms of sets, we write $U = \bigcup_i U_i$ and require all the U_i's to be open. However, if we want to free ourselves from the context of topological space, we need to focus on the properties that are natural in the categorical context, i.e. on the arrows, and forget about set-theoretic operations. Notice first that since an open cover of U is a family of open subsets of U, for each U_i we have an arrow $U_i \to U$ in $\mathscr{B}$. As a property distinguishing a covering $S_U = \{U_i \in \mathscr{O}(B) \mid i \in I, \bigcup_i U_i = U\} \in Cov(U)$ from an arbitrary collection of open subsets of U we can choose the following: for any open subset $V \subseteq U$, we have the *intersection* $S_U \cap V = \{U_i \cap V \mid i \in I\}$ that *covers* V. In other words, for any $S_U \in Cov(U)$ and $V \subseteq U$ we have $S_U \cap V \in Cov(V)$. Clearly, since U is open itself, if the above holds it follows that S_U covers U. Moreover, for each element $U_i \cap V$ of the covering $S_U \cap V$ we have a natural inclusion $U_i \cap V \subseteq U_i$. The idea is now to reformulate this property in terms of arrows in $\mathscr{B}_1$ alone, i.e. without referring to the set-theoretic intersection, and treat it as an essential characteristic of the notion of an open cover. Writing $g^*(S_U)$ for $S_U \cap V$ and V_i for $U_i \cap V$, the above property may be then formulated in abstract terms as follows:

For any covering $S_U \in Cov(U)$ and a subset $g : V \subseteq U$ there exist a covering $g^*(S_U) \in Cov(V)$ such that for all of its members $\{i_V : V_i \subseteq V\} \in g^*(S_U)$ we have a member of the original covering $\{i : U_i \subseteq U\} \in S_U$ and an inclusion $f_i : V_i \subseteq U_i$ such that $g \circ i_v = i \circ f_i$, i.e we have[1]:
$$\begin{array}{ccc} V_i & \xrightarrow{f_i} & U_i \\ \downarrow{\scriptstyle i_V} & & \downarrow{\scriptstyle i} \\ V & \xrightarrow{g} & U \end{array} .$$

We call $g^*(S_U)$ a *pull-back* of S_U along g and refer the above property as *stability under pull-back*. Now, equipping an arbitrary category with a *coverage* is nothing but declaring which of the families of morphisms we consider to be *covering*. Since the characterization above generalizes straightforwardly to an arbitrary category [16], we can simply define:

Definition 1.25. *A coverage on a category $\mathscr{C}$, denoted by Cov, is a pull-back stable assignment of a* collection *$Cov(C)$ of families of morphisms into C to each of the objects $C \in \mathscr{C}_0$, i.e. we have:*
$$\forall \{S_C \in Cov(C),\ \ g : A \to C\}\ \ \exists\, g^*(S_C) \in Cov(A) : \forall j \in g^*(S_U)\ \exists\, \{i_j \in S_C, f_j \in \mathscr{C}_1\} : \quad \begin{array}{ccc} A_j & \xrightarrow{f_j} & C_i \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i_j} \\ A & \xrightarrow{g} & C \end{array} .$$
Elements of $Cov(C)$ are called coverings of C, a pair $(\mathscr{C}, Cov)$ is called a site.

Remark. The category $\mathscr{B}$ is naturally a site if we declare coverings to be the usual open covers.

Now, when we know what means for a family of morphisms in a site to be covering, we are almost ready to state the condition for a presheaf on $(\mathscr{C}, Cov)$ to be a *sheaf*. But first, because we decided to forget about functions, we need to state what means for a family of elements $x_i \in F(C_i)$, where F is a pre-sheaf on $\mathscr{C}$ and $C_i \in \mathscr{C}_0$ for all i, to be *compatible*. Again, we need to get rid of a set-theoretic operation of intersecting elements of an open cover that appear in the definition of a covering family in the context of $\mathscr{B}$. The idea here is to exchange the requirement that the elements of the family agree on all *intersections* of the members of a covering for the *equivalent* one, namely that they agree on *all* of their *common subsets*:

Definition 1.26. *Given a pre-sheaf $F : \mathscr{C} \to \{Set\}$ on a site $(\mathscr{C}, J)$, a compatible family for a covering $\{i : C_i \to C \mid i \in I\} \in Cov(C)$ is a collection of elements $x_i \in F(C_i)$, such that for any pair of morphisms $a_i : A \to C_i$ and $a_j : A \to C_j$ with $i \circ a_i = j \circ a_j$ we have $F_1(a_i)(x_i) = F_1(a_j)(x_j) \in F(A)$.*

[1] We believe that only the commutative diagrams are worth drawing, and hence do not indicate the commutativity in any way.

Remark. The pair of morphisms in the above definition corresponds to a subset of an intersection of $U_i \cap U_j$, while $F_1(a_i)$ and $F_1(a_j)$ are the relevant restrictions.

Just like for a sheaf of continuous functions, we define an *amalgamation* of a compatible family:

Definition 1.27. *Given a pre-sheaf $F : \mathscr{C} \to \{Set\}$ on a site $(\mathscr{C}, Cov)$ and a compatible family $x_i \in F(C_i)$ for a covering $\{i : C_i \to C \mid i \in I\} \in Cov(C)$, is an amalgamation of $x_i \in F(C_i)$ an element $x \in F(C)$ such that $F_1(i)(x) = x_i$ for all i.*

Finally, we define sheaves on $(\mathscr{C}, Cov)$ to be those pre-sheaves that satisfy the *sheaf property*:

Definition 1.28. *A pre-sheaf $F : \mathscr{C}^{op} \to \{Set\}$ on a site $(\mathscr{C}, Cov)$ is called a sheaf iff it satisfies the sheaf property, i.e. for* any *covering $\{i : C_i \to C \mid i \in I\} \in Cov(C)$ and a compatible family $x_i \in F(C_i)$, there always exists a* unique *amalgamation.*

1.2.3 Category of diffeological spaces as a sheaf category

We are really close now to being able to see that a diffeological space is *actually* a sheaf on a site, and how the smooth functions can be understood from this point of view, which is the focus of this paragraph.

The site we are interested in is formed on $\{Eucl\}$ – the category of the domains of the plots, which is in a way very close to being a topological space: each object $U \in \{Eucl\}_0$ *is* an *open subset* of a Euclidean space, $U \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$, and hence admits a natural notion of an open cover coming from the topology on $\mathbb{R}^n$:

Definition 1.29. *Since the inclusions are smooth, we can define a coverage on $\{Eucl\}$, end hence the Euclidean site, by simply declaring the coverings for $\mathbb{R}^n \supseteq U \in \{Eucl\}_0$ to consist of the topological open covers of U in $\mathbb{R}^n$.*

Lemma 1.15. *The coverage on $\{Eucl\}$, as just defined, satisfies the pull-back stability requirement.*

Proof. Let's fix $U, U' \in \mathscr{O}(\mathbb{R}^n)$, an open cover $\{U_i \mid i \in I\} \in Cov(U)$, i.e. a collection of open subsets of U such that $U = \bigcup_i U_i$, and a smooth function $g : U' \to U$. We can define the pull-back of $\{U_i \mid i \in I\}$ along g to be:

$$g^*(\{U_i \mid i \in I\}) := \{g^{-1}(U_i) \mid i \in I\}.$$

They are open since g is smooth and hence continuous and U_i's are open. They cover U' because it's a domain of g and U_i's cover the whole U. The maps j and f_i from the Definition 1.25 are constructed as follows: $f_i := g|_{g^{-1}(U_i)}$ and j is the inclusion $g^{-1}(U_i) \subseteq U'$. □

A diffeological space is defined as a *set X* equipped with some additional structure. If it is about to correspond to a sheaf, the set itself has to be a part of its definition. Notice first, that the objects of $\{Eucl\}$ have an underlying structure of sets – they are *subsets* of Euclidean spaces. Moreover, a one-point space $\{*\} = \mathbb{R}^0$ is clopen and hence an object of the the category[1] $\{Eucl\}$ – it is its *terminal object*. We call such a category – with objects formed on sets and admitting a terminal object on the one-point set – concrete[2]. For any *concrete* category $\mathscr{C}$ and a fixed set X, we have a very natural[3] pre-sheaf on $\mathscr{C}$ which assigns to each object the set of all functions from its underlying set to X and the pre-composition operation to the morphisms:

[1] This is compatible with the convention that the one-point space is a manifolds.

[2] There is some ambiguity in defining the notion of a concrete category, a concrete site and a concrete pre-sheaf – namely in the place where we put the requirements connected to the terminal object. However, it does not matter for the notion of a concrete sheaf, which is the object we are really interested in.

[3] It is almost the same notion as that of a *contravariant Hom functor*, the difference being that the sets of arrows are taken in $\{Set\}$, and not as usually in the category itself.

$$F_X : \mathscr{C}^{op} \to \{Set\}$$
$$C \mapsto \{Set\}(C,X),$$
$$\{g : C' \to C\} \mapsto \big\{\_ \circ g : \{Set\}(C,X) \to \{Set\}(C',X)\big\}.$$

We already see that when applied to $\{Eucl\}$ this gives *all* functions from open subsets to our chosen set, i.e. *all* possible parametrizations. Lets then take a look at something smaller:

Definition 1.30. *Given a set X, a concrete pre-sheaf $D_X : \mathscr{C}^{op} \to \{Set\}$ on a concrete category $\mathscr{C}$ is a sub-presheaf $D_X \subseteq F_X$ that agrees on the terminal object $\mathscr{C}$ to X, i.e. $D_X(\{*\}) \cong F_X(\{*\})$*[1].

Remark. Notice that, since points of X correspond one-to-one to functions pointing at them from the one-point set, we have $F_X\{*\} \cong X$. The compatibility on the one-point-set requirement is imposed in order to be able to extract the set X from a concrete functor via $X \cong D_X\{*\}$.

And finally we can define:

Definition 1.31. *A concrete sheaf on the Euclidean site will be called a diffeological sheaf.*

And appreciate our first important result [1], [15]:

Lemma 1.16. *Diffeological spaces are in one-to-one correspondence with diffeological sheaves.*

Proof. As noted above, the idea is that $D_X(U)$ simply gives all the plots $U \to D_X\{*\} \cong X$, and hence D_X defines the whole diffeological structure on X – we will now refer to the elements of $D_X(U)$, for any $U \in \{Eucl\}_0$, as *plots*. We divide the proof into the following steps:[2]

1) *Smooth compatibility (i) is equivalent to D_X being a sub-presheaf of $F_X : \{Eucl\}^{op} \to \{Set\}$.*

   For any smooth $f : U' \to U$ we have $D_X(f) : D_X(U) \to D_X(U')$ that maps $\phi \mapsto f \circ \phi$, and hence $f \circ \phi \in D_X(U')$, i.e. for any plot $\phi : U \to X$ the map $f \circ \phi$ is a plot $U' \to X$ if only f is smooth with respect to the diffeology on (X, D_X).

2) *The sheaf condition (ii) is equivalent to the sheaf property of D_X.*

   Take an arbitrary covering $U = \bigcup_i U_i$, $i : U_i \subseteq U$ and a family of plots $\{\phi_i : U_i \to X \mid i \in I\}$ compatible in the sheaf sense, i.e for all pairs of smooth maps $a_i : V \to U_i$ and $a_j : V \to U_j$ such that $i \circ a_i = j \circ a_j$ we have:

   $$D_X(a_i)(\phi_i) = \phi_i \circ a_i = \phi_j \circ a_j = D_X(a_j)(\phi_j).$$

   The *sheaf condition* on D_X says that we then have a unique amalgamation, i.e. a plot $\phi \in D_X(U)$ such that:

   $$D_X(i) = \phi_i \ \forall_{i \in I}.$$

   We will show that this is exactly the same statement as the sheaf condition (ii) by comparing the relevant definitions.

   (a) (*coverings*) Because of the natural coverage that we put on $\{Eucl\}$, the notions of a covering in $\{Eucl\}$ and the one we standard one we use in (ii) are exactly the same.

   (b) (*compatible families*) The condition $i \circ a_i = j \circ a_j$ gives us that the functions a_i and a_j have a common image $V' := a_i(V) = a_j(V)$ contained in the intersection $V' \subseteq U_i \cap U_j$, and that they agree on it, i.e. as functions into V' they are the same

[1] By $\cong$ we denote here the bijection of sets.

[2] The small Roman numbering refers to the Definition 1.2 of a diffeological space.

– let us denote the surjective function that they both represent by $a : V \to V'$. The sheaf-theoretic compatibility condition then reads:

$$D_X(a_i)(\phi_i) = \phi_i \circ a = \phi_i|_{V'} \circ a = \phi_j|_{V'} \circ a = \phi_j \circ a = D_X(a_j)(\phi_j),$$

and since a is an *arbitrary* map with an image $V' \subseteq U_i \cap U_j$, this means that the plots ϕ_i and ϕ_j agree on all of the common subsets of U_i and U_j:

$$\phi_i|_V = \phi_j|_V \ \ \forall V \subseteq U_i \cap U_j,$$

which is equivalent to the compatibility on in the usual sense: $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$.

(c) (*amalgamations*) We have $D_X(i) = D_X(U_i \subseteq U)(\phi) = \rceil_{U_i}\phi = \phi|_{U_i}$, and hence the properties of being an amalgamation also agree in both languages.

We then see that all of the notions that appear in the formulations of a sheaf property coincide, and since statements have exactly the same form the sheaf properties are indeed equivalent.

3) *The covering condition (iii) is equivalent to the requirement $D_X(\{*\}) = F_X(\{*\})$ (plus the smooth compatibility property (i)).*

The set $D_X\{*\}$ consists of plots $\{*\} \to X$, which need to be constant. Moreover, any constant parametrization (uniquely) factors through $\{*\}$, and hence, because of the smooth compatibility condition, the set $D_X\{*\}$ is isomorphic to the subset of X for which constant maps are plots. Then, requiring $D_X(\{*\}) \cong F_X(\{*\})$ means that *all* the constant maps are plots.

□

We now see that the concept of a diffeological sheaf is precisely what we want if the generalized smooth space that it represents is to be modelled on a set (*concrete*) with the smooth structure compatible with the change of parametrizations (*pre-sheaf condition*) and the topological structure (*sheaf condition*) of Euclidean spaces.

The collection of diffeological sheaves is naturally equipped with the structure of a category, the arrows being given by *natural transformations*. Let us define:

Definition 1.32. *Given two functors $F, H : \mathscr{C} \to \mathscr{D}$, a natural transformation, denoted $\eta : F \Rightarrow H$ is an assignment to any object $C \in \mathscr{C}_0$ an arrow $\{\eta_C : F(C) \to H(C)\} \in \mathscr{D}_1$ such that for any $\{f : C \to C'\} \in \mathscr{C}_1$ we have the (commuting) naturality square:*

$$\begin{array}{ccc} F_0(C) & \xrightarrow{F_1(f)} & F_0(C') \\ \big\downarrow \eta_C & & \big\downarrow \eta_{C'} \\ H_0(C) & \xrightarrow{H_1(f)} & H_0(C') \end{array} .$$

It is an easy exercise to check that taking the functors $\mathscr{C} \to \mathscr{D}$ as objects and the natural transformations as arrows we get a category structure – composition is composition of the naturality squares, associativity follows from their commutativity and the identity functors serve as the identities – we will refer to such categories as functor categories. We then have a category structure on the collection of pre-sheaves on a category $\mathscr{C}$, denoted $PSh(\mathscr{C})$ and its full subcategory $Sh_{Cov}(\mathscr{C})$ of sheaves.

Since diffeological spaces are special kinds of sheaves, we can consider the *category* of diffeological sheaves with the arrows given by natural transformations. More *concretely*, given two diffeological sheaves $D_X, D_Y : \{Eucl\} \to \{Set\}$, a natural transformation between them is an assignment of plots

$U \to Y$ to plots $U \to X$. Indeed, $\eta : D_X \Rightarrow D_Y$ is an assignment to any object $U \in \{Eucl\}_0$ a map $\eta_U : D_X(U) \Rightarrow D_Y(U)$ such that for any smooth map[1] $g : U' \to U$ we have the naturality square:

$$\begin{array}{ccc} D_X(U) & \xrightarrow{\_\circ g} & D_X(U') \\ \downarrow{\eta_U} & & \downarrow{\eta_{U'}} \\ D_Y(U) & \xrightarrow{\_\circ g} & D_Y(U') \end{array} .$$

Hence we have a *functor category* of diffeological sheaves and natural transformations between them – let us denote it by $cSh_{Cov}\{Eucl\}$, where c comes from *concrete* and Sh denotes sheaves taken with respect to the coverage Cov on $\{Eucl\}$. We can now state and prove our second important result [1]:

Lemma 1.17. *Smooth maps between diffeological spaces are in one-to-one correspondence with natural transformations in the functor category of diffeological sheaves:*

$$cSh_{Cov}\{Eucl\}(D_X, D_Y) \cong C^\infty(X,Y).$$

Proof. A smooth function between two diffeological spaces $f \in C^\infty(X,Y)$ gives a natural transformation $\tilde{f} : D_X \Rightarrow D_Y$ via:

$$\tilde{f}_U := f \circ \_ : D_X(U) \ni \phi \mapsto f \circ \phi \in D_Y(U) \quad \forall\, U \in \{Eucl\}_0,$$

where $f \circ \phi \in D_Y(U)$ iff $f \in C^\infty(X,Y)$ and the naturality square is just the associativity of composition:

$$f \circ (\phi \circ g) = (f \circ \phi) \circ g.$$

Conversely, a natural transformation $\eta : D_X \Rightarrow D_Y$ defines a smooth function $\eta_* : X \to Y$ when we (again) identify plots from the one-point set with their images, e.g $\phi_x : \{*\} \mapsto x \in X$ with $x \in X$ – we will refer to this as the *point-map bijection*. The function η_* is then given by taking $x \in X$ to the element in Y corresponding to $\eta_{\{*\}}(\phi_x) : \{*\} \to Y$. We will show that these operations are each others inverses.

Notice first that $(\tilde{f})_{\{*\}}$ takes a plot $\phi_x \in D_X\{*\}$ to $f \circ \phi_x : \{*\} \mapsto f(x)$, and hence $(\tilde{f})_*(x) = f(x) \in Y$. Since x was arbitrary, the covering condition gives $(\tilde{f})_* = f$.

For the other direction, notice first that $\tilde{\eta}_* = \eta$ holds iff:

$$\tilde{\eta}_*(\psi)(u) = (\eta_* \circ \psi)(u) = \eta_*(\psi(u)) = \eta_U(\psi)(u) \in Y$$

for all $\psi \in D_X(U)$ and an arbitrary $u \in U$, where the first two equalities is just unpacking the definition of $\tilde{\eta}_*(\psi)$. We then need to show that $\eta_*(\psi(u)) = \eta_U(\psi)(u)$. To see why this is true, let us consider the following naturality square:

$$\begin{array}{ccc} D_X\{*\} & \xrightarrow{\_\circ g} & D_X(U) \\ \downarrow{\eta_{\{*\}}} & & \downarrow{\eta_U} \\ D_Y\{*\} & \xrightarrow{\_\circ g} & D_Y(U) \end{array} ,$$

where $g :=!_U$ is the unique function $U \to \{*\}$. For the indicating plot of $\psi(u) \in X$, i.e. $\phi_{\psi(u)} \in D_X\{*\}$, the commutativity of the above naturality square gives:

$$\eta_{\{*\}}(\phi_{\psi(u)}) \circ g = \eta_U\big(\phi_{\psi(u)} \circ g\big).$$

[1]Notice the change of direction of g comparing to the definition of a natural transformation – this is because we are now dealing with *contravariant* functors.

But notice that on the left hand side we have a function that takes $u \in U$ to the point in Y indicated by the plot $\eta_{\{*\}}(\phi_{\psi(u)})$, i.e.:

$$\eta_{\{*\}}(\phi_{\psi(u)}) \circ g : U \ni u \mapsto \eta_*(\psi(u)) \in Y.$$

Further, on the right hand side we take η_U of the plot on X that takes $u \in U$ to $\phi(u)$:

$$\phi_{\psi(u)} \circ g : U \ni u \mapsto \psi(u) \in X,$$

and hence we have $\eta_U\left(\phi_{\psi(u)} \circ g\right) = \eta_U(\psi)(u)$. We then see that $\tilde{\eta}_*(\psi)(u) = \eta_U(\psi)(u) \in Y$ and since $\psi \in D_X(U)$ and $u \in U$ were arbitrary, this finishes the proof. $\square$

We have just seen that diffeological sheaves and natural transformations between them *correspond* to diffeological spaces and their smooth maps. We will now make this statement firmer by stating what exactly does it mean for two *categories* to be *essentially the same*. Let us first define:

Definition 1.33. *We say that two objects $C, C' \in \mathscr{C}_0$ are* <u>*isomorphic*</u>*, denoted $C \cong C'$, iff they are connected by an* <u>*isomorphism*</u>*, i.e. an invertible arrow:*

$$C \cong C' \;\; \Leftrightarrow \;\; \exists \{f : C \to C'\}, \{g : C' \to C\} \in \mathscr{C}_1 : \; g \circ f = Id_C \;\&\; f \circ g = Id_{C'}.$$

Remark. The isomorphism of objects in $\{Set\}$ is a *bijection* of sets, in $\{Vect\}$ it is an *invertible linear map*, in $\{Mfld\}$ and $\{Diffeo\}$ the relevant *diffeomorphisms*, and so on, i.e. the objects are isomorphic iff they are indistinguishable from the point of view of a given categorical structure.

Definition 1.34. *Given two functors, $F, H : \mathscr{C} \to \mathscr{D}$, we call a natural transformation $\eta : F \Rightarrow H$ a* <u>*natural isomorphism*</u> *iff each of its components $\eta_C : F(C) \to H(C)$ is an isomorphism, i.e. it is invertible as an arrow in the functor category and we have $F(C) \cong H(C)$ via η_C for all $C \in \mathscr{C}$.*

Remark. Natural isomorphisms play the role of isomorphisms between functors – since we do not distinguish between isomorphic objects, we can consider the two functors connected by a natural isomorphism *the same*, and hence we will write $F \cong G$ in such a situation.

We consider two categories *equivalent* iff there is a relevant of a kind of an *isomorphism* between them:

Definition 1.35. *We say that two categories, $\mathscr{C}$ and $\mathscr{D}$, are* <u>*equivalent*</u>*, denoted $\mathscr{C} \cong \mathscr{D}$, iff we have a pair of functors:*

$$F : \mathscr{C} \to \mathscr{D}, \;\; G : \mathscr{D} \to \mathscr{C}$$

that are inverses *to each other up to a natural isomorphism, which means precisely that we have:*

$$F \circ G \cong Id_{\mathscr{D}}, \;\; G \circ F \cong Id_{\mathscr{C}}.$$

Let us now state the most important result of this section, which summarizes most of the effort that we have made so far, and prove it using the last two Lemmas:

Theorem 1.18. *The category of diffeological spaces and smooth maps is equivalent to the functor category of diffeological sheaves:*

$$\{Diffeo\} \cong cSh_{Cov}\{Eucl\}.$$

Proof. In the light of the Lemma 1.16, we define the functor $F : \{Diffeo\} \to cSh_{Cov}\{Eucl\}$ to be the one that takes a diffeological space (X, D_X) to a pre-sheaf D_X on the Euclidean site that assigns the plots $U \to X$ to the open subsets U and the pre-composition operation to the smooth maps between them. The smooth maps between diffeological spaces are, of course, taken to the natural transformations vie the construction from the Lemma 1.17. The first functor then looks like this:

$$\begin{aligned} F : \{\mathit{Diffeo}\} &\to \mathit{cSh}_{Cov}\{\mathit{Eucl}\} \\ (X, \mathsf{D}_X) &\mapsto D_X, \\ \{f : X \to Y\} &\mapsto \tilde{f} := f \circ \_ : D_X \Rightarrow D_Y. \end{aligned}$$

Let us define the inverse functor $G : \mathit{cSh}_{Cov}\{\mathit{Eucl}\} \to \{\mathit{Diffeo}\}$ to take the sheaf D_X to a diffeological space formed on its image on the one-point set with the disjoint union of *all* the plots as D_X, and a natural transformation to the corresponding functions as in the Lemma 1.17.:

$$\begin{aligned} G : \mathit{cSh}_{Cov}\{\mathit{Eucl}\} &\to \{\mathit{Diffeo}\} \\ D_X &\mapsto \big(D_X\{*\}, \bigsqcup_{U \in \{\mathit{Eucl}\}_0} D_X(U)\big), \\ \{\eta : D_X \Rightarrow D_Y\} &\mapsto \{\eta_* : X \to Y\}. \end{aligned}$$

The claim is now that $G \circ F \cong Id_{\{\mathit{Diffeo}\}}$ and $F \circ G \cong Id_{\mathit{cSh}_{Cov}\{\mathit{Eucl}\}}$. Notice here, that from the Lemma 1.16 it follows that replacing the diffeological space with the corresponding sheaf and back gives the same diffeological space, i.e. we always have that $G \circ F(X, \mathsf{D}_X) \cong (X, \mathsf{D}_X)$, and similarly $F \circ G(D_X) \cong D_X$. Let us denote these isomorphisms by $\xi_{(X,\mathsf{D}_X)}$ and ρ_{D_X}, respectively. Further, in the Lemma 1.17 we have shown that $(\tilde{f})_* = f$ and $\tilde{\eta}_* = \eta$, which gives the commutativity of the naturality squares for $\xi : G \circ F \Rightarrow Id_{\{\mathit{Diffeo}\}}$ and $\rho : F \circ G \Rightarrow Id_{\mathit{cSh}_{Cov}\{\mathit{Eucl}\}}$:

$$\begin{array}{ccc} G \circ F(X, \mathsf{D}_X) & \xrightarrow{G \circ F(f)} & G \circ F(Y, \mathsf{D}_Y) \\ \downarrow \xi_{(X,\mathsf{D}_X)} & & \downarrow \xi_{(Y,\mathsf{D}_Y)} \\ (X, \mathsf{D}_X) & \xrightarrow{f} & (Y, \mathsf{D}_Y) \end{array}, \qquad \begin{array}{ccc} F \circ G(D_X) & \xrightarrow{F \circ G(f)} & F \circ G(D_Y) \\ \downarrow \rho_{D_X} & & \downarrow \rho_{D_Y} \\ D_X & \xrightarrow{\eta} & D_Y \end{array},$$

and hence $G \circ F$ and $F \circ G$ are indeed naturally isomorphic to the identity functors. □

Remark. From now on, we will use those two equivalent definitions interchangeably, i.e. a diffeological sheaf might be referred to as a diffeological space or a natural transformation as a smooth map, along the line of the equivalence just described.

We have thus shown that the theory of diffeological spaces is actually a theory of concrete sheaves on the Euclidean site. We find this very interesting, for various reasons. Firstly, as we will shortly see, from this perspective the origin of some of the nice categorical properties of $\{\mathit{Diffeo}\}$ – namely the existence of the diffeological structure on all kinds of derived spaces – can be understood. Secondly, as seen by many to be the main purpose of speaking about mathematical structures in categorical terms, it allows for generalizations: we could now change the underlying site for something more general[1] or simply different,[2] forget about sets and just deal with abstract, not necessary concrete sheaves[3] etc., and as long as we keep to the *sheaf on a site* framework, there is a fairly good chance that the categorical properties will continue to hold.

[1] A friend of mine, Nesta van der Shaaf, is aiming for the category of *super cartesian spaces*.

[2] We are wondering, what would be the result of restricting to the *one-dimensional* plots, like for Frölicher spaces, by taking the site to be formed on the first two Euclidean spaces: the one-point set $\{*\}$ and $\mathbb{R}$ (plus the empty set), with the same coverage.

[3] This is the idea of a *smooth set* [17].

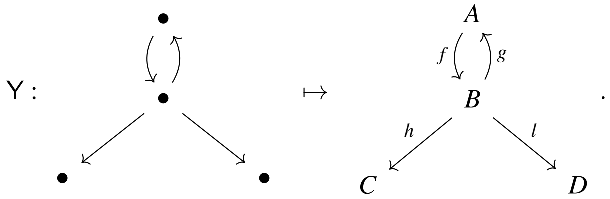
1.2.4 Sheaf-theoretic origin of categorical properties

In this paragraph we are going to show how the fact that we have natural diffeological structure on different kinds of derived spaces can be understood as a consequence or in the light of the sheaf-theoretic perspective.

Limits and colimits

Let us first broaden a bit more our categorical language by introducing the concepts of *limits* and *colimits*, which we will need not only to understand why products, coproducts, fibered products, pushout sets, one-point set and an empty set come naturally equipped with inherited diffeological structures, but also for the definitions of the tangent spaces in the coming section.

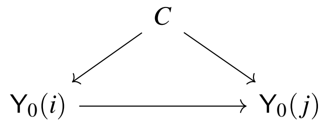
The categories that we have seen so far were all very rich in the sense of having many objects and arrows. However, it is useful to realize that simple diagrammatic structures can also be thought as categories. For example, a one object category with the identity arrow alone is a category. For two objects, we have a *discrete* category – the one with the identity arrows alone, the indiscrete category, where we have all the possible arrows, i.e. also an arrow between the objects in both directions and two other categories with one of the non-identity arrows missing, etc.[1] The functors from such *simple* categories can then be thought of as mimicking their *shape* in the target category:



We will thus refer to a functor[2] $Y : \mathscr{I} \to \mathscr{C}$ as a diagram of shape $\mathscr{I}$ in $\mathscr{C}$. It is a very useful picture to keep in mind, especially because we will now be actually interested in such *simple* diagrams only. Let us now define:

Definition 1.36. *A cone for a diagram $\mathsf{Y} : \mathscr{I} \to \mathscr{C}$ is a natural transformation $\eta : \Delta_C \Rightarrow \mathsf{Y}$, where $\Delta_C : \mathscr{I} \to \mathscr{C}$ is the constant functor that sends every object of $\mathscr{I}$ to $C \in \mathscr{C}$ and each arrow to Id_C.*

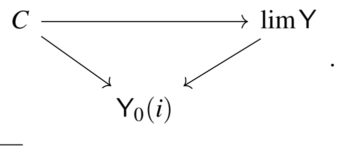
Remark. A cone for Y can be then represented as a diagram in $\mathscr{C}$ with the object C on the top and an arrow $C \to \mathsf{Y}_0(i)$ for any object $i \in \mathscr{I}_0$, such that we have:



for all the arrows $\mathsf{Y}_0(i) \to \mathsf{Y}_0(j)$ in the image of Y_1. This justifies the terminology.

In general, we have multiple cones for a given functor, i.e. for a diagram of a given shape. However, there might be a very special one:

Definition 1.37. *A cone for a diagram $\mathsf{Y} : \mathscr{I} \to \mathscr{C}$ is called limiting, denoted $\eta_{\lim} : \Delta_{\lim I} \Rightarrow \mathsf{Y}$ iff for any other cone $\eta : \Delta_C \Rightarrow \mathsf{Y}$ we have a* unique *arrow $C \to \lim \mathsf{Y}$ such that for all $i \in \mathscr{I}_0$ we have:*



[1] We can of course always add as many arrows as we want.

[2] Usually the category $\mathscr{I}$ is required to be *small*, but those are the only categories we are considering (see the Definition 1.18).

This is referred to as the universal property *of a limiting cone.*

Definition 1.38. *A* limit of a diagram $\mathsf{Y} : \mathscr{I} \to \mathscr{C}$*, denoted* $\lim \mathsf{Y}$*, is the vertex of the limiting cone.*

Remark. If we had two such limiting cones, we would have an arrow between their vertices both ways, and hence they need to be *isomorphic*, and because of the above commutativity requirement the cones coincide. Notice also, that such limiting cones do not need to exist.

As usually with the category-theoretic notions, we also have the *dual* one – that of a *colimit*, and just like with the covariant and contravariant functors, it's all about the direction of the arrows. Keeping the notation as above, we define:

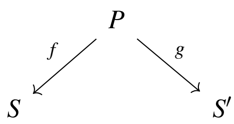
- the cocones the same way as we defined cones with the arrows now pointing *into* C,
- the limiting cocone to be the one that comes with an arrow $\text{colim}\mathsf{Y} \to C$ and
- the colimit of a functor to be its vertex.

In other words, colimits are limits in the opposite category, and vice versa.

The concepts of a limit and colimit are useful for us at this point because they allow for a generalization of notions such as products, direct sums, pull-back sets and fibered products from the category of sets and functions to an arbitrary one. We will point out some examples of the relevant constructions, leaving some of the details to the reader. Let us first consider a functor from the discrete 2-object category to $\{Set\}$:

$$\mathsf{Y}_\times : \{\bullet\ \bullet\} \ \mapsto\ \{S\ S'\}$$

A cone $\Delta_P \Rightarrow \mathsf{Y}_\times$ for such a diagram is then of the form:



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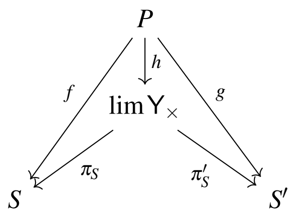
i.e. we have a pair of functions from $f : P \to S$ and $g : P \to S'$. The universal property of the limiting cone means that for such a cone we always have a unique function $h : P \to lim\mathsf{Y}_\times$ such that:



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i.e. any pair of functions to S and S' uniquely *factors through* the set $limF$, which is the distinguishing property of the product $S \times S'$. Indeed, it is the only set for which such a function, given by:

$$h := f \times g : P \ni p \mapsto (f(p), g(p)) \in S \times S',$$

always exists and is unique. We then have $\lim \mathsf{Y}_\times = S \times S'$. With a similar argument one can show that the colimit of this functor is the disjoint union of S and S', i.e. $\text{colim}\mathsf{Y}_\times = S \sqcup S'$. It is also easy to see that this construction generalizes to finite products and disjoint unions, which for the reason that should be clear by now will be referred to as coproducts.

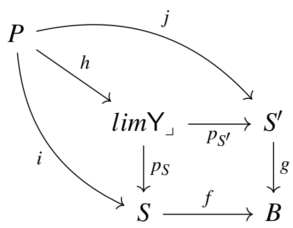
To see how the fibered products and pushout sets also arise as limits and colimits, let us consider the following functor:

$$\mathsf{Y}_{\lrcorner} : \quad \begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \quad \mapsto \quad \begin{array}{ccc} & & S' \\ & & \downarrow g \\ S & \xrightarrow{f} & B \end{array} .$$

A cone for such a diagram is of the following form:

$$\begin{array}{ccc} P & \xrightarrow{j} & S' \\ \downarrow i & & \downarrow g \\ S & \xrightarrow{f} & B \end{array} ,$$

and the universal property for the limiting one means that for any diagram as above we have a unique arrow $h : P \to \lim \mathsf{Y}_{\lrcorner}$ such that:



This is in turn a distinguishing property of a fibered product, with projections π_S and π'_S coming from the product projections π_1 and π_2. We then have $lim\mathsf{Y}_{\lrcorner} = S \times_B S'$. Similarly, the pushout set arises as a colimit of this functor, i.e. we have $colim\mathsf{Y}_{\lrcorner} = S \sqcup_B S'$.

Notice also that the limit of a functor

$$\mathsf{Y}_{\bullet} : \{\bullet\} \mapsto S,$$

is, regardless the chosen set S, the one-point set $\lim \mathsf{Y}_{\bullet} = \{*\}$. Indeed, it is the only set for which we *always* have a *unique* map *from* any other, while the the colimit is always the empty set: $\mathrm{colim}\mathsf{Y}_{\bullet} = \emptyset$ for a similar reason. We then also recover the *initial* and *final* objects of $\{Set\}$ as a colimit and a limit, respectively.

Thus, we generalize products, coproducts, fibered products and pushouts to an arbitrary category by identifying them with the, existing or not, limits or colimits of the corresponding diagrams.

We have already seen that all the above sets that can be constructed as limits or colimits of diagrams in $\{Set\}$ can be equipped with the inherited diffeological structures, and this is not a coincidence – in fact, it can be shown that the category $cSh_{Cov}\{Eucl\}$ is *complete* and *cocomplete*, meaning that it admits *all* finite[1] limits and colimits [1] – the existence of the inherited diffeological structures on the products, coproducts, fibered products, pushout sets, one-point set and the empty set can be simply seen a consequence of this fact. The completeness and cocompleteness properties are generally true for sheaf categories, and the existence of the inherited diffeological structures on those sets can be seen as a consequence of $\{Diffeo\}$ being a *sheaf category* at heart. Indeed, the existence of the relevant object in a sheaf category is basically a consequence of its existence in $\{Set\}$. There are some details we will skip here, but as an enlightening illustration of this fact, let us consider a simpler case of a pre-sheaf category:

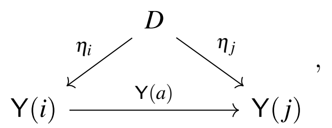
[1] *Finite* refers to the cardinality of the object set of the domain of functors we consider, i.e. we have limits and colimits to all of the diagrams that we previously referred to as *simple*

Lemma 1.19. *Limits and colimits in pre-sheaf categories can be computed* point-wisely*, meaning that for a diagram* $\mathsf{Y} : \mathscr{I} \to PSh(\mathscr{C})$*, we have:*

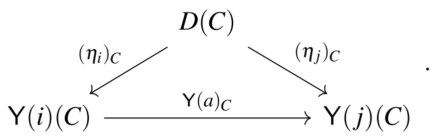
$$(\lim \mathsf{Y})(C) = \lim(\mathsf{Y}(C)) \ \ \forall_{C \in \mathscr{C}}, \ \ \mathrm{colim}(\mathsf{Y})(C) = \mathrm{colim}(\mathsf{Y}(C)) \ \ \forall_{C \in \mathscr{C}}$$

i.e the (co)limiting object for Y *is a pre-sheaf on* $\mathscr{C}$ *that takes an object* $C \in \mathscr{C}$ *to the set constructed as a (co)limit of a diagram in* $\{Set\}$ *that arises by evaluating a diagram* Y *on* $C \in \mathscr{C}$*.*

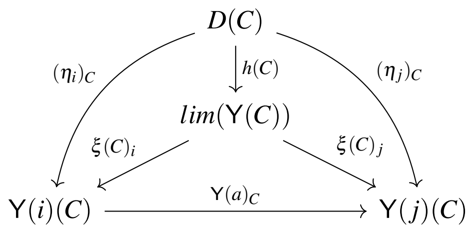
Proof. Lets consider a cone $\eta : \Delta_D \Rightarrow \mathsf{Y}$ with a vertex $D \in PSh(\mathscr{C})$. For each arrow $\{a : i \to j\} \in \mathscr{I}_1$ we then have a commutative trangle of natural transformations of the form:



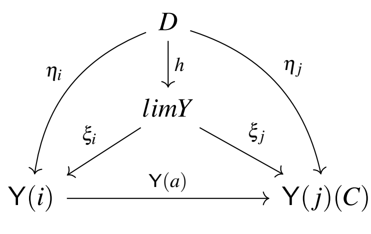
where $\mathsf{Y}(i), \mathsf{Y}(j) \in PSh(\mathscr{C})$. Evaluating this diagram on an object $C \in \mathscr{C}$ gives a diagram in $\{Set\}$:



Let us now take the limiting cone $\xi(C) : \Delta_{lim\mathsf{Y}(C)} \Rightarrow \mathsf{Y}(C)$ in $\{Set\}$, composed from the triangles as above. The universality property then says that we have a unique function $h(C) : lim(\mathsf{Y}(C)) \to D(C)$ such that for each $\{a : i \to j\} \in \mathscr{I}_1$ we have a diagram in $\{Set\}$:



We can then define a presheaf $(\lim \mathsf{Y})(C) := \lim \mathsf{Y}(C)$ and a natural transformation $h : D \Rightarrow (lim\mathsf{Y})(C)$ *point-wisely*, i.e. via $h_C := h(C)$ and similarly $\xi_i : (lim\mathsf{Y})(C) \Rightarrow \mathsf{Y}(i)$ via $(\xi_i)_C := \xi(C)_i$. It is then the unique one for which in the sheaf category we have:



An exactly analogous proof works for *colimits*. ☐

We then see that the existence of limits and colimits in pre-sheaf categories is a direct consequence of their existence in $\{Set\}$. However, if we want to apply this construction to our category $cSh_{Cov}\{Eucl\}$ things get a little more tricky – while limits of concrete sheaves are again concrete sheaves,[1] colimits might not be. However, there are operations of *sheafification* and *concretization*[2] that take the resulting pre-sheaves to the elements of $cSh_{Cov}\{Eucl\}$ that are limits of the relevant diagrams in there. Since we want to keep our presentation elementary, instead of going into details here we will show how this works on some examples – we will construct the product and the terminal object – and refer an interested reader to [1].

Lets consider a functor Y as before but with $cSh_{Cov}\{Eucl\}$ as its codomain:

$$\underline{\mathsf{Y}}_\times : \{\bullet\ \bullet\} \ \mapsto \ \{D_X \ \ D_Y\},$$

which is now a diagram in $\{Diffeo\}$. A cone for this diagram is of the form:

$$\begin{array}{ccc} & D_Z & \\ {\scriptstyle \eta}\swarrow & & \searrow{\scriptstyle \xi} \\ D_X & & D_Y \end{array} ,$$

where D_Z is an arbitrary diffeological sheaf with natural transformations to D_X and D_Y. Following the prescription above, we define the limiting sheaf as follows:

$$\begin{aligned} D_X \times D_Y : \{Eucl\}^{op} &\to \{Set\} \\ U &\mapsto D_X(U) \times D_Y(U), \\ \{f : U' \to U\} &\mapsto \{D_X(f) \times D_Y(f) : D_X(U) \times D_Y(U) \to D_X(U') \times D_Y(U')\} \end{aligned}$$

Let us now check that the universal property holds and that this is the diffeological space we expect:

Lemma 1.20. *The functor $D_X \times D_Y$ is a limit of Y in $cSh_{Cov}\{Eucl\}$ and corresponds to the product $X \times Y$ equipped with the product diffeology, i.e. we have:*

$$\lim \underline{\mathsf{Y}}_\times = D_X \times D_Y \cong D_{X\times Y}.$$

Proof. Notice first that since the underlying set is given by evaluation on the terminal object, we have:

$$D_X \times D_Y\{*\} = D_X\{*\} \times D_Y\{*\} \simeq X \times Y.$$

Moreover, evaluating the limiting cone on $\{*\}$ by definition gives a limiting cone in $\{Set\}$:

$$\begin{array}{ccc} & X \times Y & \\ {\scriptstyle \pi_X}\swarrow & & \searrow{\scriptstyle \pi_Y} \\ X & & Y \end{array} ,$$

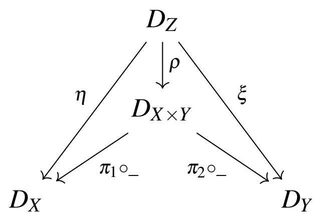
and hence, because of the bijection of the Lemma 1.17, the natural transformations of the limiting cone in $\{Diffeo\}$ are given by:

$$\pi_X \circ \_ : D_X \times D_Y \Rightarrow D_X\,, \ \ \pi_Y \circ \_ : D_X \times D_Y \Rightarrow D_Y\,.$$

[1] The sheaf property is a consequence of the fact that the inclusion $Sh_{Cov}(\mathscr{C}) \hookrightarrow PSh(\mathscr{C})$ is a *left adjoint*, the proof of the concreteness property can be found in [1].

[2] One can see the effect of those operations in the definition of a *coproduct diffeology*, where we need to deal with the *connected components* of the open subsets separately.

The universal property means here that we should always have a unique natural transformation $\rho : D \Rightarrow D_X \times D_Y$ such that:



.

But again, ρ is given by a composition with a function $h : Z \to X \times Y$, and similarly $\eta = f \circ \_$ for $f : Z \to X$ and $\xi = g \circ \_$ for $g : Z \to Y$. Hence, evaluating the above diagram on $\{*\}$ gives that indeed there is always a unique such ρ, namely we need to take $h := f \times g$.

Further, since plots are the same things as smooth parametrizations, taking (Z, D_Z) to be an open subset of a Euclidean space equipped with the manifold diffeology, the universality property, once again evaluated on $\{*\}$, gives us a unique plot $\phi : U \to X \times Y$ for any pair of plots $\phi_X \in D_X(U)$ and $\phi_Y \in D_Y(U)$ such that $\pi_X \circ \phi = \phi_X$ and $\pi_Y \circ \phi = \phi_Y$. Conversely, since composition of smooth maps is smooth, any plot on $X \times Y$ defines such a pair of plots on X and Y and hence *all* of the plots on $X \times Y$ arise in this way.

We then see, that the sheaf thus defined is the same as the one given by the product set equipped with the product diffeology, i.e. $D_X \times D_Y \simeq D_{X\times Y}$. From the Lemma 1.16 it then follows that it is an object in $cSh_{Cov}\{Eucl\}$, which finishes the proof. $\square$

Let us now see how a terminal object of $\{Diffeo\}$, i.e. the one-point set with all the parametrizations smooth, arises in a similar manner. Notice first, that the corresponding diffeological sheaf is of the form:

$$\begin{aligned} D_{\{*\}} : \{Eucl\}^{op} &\to \{Set\} \\ U &\mapsto \big\{ !_U : U \to \{*\} \big\}, \\ \{f : U' \to U\} &\mapsto \{ !_U \mapsto !_U \circ f = !_{U'} \}, \end{aligned}$$

where with "!" we denoted the unique functions taking all of the points in the domain to the only element of $\{*\}$. But since the one-point sets are all isomorphic, it is the same thing as a constant sheaf $D_{\{*\}} = \Delta_{\{*\}}$. Notice now, that when we consider the functor:

$$\underline{\mathsf{Y}}_{\bullet} : \{\bullet\} \mapsto D_X,$$

the point-wisely calculated limit takes every object to the limit in $\{Set\}$, i.e. the one-point set and we have $\Delta_{\{*\}} = lim \underline{\mathsf{Y}}_{\bullet}$.

Similar analysis can be done to show that the diffeological structure on fibered products is also a consequence of a sheaf-theoretic structure of $\{Diffeo\}$ and the fact that those are spaces that arise as limits in $\{Set\}$. As mentioned before, *colimits* arise in a similar manner – we just need to make sure in the end that the resulting pre-sheaves are concrete sheaves. Hence, the origin of the natural diffeological structure on disjoint unions, pushout sets and an empty set can be recognized to be the fact that $\{Diffeo\}$ is a sheaf category at heart *plus* that we have the *sheafification* and *concretization* functors with some nice structure-preserving properties.[1]

[1] See [1].

Cartesian closedness

Except for the inherited diffeological structures on the sets that arise as limits and colimits in $\{Set\}$, we have also seen that the category $\{Diffeo\}$ is *closed*, i.e. it admits an *exponential object* for any pair of diffeological spaces (X, D_X) and (Y, D_Y) given by the diffeological space of smooth maps $C^\infty(Y,X)$ equipped with the functional diffeology. We then would like to see how $C^\infty(Y,X)$ arises parallelly to the exponential object in $\{Set\}$. Let us recall:

In $\{Set\}$, the exponential object of S and T is the set of functions $T \to S$ denoted by $S^T = \{Set\}(T,S)$. It comes naturally equipped with the *evaluation map*:

$$\{f : P \to S^T\} \ \mapsto \ ev(f) : (p,t) \mapsto f(p)(t) \in S,$$

which provides a bijection $\{Set\}(P,S^T) \cong \{Set\}(P \times T, S)$ that we can treat as a defining property of the set S^T – clearly, this is the unique set with such a property, and it always exists. We will now use the coresponing version of this isomorphism to *define* the sheaf $D_Y^{D_X}$ and then show that it is indeed the diffeological space of smooth maps with functional diffeology:

Lemma 1.21. *We have a unique object $D_Y^{D_X} \in cSh_{Cov}\{Eucl\}_0$ for which the following holds:*

$$cSh_{Cov}\{Eucl\}(D_Z, D_Y^{D_X}) \cong cSh_{Cov}\{Eucl\}(D_Z \times D_X, D_Y). \qquad (*)$$

Moreover, the isomorphism above is necessarily given by the set-theoretic evaluation map applied to the functions corresponding to the natural transformations, i.e. we have:

$$cSh_{Cov}\{Eucl\}(D_Z, D_Y^{D_X}) \ni \eta = (\eta_* \circ \_) \ \mapsto \ (ev(\eta_*) \circ \_) \in cSh_{Cov}\{Eucl\}(D_Z \times D_X, D_Y).$$

Further, the diffeological space corresponding to $D_Y^{D_X}$ is isomorphic to the space of smooth functions between X and Y equipped with the functional diffeology, i.e. we have an isomorphism in $cSh_{Cov}\{Eucl\}$:

$$D_Y^{D_X} \cong (C^\infty(X,Y), \mathsf{D}_{C^\infty(X,Y)}).$$

Before we proceed to the proof, let us recall a few facts about the one-point set and show how they generalize to the corresponding facts concerning the terminal object $\Delta_{\{*\}} \in cSh_{Cov}\{Eucl\}$. Notice now, that the point-map bijection that we have seen and used, i.e. the correspondence between the constant functions $\phi_x : \{*\} \mapsto x \in X$ and the elements $x \in X$ that we have seen, gives us the following bijection:

$$\{Set\}(\{*\}, X) \cong X.$$

Similarly, an object in a category can be identified with the set of arrows pointing at it from the terminal object – this is generally true for concrete categories[1], and especially in $cSh_{Cov}\{Eucl\}$. We then have[2] that for any $D_X \in cSh_{Cov}\{Eucl\}_0$:

$$cSh_{Cov}\{Eucl\}(\Delta_{\{*\}}, D_X) \cong D_X.$$

Indeed, any natural transformation $\eta : \Delta_{\{*\}} \Rightarrow D_X$ picks an element of $D_X(U)$ for each $U \in \{Eucl\}_0$ in a way compatible with composition with smooth functions between them, and considering *all* such natural transformations gives the whole D_X, and thus we recover the diffeological structure of (X, D_X). Moreover, in $\{Set\}$ we also have:

$$\{*\} \times X \cong X,$$

[1] In fact morphisms from the terminal object *generalize* the notion of an element of an object from $\{Set\}$ to arbitrary categories (admitting a terminal object).

[2] This is in fact an incarnation of the Yoneda lemma.

which holds more generally as well. In our case, the presheaves D_X and $\Delta_{\{*\}} \times D_X$ agree *point-wisely*, precisely because of the above isomorphism, and hence they define the same diffeological spaces, i.e. for any $D_X \in cSh_{Cov}\{Eucl\}_0$ we also have:

$$\Delta_{\{*\}} \times D_X \cong D_X.$$

Equipped with this very useful observations, we are now ready to prove the Lemma 1.21:

Proof. Using the above observations, setting $D_Z = \Delta_{\{*\}}$ in $(*)$ gives:

$$D_Y^{D_X} \cong cSh_{Cov}\{Eucl\}(\Delta_{\{*\}} \times D_X, D_Y) \cong cSh_{Cov}\{Eucl\}(D_X, D_Y) \simeq C^\infty(X,Y),$$

which was one of our claims. Notice however, that by now the last equivalence, denoted by "$\simeq$" is just the set-theoretic one – we also want to show that the diffeology on $D_Y^{D_X}$ and $C^\infty(X,Y)$ agree. But let us first take a closer look at the isomorphism $(*)$. Notice, that since the above holds, we have that the underlying set of $D_Y^{D_X}$ is contained in the set (isomorphic to the set) of functions Y^X. Now, considering a general D_Z again, in the light of the Lemma 1.17, a natural transformation $\eta : D_Z \Rightarrow D_Y^{D_X}$ has to be of the form $\eta = \eta_* \circ \_$ where $\eta_* : Z \to Y^X$. Similarly, since we have $D_Z \times D_X \cong D_{Z \times X}$, a natural transformation $\xi : D_Z \times D_X \Rightarrow D_Y$ is of the form $\xi = \xi_* \circ \_$ for $\xi_* : Z \times X \to Y$. If we now take a pair of constant plots:

$$\phi_z : \{*\} \mapsto z \in Z, \;\; \phi_{z,y} : \{*\} \mapsto (z,y) \in Z \times Y,$$

the isomorphism $(*)$ relates $\eta(\phi_z) = \phi_z \circ \eta_*$, which corresponds to $\eta_*(z) \in Y^X$ via the point-map bijection, to $\xi(\phi_{z,x}) = \phi_{z,x} \circ \xi_*$, which in turn corresponds to $\xi_*(z,x) \in Y$, and hence we can identify $\xi_*(z,x) = \eta_*(z)(x)$. The $(*)$ is then indeed given through the set-theoretic evaluation map on the underlying functions.

Using this and the Lemma 1.17 it is not difficult to see now that the diffeology defined by $D_Y^{D_X}$ is actually the functional diffeology. Indeed, take D_Z to be a subset U of the Euclidean space with its manifold diffeology and notice that since plots are the smooth parametrizations, we have:

- by the Lemma 1.17, the map $\phi : U \to D_Y^{D_X}$ is a plot iff the transformation $\tilde{\phi} = \phi \circ \_$ is natural,
- the transformation $\tilde{\phi} = \phi \circ \_$ is natural iff $= ev(\phi) \circ \_$ is and
- again by the Lemma 1.17 the transformation $ev(\phi) \circ \_$ is natural iff $ev(\phi)$ is smooth.

Hence, recognizing $ev(\phi) = ev_\phi$, we recover the Definition 1.12 of the functional diffeology:

$$\{\phi : U \to C^\infty(X,Y)\} \in D_Y^{D_X}(U) \;\; \Leftrightarrow \;\; ev_\phi \in C^\infty(U \times X, Y).$$

We have thus established that the diffeological space corresponding to the diffeological sheaf $D_Y^{D_X}$, defined by requiring $(*)$, is indeed isomorphic in $\{Diffeo\}$ to $C^\infty(X,Y)$ with functional diffeology. □

We then see, that the existence of the exponential objects in $\{Diffeo\}$ is also strongly tightened to the corresponding property of the category $\{Set\}$ and the fact that it is a category of sheaves.

Subsets and quotients

We would also like to see the diffeological spaces that arise as subsets and quotients in the context of the the category $cSh_{Cov}\{Eucl\}$. In order to do that, we need to generalize the concepts of an *inclusion* of sets and a *quotient projection* - i.e. one that comes while taking a quotient with respect to an equivalence relation on a set – to the more general context of arbitrary categories. Since the mentioned functions are special kinds of surjections and injections, we will first generalize those. Let us then make here a simple observation: if two functions composed with an *injective* function agree, they are the same, and similarly if they agree when pre-composed with a *surjective* function. Conversely, if a function has one of those properties, we can conclude that it is injective or surjective, respectively. Injections and surjections can be then generalized to arbitrary categories as follows:

Definition 1.39. *An arrow in a category $\mathscr{C}$ is called a monomorphism, denoted by $m : M \hookrightarrow C$, iff for any pair of arrows $f, g : C' \to C$ we have:*

$$m \circ f = m \circ g \quad \Rightarrow f = g.$$

Definition 1.40. *An arrow in a category $\mathscr{C}$ is called an epimorphism, denoted by $e : E \twoheadrightarrow C$, iff for any pair of arrows $f, g : C \to \mathscr{C}'$ we have:*

$$f \circ e = g \circ e \quad \Rightarrow f = g$$

Remark. The notions just defined are *dual* to each other in the sense that monomorphisms are exactly epimorphisms in the opposite category, and vice versa.

Notice here, that since the natural transformations in sheaf categories are functions in $\{Set\}$ at each component and their composition is a composition of those, an arrow in $Sh(\mathscr{C})$ is a monomorphism or an epimorphism precisely if *all* those functions are injective or surjective, respectively. We then have the following nice Lemmas for our sheaf category $cSh_{Cov}\{Eucl\}$:

Lemma 1.22. *A natural transformation $\eta : D_X \Rightarrow D_Y$ is an* epimorphism *iff the corresponding function η_* is* surjective.

Proof. By definition, such η is an epimorphism iff for *any* pair of arrows $\xi, \rho : D_Y \Rightarrow D_Z$ from $\xi \circ \eta = \rho \circ \eta$ it follows that $\xi = \rho$. But notice, that $\xi \circ \eta = \rho \circ \eta$ reads:

$$\xi_* \circ \eta_* \circ \phi = \rho_* \circ \eta_* \circ \phi \;\; \forall_{\phi \in D_Y},$$

which, because of the covering property, holds iff $\xi_* \circ \eta_* = \rho_* \circ \eta_*$. Similarly, $\xi = \rho$ holds iff $\xi_* = \rho_*$, and we can conclude that the natural transformation η is an epimorphism in $cSh_{Cov}\{Eucl\}$ iff the function η_* is an epimorphism in $\{Set\}$, i.e. a surjection. □

By a very similar analysis we also have:

Lemma 1.23. *A natural transformation $\eta : D_X \Rightarrow D_Y$ is a* monomorphism *iff the corresponding function η_* is* injective.

While generalizing inclusions and quotient projections we will take a slightly different approach than we took so far. Instead of finding the properties that distinguish them from arbitrary functions and can be spelled out solely in terms of the arrows in $\{Set\}$, i.e. functions, in order to use use them to define the generalized notions, we will state some categorical definitions and then prove that they collapse to the ones we are interested in when considering $\{Set\}$. Focusing on the inclusions for a while, let us define:

Definition 1.41. *An arrow in a category $\mathscr{C}$ is called a* <u>*strong monomorphism*</u> *iff it is a monomorphism $i : M \hookrightarrow C$ such that for any diagram of the form:*

$$\begin{array}{ccc} E & \xrightarrow{f} & M \\ {\scriptstyle e}\downarrow\!\!\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{g} & C \end{array},$$

where e is an epimorphism and f, g are arbitrary, there is a unique *diagonal arrow $h : B \to M$ making the whole thing commute.*

Let us now argue that the notion of a *strong monomorphism* is a legitimate candidate for a generalization of the set-theoretic inclusions:

Lemma 1.24. *An function between sets is an inclusion iff it is a strong monomorphism in $\{Set\}$.*

Proof. Notice first, that since i is injective, it is a bijection onto its image and we have $i(M) \cong M$. Further, since e is surjective and the square commutes, we also have $g(D) \cong i(M)$. The function g then gives a unique function into M simply by $\underline{g} : B \ni d \mapsto g(d) \in M$. The square from the definition then collapses to:

$$\begin{array}{ccc} E & & \\ {\scriptstyle e}\downarrow\!\!\downarrow & \searrow^{f} & \\ B & \xrightarrow{\underline{g}} & M \end{array},$$

and we have no other choice than to take $h := \underline{g}$, which is always there, makes everything commute and is uniquely determined. Clearly, such $\underline{g}$ is at hand for an arbitrary g and fixed i only if the injection i is in fact an inclusion. □

The following Lemma describes the relation between the *internal inclusions*, i.e. strong monomorphisms in $cSh_{Cov}\{Eucl\}$, and the notion of the subspace diffeology:

Lemma 1.25. *A natural transformation $\eta : D_A \Rightarrow D_X$ in $cSh_{Cov}\{Eucl\}$ is a strong monomorphism iff $\eta_* : A \hookrightarrow X$ is an inclusion of sets and D_A is equipped with the subspace diffeology:*

$$D_A \cong (A, \mathsf{D}_{A \subseteq X}).$$

Proof. By definition, a natural transformation η is a strong monomorphism in $cSh_{Cov}\{Eucl\}$ iff it is a monomorphism and for any diagram of the form:

$$\begin{array}{ccc} D_Y & \xrightarrow{\lambda} & D_A \\ {\scriptstyle \xi}\downarrow\!\!\downarrow & & \downarrow{\scriptstyle \eta} \\ D_Z & \xrightarrow{\rho} & D_X \end{array},$$

where λ and ρ are arbitrary and ξ is an epimorphism there is a unique $\mu : D_Z \Rightarrow D_A$ making the whole diagram commute. Our, favourite by now, Lemma 1.17 tells us that we actually have the four smooth functions that define the natural transformations above via pre-composition:

$$\begin{array}{ccc} D_Y & \xrightarrow{\lambda_* \circ \_} & D_A \\ {\scriptstyle \xi_* \circ \_}\downarrow\!\!\downarrow & & \downarrow{\scriptstyle \eta_* \circ \_} \\ D_Z & \xrightarrow{\rho_* \circ \_} & D_X \end{array}.$$

Further, from the Lemmas 1.23 and 1.22 we know that η_* is injective and $\xi_* : Y \to Z$ is surjective. We then have the following diagram in $\{Set\}$:

$$\begin{array}{ccc} Y & \xrightarrow{\lambda_*} & A \\ \downarrow\xi_* & & \downarrow\eta_* \\ Z & \xrightarrow{\rho_*} & X \end{array},$$

and the existence and uniqueness of μ as above is equivalent to the existence of a unique $\mu_* : Z \to A$ making the last diagram commute, i.e. by the Lemma 1.24 it is there iff the function η_* is actually an inclusion. Moreover, requiring μ and η to be natural fixes the diffeology on A. To see this, take $Z = U$ to be an open subset of a Euclidean spaces with the manifold diffeology, $\xi_* = Id_U$ and $\rho_* = \{\psi : U \to X\} \in D_X(U)$ with an image contained in A so that the square commutes. It then collapses to:

$$\begin{array}{ccc} & & A \\ & \nearrow^{h} & \downarrow i \\ U & \xrightarrow{\phi} & X \end{array},$$

where $h = \lambda_* = \mu_* : U \to A$ is unique and since we need $\tilde{h}$ to be natural, it is actually a plot on A. We then have that every plot on A comes from a composition with the inclusion map, i.e. $\mathsf{D}_A \subseteq \mathsf{D}_{A\subseteq X}$, while the other inclusion holds iff η is natural. Indeed, notice that η is natural iff for any plot on $\phi : U \to A$, the parametrization $i \circ \phi$ is a plot on X and hence $\mathsf{D}_{A\subseteq X} \subseteq \mathsf{D}_A$. We can then conclude that in order for η to be a strong monomorphism, the diffeology on A needs to be exactly the subspace diffeology inherited from X. ☐

The fact that any subset of a diffeological space inherits a unique natural diffeological structure can be phrased as follows:

Definition 1.42. *We say that a* concrete *category $\mathscr{C}$ is* closed for subsets *iff for any inclusion function* $i : \{Set\}_0 \ni A \hookrightarrow C \in \mathscr{C}_0$ *there is a* unique *object in $\mathscr{C}$ with the underlying set A for which i becomes a* strong monomorphism.

Lemma 1.26. *The category $cSh_{Cov}\{Eucl\}$ is closed for subsets.*

Proof. From the Lemma 1.25 we know that such an object is always uniquely given by the diffeological space equipped with the subspace diffeology. ☐

Let us now finally take a look at the quotient diffeological structures. As we will see, the notions of an inclusion and a quotient projection are closely related which will allow us to take a very similar approach. We thus define the *strong epimorphisms*:

Definition 1.43. *An arrow in a category $\mathscr{C}$ is called a* strong epimorphism *iff it is an epimorphism $e : E \hookrightarrow D$ such that for any diagram of the form:*

$$\begin{array}{ccc} E & \xrightarrow{f} & M \\ \downarrow e & & \downarrow i \\ B & \xrightarrow{g} & C \end{array},$$

where m is a monomorphism and f, g are arbitrary, there is a unique *diagonal arrow $h : B \to M$ making the whole thing commute.*

Remark. A *strong epimorphism* is one for which *any* monomorphism is strong – and vice versa.

Let us first argue, that the notion of a *strong epimorphism* is a legitimate candidate for a generalization of the quotient projections:

Lemma 1.27. *A function between sets is a quotient projection iff it is a strong epimorphism in* $\{Set\}$.

Proof. Consider an equivalence relation on a set E and take e to be the corresponding projection so that $e(x) = e(x')$ iff $x \sim x'$ for $x, x' \in E$ and $B \cong E/\sim$. Since i is injective, the commutativity of the square:

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & M \\ \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle i} \\ E/\sim & \xrightarrow{\ g\ } & C \end{array}$$

means that f is constant on the equivalence classes and hence defines a function from the quotient $\underline{f} : E/\sim \to M$, which makes the upper triangle commute. Further, since the square commutes, we also have $i \circ \underline{f} = g$ and we need to take $h := \underline{f}$ which always exists and is uniquely determined.

Conversely, such h is always at hand only if the surjection e is in fact a quotient projection. Indeed, consider $i = Id_M$. The square then collapses to:

$$\begin{array}{ccc} E & & \\ \Big\downarrow{\scriptstyle e} & \searrow^{f} & \\ B & \xrightarrow{\ g\ } & M \end{array} \quad ,$$

and hence $h = g$. But notice, that the commutativity of this triangle means that for any $x \in E$ we have $g(e(x)) = f(x)$, i.e. f is constant on the equivalence classes defined by $x \sim x'$ iff $e(x) = e(x')$ and since f is arbitrary and e is surjective, the domain of g needs to be the quotient $E/\sim$ and e becomes the quotient projection. □

The following Lemma describes the relation between the *internal quotient projections*, i.e. strong epimorphisms in $cSh_{Cov}\{Eucl\}$, and the notion of the quotient diffeology:

Lemma 1.28. *A natural transformation* $\eta : D_Y \Rightarrow D_Z$ *in* $cSh_{Cov}\{Eucl\}$ *is a strong epimorphism iff* $\eta_* : Y \twoheadrightarrow Z \cong Y/\sim$ *is a quotient projection and* $Y/\sim$ *is the equipped with the quotient diffeology:*

$$D_Z \cong (Y/\sim, \mathsf{D}_{\eta_*(Y)}).$$

Proof. Exactly like in the proof of the Lemma 1.25, using the notation from there, the square which is used to define the *strong* is actually given by a diagram in $\{Set\}$:

$$\begin{array}{ccc} Y & \xrightarrow{\ \lambda_*\ } & A \\ \Big\downarrow{\scriptstyle \xi_*} & & \Big\downarrow{\scriptstyle \eta_*} \\ Z & \xrightarrow{\ \rho_*\ } & X \end{array} \quad .$$

with all the functions smooth, η_* injective and ξ_* surjective. The natural transformation $\mu : D_Z \to D_Z$ is once again given through the unique $\mu_* : Z \to A$ making the above diagram commute, which by the Lemma 1.27 is there iff the function ξ_* is actually a quotient projection and we get[1] $Z \cong Y/\sim$. We will now see that the smoothness of ξ_* and μ_* fixes the quotient diffeology on Z. Notice first, that the smoothness of ξ_* means that for any $\phi \in \mathsf{D}_Y$ we have $\xi_* \circ \phi \in \mathsf{D}_Z$, i.e. $\mathsf{D}_{\xi_*(Y)} \subseteq \mathsf{D}_Z$. For the other

[1] We will stick to Z for now to keep the notation cleaner.

inclusion, we need to show that each $\phi \in \mathsf{D}_Z$ factors through a plot on Y. To see this, take $A = Y$ and $\eta_* = Id_Y$ i.e. consider the square:

$$\begin{array}{ccc} Y & \xrightarrow{\lambda_*} & Y \\ \big\downarrow \xi_* & & \big\updownarrow = \\ Z \cong Y/\sim & \xrightarrow{\rho_*} & Y \end{array} .$$

Notice first, that Id_Y is of course injective, $\mu_* = \rho_*$ picks an element of Y for each class in $Y/\sim$ and λ_* projects all the elements of a class to the chosen one, for all classes. We then have $\xi_* \circ \mu_* = Id_Z$. Further, the smoothness of μ_* gives that $\psi := \mu_* \circ \phi \in \mathsf{D}_Y$ for any plot $\phi \in \mathsf{D}_Z$. We then have:

$$\xi_* \circ \psi = \xi_* \circ \mu_* \circ \phi = Id_Z \circ \phi = \phi,$$

which finishes the proof. □

The fact that any quotient of a diffeological space inherits a unique natural diffeological structure can be phrased as follows:

Definition 1.44. *We say that a* concrete *category $\mathscr{C}$ is* <u>*closed for quotients*</u> *iff for any quotient projection $e : \{Set\}_0 \ni Z \twoheadrightarrow C \in \mathscr{C}_0$ there is a* unique *object in $\mathscr{C}$ with the underlying set Z for which e becomes a* strong epimorphism.

Lemma 1.29. *The category $cSh_{Cov}\{Eucl\}$ is closed for quotients.*

Proof. From the Lemma 1.28 we know that such an object is always uniquely given by the diffeological space equipped with the quotient diffeology. □

We have thus seen how all of the natural constructions that we presented in the first section can be re-understood in the light of the sheaf-theoretic perspective on the category of diffeological spaces and smooth maps between them. This is the only detailed and accessible presentation of this facts that we are aware of.

1.3 Tangent spaces

Let us begin with a quotation from [10]:

> "There are indeed many ways to think about tangent spaces, which are equivalent for manifolds but not when applied to diffeological spaces [...] different kinds of questions may need different versions of tangent bundles."

We are then going to introduce two approaches to tangent spaces – *internal* and *external*. As we will see, the ambiguity of diffeological tangent spaces doeas not influence our analysis, and hence it will not be as exhaustive as in the previous section – we will just introduce the notions, sketch the proofs and infer what is relevant. We refer an interested reader to [5].

1.3.1 Internal

This paragraph deals with the concept of an *internal tangent space*, which is defined solely by the plots centered at a given point of a diffeological spaces. Let us first introduce yet another perspective on a diffeological space (X, D_X): just like manifolds can be seen as open subsets of a Euclidean space glued together, a diffeological space can also be seen as being glued from the domains of its plots in the following sense:

Definition 1.45. *For a given diffeological space (X, D_X), a category of plots on X, denoted $\{Eucl\}/X$, has plots as objects and triangles of the form:*

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X & \end{array} ,$$

where $\phi, \psi \in \mathsf{D}_X$ and $f \in C^\infty(U,V)$ as arrows.

Remark. This is an example of a *slice category*, i.e. one given by arrows into a fixed element of a bigger category, here the category of diffeological spaces $\{Diffeo\}$.

Lemma 1.30. *A diffeological space (X, D_X) can be recovered from its category of plots $\{Eucl\}/X$ as a colimit of the forgetful functor:*

$$U_X : \{Eucl\}/X_1 \ni \begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X & \end{array} \quad \mapsto \quad \{f : U \to V\} \in \{Eucl\}_1 \subset \{Diffeo\}_1 .$$

Proof. (sketch) A natural transformation $\eta : U_X \Rightarrow \Delta_X$ is given by plots that were originally the elements of $\{Eucl\}/X$, i.e. for $\phi : U \to X$ we have $U_X(\phi) = U$ and hence we can put:

$$\eta_U := \phi : U \to X.$$

For any other cocone $\xi : U_X \to Y$, we have:

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ {\scriptstyle\xi_U}\downarrow & X & \downarrow{\scriptstyle\xi_V} \\ & Y & \end{array} ,$$

and since the diagram is in $\{Diffeo\}$, all of the components of ξ have to be plots, i.e. $\xi_U \in D_Y(U)$, and hence the map $\phi \mapsto \xi_U$ can be interpreted as a natural transformation $h : D_X(U) \Rightarrow D_Y(U)$, which then gives a unique smooth map $\{h_* : X \to Y\} \in C^\infty(X,Y)$ making everything commute. □

To define an internal tangent space T_xX we need to restrict the category $\{Eucl\}/X$ to the plots *centered at* $x \in X$:

Definition 1.46. *The category of pointed diffeological spaces, denoted* $\{Diffeo\}_*$*, is a category of diffeological spaces together with distinguished points* (X,x) *as objects, and smooth functions respecting them as arrows:*

$$\{f : (X,x) \to (Y,y)\} \in \{Diffeo\}_*((X,x),(Y,y)) \;\; \Leftrightarrow \;\; f \in C^\infty(X,Y) \;\; \& \;\; f(x) = y.$$

Definition 1.47. *We denote by* $\{Eucl\}_0$ *a subcategory of* $\{Eucl\}$ *of connected Euclidean spaces pointed at zero.*

Definition 1.48. *For a given pointed diffeological space* (X,x)*, a category of plots centered at* $x \in X$*, denoted by* $\{Eucl\}_0/(X,x)$*, has plots defined on objects of* $\{Eucl\}_0$ *such that* $\phi(0) = x \in X$*, as objects and triangles of the form:*

$$\begin{array}{ccc} (U,0) & \xrightarrow{\quad f \quad} & (V,0) \\ & {}_{\phi}\searrow \quad \swarrow_{\psi} & \\ & (X,x) & \end{array} ,$$

where $\phi, \psi \in \mathsf{D}_X$*,* $f \in C^\infty(U,V)$*, and* $f(0) = 0$*, as arrows.*

Remark. The category of plots centered at $x \in X$ is a subcategory of the comma category in $\{Diffeo\}_*$.

Definition 1.49. *We denote the forgetful functor* $\{Eucl\}_0/(X,x) \to \{Eucl\}_0$ *by* $U_{(X,x)}$.

We can now define the *internal tangent space* as follows [5]:

Definition 1.50. *An internal tangent space* T_xX *to a diffeological space* (X, D_X) *at a point* $x \in X$ *is the vector space given as a colimit of the functor:*

$$\{Eucl\}_0/(X,x) \xrightarrow{U_{(X,x)}} \{Eucl\}_0 \xrightarrow{T_0} \{Vect\} \;\; ,$$

where T_0 *takes elements of* $\{Eucl\}_0$ *to their tangent spaces at zero and smooth maps to derivatives:*

$$T_0 : \{f : (U,0) \to (V,0)\} \;\; \mapsto \;\; \{T_0 f : T_0U \to T_0V\}.$$

Remark. We see that the internal tangent space is a vector space by definition – it is an object in $\{Vect\}$. The operation of taking a tangent spaces can be extended to a functor $T : \{Diffeo\}_* \to \{Vect\}$ as a left Kan extension of T_0 along the inclusion of $\{Eucl\}_0$ into pointed diffeological spaces $\{Diffeo\}_*$:

$$\begin{array}{ccc} \{Eucl\}_0 & \xrightarrow{T_0} & \{Vect\} \\ \big\downarrow{\scriptstyle i} & \nearrow_{T} & \\ \{Diffeo\}_* & & \end{array} ,$$

but we will not go into detail here.

1.3.2 External

In the manifold framework, tangent vectors can be defined as derivations on the algebra of germs of smooth functions. This perspective can be naturally generalised to the diffeological setting.

Definition 1.51. *We will denote by $\mathscr{O}_{\mathsf{D}}(X)$ the category of open subsets and inclusions on a topological space given by equipping a diffeological space (X, D_X) with the* D*-topology. The full subcategory of $\mathscr{O}_{\mathsf{D}}(X)$ on those opens that contain $x \in X$ will be denoted by $\mathscr{O}_{\mathsf{D}}(X,x)$.*

We now have a natural sheaf of diffeologically smooth functions on $\mathscr{O}_{\mathsf{D}}(X)$ assigning to D-opens containing $x \in X$ the spaces of smooth real functions on them, equipped with functional diffeologies:

$$\begin{aligned} G_X : \mathscr{O}_{\mathsf{D}}(X) &\to \{Diffeo\} \\ B' \subseteq B \quad &\mapsto]_{B,B'} : C^\infty(B,\mathbb{R}) \to C^\infty(B',\mathbb{R}), \end{aligned}$$

and its *pointed* version:

$$\begin{aligned} G_{(X,x)} : \mathscr{O}_{\mathsf{D}}(X,x) &\to \{Diffeo\} \\ x \in B' \subseteq B \quad &\mapsto]_{B,B'} : C^\infty(B,\mathbb{R}) \to C^\infty(B',\mathbb{R}), \end{aligned}$$

where each element $f \in G_{(X,x)}(B)$ is a smooth map, for a manifold diffeology on $\mathbb{R}$, defined on a D-open neighbourhood of $x \in X$ and has a well-defined value $f(x) \in \mathbb{R}$. We now define the *space of germs at $x \in X$*:

Definition 1.52. *For a pointed diffeological space (X,x) the space of germs at $x \in X$, denoted G_xX, is defined to be the colimit of $G_{(X,x)}$.*

Remark. The space G_xX can be understood as $C^\infty(B_0,\mathbb{R})$ for the smallest possible $B_0 \in \mathscr{O}_{\mathsf{D}}(X)$ containing $x \in X$. Indeed, the colimiting element is the one for which we have an inclusion of B_0 to any other element of $\mathscr{O}_{\mathsf{D}}(X,x)$.

Notice here that an element $[f] \in G_xX$ still has a well-defined value $[f(x)]$, which makes G_xX an $\mathbb{R}$-algebra under point-wise defined operations of addition, multiplication and scalar multiplication. Moreover, all those operations are smooth for the diffeology on G_xX coming from the functional diffeology on the spaces $C^\infty(B,\mathbb{R})$.

Just like for manifolds, the tangent vectors, which we now call *external* since they can be defined through not only the diffeological structure of X but also via the manifold and ring structures of real numbers, are *derivations* on the algebra of germs:

Definition 1.53. *A derivation on the diffeological $\mathbb{R}$-algebra G_xX, i.e. a smooth $\mathbb{R}$-linear map $D : G_xX \to \mathbb{R}$ satisfying the Leibniz's rule:*

$$D([f][g]) = D([f])g(x) + f(x)D([g]),$$

will be called an external tangent vector at $x \in X$. The space of all external tangent vectors is the external tangent space and is denoted by $\hat{T}_xX$.

Remark. The external tangent space $\hat{T}_xX$ is a real vector space under point-wise addition and scalar multiplication; the Leibniz's rule implies that $D([c]) = 0$ for any $D \in \hat{T}_xX$ and a germ constant function $c : B \to \mathbb{R}$. The operation of taking an external tangent space can be easily extended to a functor:

$$\begin{aligned} \hat{T} : \{Diffeo\}_* &\to \{Vect\} \\ \{f : (X,x) \to (Y,y)\} &\mapsto \{\hat{T}_xX \ni D \mapsto f_*(D) \in \hat{T}_yY\}, \end{aligned}$$

where[1] $f_*(D)([g]) := D([g \circ f])$ for a germ $[g] \in G_yY$.

[1] Even though g might not be defined on the whole image of f, since the latter is smooth and hence D-continuous and takes x to y, the germ $[g \circ f]$ at $x \in X$ is well-defined.

1.3.3 Comparison

The computations of tangent spaces for different, more or less pathological from the manifold point of view, examples done in [5] are summarised there in the following table:

Diffeological space	**Base point**	**Internal**	**External**
discrete diffeological space	any point	$\mathbb{R}^0$	$\mathbb{R}^0$
indiscrete diffeological space	any point	$\mathbb{R}^0$	$\mathbb{R}^0$
topological space with continuous diffeology	any point	$\mathbb{R}^0$	$\mathbb{R}^0$
smooth manifold of dimension n	any point	$\mathbb{R}^n$	$\mathbb{R}^n$
axes in $\mathbb{R}^2$ with the pushout diffeology	0	$\mathbb{R}^2$	$\mathbb{R}^2$
axes in $\mathbb{R}^2$ with the sub-diffeology	0	$\mathbb{R}^2$	$\mathbb{R}^2$
three lines intersecting at 0 in $\mathbb{R}^2$ with the sub-diffeology	0	$\mathbb{R}^3$	$\mathbb{R}^3$
$\mathbb{R}^n$ with wire diffeology $(n \geq 2)$	any point	uncountable dimension	$\mathbb{R}^n$
1-dimensional irrational torus	any point	$\mathbb{R}$	$\mathbb{R}^0$
quotient space $\mathbb{R}^n/O(n)$	[0]	$\mathbb{R}^0$	$\mathbb{R}$
$[0, \infty)$ with the sub-diffeology of $\mathbb{R}$	0	$\mathbb{R}^0$	$\mathbb{R}$
vector space V with fine diffeology	any point	V	
diffeomorphism group of a compact smooth manifold M	1_M	C^∞ vector fields on M	

.

The two concepts disagree in some very singular cases, but they *often agree*, including of course for *manifolds* and, what is important for our analysis, also for *diffeological groups*. This frees our analysis of the ambiguity of the definition of the diffeological tangent space. The best intuition to use for our purpose is that the tangent vectors are *pushed forward* to the diffeological space by the one-dimensional plots.

2 Groupoids and algebroids

Another beautiful concept that we will need for our analysis of the symmetry structure of the initial value formulation of General Relativity is that of a *groupoid*. This is a more general notion than that of a group structure: while groups can be understood as representing symmetry transformations of an object, the framework of groupoids is reach enough to include the invertible transformations *between* possibly different objects.

Just like groups, groupoids can also be equipped with manifold structure, giving rise to *Lie groupoids*. Moreover, parallel to the construction of a Lie algebra from a Lie group, from a given Lie groupoid we can construct a *Lie algebroid* which is a very general and natural object generalizing both Lie algebras and tangent bundles.

Since the symmetry structure that we will be studying in detail in the coming chapter is not modelled by a *Lie* groupoid but by a *diffeological* groupoid, and what we will be actually interested in is the associated algebroid structure, we will attempt to generalize some of the constructions from the theory of Lie groupoids and their associated algebroids to the diffeological setting. To the best of our knowledge, diffeological algebroids have not been treated in detail in the literature before – we do *not* aspire to fill this gap exhaustively, but merely point at some strategies of how the generalization might be approached. We aim to establish a ground firm enough to fully support the analysis of the next chapter.

We start by defining groupoids as purely algebraic structures and give some examples. Next, we put some smoothness on their underlying sets and maps and hence arrive at a notion of a Lie groupoid. Further, we generalize the notion of a Lie algebra to that of a Lie algebroid to finally discuss the generalization of this construction to the diffeological setting.

We refer to [13] and [12] for the general theory of Lie groupoids and algebroids and to [10] for the theory of diffeological groupoids.

2.1 Groupoids

The notion of a groupoid is such a general and natural one that the neatest way to define it is to distinguish groupoids among general categories:[1]

Definition 2.1. *A groupoid $\mathfrak{G}$ is a category where all arrows are isomorphisms.*

Remark. As noted in the previous section (see "Categorical preliminaries"), groups can be characterised as one-object categories with all the arrows invertible. Hence, the above is a straightforward generalization where we simply drop the one-object requirement. A groupoid can be also understood as a group for which an arbitrary pair of elements might not be composable, and when it is, the operation is associative.

The definition above can be *unpacked* to an equivalent one:

Definition 2.2. *A groupoid $\{G \rightrightarrows Q\}$ consists of two sets, G and Q, which will be called the arrow space and the object space/base, respectively, together with:*

- *two maps $s,t : G \twoheadrightarrow Q$, called the source projection and the target projection,*
- *the inclusion map $\varepsilon : Q \hookrightarrow G$, taking $Q \ni q \mapsto \varepsilon(q) := Id_q \in G$,*
- *the partial multiplication $\mu : G * G \to G$, taking $G * G \ni (g,h) \mapsto \mu(g,h) =: gh$ for:*

  $$G * G := \{(g,h) \in G \times G : s(h) = t(g)\}$$

  is the space of composable arrows,
- *the inversion map $I : G \to G$, taking $g \mapsto g^{-1} =: I(g)$,*

which are subject to the following conditions:

1. *$s(gh) = s(h)$ and $t(gh) = t(g)$ for all $(g,h) \in G * G$,*
2. *$g(hl) = (gh)l$ for all $g,h,l \in G$ such that $s(g) = t(h)$ and $s(h) = t(l)$,*
3. *$s \circ \varepsilon = Id_Q = t \circ \varepsilon$, i.e. $s(Id_q) = q = t(Id_q)$ for all $q \in Q$*
4. *$gId_{s(g)} = g = Id_{t(g)}g$ for all $g \in G$*
5. *$I(g) = g^{-1} \in G$ is a two-sided inverse of $g \in G$: $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$ so that both (g,g^{-1}) and (g^{-1},g) are in $G * G$ and we have $g^{-1}g = Id_{s(g)}$ and $gg^{-1} = Id_{t(g)}$.*

Lemma 2.1. *The two definitions given above coincide.*

Proof. We simply need to take $G := \mathfrak{G}_1$ and $Q := \mathfrak{G}_0$. Indeed, the projections s and t are assigning to each arrow its starting point, i.e. a *source* and the end-point, i.e. *target*, respectively. Moreover, the inclusion map ε is well defined since we always have a unique identity arrow for any object of a category. Further, the set of *composable arrows* simply consists of the arrows that meet and hence can be composed and finally the inversion map exists iff the arrows in $\mathfrak{G}$ are invertible. The requirements $(1-5)$ are simple properties of categorical composition of arrows. □

Remark. If the base has only one element, all the arrows are composable, i.e. $G * G = G \times G$ and we indeed get an ordinary group: $\{G \rightrightarrows \{*\}\} \simeq G$.

For any element of the base we have a natural group associated to it:

[1] We again do not worry about the "big" structures, see the Definition (1.18) of a category that we use.

Definition 2.3. *The isotropy group, or automorphism group of $q \in Q$ is defined by*

$$G_q := s^{-1}(q) \cap t^{-1}(q),$$

i.e. G_q consists of the arrows that start and end at q and hence can all be composed. This is indeed a group with multiplication and inverse map inherited from $\{G \rightrightarrows Q\}$ and Id_q acting as a unit.

Remark. An isotropy group of a group seen as a groupoid is the group itself: $\{G \rightrightarrows \{*\}\} \cong G_{\{*\}} \cong G$.

Before putting more structure on groupoids, we will give a few examples to illustrate this notion:

Definition 2.4. *The simplest groupoid that is not a group is pair groupoid over the base Q, denoted $\mathfrak{G}_{Q\times Q} := \{Q \times Q \rightrightarrows Q\}$. The arrows are simply ordered pairs $G = Q \times Q \ni (q, q')$, which represent the arrows connecting them that can be composed iff they meet, i.e.:*

$$G * G := \{((q,q'),(q'',q''')) \mid q' = q''\},$$

the result of the composition being, of course, the arrow (q, q'''), i.e.:

$$\mu : G * G \ni ((q,q'),(q',q'')) \mapsto (q,q')(q',q'') = (q,q'').$$

The inclusion is simply given by $\varepsilon(q) := (q,q)$, the inversion map switches the order: $I(q,q') := (q',q)$, while the source and target maps are the product projections: $s(q,q') = q'$ and $t(q,q') = q$.

Definition 2.5. *Another example can be formed when we take the base $Q = Sub(V)$ to be the set of sub-spaces $H \subseteq V$ of a fixed vector space V with the invertible linear maps between them as arrows, $\mathfrak{G}(H,H') = Inv(H,H')$ and composition as partial multiplication. The arrows are composable when the codomain of the first map coincides with the domain of the second, there are identities Id_H corresponding to each subspace, and each morphism is invertible. Isotrophy groups of H is its group of endomorphisms $End(H)$.*

Definition 2.6. *The archetypal example of a groupoid is a fundamental groupoid or Poincaré groupoid of a topological space $(B, \mathscr{T})$, denoted $\Pi_1(B)$, with objects being points of the space B and morphisms the endpoint-preserving homotopy classes of paths between them.*

We are now ready to equip our groupoids with smooth structures and generalize relevant the notions from Lie theory.

2.2 Lie groupoids and Lie algebroids

If the arrow and object spaces of a groupoid are equipped with manifold structures compatible with given maps, we arrive at the notion of a Lie groupoid, generalizing that of a Lie group:

Definition 2.7. *A Lie groupoid is a groupoid such that:*

1. *the sets G and Q are manifolds,*
2. *the projections s and t are surjective submersions,*
3. *the partial multiplication $(g,h) \mapsto gh$ and the inclusion map ε are smooth.*

From this definition it follows [12] that in a Lie groupoid:

- the inclusion map ε is an *immersion*,
- the inversion map I is a *diffeomorphism*,
- the set $G * G$ is a *closed submanifold* of $G \times G$,
- for each $q \in Q$ the fibers $s^{-1}(q)$ and $t^{-1}(q)$ are *submanifolds* of G,
- for each $q \in Q$ the isotrophy group G_q is a *Lie group*.

Lemma 2.2. *A Lie group is a Lie groupoid over a one-point base.*

Proof. Indeed, for a Lie group the corresponding groupoid $\{G \rightrightarrows \{*\}\} \simeq G$ is a Lie groupoid: a one-point set is trivially a manifold, projections s and t are trivial $s,t = !_G : G \ni g \mapsto \{*\}$ and hence smooth, the inclusion map points at the identity: $\varepsilon : \{*\} \mapsto e \in G$, i.e. is constant and hence smooth. □

Another example of a Lie groupoid is a pair groupoid $\mathfrak{G}_{M\times M} = \{M \times M \rightrightarrows M\}$ formed on manifold M. Indeed, the product $M \times M$ is naturally equipped with a manifold structure which makes the projections submersions and the smooth inclusion $\varepsilon : M \ni x \mapsto (x,x) \in M \times M$. The multiplication can be understood as a projection $M \times M \times M \ni (x,y,z) \simeq \big((x,y),(y,z)\big) \mapsto (x,z) \in M \times M$, and hence is also smooth.

Just as a Lie algebra associated to a Lie group can be defined in two equivalent ways – as a tangent space at the identity with a commutator given through the tangent map to the adjoint representation *or* as an algebra of left-invariant vector fields on the group – a Lie algebroid can also be viewed from two different yet equivalent perspectives.

Notice first that the tangent space at the identity of a group is the same thing as a *normal bundle* taken with respect to the inclusion $\varepsilon : \{*\} \mapsto e \in G$. Indeed, for an immersion $i : A \hookrightarrow M$ the normal bundle is defined as a quotient bundle:

$$N^i A := \frac{TM|_{i(A)}}{i^*(TA)}.$$

Hence, for $\varepsilon : Q \hookrightarrow G$ and $Q = \{*\}$ it is a bundle over the one-point manifold, i.e. a vector space. Moreover, $T\{*\} \cong \{*\} \cong \varepsilon^*(T\{*\})$ and we get:

$$N^\varepsilon Q = \frac{TG|_{\varepsilon(Q)}}{\varepsilon^*(TQ)} \cong TG|\varepsilon(*) = T_e G =: \mathfrak{g}.$$

Let us now generalize the notion of a left-invariant vector field to the context of groupoids:

Definition 2.8. *A left-invariant vector field on a Lie groupoid $\{G \rightrightarrows Q\}$ is a vector field $X^L \in \mathscr{X}^L(G)$ on the arrow space satisfying the following conditions:*

$$s_* X^L = 0 \ \ \& \ \ X^L(hg) = (L_g)_* X^L(h) \ \ \forall (g,h) \in G * G,$$

where L_g is the left[1] *multiplication $L_g(h) := hg$.*

Notice that for a Lie groupoid $\{G \rightrightarrows \{*\}\}$ the projection t is constant and hence t_* is identically zero and since $G * G = G \times G$ the second requirement is the usual one for left-invariant vector fields on G. Moreover, a left-invariant vector field on a Lie groupoid is fully determined by its value along $\varepsilon(Q)$. Indeed, we have:

$$X^L(g) = X^L(Id_{t(g)}g) = (L_g)_* X^L(Id_{t(g)})$$

Further, since $s \circ \varepsilon = Id_Q$ and we require $s_* X^L = 0$, the tangent part of X^L along $\varepsilon(Q)$ needs to vanish and we get:

$$\Gamma(N^\varepsilon(Q)) \cong \mathscr{X}^L(G).$$

A natural generalization of the notion of a Lie algebra to the theory of Lie groupoids is then the space of sections of the normal bundle $N^\varepsilon(Q)$ equipped with the bracket coming from local left-invariant extensions to the neighbourhood of $\varepsilon(Q)$, i.e. for any pair of sections $\sigma_1, \sigma_2 : Q \to N^\varepsilon(Q)$ we have:[2]

$$\{\sigma_1, \sigma_2\} := \left[\sigma_1^L, \sigma_2^L\right],$$

where $\{\cdot,\cdot\}$ is the bracket on $\Gamma(N^\varepsilon(Q))$ and the superscript L denotes the left-invariant extension. Notice however, that unlike in Lie algebras, the bracket structure on $\Gamma(N^\varepsilon(Q))$ is now in general dependent on the point on $\varepsilon(Q)$ – structure "constants" are in general *functions* on the base space. Notice also, that since ε is an immersion, $N^\varepsilon(Q)$ can be seen as a vector bundle over Q itself, and hence we also have a $C^\infty(Q,\mathbb{R})$-module structure on the space of sections. Moreover, we have a natural vector bundle map:

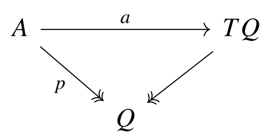
$$a := t_*|_{N^\varepsilon(Q)} : N^\varepsilon(Q) \to TQ,$$

such that for any pair of sections $\sigma_1, \sigma_2 : Q \to N^\varepsilon(Q)$ and a smooth function $f \in C^\infty(Q,\mathbb{R})$ we have:

$$a \circ \{\sigma_1, \sigma_2\} = a \circ \left[\sigma_1^L, \sigma_2^L\right]\big|_{\Gamma(TG)} = t_*\left[\sigma_1^L, \sigma_2^L\right]\big|_{\Gamma(TG)} = \left[t_*\sigma_1^L, t_*\sigma_2^L\right]_{\Gamma(TQ)} = [a \circ \sigma_1, a \circ \sigma_2]_{\Gamma(TQ)},$$
$$\{\sigma_1, f\sigma_2\} = [\sigma_1^L, f^L\sigma_2^L]\big|_{\Gamma(TG)} = f^L[\sigma_1^L, \sigma_2^L]\big|_{\Gamma(TG)} + \sigma_1^L(f^L)\sigma_2^L = f\{\sigma_1, \sigma_2\} + (a \circ \sigma_1)(f)\sigma_2,$$

where by f^L we denote the extension of f compatible with $(f\sigma)^L = f^L\sigma^L$, i.e. a function on G that is constant on orbits of the adjoint action and agrees with f on $\varepsilon(Q)$. Such a structure is called a *Lie algebroid*:

Definition 2.9. *A Lie algebroid, denoted $\{A_{\{\cdot,\cdot\}} \twoheadrightarrow Q\}$, is a vector bundle $p : A \twoheadrightarrow Q$ over a base manifold Q equipped with a bundle map $a : A \to TQ$:*



and a Lie bracket $\{\cdot,\cdot\}_A$ on the space $\Gamma(A)$ of smooth, compactly supported sections of A and we require these structures to be compatible in the following sense:[3]

i) $a \circ \{\sigma_1, \sigma_2\}_A = [a \circ \sigma_1, a \circ \sigma_2]_{\Gamma(TQ)}$ *for all* $\sigma_1, \sigma_2 \in \Gamma(A)$,

ii) $\{\sigma_1, f\sigma_2\}_A = f\{\sigma_1, \sigma_2\}_A + (a \circ \sigma_1)(f)\sigma_2$,

where $[\cdot,\cdot]$ denotes the commutator of vector fields over Q and $f \in C^\infty(Q,\mathbb{R})$.

[1] In groupoid notation composition puts the first element on the *right*.
[2] The space of left-invariant vector fields on G is closed under the commutator.
[3] Actually, the first requirement follows from the second one and hence could be skipped in the definition.

Wrapping up what was said so far, we define:

Definition 2.10. *The Lie algebroid* $\{A(\mathfrak{G})_{\{\cdot,\cdot\}} \twoheadrightarrow Q\}$ *of a Lie groupoid* $\mathfrak{G} \cong \{G \rightrightarrows Q\}$ *is given by:*

- *the vector bundle* $A(\mathfrak{G}) := N^{\varepsilon}Q$,
- *the anchor map* $a := (t_*)|_{N^{\varepsilon}(Q)}$,
- *the bracket* $\{\cdot,\cdot\}_{A(\mathfrak{G})}$ *inherited from* $\mathscr{X}^L(G)$.

$$\begin{array}{ccc} N^{\varepsilon}(Q) & \xrightarrow{t_*|_{N^{\varepsilon}(Q)}} & TQ \\ & \searrow \quad \swarrow & \\ & Q & \end{array} .$$

We have seen that a Lie algebroid of a Lie group is the usual Lie algebra. For our favourite example of a pair groupoid $\mathfrak{G}_{M\times M} = \{M \times M \rightrightarrows M\}$, the associated algebroid is the tangent bundle TM. Indeed, the normal bundle with respect to the inclusion $\varepsilon : M \ni x \mapsto (x,x) \in M \times M$ gives:

$$N^{\varepsilon}Q = \frac{TG|_{\varepsilon(Q)}}{\varepsilon^*(TQ)} = \frac{(TM \times TM)|_{\varepsilon(M)}}{\varepsilon^*(TM)} \cong \frac{TM \oplus TM}{TM} \cong TM.$$

Another way to see this is to notice that a vector field on $\varepsilon(M)$ is a pair of vector fields:

$$\mathscr{X}(M \times M)|_{\varepsilon(M)} \ni X = (X_1, X_0) \in \Gamma(M \oplus M),$$

and if we want it to be left-invariant we need $s_*X = X_1 = 0$ with X_0 arbitrary, which also makes the bracket trivial. The anchor is then the identity Id_{TM} and the projection $p : TM \twoheadrightarrow M$ the usual one.

We have thus seen how the concept of a Lie group and its associated Lie algebra can be naturally generalised to Lie groupoids and algebroids.

2.3 Notes on diffeological Lie algebroids

The construction of a Lie algebroid from a Lie groupoid, as presented in the previous section, relies heavily on tools of differential geometry that we have not developed for general diffeological spaces. It seems that the easiest way to generalize it to the diffeological setting is to slightly change the perspective and find a different characterisation of the algebroid bundle. As we will see, the space of sections of a normal bundle $N^{\varepsilon}Q$ can be seen as a Lie algebra of a Lie group associated to the Lie groupoid. Let us introduce the following [18]:

Definition 2.11. *The group of bisections, denoted $B(\mathfrak{G})$, of a Lie groupoid $\mathfrak{G} \cong \{G \rightrightarrows Q\}$ is given by smooth sections of G with respect to the source projection $s : G \twoheadrightarrow Q$ for which composition with the target projection is a diffeomorphism:*

$$B(\mathfrak{G}) := \big\{\{\sigma : Q \to G\} \in C^{\infty}(Q,G) : \ s \circ \sigma = Id_Q, \ \ t \circ \sigma \in Diff(Q)\big\},$$

with the group structure inherited from the groupoid as follows:

- *multiplication is given by $\sigma * \rho := \mu \circ (\sigma \cdot \rho, \rho)$, where $\sigma \cdot \rho := \sigma \circ t \circ \rho$,*
- *inverse map is given by $\sigma^{-1} := I\big(\sigma \circ (t \circ \sigma)^{-1}\big)$,*
- *identity is given by the inclusion map: $\sigma * \varepsilon = \sigma = \varepsilon * \sigma$.*

Remark. For a one-point base $Q = \{*\}$, sections are just elements of the group and we have $B(\mathfrak{G}) \simeq G$. It is illustrative to see that the inverse and multiplication defined above are indeed compatible:

$$\begin{aligned}\sigma^{-1} * \sigma(x) &= I\big(\sigma \circ (t \circ \sigma)^{-1}\big) * \sigma(x) = \mu\big(I(\sigma \circ (t \circ \sigma)^{-1} \cdot \sigma(x)), \sigma(x)\big) \\ &= \mu\big(I(\sigma \circ (t \circ \sigma)^{-1} \circ t \circ \sigma(x)), \sigma(x)\big) = \mu\big(I(\sigma(x)), \sigma(x)\big) = \varepsilon(x).\end{aligned}$$

An element of $B(\mathfrak{G})$ is a map assigning to each point $q \in Q$ an arrow in $\mathfrak{G}$ that starts at $\varepsilon(q)$ in such a way that the ending point depends smoothly on q with multiplication of sections coming from the composition of arrows in $\mathfrak{G}$ but *not* in a straight-forward, point-wise way – we would then miss out on the inter-connectivity of the base of the groupoid and could grasp only something like a bundle of groups, which is not nearly as natural or interesting.

To illustrate this notion, let us take a look again at the group of bisections of a pair groupoid $\mathfrak{G}_{M\times M}\}$ over a manifold M. It consists of maps of the form:

$$B(\mathfrak{G}_{M\times M}) \ni \hat{\sigma} : M \ni x \mapsto (\sigma(x), x) \in M \times M,$$

and hence is fully determined by $\sigma \in Diff(M)$. Multiplication is simply given by composition of those diffeomorphisms:

$$\hat{\sigma} * \hat{\rho}(x) = \mu(\hat{\sigma} \circ t \circ \hat{\rho}(x), \rho(x)) = \big(\rho \circ \sigma(x), \rho(x)\big)\big(\rho(x), x\big) = (\rho \circ \sigma(x), x),$$

the identity corresponds to the identity Id_M and the inverse map inverts the diffeomorphisms:

$$\varepsilon(x) = (Id_M(x), x), \quad \hat{\sigma}^{-1}(x) = (\sigma^{-1}(x), x).$$

The group of bisections of a pair groupoid on a manifold M is therefore isomorphic to its diffeomorphisms group:

$$B(\mathfrak{G}_{M\times M}) \cong Diff(M).$$

It has only recently been shown that a group of bisections of any Lie groupoid $\mathfrak{G}$ on a *compact base* admits a natural locally convex Lie group structure for whose Lie algebra is isomorphic to the Lie algebra $\Gamma(A(\mathfrak{G}))$ of sections of the associated Lie algebroid [18]:

Lemma 2.3. *The Lie algebra $T_\varepsilon B(\mathfrak{G})$ of the group of bisections $B(\mathfrak{G})$ of a Lie groupoid $\{G \rightrightarrows Q\}$ with a compact base Q is isomorphic to the space of sections $\Gamma(A(\mathfrak{G}))$ of the Lie algebroid $A(\mathfrak{G})$ of $\mathfrak{G}$ with the algebroid bracket.*

Proof. We will *not* give the full proof here, but only sketch how this can be true. An element of the Lie algebra $v \in B(\mathfrak{G})$ is given by a tangent vector at the identity $\varepsilon \in B(\mathfrak{G})$, i.e. corresponds to a class of curves $V = [c]$. Since ε is injective, such a curve:

$$c : I \ni \tau \mapsto c(\tau) = \hat{\sigma}_\tau \in B(\mathfrak{G})$$

corresponds to a one-parameter family of diffeomorphisms of the neighbourhood of $\varepsilon(Q)$ in G via:

$$\check{\sigma}_\tau : \varepsilon(Q) \ni \varepsilon(x) \mapsto \hat{\sigma}_\tau(x) \in G$$

such that $\check{\sigma}_0 = Id_{\varepsilon(Q)}$ and hence $\hat{\sigma}_0(x) = \varepsilon(x)$ for all $x \in Q$.[1] The tangent vector that it represents gives then a section of the normal bundle:

$$V = \dot{c}(0) = \frac{d}{d\tau}\check{\sigma}_\tau\big|_{\tau=0} \in \Gamma(N^\varepsilon Q),$$

and hence we have:

$$T_\varepsilon B(\mathfrak{G}) \cong \Gamma(N^\varepsilon(Q)) \cong \Gamma(A(\mathfrak{G})).$$

Notice here that the space of maps $\check{\sigma} : \varepsilon(Q) \to \mathfrak{E}(\Sigma)$ associated to sections $\hat{\sigma} \in B(\mathfrak{G})$ naturally inherits a group structure from $B(\mathfrak{G})$:

$$\varepsilon(Q) \ni \varepsilon(x) \mapsto \check{\rho} \cdot \check{\sigma}(x) := (\check{\rho} \circ \varepsilon) * \hat{\sigma}(x) = \check{\rho} \circ \varepsilon \circ t \circ \hat{\sigma}(x) = \hat{\rho} * \hat{\sigma}(x) \in G.$$

Moreover, the left-invariant extension of the vector $V \in T_\varepsilon B(\mathfrak{G})$ to a vector field V^L on the neighbourhood of the identity corresponds to the left-invariant extension of the vector field $V \in \Gamma(N^\varepsilon(Q))$ to a vector field V_L on the groupoid G, and hence the Lie algebra structure of $T_\varepsilon B(\mathfrak{G})$ is equivalent to the algebroid structure of $A(\mathfrak{G})$. Indeed, the left-invariant extension of V on $B(\mathfrak{G})$ is given by:

$$V^L(\hat{\rho}) = (L_{\hat{\rho}})_* v(\varepsilon) = (L_{\hat{\rho}})_* \big(\frac{d}{d\tau}\check{\sigma}_\tau\big|_{\tau=0}\big) = \frac{d}{d\tau}(\check{\sigma}_\tau \cdot \check{\rho})\big|_{\tau=0},$$

while its left-invariant extension on G satisfies:

$$s_* V_L = \frac{d}{d\tau}(s \circ \check{\sigma}_\tau)\big|_{\tau=0} = \frac{d}{d\tau} Id_{\varepsilon(Q)}\big|_{\tau=0} = 0,$$
$$V_L(g) = (L_g)_* X^v(Id_{t(g)}) = \frac{d}{d\tau}\big((\hat{\sigma}_\tau \circ t(g))g\big)\big|_{\tau=0}.$$

But notice that for $\{g : x \to t(g)\} \in G$ and any $\hat{\rho}_g$ such that $\hat{\rho}_g(x) = g$ we have:

$$\check{\sigma}_\tau \cdot \check{\rho}_g : \varepsilon(x) \mapsto \hat{\sigma}_\tau * \hat{\rho}_g(x) = (\hat{\sigma}_\tau \circ t(g))g,$$

and since $s_* V_L = 0$ is always satisfied, the two characterisations agree: $V_L(\hat{\rho}_g)(x) = V^L(g)$. □

The space of sections of a Lie algebroid of a Lie groupoid over a compact base can then be seen as the Lie algebra of its Lie group of bisections. Moreover, again for the compact base, the groupoid itself, and hence the whole algebroid structure with the algebroid bundle and the anchor map, can be reconstructed from a given group of bisection [19]. The idea is now to "project" the problem of defining

[1] The map $\check{\sigma}$ can be understood as the one that pre-composed with ε gives $\hat{\sigma}$, i.e. $\check{\sigma} \circ \varepsilon = \hat{\sigma}$, and as such is uniquely determined by $\hat{\sigma}$.

general diffeological algebroids associated to diffeological groupoids to a much simpler one, namely defining diffeological Lie algebras of diffeological *groups* and diffeological Lie algebroids through the diffeological Lie algebras of their groups of bisections. It is not clear today how this program can be completed, or even how the diffeological Lie algebroids should be defined in general, and what needs to be assumed to assure that the whole structure is encoded in the group of bisections. However, as we will shortly see, being a diffeological group is not such a strong requirement and for any *diffeological groupoid*, its groups of bisections *always* inherits a *diffeological group structure* – in a way things get simpler than in the Lie framework. When it comes to the anchor map, it is not even clear if the sections of the diffeological tangent bundle always admit a commutator bracket or an action on functions on the base – it may depend on the chosen definition of a tangent bundle in the first place. To the best of our knowledge, these things have not yet been fully investigated; since we were striving for the application to General Relativity and the groupoid that we wanted to analyze, although far from being a Lie groupoid, still did not seem to require a full generality of the *missing* framework, we did not focus on answering those questions, which also from today's perspective seem at least big enough for a full Master's project. We have a feeling, that defining those notions properly could give a beautiful and rich theory of *diffeological Lie algebroids*.

In the application we are concerned with, the diffeological groupoid is formed on quotient spaces of mappings between manifolds and hence, as we will see, the tangent vectors to its group of bisections are actually *vector fields*, and so are the sections of the diffeological tangent bundle of the base. The algebroid bracket is then defined through the commutator of their left-invariant extensions, and the anchor is a simple map between vector bundles over a diffeological space, which is not a manifold but it *is* a topological space so we can spell out the trivialisation condition.[1] We leave the question of under which assumption on a diffeological groupoid such a rich structure of the tangent space at the identity to the group of bisections can be derived, open for further investigation.

Let us now get back to the diffeological setting and define what we can – diffeological groupoids and their groups of bisections.

Definition 2.12. *A diffeological group is a group equipped with a diffeology compatible with the group operations, i.e. multiplication $G \times G \to G$ and the inverse map $G \to G$ are smooth maps.*

Remark. A Lie group is a diffeological group for the manifold diffeology.

Naturally, we also have a natural notion of a *diffeological groupoid*[2] – its just a groupoid with G and Q equipped with diffeologies such that the structure maps are smooth:

Definition 2.13. *A diffeological groupoid is a groupoid such that:*

i) the arrow space G and a base Q are diffeological spaces,

ii) the projections s and t, partial multiplication $(g,h) \mapsto gh$, the inclusion map ε and the inverse map I are smooth.

Remark. Lie groupoids are diffeological groupoids when G and Q are equipped with the manifold diffeologies. Moreover, similarly to the manifold case, diffeological groupoids over the one-point space are simply diffeological groups, and so are the isotropy groups.

The definition of the group of bisections of a Lie groupoid makes perfect sense in the diffeological setting. We the have:

Lemma 2.4. *The group of bisections $B(\mathfrak{G})$ of a diffeological groupoid is a diffeological group.*

[1] Which we do not do in the end for time-related reason.

[2] A diffeological groupoid is an *internal groupoid* in the category of diffeological spaces, unlike the Lie groupoids (because of the submersion requirement on the projections).

Proof. We put on a diffeological groupoid $B(\mathfrak{G}) = \{G \rightrightarrows Q\}$ the subspace-functional diffeology restricted by the requirement that plots must remain plots when composed with the inverse map:

$$\{\phi : U \ni u \mapsto \hat{\sigma}_u \in B(\mathfrak{G})\} \in \mathsf{D}_{B(\mathfrak{G})} \quad \Leftrightarrow \quad \begin{array}{c} \{ev_\phi : U \times X \ni (u,x) \mapsto \hat{\sigma}_u(x)\} \in C^\infty(U \times Q, G) \\ \& \\ \{ev_\phi : U \times X \ni (u,x) \mapsto I \circ \hat{\sigma}_u(x) \in G\} \in C^\infty(U \times Q, G). \end{array}$$

Since composition of smooth maps is a smooth operation for the functional diffeology, so is multiplication in $B(\mathfrak{G})$, smoothness of the inverse map being forced by the above definition. □

Hence the program that we sketch here seems to make sense. Diffeological algebras of diffeological groups also have not yet been properly worked out in details, however it seems to be a reasonable claim that, under some mild assumptions on the diffeological structure of a group,[1] diffeological groups admit diffeological algebras which are diffeological vector spaces, i.e. diffeological spaces with smooth addition and scalar multiplication, equipped with a bracket emergent from the group structure like for the manifold case. For example, it has been shown [5] that the internal and external tangent spaces at the identity of a diffeological group are isomorphic.[2] Moreover, taking X to be a manifold M, we get the algebra of vector fields:

$$T_{Id_M} Diff(M) \cong \mathscr{X}(M),$$

and hence $T_{Id_M} B(\mathfrak{G}_{M\times M}) \cong \Gamma(A(\mathfrak{G}_{M\times M}))$, as expected.

The isomorphism above can be intuitively understood in the light of the Proposition 3.3 of [5], which says that the internal tangent space is spanned by vectors "pushed forward" from the neighbourhood of zero in $\mathbb{R}$ by the one-dimensional pointed plots. Indeed, a one-dimensional pointed plot:

$$\phi : (I,0) \to Diff(M), \;\; \phi(0) = Id_M,$$

corresponds to a one-parameter family of diffeomorphisms:

$$(I,0) \ni \tau \mapsto \alpha_\tau \in Diff(M), \;\; \alpha_0 = Id_M,$$

and pushing-forward the tangent vector at $0 \in \mathbb{R}$ simply means:

$$\phi^*\left(\frac{d}{d\tau}\right) = \frac{d}{d\tau}\alpha_\tau|_{\tau=0} \in \mathscr{X}(M).$$

A complete proof of this fact, relying on the compact manifold structure of M, can be found in [5]. We will use this kind of approach in our analysis of the diffeological algebroid of hypersurface deformations in the second part.

Let us also mention here a different approach to generalizing the notion of a Lie algebroid to the diffeological setting that we came up with. Notice that (in the manifold setting) the tangent bundle along $\varepsilon(Q)$ can be seen as a pullback [13]:

$$\begin{array}{ccc} N^\varepsilon Q = ker(Ts)|_{\varepsilon(Q)} & \longrightarrow & ker(Ts) \subset TG \\ \downarrow & & \downarrow \pi|_{ker(Ts)} \\ Q & \xrightarrow{\varepsilon} & G \end{array},$$

[1] We need to make sure that there are not *too many* plots of a group.

[2] Also, the simpler notion of a *tangent cone* [5] seem to agree here (e.g. one get a proper vector space).

and since ε is an immersion, it splits for $TG|_{\varepsilon^*(TQ)} \cong TQ \oplus N^{\varepsilon}(Q)$. Moreover, since s and t are surjective submersions and $t \circ \varepsilon = Id_Q = s \circ \varepsilon$, we have:[1]

$$ker\big(s_*|_{\varepsilon^*(TQ)}\big) \cong \frac{TG|_{\varepsilon^*(TQ)}}{TQ} \cong N^{\varepsilon}(Q).$$

Tangent maps and their kernels, which are vector bundles, are easily defined for both approaches to the tangent spaces that we described, and hence the algebroid bundle could be *defined* in the diffeological context as $A(\mathfrak{G}) := ker\big(s_*|_{\varepsilon^*(TQ)}\big) \subset TG|_{\varepsilon(Q)}$. We believe it should not be difficult to show that $T_{\varepsilon}B(\mathfrak{G}) \cong \Gamma\big(ker\big(s_*|_{\varepsilon^*(TQ)}\big)\big)$. However, there is still a problem of defining the bracket, which should now come from the commutator of *diffeological vector fields*, that also, to the best of our knowledge, have not yet been studied in full generality.

There are many interesting and unsolved issues in the context of diffeological spaces, their tangent structures, diffeological groups, groupoids and algebroids, and we are glad that we had a chance to discover these missing aspects of our knowledge on smooth spaces and symmetry structures. However, we will leave it here for now and move to the beautiful theory of General Relativity, where we hope to make an enlightening use of these concepts.

[1] It is a matter of convention now if we use s_* or t_* here.

Part II

General Relativity

We are now going to apply the framework that we have developed in the previous chapter to the theory of General Relativity, especially to the problem of deriving the constraints bracket relation from the global symmetry structure of the initial value formulation of the Einstein equation. The idea is to follow the path of [2] and analyse the algebroid associated to the groupoid presented there and describing evolutions of space-like hypersurfaces in a diffeomorphism invariant way.

First we are going to introduce the dynamical approach to the Einstein equation, which as such is *timeless*, and explain how the constraints and their bracket relations can be derived. In the next section we describe the mentioned groupoid and equip it with a simple diffeological structure, different from the one used in [2]. We then analyse in detail the group of bisections that it gives rise to and its tangent space at the identity, which we identify with the sections of the algebroid bundle. Finally, we show that the bracket of constant sections of the algebroid bundle is identical to the bracket structure between the constraints.

1 Dynamical approach to the Einstein equation

The theory of General Relativity is *timeless* in spirit – the Einstein equation:

$$G(g)_{\mu,\nu} := R(g)_{\mu,\nu} - \frac{1}{2}R(g)g_{\mu,\nu} + \Lambda g_{\mu,\nu} = \frac{8\pi G}{c^4}T_{\mu,\nu}$$

binds the space-time metric tensor g with energy-momentum tensor $T_{\mu,\nu}$, which are defined *on the whole space-time M* at once, so to say. To make it into a dynamical field theory that we are used to,[1] one can *choose a foliation* of the space-times and *project* the equation with respect to it. When one further assumes that the space-time M is *globally hyperbolic* in the first place, meaning that it admits a foliation by *Cauchy* hypersurfaces and pick one of those, the result of such an approach is a field theory of *Riemannian* metrics $\gamma \in \mathsf{Riem}(\Sigma)$ on Σ and their "time derivatives", which are now dependent not only on the symmetric tensor $\dot{\gamma}$ which measures how the metric is changing when we change a Cauchy surface, but also on the way in which the neighbouring slices are *glued* together into M, grasped by a pair (S^i, L) of a *shift* vector field and a *lapse* function on Σ, which are then additional *degrees of freedom* if we want our new, projected equations to be *covariant* as was the original one. We then get a field theory of $(\gamma_{\mu,\nu}, k, S^i, L)$ for which (S^i, L) are *not* dynamical variables. When we perform a Legandre transformation of this system, we get the so-called *ADM* formalism,[2] which is a Hamiltonian approach to the projected Einstein field equations. Due to the additional degrees of freedom, it is a *constrained* system with a vector constraint associated to S^i called the *momentum constraint* and a function associated to L called the *energy constraint*. If we try hard enough, we can put a well-defined *Poisson* structure on the phase space of the system, which gives us the non-trivial *bracket relations* between the constraints, the so-called *constraints bracket*. What is striking, is that the structure "constants" are *not* constant but depend on the *point* in $\mathsf{Riem}(\Sigma)$. Understanding this structure is the goal of the research program to which this work aims to contribute. As an introduction, we will now give some details on the constructions sketched above. This is standard material, we refer to [8] and [20] for the missing details.

[1] The question if this is the *right* thing to do is a valid one.

[2] In tribute to R. Arnowitt, S. Deser and C. W. Misner.

1.1 Initial value formulation

To consider the Einstein equation as an evolution problem, we need to choose a *foliation* of the manifold M that is going to be our space-time, i.e. a one-parameter family of embeddings:

$$X : \mathbb{R} \times \Sigma \to M, \;\; X_t(x) \equiv X(x,t), \;\; X_t : \Sigma \hookrightarrow M,$$

such that for each $t \in \mathbb{R}$ the map:

$$X_t : \Sigma \to X_t(\Sigma) =: \check{\Sigma}_t \subset M,$$

is a homeomorphism, $\check{\Sigma}_t$ are Cauchy surfaces in (M,g) and X provides a topological isomorphism spaces $\Sigma \times \mathbb{R} \simeq M$. The foliation X, together with the choice of coordinates on Σ, gives us global coordinates on M via $p = (x,t) \in \check{\Sigma}_t \subset M$, with the initial value Cauchy surface being

$$\check{\Sigma} := \check{\Sigma}_0 = \{(x,t) \in M \mid t = 0)\} \subset M.$$

We can then define the *evolution vector field*:

Definition 1.1. *The evolution vector field $\dot{X}$ of the foliation $X : \mathbb{R} \times \Sigma \to M$ is given at $p \in M$ by the differential of X with respect to t:*

$$\dot{X}(p) := \frac{\partial}{\partial t} X(x,t)|_{p=(x,t)}.$$

Thanks to the Lorentzian metric g on M, the evolution vector field $\dot{X}$ can be decomposed into a tangential and a normal part with respect to the foliation leave at any given point $p \in M$:

$$\dot{X}^\mu(p) = L(p)\hat{n}^\mu(p) + S^i(p), \tag{1}$$

where $\hat{n}(p)$ is the unit normal, so that $g_{\mu,\nu}\hat{n}^\mu\hat{n}^\nu = -1$.

Definition 1.2. *We call the function L defined above the lapse function and the accompanying vector field $S^i \in \mathscr{X}(M)$ the shift vector field.*

Remark. The shift vector field $S^i(x,t)$ at the point $p = (x,t)$ is then by definition *tangent* to the hypersurface $\check{\Sigma}_t$.

If we further require the foliation to be oriented accordingly to the embedded Cauchy surfaces, $L : \Sigma \to \mathbb{R}_+$ needs to be positive and monotonically increasing. Further, since each $\check{\Sigma}_t$ is space-like, the vector field $\dot{X}$ is time-like and we have:

$$g_{\mu,\nu}\dot{X}^\mu\dot{X}^\nu = -L^2(p) + g_{i,j}(p)S^i(p)S^j(p) < 0.$$

Remark. Lapse and shift can be thought of as a convenient way to parametrize the "$3+1$" diffeomorphism invariance of (M,g). For given local coordinates on $\check{\Sigma}_t$ for some $t \in \mathbb{R}$, we can still decide on the parametrization of the orthogonal direction through the lapse function and also choose coordinates on all the other slices, but only such that we get a smooth parametrization of neighbourhoods in M, i.e., the transition functions on M need to be smooth, and hence coordinates on neighbouring slices are given by specifying a vector field that describes how the coordinates shift as we move along the increasing t parameter.

The evolution vector field is thought of as describing (not surprisingly) the *evolution* of the data given on the Cauchy surface $\check{\Sigma}$. The (local) time can be then interpreted as the function $\hat{t} : M \to \mathbb{R}$ that gives rise to the one-form $d\hat{t}$ and hence (because of the the duality given by g) to a vector field $\vec{\nabla}\hat{t} := d\hat{t}^\#$, such that:

$$\dot{X}(p) \cdot \vec{\nabla}\hat{t} = -1,$$

since then for $p=(x,t)$ and $p'=(x,t+\delta t)$, we have:

$$\hat{t}(p') = \hat{t}(p) - \vec{\nabla}\hat{t}\cdot\dot{X}(p)\delta t = \hat{t}(p) + \delta t.$$

Hence we can think of the foliation leaves as time slices and setting $\hat{t}|_{\check{\Sigma}_0} = 0$ we have:

$$\check{\Sigma}_t = \hat{t}^{-1}(t), \;\; \hat{t} = \pi_1 \circ X^{-1},$$

where π_1 is the projection onto $\mathbb{R}$ from the domain of $X : \mathbb{R}\times\Sigma \to M$. Since we want to think of the Einstein equation as an initial value problem for data defined on Σ, we need to find a way to think of $g|_{\check{\Sigma}}$ restricted to $T\check{\Sigma}$ as a Riemannian metric on Σ alone. We then define:

Definition 1.3. *The* 1^{st} *fundamental form* γ *is the Riemannian metric tensor on* Σ *pulled back from* (M,g) *via the embedding* $X_0 : \Sigma \hookrightarrow M$*:*

$$\gamma := (X_0)_* g \in \mathsf{Riem}(\Sigma),$$

where $\mathsf{Riem}(\Sigma)$ denotes the space of 3-dimensional Riemannian metrics on Σ. A metric γ above is given for any value of $t\in\mathbb{R}$, and hence we have one on any slice:

$$\gamma(t) := (X_t)_* g \in \mathsf{Riem}(\Sigma),$$

and since X is a foliation of the whole M, we have thus defined γ on each point $p=(x,t)\in M$. Furthermore, it can be extended by zero to a 4-dimensional tensor field $\tilde{\gamma}\in S^2(M)$.[1] It is then related to the Lorentzian metric g on M via [20]:

$$\tilde{\gamma}_{\mu,\nu}(x,t) = g_{\mu,\nu}(x,t) + \hat{n}^{\mu}(x,t)\hat{n}^{\nu}(x,t).$$

Notice now that thanks to (1) we have $\hat{n}^{\mu} = L^{-1}(\dot{X}^{\mu} - S^i)$, and the relation (1.1) gives:

$$g_{\mu,\nu} = \gamma_{\mu,\nu} - \hat{n}_{\mu}\hat{n}_{\nu} = \gamma_{\mu,\nu} - L^{-2}(\dot{X}_{\mu} - S_{\mu})(\dot{X}_{\nu} - S_{\nu}),$$

so that the Lorentzian metric tensor takes the form:

$$g_{\mu,\nu} = \begin{pmatrix} -L^2 + \gamma_{ij}S^iS^j & S^1 & S^2 & S^3 \\ S^1 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ S^2 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ S^3 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}.$$

Hence knowing the projected metrics $\gamma(t)$ together with the lapse function $L(x,t)$ and the shift vector field $S(x,t)$ on each time slice is equivalent to knowing the whole metric tensor g on M:

$$g = g(\gamma, L, S^i).$$

The vacuum Einstein equation can be understood as a field equation for the Einstein-Hilbert *action*:

$$S = \int_M \sqrt{-g}\,{}^4R(g),$$

which thanks to the Gauss-Codazzi equations can be rewritten in a *projected* form as follows:

$$S = \int_M \sqrt{-g}\,{}^4R(g) = \int_{\mathbb{R}} dt \int_{\Sigma_t} L\sqrt{\gamma}\{{}^3R(\gamma) + tr(k^2) - (tr(k))^2\},$$

[1] We denote by $S^2(M)$ the space of symmetric covariant 2-tensor fields on the manifold M.

where $k \in S^2(\Sigma)$ is the <u>2^{nd} fundamental form</u> which plays the role of the "time derivative" and is given as a function of the lapse, shift, γ and $\dot{\gamma}$ through:

$$k(L,S,\gamma,\dot{\gamma}) := -\frac{1}{2}\mathscr{L}_{\hat{n}(\tau)}\tilde{\gamma}(\tau) = \frac{1}{2L(\tau)}\left\{\mathscr{L}_{S(\tau)}\gamma(\tau) - \dot{\gamma}(\tau)\right\},$$

The Lagrangian of General Relativity is then a real functional of the form:

$$L : T\big(Riem(\Sigma) \times \big(\mathscr{X}(\Sigma) \oplus \mathscr{F}(\Sigma)\big)\big) \ni (\gamma,\dot{\gamma},S,\dot{S},L,\dot{L}) \mapsto \int_\Sigma L\sqrt{\gamma}\{{}^3R(\gamma) + tr(k^2) - (tr(k))^2\} \in \mathbb{R}.$$

1.2 Constraints bracket

Performing the Legendre transform on the projected Lagrangian of General Relativity gives the so-called ADM Hamiltonian. The lapse function and shift vector field are then treated as additional variables, and since their time derivatives do not appear in the Lagrangian, they give rise to the constraints:

$$C_{mo}(\gamma,\pi) = -2div(\pi) = 0,$$
$$C_{en}(\gamma,\pi) = -{}^3R(\gamma) + tr(\pi^2) - \frac{1}{2}(tr(\pi))^2 = 0,$$

where $\pi \in S^2(\Sigma)$ is the variable conjugate to $\dot{\gamma}$. These constraints can be made into functions on the phase space $T^*Riem(\Sigma)$ by pairing with vector field X and a function ϕ on Σ [2]:

$$C_{(X,\phi)} : (\gamma,\pi) \mapsto \int_\Sigma \sqrt{\gamma}\left\{\gamma(X, C_{mo}(\gamma,\pi)) + \phi \cdot C_{em}(\gamma,\pi)\right\}. \tag{2}$$

Defying $C_{(X,\phi)} = C_{(X,0)} + C_{(0,\phi)} =: C_X + C_\phi$, the Poisson bracket structure of the constraints can be shown to be [2]:[1]

$$\{C_X, C_Y\} = C_{[X,Y]}, \tag{3}$$
$$\{C_X, C_\phi\} = C_{X\cdot\phi}, \tag{4}$$
$$\{C_\phi, C_\psi\} = C_{\phi grad_\gamma \psi - \psi grad_\gamma \phi}. \tag{5}$$

This is the structure we will be interested in deriving in the coming section. What is interesting is that in the last equation we see the dependence on the metric $\gamma \in \mathsf{Riem}(\Sigma)$. For this reason it is a plausible idea to look at a *groupoid* as a potential source of this bracket structure, since the algebroid bracket can depend on a point in the base, which will be related (in a non-trivial way) to the space of Riemannian metrics $\mathsf{Riem}(\Sigma)$.

[1] Note here that, even though the bracket structure above is derived for the vacuum case, introducing fields on the space-time does not influence it [9].

2 Diffeological algebroid of hypersurface deformations

The aim of this section, and indeed the main goal of this thesis, is to improve the derivation of the algebroid structure associated to the groupoid describing the symmetry of the initial value formulation of the Einstein equation proposed in [2] by Weinstein, Blohmann and Fernandes. In the previous chapter we have provided a framework sufficient for this purpose.

It is a legitimate question to ask here *why* we should be interested in this groupoid structure, for which we would like to give the following answers:

1. Because it is *beautiful* and *naturally* associated to the conceptual and mathematical framework of General Relativity.
2. Because the algebroid structure that we get is *identical* to the Poisson bracket structure that appears when one considers the ADM approach to Einstein equation.
   Further, one may ask why this bracket structure is *important*. My answers are as follows:
   (a) Because it *is there*.
   (b) Because, as claimed in [9], under some plausible assumptions the constraints *themselves* can be *recovered* from this bracket structure alone.

We find each of the mentioned reasons convincing, however the last one seems to be important not only from an aesthetic or idealistic perspective but also from the conceptual point of view. Let us mention here the main points of the 1976 paper [9] by Hojman, Kuchar and Teitelboim. The authors show that when we interpret the deformation vector field as the *generator of evolution* of fields on the space-time, which we find a very plausible thing to do, and apply it *to the space-time metric itself*, we can conclude that the lapse and shift, seen as components of this generator, has to satisfy commutation relations of the constraints. Furthermore, they prove that if we want the change of an arbitrary functional on the phase space to be generated via a the Poisson bracket with the "hamiltonian", which is now understood to be a pair: a vector field called *super-momentum* and a function called super-hamiltonian defined on the space-time, as quantities dependent on the canonical variables *and* require the Poisson bracket to be of the same form as the commutator of the generators of the deformations, we *inevitably* end up with the *ADM* hamiltonian. Therefore, if we can show that the kinematical setting of the theory of gravity, which can be understood to be given by diffeomorphism invariant evolutions of 3-dimensional hypersurfaces in 4-dimensional, Lorentzian, globally hyperbolic space-times and grasped by the groupoid of [2], *gives rise* to the bracket structure in question, the interpretation of [9] allows us to recover the full Einstein equation. We can then conclude that the Einstein's theory is somehow dynamically empty – everything is already present in the chosen *kinematical* framework. Making their points clear in the context of the mathematical framework put forward by Weinstein, Blohmann and Fernandes and addressed in this work would be one of the directions of development of the old program aiming to understand the dynamics of General Relativity.

The plan of this section is as follows. After presenting the groupoid proposed in [2], we take the strategy of deriving its associated algebroid that was suggested in the Appendix there and described in some detail in the first part of this work. Namely, we put a natural diffeological structure on the groupoid that allows for a detailed analysis of the associated group of bisections. Next we argue that the Lie algebra of this group is indeed given by the commutator of g-gaussian[1] extensions of vector fields given solely *on* the embedded surfaces. They can be identified with pairs consisting of a function and a vector field on the non-embedded hypersurface, i.e. the "space" itself, and constitute the fibers of the algebroid.[2] We also show that the algebroid bundle is a trivial diffeological

[1] We change the terminology a little bit and call them simply *Gaussian*.

[2] Here we fully agree with [2].

fiber bundle.[1] We also present another argument that leads to the same conclusion. Namely, we argue independently that the algebroid bundle is of the form as just explained when we identify it with the *normal bundle* of the groupoid, and then show that left-invariant vector fields that the group structure of the group of bisections generates can be understood as g-gaussian extensions.[2]

[1]This was missing in [2].

[2]This kind of reasoning was also suggested in the Appendix of [2].

2.1 Groupoid symmetry

In this section we are going to present the groupoid that naturally appears when one considers the dynamical approach to the Einstein equation, proposed by Weinstein, Blohmann and Fernandes in [2]. We provide some interesting details about well-definiteness of multiplication missing in the original presentation.

As we have just seen, in order to consider the Einstein equation as an evolution problem we *break* the 4-dimensional diffeomorphism invariance of the theory by *choosing* a foliation, which is then somehow *restored* by the freedom of choosing the lapse function and shift vector field that together fix the way in which the neighbouring time slices are to be glued into a 4-dimensional, Lorentzian manifold. This "$3+1$" diffeomorphism invariance of the problem forces us to consider *classes* of embedded hypersurfaces as *solutions* to the initial value problem of the projected Einstein equation. Indeed, any two Cauchy developments of the same initial data are isometric *in the neighbourhood of the initial value hypersurface*, and obviously agree in it. The solutions of the initial value problem are then only given *up to the isometries preserving the embedded initial value hypersurface*. Following [2], given a (fixed) 3-dimensional manifold Σ, we define the space of Σ-*universes*, which consists of all such a classes:[1]

Definition 2.1. *The space of* <u>Σ*-universes*</u>, *denoted* $\mathfrak{U}(\Sigma)$, *consists of equivalence classes of embeddings*[2] $i:\ \Sigma\ \hookrightarrow\ (M,g)$ *of* Σ *such that:*

i) (M,g) *is a* 4*-dimensional, connected, Lorentzian manifold,*

ii) $i:\Sigma\hookrightarrow M$ *is a proper embedding of* Σ *into* M *as a space-like hypersurface,*

and two embeddings are equivalent iff there is an orientation-preserving isometry making

$$\begin{array}{ccc} & \Sigma & \\ {}^{i}\swarrow & & \searrow^{i'} \\ M & \xrightarrow{\ \psi\ } & M' \end{array}$$

a commutative diagram. We will write $i\sim i'$ *and* $[i]=[i']\in\mathfrak{U}(\Sigma)$.

Remark. Note that to each class a unique Riemannian metric tensor $\gamma_{[i]}\in Riem(\Sigma)$ is associated.[3] Indeed, for $i\sim i'$ we have:

$$i'^{*}g'=(\psi\circ i)^{*}g'=(i)^{*}(\psi^{*}g')=(i)^{*}g,$$

We want to construct a groupoid over $\mathfrak{U}(\Sigma)$ that represents possible global evolutions of an embedded hypersurface *regardless* of the details of the ambient space-time. It is then natural[4] to consider the *classes of pairs* of such embeddings, and again following [2] we define:

Definition 2.2. *A groupoid of* <u>Σ*-evolutions*</u>, *denoted* $\mathfrak{G}_\Sigma=\mathfrak{E}(\Sigma)\rightrightarrows\mathfrak{U}(\Sigma)$, *is given by:*

1. *The space* $\mathfrak{E}(\Sigma)$ *consists of classes of pairs of embeddings* (i_1,i_0), *where* $i_1,i_0:\Sigma\hookrightarrow M$ *are embeddings as before and two pairs are equivalent iff there is an orientation-preserving*

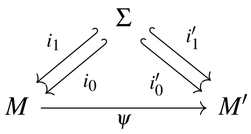
[1] *Without* assuming that the Einstein equation is satisfied.

[2] The space of all such embeddings in the *hyperspace* of [9].

[3] The 2^{nd} fundamental form $k:=-\frac{1}{2}\mathscr{L}_{\hat{n}}\gamma$ is not necessary preserved by such isometries [4].

[4] A pair groupoid does not make a lot of sense here since the evolution must be *representable* in a one fixed space-time manifold. This is really a brilliant construction of the authors of [2] that we learned to appreciate.

isometry making the inner and outer triangles in the following diagram commute:



We will write[1] $(i_1,i_0)\sim(i_1',i_0')$ *and* $\big[i_1,i_0\big]=\big[i_1',i_0'\big]\in\mathfrak{E}(\Sigma)$.

2. *The structure maps are defined similarly to the pair groupoid, defining:*

   i) *the source and target projections* $s,t:\mathfrak{E}(\Sigma)\to\mathfrak{U}(\Sigma)$ *by* $[i_1]\xleftarrow{\ t\ }\big[i_1,i_0\big]\xrightarrow{\ s\ }[i_0]$,

   ii) *the inclusion* $\varepsilon:\mathfrak{U}(\Sigma)\hookrightarrow\mathfrak{E}(\Sigma)$ *by* $\varepsilon([i]):=\big[i,i\big]$,

   iii) *the inverse map by* $\big[i_1,i_0\big]^{-1}:=\big[i_0,i_1\big]$,

   iv) *the partial multiplication by:*

$$\mathfrak{E}(\Sigma)*\mathfrak{E}(\Sigma):=\big\{\big(\big[i_2,\tilde{i}_1\big],\big[i_1,i_0\big]\big)\mid[\tilde{i}_1]=[i_1]\big\},\quad \mu\big(\big[i_2,\tilde{i}_1\big],\big[i_1,i_0\big]\big):=\big[i_2,i_0\big].$$

Remark. Since the classes $\big[i,i\big]$ and $\big[i\big]$ are essentially the same, the map ε is injective; and since $\big[i,i\big]$ is always in $\mathfrak{E}(\Sigma)$, the maps s and t are surjective.

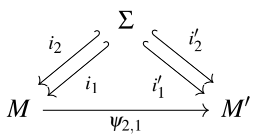
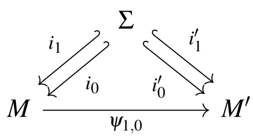
It is illustrative to see that this simple multiplication in $\mathfrak{E}(\Sigma)$ does not depend on the representatives:

Lemma 2.1. *The multiplication in* $\mathfrak{E}(\Sigma)$ *is well-defined.*

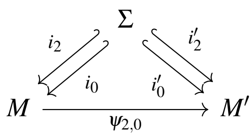
Proof. Notice first that without loss of generality we can assume that $i_1=\tilde{i}_1$ so that all three embeddings have the same target manifold M. Indeed, since $\tilde{i}_1\sim i_1$, we have an isometry $\psi: M_{\tilde{i}_1}\to M_{i_1}$ and hence $[i_2,\tilde{i}_1]=[\psi\circ i_2,i_1]$. If we now take another representative for each of the elements of $\mathfrak{E}(\Sigma)$ that we want to multiply:

$$(i_2,i_1)\sim(i_2',i_1'),\qquad (i_1,i_0)\sim(i_1',i_0'),$$

we are given two pairs of commuting triangles:

 ,  .

Notice now, that $\psi_{2,1}$ and $\psi_{1,0}$ both preserve $i_1(\Sigma)$, i.e. agree on this hypersurface, and hence can be *glued* there to give a smooth isometry $\psi_{2,0}$ preserving $i_0(\Sigma)$ and $i_2(\Sigma)$ so that we have:

 .

□

Notice here that since the embeddings are proper, a pair of those $i_0,i_1:\Sigma\hookrightarrow M$ into the same target space-times M induces a diffeomorphism:

$$ev_{(1,0)}=i_1\circ i_0^{-1}:i_0(\Sigma)\to i_1(\Sigma).$$

[1]We write the pairs in $(1,0)$-order for better compatibility with the groupoid multiplication.

Further, the partial multiplication of Σ-evolutions can understood as a composition of these diffeomorphisms. Indeed, the multiplication operation corresponds to:

$$\left(i_2 \circ i_1^{-1},\ i_1 \circ i_0^{-1}\right) \mapsto (i_2 \circ i_0^{-1}) = (i_2 \circ i_1^{-1}) \circ (i_1 \circ i_0^{-1}).$$

Let us also take a look here at the isotropy group $\mathfrak{E}(\Sigma)_{[i]}$ of an element $[i] \in \mathfrak{U}(\Sigma)$. It is given by those elements $g = [i_1, i_0] \in \mathfrak{E}(\Sigma)$ for which:

$$s([i_1, i_0]) = [i_0] = [i_1] = t([i_1, i_0]),$$

with multiplication corresponding to composition of the isometries. Since i_1 and i_0 have the same target: $i_1, i_0 : \Sigma \hookrightarrow M_i$, the equality of classes $[i_0] = [i_1]$ means that we have an isometry $\psi \in Iso(M_i, g_i)$ such that $i_1 = \psi \circ i_0$. Notice that if for a given pair of embeddings such an isometry exists, it is unique. Moreover, having fixed an isometry, for any representative $[i] \ni i_0 : \Sigma \hookrightarrow \Sigma$ we get $i_1 := \phi \circ i_0$ so that $[i_0] = [i_1]$. The isometries of M_i are then in one-to-one correspondence with the elements of the isotropy group. For elements of $\mathfrak{E}(\Sigma)_{[i]}$ we have $i_1 \circ i_0^{-1} = \psi$: and hence the composition corresponds to the composition of isometries and we can conclude:

$$\mathfrak{E}(\Sigma)_{[i]} = Iso(M_i, g_i).$$

2.2 Diffeological groupoid

Since we want to analyze the algebroid associated to the groupoid of Σ-evolutions $\mathfrak{G}_\Sigma$, let us first explain how it can be naturally made into a diffeological groupoid. The choice of a particular diffeological structure that we make is *different* than the one made in [2]. We find our approach much simpler and much more natural, which are important advantages for the following reasons. Firstly, since this helps us make our analysis simpler and clearer, at least in our view, than the original one. Secondly, the algebroid structure essentially depends on this choice and since we would like to think of the bracket structure as really *emergent* from this symmetry structure, it is very important to assure that we are not implicitly assume anything on the way. We also provide details missing in the original paper, e.g. prove that the resulting structure is a diffeological groupoid.

The space of Σ-universes is a quotient of a subspace of the space of mappings from Σ with varying codomains. It is then natural to consider the quotient diffeology. For the space of mappings itself, we can think of a functional diffeology if we consider manifold diffeologies on Σ and the target space-times and restrict ourselves to those parametrizations, for which locally the latter does not change. We define:

Definition 2.3. *A parametrization $\phi : U \to \mathfrak{U}(\Sigma)$ is a plot iff:*

1. $\phi : u \mapsto [i_u]$ *(quotient diffeology),*
2. $\forall u \in U \;\; \exists u \in V_u \in \mathscr{O}(U) : \;\; M_{i_v} = M_{i_u} \;\; \forall v \in V_u$ *(fix the target),*
3. $\forall u \in U$ *the map* $ev_\phi : V_u \times \Sigma \ni (v,x) \mapsto i_v(x) \in M_{i_u}$ *is smooth (functional diffeology).*

To equip the space of arrows $\mathfrak{E}(\Sigma)$ with a diffeological structure, we will go along similar lines: before taking the quotient, we consider the diffeology on the space of pairs of embeddings by assuming that the target space-time is locally fixed and that the local evaluation maps are smooth:

Definition 2.4. *A parametrization $\phi : U \to \mathfrak{E}(\Sigma)$ is a plot iff:*

1. $\phi : u \mapsto \left[(i_1)_u, (i_0)_u\right]$ *(quotient diffeology),*
2. $\forall u \in U \;\; \exists u \in V_u \in \mathscr{O}(U) : \;\; M_{(i_1)_v} = M_{(i_0)_v} = M_{(i_1)_u} = M_{(i_0)_u} \;\; \forall v \in V_u$ *(fix the target),*
3. $\forall u \in U$ *the map* $ev_\phi : V_u \times \Sigma \ni (v,x) \mapsto ((i_1)_v(x), (i_0)_v(x)) \in M_{i_u} \times M_{i_u}$ *is smooth (functional diffeology).*

Smoothness of the structure maps of $\mathfrak{E}(\Sigma) \rightrightarrows \mathfrak{U}(\Sigma)$ then becomes trivial:

Lemma 2.2. *With the diffeologies just defined, $\mathfrak{G}_\Sigma$ becomes a diffeological groupoid.*

Proof. Notice first that since we are using quotient diffeologies, all plots come from compositions with the projections and hence we can focus on smooth parametrizations of the space of embeddings and the space of pairs of embeddings ($\mathfrak{U}(\Sigma)$ and $\mathfrak{E}(\Sigma)$ before taking quotients. Then we see that:

- the projections $s,t : \mathfrak{E}(\Sigma) \to \mathfrak{U}(\Sigma)$ are smooth, since for any parametrization ϕ of the space of pairs of embeddings we have $ev_{s\circ\phi} = \pi_0 \circ ev_\phi$ and $ev_{t\circ\phi} = \pi_1 \circ ev_\phi$, which are smooth if only ev_ϕ is;
- the inclusion is smooth since for any parametrization ϕ of the space of embeddings we have $ev_{\varepsilon\circ\phi} = ev_\phi \times ev_\phi$, which is smooth iff ev_ϕ is;
- the inversion map is smooth, since for any parametrization ϕ of the space of pairs of embeddings we have $ev_{I\circ\phi} = inv \circ ev_\phi$, where inv denotes exchanging the product factors, which is smooth iff ev_ϕ is;

- since the set of composable arrows in $\mathfrak{E}(\Sigma)$ is naturally equipped with the product-subspace diffeology, any parametrization ϕ of this space is a plot iff the 3-factor evaluation maps

$$ev_\phi : V_u \times \Sigma \ni (v,x) \mapsto \big((i_1)_v(x), (i_0)_v(x)), (j_0)_v(x)\big) \in M_{(i_1)_u} \times M_{(i_0)_u} \times M_{(j_0)_u}$$

are smooth, and the map $ev_{*\circ\phi}$ is smooth iff $\pi_0 \circ ev_\phi$ and $\pi_2 \circ ev_\phi$ are.

□

Let us here also take a look at the D-topologies on $\mathfrak{U}(\Sigma)$ and $\mathfrak{E}(\Sigma)$. By definition, a subset of a diffeological space is D-open iff it is an image of a plot. Because of the diffeology we put on $\mathfrak{U}(\Sigma)$, an open subset there consists of the *classes* of those embeddings that can be represented in a *single* manifold in such a way that these representatives are all connected by a *smooth* deformation given by the evaluation map of a plot. Similarly, D-opens in $\mathfrak{E}(\Sigma)$ consist of classes of pairs of such embeddings.

Notice also that a plot on $\mathfrak{U}(\Sigma)$ determines a plot on the space of Riemannian metrics on Σ. Indeed, for a smooth parametrization $\phi : U \ni u \mapsto [i_u] \in \mathfrak{U}(\Sigma)$, the map:

$$U \ni u \mapsto \gamma_{[i_\tau]} = i_\tau^* g_i \in \mathsf{Riem}(\Sigma)$$

is well defined by the Remark (2.1) and smooth thanks to the smoothness of the evaluation map.[1]

[1]This looks a lot like the requirement 4. in the definition of the diffeological structure on the groupoid given in [2].

2.3 Group of bisections

In this paragraph we are going to take a close look at the space of bisections associated to $\mathfrak{G}_\Sigma$, given by:

$$B(\mathfrak{G}_\Sigma) = \left\{\hat{\sigma} \in C^\infty(\mathfrak{U}(\Sigma), \mathfrak{E}(\Sigma)) : \ s \circ \hat{\sigma} = Id_{\mathfrak{U}(\Sigma)}, \ t \circ \hat{\sigma} \in \mathit{Diff}(\mathfrak{U}(\Sigma))\right\}$$

and the multiplication defined in (2.11). The definition above means that $\hat{\sigma}$ is smooth, $s \circ \hat{\sigma}[i] = [i]$ for all $[i] \in \mathfrak{U}(\Sigma)$ and $[i] \mapsto t \circ \hat{\sigma}[i]$ is a diffeologically smooth, invertible map with a smooth inverse. Notice first that smoothness of $\hat{\sigma}$ means:

$$\{\phi : U \ni u \mapsto [i_u] \in \mathfrak{U}(\Sigma)\} \in \mathsf{D}_{\mathfrak{U}(\Sigma)} \ \Rightarrow \ \{\hat{\sigma} \circ \phi : U \ni u \mapsto \hat{\sigma}[i_u] \in \mathfrak{E}(\Sigma)\} \in \mathsf{D}_{\mathfrak{E}(\Sigma)}.$$

It follows that for a smooth section $\hat{\sigma}$ we have:

1. The section $\hat{\sigma}$ is actually given on the level of the embeddings: $\hat{\sigma}[i_u] =: \left[\sigma^1(i_u), \sigma^0(i_u)\right]$,
2. If $\phi \in \mathsf{D}_{\mathfrak{U}(\Sigma)}$, i.e. the local[1] evaluation map:

$$ev_\phi : V_u \times \Sigma \ni (u,x) \mapsto i_u(x) \in M_{i_u}$$

is smooth, from the smoothness of $\hat{\sigma}$ and the form of the diffeological structure on $\mathfrak{E}(\Sigma)$ it follows that the evaluation maps associated to $s \circ \hat{\sigma}$ and $t \circ \hat{\sigma}$ are also smooth:

$$ev_{\sigma^0 \circ \phi} : V_u \times \Sigma \ni (v,x) \mapsto \sigma^0(i_v)(x) \in M_{\sigma(i_u)},$$
$$ev_{\sigma^1 \circ \phi} : V_u \times \Sigma \ni (v,x) \mapsto \sigma^1(i_v)(x) \in M_{\sigma(i_u)},$$

where since classes in $\mathfrak{E}(\Sigma)$ are formed on the pairs of embeddings with *the same target manifold* we were allowed to put $M_{\sigma(i_u)} := M_{\sigma^1(i_u)} = M_{\sigma^0(i_u)}$.

Hence we have a pair of maps:

$$\sigma^0, \sigma^1 : \mathfrak{Emb}(\Sigma) \to \mathfrak{Emb}(\Sigma),$$

where $\mathfrak{Emb}(\Sigma)$ denotes the space of embeddings. To handle these, let us take look at the section $\hat{\sigma}$ *locally* in the sense of D-topology on $\mathfrak{U}(\Sigma)$, i.e. fix an open subset $A \in \mathscr{O}_{\mathsf{D}}(\mathfrak{U}(\Sigma))$ given by the image of a plot $\phi : U \ni u \mapsto [i_u] \in \mathfrak{U}(\Sigma)$. Let us further restrict it to $V_u \subseteq U$ for some $u \in U$ and fix a representative $i := i_u : \Sigma \hookrightarrow M := M_{i_u}$. Such an open set $A_i \in \mathscr{O}_{\mathsf{D}}(\mathfrak{U}(\Sigma))$ is then given by a subset:

$$U_i := \{i_v(x) \mid v \in V_u, \ x \in \Sigma\} \subseteq M,$$

up to the choice of the representative $i \in [i]$, i.e. up to the isometries on U_i preserving $i(\Sigma)$. Let us forget for a while about this invariance[2] and see how far we can get after fixing such U_i. The maps σ^1 and σ^0 give us a pair of maps:

$$\Psi_i^0 : M \supseteq U_i \ni i_v(x) \mapsto \sigma^0(i_v)(x) \in M_{\sigma(i)},$$
$$\Psi_i^1 : M \supseteq U_i \ni i_v(x) \mapsto \sigma^1(i_v)(x) \in M_{\sigma(i)},$$

and hence locally σ^0 and σ^1 are given by composition with the maps just defined:

$$\sigma^0(i_v) = \Psi_i^0 \circ i_v \qquad \sigma^1(i_v) = \Psi_i^1 \circ i_v \quad \forall_{v \in V_u}.$$

[1]In the sense of the definition (2.4).

[2]We can do it since we are working with *quotient diffeologies* and hence all plots arise as compositions of plots on bigger spaces with the projections.

Clearly, the maps Ψ need to be *smooth* for the section $\hat{\sigma}$ to be smooth. Moreover, we will show that they are diffeomorphic onto their images. To see this, notice first that since σ^1 induce a diffeomorphism on $\mathfrak{U}(\Sigma)$ via:

$$t \circ \hat{\sigma} : \mathfrak{U}(\Sigma) \ni [i] \mapsto [\sigma^1(i)] \in \mathfrak{U}(\Sigma).$$

The smooth inverse $(t \circ \hat{\sigma})^{-1}$, similarly to the original map, needs to be given on the level of embeddings:

$$(t \circ \hat{\sigma})^{-1}[i] = \left[(\sigma^1)^{-1}(i)\right].$$

The map $(\sigma^1)^{-1} : \mathfrak{Emb}(\Sigma) \to \mathfrak{Emb}(\Sigma)$ is again smooth and hence locally we can represent it by the manifold map as we did with σ^1:

$$(\sigma^1)^{-1} : \sigma^1(i_v) \mapsto \Phi^1_i \circ \sigma^1(i_v), \quad \Phi^1_i : M_{\sigma(i)} \to M \quad \forall_{v \in V_u}.$$

For $(t \circ \hat{\sigma})^{-1}$ to be the relevant inverse, we need:

$$\left[(\sigma^1)^{-1} \circ \sigma^1(i_v)\right] = [i_v] \quad \forall_{v \in V_u},$$

and hence there has to be an isometry $\psi \in Iso(M)$ such that:

$$\psi \circ \Phi^1_i \circ \Psi^1_i \circ i_v = i_v \quad \forall_{v \in V_u}.$$

The map $\psi \circ \Phi^1_i$ provides then the smooth local inverse to Ψ^1_i. Let us now take a look at the requirement $s \circ \hat{\sigma} = Id_{\mathfrak{U}(\Sigma)}$. In terms of the maps Ψ, this means that:

$$s \circ \hat{\sigma}[i_v] = [\sigma^0(i_v)] = \left[\Psi^0_i \circ i_v\right] = [i_v] \quad \forall_{v \in V_u},$$

and hence Ψ^0_i is actually a locally defined *isometry*. We can then *invert* Ψ^0_i and define a local *diffeomorphism* via:

$$F_i(\hat{\sigma}) := (\Psi^0_i)^{-1} \circ \Psi^1_i : M \supseteq U \to U' \subseteq M,$$

where $U' = (\Psi^0_i)^{-1}(\Psi^1_i(U))$. We have then managed to associate a diffeomorphism between two subsets of a Lorentzian manifold to the section $\hat{\sigma} \in B(\mathfrak{G}_\Sigma)$ at the cost of restricting our attention to the small neighbourhood of $[i] \in A_i$ and choosing a representative $i \in [i]$. The section $\hat{\sigma}$ can be then locally understood as $F_i(\hat{\sigma})$ given *up to the isometries on U preserving* $i(\Sigma)$, and hence essentially given by $\Psi^1_i(\hat{\sigma})$. A section $\hat{\sigma} \in B(\mathfrak{G}_\Sigma)$ restricted to A_i corresponds then to a class of diffeomorphisms of U_i:[1]

$$[F_i(\hat{\sigma})] \in \mathit{Diff}(U_i, g, i) := \mathit{Diff}(U_i) \big/ Iso(U_i, g, i),$$

where $\mathit{Diff}(U_i)$ denotes the group of diffeomorphisms of a neighbourhood U_i of $i(\Sigma) \subseteq M$, and $Iso(U_i, g, i)$ denotes the group of isometries of (U_i, g) fixing the embedded hypersurface $i(\Sigma)$. Let us now take a look at multiplication in $B(\mathfrak{G}_\Sigma)$:

$$\begin{aligned}\hat{\rho} * \hat{\sigma}[i] &= \mu\left(\hat{\rho} \circ t \circ \hat{\sigma}[i], \hat{\sigma}[i]\right) = \mu\left(\hat{\rho} \circ t \circ \left[\sigma^1(i), \sigma^0(i)\right], [\sigma^1(i), \sigma^0(i)]\right) \\ &= \mu\left(\hat{\rho}[\sigma^1(i)], [\sigma^1(i), \sigma^0(i)]\right) = \left[\rho^1(\sigma^1(i)), \rho^0(\sigma^1(i))\right]\left[\sigma^1(i), \sigma^0(i)\right] = \left[\rho^1 \circ \sigma^1(i), \sigma^0(i)\right].\end{aligned}$$

Note that these two elements can be multiplied because $[\rho^0(i)] = [i]$ for all embedding. For $\hat{\sigma}$ close to the identity $\varepsilon \in B(\mathfrak{G}_\Sigma)$ in the sense that $[\sigma^1(i)] \in A_i$ so that we can apply $\Psi^1_i(\hat{\rho})$ to $\sigma^1(i)$, and sufficiently small V_u, at the level of the maps Ψ we get:

$$\rho^1 \circ \sigma^1(i_v) = \Psi^1_i(\hat{\rho}) \circ \Psi^1_i(\hat{\sigma}) \circ i_v, \quad \sigma^0(i_v) = \Psi^0_i(\hat{\sigma}) \circ i_v,$$

[1]There might be some difficulties in making this quotient into a well-defined group since $Iso_{loc(i)}(M)$ is *not* a normal subgroup of $\mathit{Diff}_{loc}(M, g, i)$.

which can be represented as follows:

$$\Psi^1_i(\hat{\rho} * \hat{\sigma}) = \Psi^1_i(\hat{\rho}) \circ \Psi^1_i(\hat{\sigma}), \quad \Psi^0_i(\hat{\rho} * \hat{\sigma}) = \Psi^0_i(\hat{\rho}) \circ \Psi^0_i(\hat{\sigma}),$$

and at the level of diffeomorphisms F reads:

$$F_i(\hat{\rho} * \hat{\sigma}) = \left(\Psi^0_i(\hat{\sigma})\right)^{-1} \circ (\Psi^0_i(\hat{\rho}))^{-1} \circ \Psi^1_i(\hat{\rho}) \circ \Psi^1_i(\hat{\sigma}) : M \supseteq U \to U'' \subseteq M,$$

where $U'' = (\Psi^0_i(\hat{\rho} * \hat{\sigma}))^{-1} \circ (\Psi^1_i(\hat{\rho} * \hat{\sigma})(U)$. Hence, at the level of *classes* in $\mathit{Diff}_{loc}(M,g,i)$, the composition in $B(\mathfrak{G}_\Sigma)$ for $\hat{\sigma}$ close to the identity ε and at the neighbourhood of $[i] \in \mathfrak{U}(\Sigma)$ simply corresponds to composition of diffeomorphisms:

$$[F_i(\hat{\rho} * \hat{\sigma})] = [F_i(\hat{\rho}) \circ F_i(\hat{\sigma})].$$

Notice here that we have a natural quotient diffeological structure on $\mathit{Diff}_{loc}(M,g,i)$ coming form the one on $B(\mathfrak{G}_\Sigma)$ and associated to this construction – plots on $\mathit{Diff}_{loc}(M,g,i)$ simply come from those on $\mathit{Diff}_{loc(i)}(M)$ through the maps $F_i(\hat{\sigma})$. Furthermore, since composition in $\mathit{Diff}_{loc}(M,g,i)$ comes from composition of F_i's which is a smooth operation between functional spaces, it is smooth for this diffeology. Finally, the additional requirement that we put on $B(\mathfrak{G}_\Sigma)$ assures the inverse map to be smooth as well, and hence we can conclude that $\mathit{Diff}_{loc}(M,g,i)$ equipped with this diffeology is a *diffeological group*.[1] In a small neighbourhood U_ε of the identity we then have:[2]

$$\boxed{B(\mathfrak{G}_\Sigma) \supseteq U_\varepsilon\big|_{A_i} \cong \mathit{Diff}(U_i,g,i) = \mathit{Diff}(U_i) \Big/ \mathit{Iso}(U_i,g,i)\,.}$$

Notice that the neighbourhoods $U_\varepsilon \in \mathscr{O}_{\mathrm{D}}(B(\mathfrak{G}_\Sigma))$ and $A_i \in \mathscr{O}_{\mathrm{D}}(\mathfrak{U}(\Sigma))$ and $U_i \subseteq M_i$ are interlinked in the sense that U_ε is determined by A_i, which is in turn given by U_i, i.e. an image of a plot going through $[i] \in \mathfrak{U}(\Sigma)$. Since the plot is arbitrary, they should all be thought of as *infinitesimally small*.

[1]See the footnote at the previous page.

[2]It would be interesting to see the connection of this realisation to the ideas described in §11.6 of [3].

2.4 Algebroid

We are now going to determine the tangent space at the identity to the group of bisections $B(\mathfrak{G}_\Sigma)$, whose elements correspond to sections of the algebroid bundle. From [5] we know that it should be determined by the one-dimensional plots.

Since we have the functional-subspace diffeology[1] on $B(\mathfrak{G}_\Sigma)$, such a curve centered at the identity $\varepsilon : \mathfrak{U}(\Sigma) \hookrightarrow \mathfrak{E}(\Sigma)$ is a one-parameter family of sections:

$$c : (-a,a) := I \ni \tau \mapsto c(\tau) = \hat{\sigma}_\tau \in B(\mathfrak{G}_\Sigma),$$

such that the evaluation map:[2]

$$ev_c : I \times \mathfrak{U}(\Sigma) \ni (\tau, [i]) \mapsto \hat{\sigma}_\tau[i] \in \mathfrak{E}(\Sigma)$$

is smooth and $s \circ \hat{\sigma}_\tau[i] = t \circ \hat{\sigma}_0[i] = [i]$. An analysis similar to the one presented in the previous paragraph allows to conclude that, again locally in the sense of picking a small neighbourhood $[i] \in A_i$ and $\tau \in I$, for I being an arbitrarily small neighbourhood of $0 \in \mathbb{R}$, we have a neighbourhood of $\varepsilon \in U_\varepsilon \subseteq B(\mathfrak{G}_\Sigma)$ for which the curve c corresponds to a smooth with respect to τ family of diffeomorphisms $F_i(\tau)$ of an small enough neighbourhood of $i(\Sigma) \subseteq M$, given up the the isometries fixing $i(\Sigma)$. Moreover, for $F_i(0)$ is such an isometry itself and hence $[F_i(0)] = [Id_{U_i}]$. We then have a curve through the identity $[Id_{U_i}] \in Diff(U_i, g, i)$:

$$\left\{ I \ni \tau \mapsto \hat{\sigma}_\tau \in U_\varepsilon \subseteq B(\mathfrak{G}_\Sigma)\big|_{A_i} \right\} \cong \left\{ I \ni \tau \mapsto [F_i(\tau)] \in V_{[Id_{U_i}]} \subseteq Diff(U_i, g, i) \right\}.$$

A tangent vector that is represented by a one-parameter group of diffeomorphisms $[F_i(\tau)]$ is given by pushing forward the tangent vector $\frac{d}{d\tau} \in T_0\mathbb{R}$, and hence can be identified with a vector field on U_i:

$$c^* \left(\frac{d}{d\tau} \right) \bigg|_{A_i} = \frac{d}{d\tau} F_i(\tau)(x) \big|_{\tau=0} \in \mathscr{X}(U)$$

given *up to the isometries preserving* $i(\Sigma) \subseteq U_i$. We can exploit the freedom given by the group $Iso(M, g, i)$ by *partly* fixing coordinates on U_i. Indeed, since U_i can be arbitrarily small, we can put *normal Gaussian coordinates* there. We still have a freedom of choosing coordinates on Σ but when this is done, the coordinates on U_i will be given. The Gaussian normal coordinates are defined as follows. First, we take a normal, future directed vector field along $i(\Sigma)$ and extend it to the neighbourhood by *parallel transport* along the *geodesics* that they define. The flow of this vector field defines a *foliation* of U_i which we can use to define coordinates: $p = (x,t) \in U$ iff p is the point at which we end up when we move along the integral curve of the extended normal vector field that crosses $i(x)$ by the parameter value t. This construction gives a *canonical* isometry:

$$U_i \cong \bar{U}_i \subseteq \Sigma \times \mathbb{R}$$

fixing $i(\Sigma)$. The metric in this representation takes the *Gaussian* form:

$$g(t) = \hat{n} \otimes \hat{n} + \gamma(t),$$

and since $i(\Sigma)$ is fixed by $i(\Sigma) \cong \{(x,0) \in \bar{U}\}$, we have $\gamma(0) = \gamma_{[i]}$. A vector field $\vec{V} \in \mathscr{X}(U_i)$ can be thought of as being given *up to the isometries preserving* $i(\Sigma) \subseteq U_i$ iff it *respects the Gaussian normal coordinates*, which can be grasped in various equivalent ways:

[1] Restricted by the requirement that assures smoothness of the inverse map, see (2.4).

[2] Since we are interested in the curve on the infinitesimal neighbourhood of $0 \in I$, we can take $a > 0$ small enough so that $V_\tau = I$ for all $\tau \in I$, i.e. the evaluation map is well-defined for the whole interval $I = (-a,a)$.

1. It is generated by a diffeomorphism that intertwines the Gaussian time flow, i.e.:
$$\vec{V} = \frac{d}{d\tau}\alpha_\tau\big|_{\tau=0}, \quad \alpha_\tau \circ \Phi^{\hat{n}}_{\Delta t} = \Phi^{\hat{n}}_{\Delta t} \circ \alpha_\tau,$$
where $\hat{n}$ denotes the *extended* normal vector field, i.e. $\Phi^{\hat{n}}_{\Delta t}$ takes (x,t) to $(x,t+\Delta t)$;
2. It respects the normal vector field along $i(\Sigma)$ in the sense that:
$$\mathscr{L}_{\vec{V}} g(\vec{W},\hat{n}) = 0 \quad \forall_{\vec{W}\in\mathscr{X}(U_i)};$$
3. It respects the Gaussian form of the metric [4], i.e. for a metric:
$$g' := g + \mathscr{L}_{\vec{V}} g,$$
we have $g'(\hat{n},\hat{n}) = -1$ and if $g(\hat{n},\vec{W}) = 0$ than also $g'(\hat{n},\vec{W}) = 0$;
4. It respects the Gaussian slicing in the sense that:
$$\mathscr{L}_{\vec{V}} g = \mathscr{L}_X \gamma + \phi\dot{\gamma},$$
where $\vec{V} = X + \phi\hat{n}$ is again the normal tangent decomposition of $\vec{V}$ along $i(\Sigma)$.

The first characterisation is quite clear – such a vector field is effectively given already in the foliated neighbourhood. The equivalence of 1. and 2. is argued in [2], the equivalence of them to 3. in [4]. In [2] it is also proved that such a vector field is determined by its values on $i(\Sigma)$ via:
$$\hat{n}(\phi) = 0, \quad [\hat{n},X] = grad_\gamma(\phi),$$
where $\vec{V} = X + \phi\hat{n}$ is the normal tangent decomposition of $\vec{V}$ along $i(\Sigma)$. This realisation is crucial for determining the algebroid bracket. We find the characterisation 4. of our own[1] very appealing, and we will show here that it is *equivalent* to the requirement just mentioned. We will call vector fields that satisfy 4. above, and hence all the other characterisations, *Gaussian*:[2]

Definition 2.5. *A vector field v defined in the neighbourhood U of a space-like hypersurface $i(\Sigma)$, where $i : \Sigma \hookrightarrow M$ is a proper embedding and (M,g) is a connected, Lorentzian manifold, is called* <u>*Gaussian*</u> *iff it satisfies:*
$$\mathscr{L}_{\vec{V}} g = \mathscr{L}_X \gamma + \phi\dot{\gamma},$$
where $\vec{V} =: X + \phi\hat{n}$ is the tangent-normal decomposition along $i(\Sigma)$ and $\dot{\gamma} := \mathscr{L}_{\hat{n}}\gamma$.

Lemma 2.3. *The Gaussian vector fields are generated by its values on $i(\Sigma)$ via:*
$$\hat{n}(\phi) = 0, \quad [\hat{n},X] = grad_\gamma(\phi).$$

Proof. We will calculate the Lie derivative of the metric for an arbitrary vector field defined in the neighbourhood of the embedded hypersurface. We will perform this analysis in *Gaussian normal coordinates* that we always have on a sufficiently small neighbourhood of $i(\Sigma)$. We then have the metric in the Gaussian form and hence $g(\partial_t,\partial_t) = -1$ so that the normal direction is given by $\hat{n} = \partial_t$. For an arbitrary vector field $\mathscr{X}(U) \ni \vec{V} = X + \phi\partial_t$ we calculate:[3]
$$\mathscr{L}_X(-dt\otimes dt) = 0,$$
$$\mathscr{L}_{\phi\partial_t}(-dt\otimes dt) = -2d\phi\otimes dt,$$
$$\mathscr{L}_X\gamma = \mathscr{L}_X\gamma + 2i_{\frac{\partial X}{\partial t}}\gamma\otimes dt,$$
$$\mathscr{L}_{\phi\partial_t}\gamma = \phi\dot{\gamma}.$$

[1] Although inspired by a similar equation and calculation found in [2].

[2] They are being referred to as "g-gaussian" in [2].

[3] This is actually a bit tricky and can be done by using the formula $\mathscr{L}_X = di_X + i_X d$. We omit "tildas" here for the clarity of notation, it should be clear from the context if we mean the 3-dimensional Riemannian metric or its zero-extension to a 4-dimensional tensor.

Hence calculating the Lie derivative of g with respect to V gives [2]:

$$\begin{aligned}\mathscr{L}_{\vec{V}}g &= \mathscr{L}_{X+\phi\partial_t}(-dt\otimes dt+\gamma)\\ &= \mathscr{L}_X(-dt\otimes dt)+\mathscr{L}_X\gamma+\mathscr{L}_{\phi\partial_t}(-dt\otimes dt)+\mathscr{L}_{\phi\partial_t}\gamma\\ &= \mathscr{L}_X\gamma+2i_{\frac{\partial X}{\partial t}}\gamma\otimes dt-2d\phi\otimes dt+\phi\dot{\gamma}.\end{aligned}$$

We then see that $\vec{V}$ is Gaussian if and only if in Gaussian normal coordinates we have [2]:

$$i_{\frac{\partial X}{\partial t}}\gamma\otimes dt = d\phi\otimes dt \quad\Leftrightarrow\quad \frac{\partial\phi}{\partial t}=0 \;\;\&\;\; \frac{\partial X}{\partial t}=grad_\gamma(\phi).$$

Therefore, any vector field $X+\phi\hat{n}$ along $i(\Sigma)$ can be uniquely extended to a Gaussian vector field on a Gaussian neighbourhood $U\supseteq i(\Sigma)$ by solving the above equations, which in coordinate-free form read:

$$\hat{n}(\phi)=0, \quad [\hat{n},X]=grad_\gamma(\phi).$$

□

The algebroid bracket is given by a Lie algebra bracket of the group of bisections. Since $B(\mathfrak{G}_\Sigma)$ is locally given by the quotient:

$$Diff(U_i)\Big/Iso(U_i,g,i),$$

the Lie bracket should come from the bracket on $Diff(U_i)$ which is just a commutator of vector fields. The algebroid bracket is given by commutator of vector fields generated by the classes of diffeomorphisms, which as we argued can be identified with Gaussian vector fields. As we have seen, such vector fields are uniquely given by their values on the embedded hypersurfaces, and hence can be represented by pairs $(X,\phi)\in\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)$. We can then conclude:

$$\boxed{A_{[i]}\mathfrak{G}_\Sigma\cong\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)}.$$

Moreover, from our construction it is clear that the algebroid bundle is locally trivializable with respect to the D-topology on $\mathfrak{U}(\Sigma)$. Indeed, we could pick any other $u\in U$ from the domain of the plot which image is the D-open in $\mathscr{O}_{\mathsf{D}}(\mathfrak{U}(\Sigma))$ that we are working on, and get another $i=i_u$ for which the analysis can be repeated with the exact same result. Moreover, even though we did not discuss diffeological vector bundles, it is clear that the one we are concerned with admits a global, nowhere vanishing section – we can take any constant section of the form:

$$\mathfrak{U}(\Sigma)\ni[i]\mapsto(X,\phi)\in\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)\quad\forall_{[i]\in\mathfrak{U}(\Sigma)},$$

and hence we actually have a *trivial bundle*:[1]

$$\boxed{A(\mathfrak{G}_\Sigma)\cong\mathfrak{U}(\Sigma)\times\big(\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)\big)}.$$

Denoting[2] by $G_\gamma(X,\phi)$ the Gaussian extension of (X,ϕ) for $\gamma\in\mathsf{Riem}(\Sigma)$ the *algebroid bracket* reads:

$$\boxed{\big\{(X,\phi),(Y,\psi)\big\}_{A(\mathfrak{G}_\Sigma)}[i]=\big[G_{\gamma_{[i]}}(X,\phi),G_{\gamma_{[i]}}(Y,\psi)\big]}$$

[1] This was stated but not proved in [2].

[2] Such an extension does depend only on the metric on $i(\Sigma)$, which is determined by $\gamma=i^*g$ and *not* on the 4-dimensional metric g in the neighbourhood – this is why we change the notation introduced in [2].

The algebroid bundle is then a *fiber vector bundle* with the typical fiber $\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)$ and the bracket given by the commutator of Gaussian extensions of the data given on Σ. Let us finally calculate the commutator of the Gaussian extensions:

$$\begin{aligned}\big[G_\gamma(X,\phi),G_\gamma(Y,\psi)\big] &= \big[X+\phi\hat{n},Y+\psi\hat{n}\big]\\ &= [X,Y]+[X,\psi\hat{n}]_\gamma+[\phi\hat{n},Y]_\gamma+[\phi\hat{n},\psi\hat{n}]\\ &= [X,Y]+X(\psi)\hat{n}-\psi[\hat{n},X]_\gamma+\phi[\hat{n},Y]_\gamma-Y(\phi)\hat{n}\\ &= [X,Y]+\phi grad_\gamma(\psi)-\psi grad_\gamma(\phi)+\big(X(\psi)-Y(\phi)\big)\hat{n},\end{aligned}$$

When we take a pair of *constant* sections, this indeed corresponds to the Poisson bracket structure on the space of constraints of the *ADM* formulation of the Einstein equation, as we have seen it in the previous section:

$$\{(X,\phi),(Y,\psi)\}_{A(\mathfrak{G}_\Sigma)}:\mathfrak{U}(\Sigma)\ni[i]\mapsto\big([X,Y]+\phi grad_{\gamma_{[i]}}(\psi)-\psi grad_{\gamma_{[i]}}(\phi)\ ,\ X(\psi)-Y(\phi)\big).$$

We also developed an alternative way of supporting the equation (2.5) that distinguish the Gaussian vector fields as the right representation of the vectors in the algebroid bundle, which goes as follows. We will first establish the form of the fibers of the algebroid by an independent reasoning – we will show that there is a well-defined normal bundle of our groupoid and that it is of the form just discovered. Let us then take a look at the tangent bundle to the space of Σ-universes. A smooth curve $c:I\to\mathfrak{U}(\Sigma)$ is of the following form:[1]

$$c:I\ni\tau\mapsto[i_\tau]\in\mathfrak{U}(\Sigma)\quad\{ev_c:(\tau,x)\mapsto i_\tau(x)\}\in C^\infty(I\times M,M).$$

Before taking the class, the evaluation map describes a deformation of $i(\Sigma)$ in M. The tangent vector that it generates is then a vector field along $i_0(\Sigma)$, given by:

$$c^*\left(\frac{d}{d\tau}\right)=\frac{d}{d\tau}i_\tau(x)\big|_{\tau=0}\in\Gamma(TM|_{i(\Sigma)}),$$

which can be decomposed into normal and tangential parts:

$$\frac{d}{d\tau}i_\tau(x)\big|_{\tau=0}=X(i(x))+\phi(i(x))\hat{n}(i(x)),$$

where $\hat{n}$ is the normal, future directed vector field along $i(\Sigma)$. Since $i(\Sigma)\cong\Sigma$, X and ϕ can be interpreted as a vector field an a function on Σ. Furthermore, neither the vector field $\frac{d}{d\tau}i_\tau(x)\big|_{\tau=0}$ nor its decomposition depends on the representative in $[i_0]$, and hence we can conclude:[2]

$$T_{[i]}\mathfrak{U}(\Sigma)\cong\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma).$$

Moreover, the tangent space $T\mathfrak{E}(\Sigma)$ is determined in a similar way to be of the form:

$$T_{[i]}\mathfrak{E}(\Sigma)\cong\big(\mathscr{X}\oplus\mathscr{F}(\Sigma)\big)\times\big(\mathscr{X}(\Sigma)\oplus\mathscr{F}(\Sigma)\big)\cong T_{[i]}\mathfrak{U}(\Sigma)\times T_{[i]}\mathfrak{U}(\Sigma),$$

and hence the embedding $\varepsilon:\mathfrak{U}(\Sigma)\ni[i]\mapsto[i,i]\in\mathfrak{E}(\Sigma)$ pushes forward the vectors via:

$$\varepsilon^*_{[i]}:T_{[i]}\mathfrak{U}(\Sigma)\ni(X,\phi)\mapsto\big((X,\phi),(X,\phi)\big)\in T_{[i,i]}\mathfrak{E}(\Sigma).$$

We then have a well-defined notion of the normal bundle:

$$N^\varepsilon\mathfrak{U}(\Sigma)=\frac{T\mathfrak{E}(\Sigma)|_{\varepsilon(\mathfrak{U}(\Sigma))}}{\varepsilon^*(T\mathfrak{U}(\Sigma))}=\frac{(T\mathfrak{U}(\Sigma)\times T\mathfrak{U}(\Sigma))|_{\varepsilon(\mathfrak{U}(\Sigma))}}{\varepsilon^*(T\mathfrak{U}(\Sigma))}\cong\frac{T\mathfrak{U}(\Sigma)\oplus T\mathfrak{U}(\Sigma)}{T\mathfrak{U}(\Sigma)}\cong T\mathfrak{U}(\Sigma).$$

[1] We again shrink I if necessary, and put $M:=M_{i_0}$.

[2] It is actually also a trivial bundle for we can take a constant, nowhere vanishing section as before.

Given that the algebroid bundle is as above, to determine the Lie algebra of $B(\mathfrak{G}_\Sigma)$, we could think of determining the *left-invariant* extensions of the sections of the algebroid bundle:

$$\mathfrak{U}(\Sigma) \ni [i] \mapsto V_{[i]} = (X_{[i]}, \phi_{[i]}) = \left(\frac{d}{d\tau} F_i(\tau)\big|_{\tau=0} \right)\Big|_{i(\Sigma)} \in \mathscr{X}(\Sigma) \oplus \mathscr{F}(\Sigma),$$

understood as vectors in $T_\varepsilon B(\mathfrak{G}_\Sigma)$. As we have seen, the multiplication in $U_\varepsilon \subseteq B(\mathfrak{G}_\Sigma)$ corresponds to multiplication of diffeomorphisms:

$$\hat{\sigma} * \hat{\rho}\big|_{A_i} \cong [F_i(\hat{\sigma}) \circ F_i(\hat{\rho})] \in Diff(U_i, g, i).$$

The left-invariant extension of $V = V^L(\varepsilon)$ at $\hat{\rho} \in B(\mathfrak{G}_\Sigma)$ is then given by:

$$V^L_{[i]}(\hat{\rho}) = (L_{\hat{\rho}})_* V^L_{[i]}(\varepsilon) = (L_{\hat{\rho}})_* V_{[i]} = (L_{\hat{\rho}})_* \frac{d}{d\tau} F_i(\tau)\big|_{\tau=0} = \frac{d}{d\tau} F_i(\hat{\rho}) \circ F_i(\tau)\big|_{\tau=0}.$$

If we now take $\hat{\rho} = \hat{\rho}_{\tau'}$ such that $F_i(\hat{\rho}_{\tau'}) := F_i(\tau')$ for some *fixed* $\tau' \in I$, the above reads:

$$V^L_{[i]}(\hat{\rho}_{\tau'}) = \frac{d}{d\tau} F_i(\hat{\rho}_{\tau'}) \circ F_i(\tau)\big|_{\tau=0} = \frac{d}{d\tau} F_i(\tau') \circ F_i(\tau)\big|_{\tau=0} = \frac{d}{d\tau} F_i(\tau' + \tau)\big|_{\tau=0} = \frac{d}{d\tau} F_i(\tau)\big|_{\tau=\tau'}.$$

Left-invariant extension of $V : [i] \mapsto (X_{[i]}, \phi_{[i]})$ can be then understood as extending *each* $X_{[i]} + \phi_{[i]}\hat{n}$ from $i(\Sigma)$ to its neighbourhood in a way compatible with the above relation:

$$V^L(\hat{\sigma}_\tau) : \mathfrak{U}(\Sigma) \ni [i] \mapsto \left(\frac{d}{d\tau} F_i(\tau) \right)\Big|_{i(\Sigma)} =: (X(\tau), \phi(\tau)) \in A_{[F_i(\tau) \circ i]}(\mathfrak{G}_\Sigma).$$

Notice however that we are only given the vector fields $\frac{d}{d\tau} F_i(\tau)$ *along* the embeddings, i.e. precisely pairs $(X(\tau), \phi(\tau))$, rather than fully fledged vector fields on the neighbourhoods of $i(\Sigma)$'s that we could commute – this is a result of the invariance hat was not already taken care of. Before we fix this, let us make a few simple observations. For simplicity, we assume here that the deformations $F_i(\tau)$ are *time-like* in the sense that the functions $\phi(\tau) \in \mathscr{F}(\Sigma)$ are *non-negative*. Notice first that the Lorentzian metrics at each slice of the deformation:

$$g_i(\tau) := F_i(\tau)^* g|_{\Sigma_\tau}, \quad \Sigma_\tau := \{F_i(\tau)(i(\Sigma)) \mid \tau \in I\} \cong \Sigma,$$

are fixed, and so are the Riemannian metrics on the slices:

$$\gamma_i(\tau) := i^*(F_i(\tau)^* g) = (F_i(\tau) \circ i)^* g \in \mathsf{Riem}(\Sigma).$$

Further, notice that at each slice the Lorentzian metric takes the form:

$$g_i(\tau) = \hat{n} \otimes \hat{n} + \gamma_i(\tau) = \hat{n} \otimes \hat{n} + F_i(\tau)^* \gamma(0),$$

Now, there is a *distinguished* way to represent the left-invariant extension of $V \in T_\varepsilon B(\mathfrak{G}_\Sigma)$ as actual vector fields defined in the neighbourhoods of $i(\Sigma)$'s – we can simply require that the slices of the deformation are glued *according to the deformation vector fields*, i.e. in such a way that the normal direction does not change.[1] The $\frac{d}{d\tau} F_i(\tau)$ becomes then a *smooth vector field* and, omitting the subscripts, we have:

$$\mathscr{L}_{\frac{d}{d\tau} F_i(\tau)} \gamma(\tau) = \mathscr{L}_{X(\tau)} \gamma(\tau) + \phi(\tau) \mathscr{L}_{\hat{n}} \gamma(\tau).$$

for each $\tau \in I$. Having represented the extension as a proper vector field, we can calculate:

$$\mathscr{L}_{\frac{d}{d\tau} F_i(\tau)} g = \frac{d}{d\tau} F(\tau)^* g = \frac{d}{d\tau} \left(\hat{n} \otimes \hat{n} + F(\tau)^* \gamma \right) = \frac{d}{d\tau} F(\tau)^* \gamma = \mathscr{L}_{X + \phi \hat{n}} \gamma = \mathscr{L}_X \gamma + \phi \mathscr{L}_{\hat{n}} \gamma.$$

We have thus established another geometrical perspective on the origin of the bracket structure in question.

[1] This is essentially the choice of a Gaussian representative.

2.5 Anchor

In this paragraph we are going to determine the anchor $a: A(\mathfrak{G}_\Sigma) \to T\mathfrak{U}(\Sigma)$ and show that it satisfies the usual requirements.

We define the anchor to be the map that associates to pairs $(X,\phi) \in A_{[i]}(\mathfrak{G}_\Sigma)$ their Gaussian extensions with respect to $\gamma_{[i]}$, i.e.:

$$a := G_{[\_]} : A_{[i]}(\mathfrak{G}_\Sigma) \ni (X_{[i]}, \phi_{[i]}) \mapsto G_{\gamma_{[i]}}(X_{[i]}, \phi_{[i]}) \cong (X_{[i]}, \phi_{[i]}) \in T_{[i]}\mathfrak{U}(\Sigma).$$

We will show that the anchor defined this way satisfies the usual requirement of the Lie algebroid anchor. Notice first that we have a bracket structure on the space of sections $\Gamma(T\mathfrak{U}(\Sigma))$ given by the commutator of the vector fields, i.e. for another curve:

$$c' : I \ni \tau \mapsto [i'_\tau] \in \mathfrak{U}(\Sigma)$$
$$\{ev_{c'} : (\tau, x) \mapsto i'_\tau(x)\} \in C^\infty(I \times M, M),$$

such that $i_0 = i'_0$ we can put:

$$\big[\frac{d}{d\tau} i_\tau(x), \frac{d}{d\tau} i'_\tau(x)\big]$$

to be the usual commutator of vector fields. Since the algebroid bracket is also given by the commutator of vector fields and the anchor simply picks the Gaussian extensions among arbitrary vector fields, for $\gamma := \gamma_{[i]}$ we indeed have:

$$a \circ \{(X,\phi),(Y,\psi)\}[i] = \big[G_\gamma(X_{[i]}, \phi_{[i]}), G_\gamma(Y_{[i]}, \psi_{[i]})\big] = \big[a \circ (X,\phi)[i], a \circ (Y,\psi)[i]\big].$$

Moreover, we can multiply the sections of the algebroid bundle by smooth functions $f \in C^\infty(\mathfrak{U}(\Sigma), \mathbb{R})$:

$$f(X,\phi) : \mathfrak{U}(\Sigma) \ni [i] \mapsto f[i](X_{[i]}, \phi_{[i]}) = (f[i]X_{[i]}, f[i]\phi_{[i]}).$$

We also have an action of the sections $\mathscr{X}(\mathfrak{U}(\Sigma))$ on those functions, given by:

$$\big(\frac{d}{d\tau} i_\tau\big)(f) := \frac{d}{d\tau} f[i_\tau] \in \mathbb{R},$$

and hence we can write:

$$\begin{aligned}
\{(X,\phi), f(Y,\psi)\}[i] &= \{(X_{[i]}, \phi_{[i]}), f[i](Y_{[i]}, \psi_{[i]})\} \\
&= \big[G_\gamma(X_{[i]}, \phi_{[i]}), G_\gamma(f[i]Y_{[i]}, f[i]\psi_{[i]})\big] \\
&= \big[G_\gamma(X_{[i]}, \phi_{[i]}), f[i]G_\gamma(Y_{[i]}, \psi_{[i]})\big] \\
&= f[i]\big[G_\gamma(X_{[i]}, \phi_{[i]}), G_\gamma(Y_{[i]}, \psi_{[i]})\big] + G_\gamma(X_{[i]}, \phi_{[i]})(f[i])G_\gamma(Y_{[i]}, \psi_{[i]}) \\
&= f\{(X,\phi),(Y,\psi)\}[i] + (a \circ (X,\phi))(f)(Y,\psi),
\end{aligned}$$

where again $\gamma := \gamma_{[i]}$. We can then conclude that the algebroid that we derived:

$$\begin{array}{ccc}
\mathfrak{U}(\Sigma) \times \big(\mathscr{X}(\Sigma) \oplus \mathscr{F}(\Sigma)\big) \cong A(\mathfrak{G}_\Sigma) & \xrightarrow{\;G_{[_]}\;} & T\mathfrak{U}(\Sigma) \cong \mathfrak{U}(\Sigma) \times \big(\mathscr{X}(\Sigma) \oplus \mathscr{F}(\Sigma)\big) \\
 & {\scriptstyle p_0} \searrow \quad \swarrow & \\
 & \mathfrak{U}(\Sigma) &
\end{array}$$

has all the properties of the usual Lie algebroid except that the base space is no longer a manifold. Because of the diffeology we put on the space of Σ-universes, the tangent bundle $T\mathfrak{U}(\Sigma)$ and the anchor are *different* than those presented in [2].

Summary

We have thus succeed in improving the details of the analysis presented in [2] that we found disturbing:

1. The framework of diffeological algebroids was discussed broadly enough for the purpose of the analysis, together with the difficulties that need to be faced when aiming for generality.
2. The diffeological structure was replaced by a simple and natural one that still led to the desired bracket structure.
3. The resulting global structure of the algebroid got simplified. We find the analysis being still far from simple, however we feel that the reasoning we present is less sophisticated and the strategy of deriving the bracket – determine the algebra of the group of bisections – is now fairly straightforward.

The crucial point for our satisfaction is the realisation that we make at the end of 2.3 where we conclude that the local – both in the sense that the sections need to be close to the identity in the sense of the D-topology on $B(\mathfrak{G}_\Sigma)$ *and* that we need to restrict them to a small neighbourhood in $\mathfrak{U}(\Sigma)$ with respect to the D-topology there – structure of the group of bisections is given by a quotient of the group of diffeomorphisms of a neighbourhood of some embedded hypersurface by the group of isometries preserving it. The realisation that we make just after this – that curves through the identity in $B(\mathfrak{G}_\Sigma)$ correspond to curves through the identity in the quotient and hence the bracket structure of the algebroid is actually determined by the *Lie algebra* of this quotient group[1] – we see as a precise justification of the use of Gaussian extensions as generating the algebroid bracket in our approach.

Besides that, basic theory of diffeological spaces was reviewed together with different approaches to diffeological tangent spaces, an introductory text on the sheaf theoretic point of view on the theory of diffeological spaces was produced. Furthermore, the theory of Lie groupoids and Lie algebroids was presented in some detail. The initial value formulation of the Einstein equation was also mentioned together with the original strategy of deriving the constraints and their Poisson bracket structure, which importance have also been discussed.

Let us mention here some of the further research perspectives that we have in mind in this context.

Relativistic dynamics

As noted before, in the light of the 1976 paper "Geometrodynamics regained" by A. Hojman, K. Kuchař and C. Teitelboim [9], where the authors claim that the ADM constraints can be recovered if we assume their bracket structure, the conclusion that might be inferred from the fact that the bracket structure is a *consequence* of the groupoid symmetry could be that the Einstein's theory is *dynamically empty* in the sense that the dynamics is already present in the kinematical setting. Making their points mathematically sound and connecting these ideas, at least on the conceptual level, to the groupoid origin of the bracket could be one of the direction of development of this approach. We find the defining equation of Gaussian vector fields and the differential equations that allow for the unique extension of the data on an embedded hypersurface especially intriguing. It would be very interesting to try to interpret them in the context of the *dynamics* of hypersurfaces deformation and maybe of the Einstein equation.

Notice also that even though we are used to thinking of Σ as a Cauchy surface for the whole spacetimes, we nowhere assume that it is *big*. A natural question would be if the *geodesic equation* could be somehow recovered as a limiting case of the mentioned differential equations.

[1] See the footnote there.

Quantization as Lie integration

The original motivation of Prof. Klaas Landsman for making me look into these structures was the quantization framework developed by him in [12].[1] Let us here introduce briefly this idea.

Consider a finite-dimensional physical system on a configuration space given by the tangent bundle TQ. The spatial, "external" symmetry of this system can be described by a pair groupoid $Q \times Q \rightrightarrows Q$, where a pair $(a,b) \in Q \times Q$ can be understood as possible initial and final 'position' of the system and taking the whole pair groupoid means "it is possible for the system to evolve from one 'position' to any other". The tangent bundle TQ can be seen as the algebroid $A(Q \times Q) \cong TQ$ of this pair groupoid, with the dynamics of the system given through a lagrangian function $L : A(Q \times Q) \to \mathbb{R}$. The quantum counterpart of this system can in turn be understood as given by the non-commutative C^*-algebra generated by the pair groupoid, since we have [12]:

$$C^*(Q \times Q) \cong \mathscr{K}(L^2(Q)),$$

where $\mathscr{K}(L^2(Q))$ is the algebra of compact operators of square-integrable functions on Q. Quantization of a simple system with a configuration space TQ, seen as replacing the Poisson algebra of classical observables by the C^*-algebra of quantum observables, can be then thought of as replacing the commutative algebra $C^\infty(A^*(G), \mathbb{R})$, where G is the groupoid describing the external symmetry of the system, by the non-commutative C^*-algebra $C^*(G)$ generated by this groupoid. Instead of $C^\infty(A^*(G), \mathbb{R})$, we can also consider the algebra of functions vanishing at infinity on the dual of the algebroid as the algebra of classical observables. It turns out[12], that it is isomorphic as a C^*-algebra to the one generated by the algebroid:

$$C_0(A^*(G)) \cong C^*(A(G)),$$

when we consider $A(G)$ as a Lie groupoid with fiber-wise addition as partial multiplication. Quantization of a system can be then understood as literally replacing the algebroid $A(G)$ by the groupoid G, and grasped by the slogan "Quantization is Lie integration". Taking the classical limit is in turn replacing the global structure of a groupoid by its linearized, infinitesimal algebroid version.

It can be argued [7] that the groupoid proposed in [2] is the right generalisation of a pair groupoid description of external symmetries of a simple one-particle system at the classical level to the case of the kinematics of hypersurface deformations. It would be then interesting to see if a construction of an algebra associated to this groupoid parallel to the construction of a C^*-algebra from a Lie groupoid can be performed. It is clear that this cannot be done straightforwardly, since there is *no* integration theory for general diffeological spaces and hence the convolution product cannot be defined in the usual way; A different approach would have to be developed for this purpose. Unfortunately, since we were fully engaged in the analysis presented in this work, we did not pursue this question so far.[2] Progress at this point may lead to an alternative approach to canonical quantization of gravity.

[1] In the context of finite dimensional systems.

[2] We would like to check if the simplified diffeological framework that we suggest at the end of the paragraph 2.2.3, given by changing the site over which the diffeological sheaves are defined to subsets of $\mathbb{R}$ (plus the one-point set and the empty set, of course), so that we only have zero- and one-dimensional plots would help in developing integration theory.

References

[1] J. Baez and A. E. Hoffnung. Convenient categories of smooth spaces. *Transactions of the American Mathematical Society*, 363, 07 2008.

[2] C. Blohmann, M. Fernandes, and A. Weinstein. Groupoid symmetry and constraints in general relativity. *Communications in Contemporary Mathematics*, 15, 03 2010.

[3] C. Blohmann and A. Weinstein. Hamiltonian Lie algebroids. *arXiv e-prints*, page arXiv:1811.11109, Nov 2018.

[4] M. Bojowald, B Suddhasattwa, U. Buyukcam, and F. D'Ambrosio. Hypersurface-deformation algebroids and effective spacetime models. *Phys. Rev.*, D94(10):104032, 2016.

[5] J. Christensen and E. Wu. Tangent spaces and tangent bundles for diffeological spaces. *arXiv e-prints*, page arXiv:1411.5425, Nov 2014.

[6] John Daniel Christensen, Gord Sinnamon, and Enxin Wu. The d-topology for diffeological spaces. *Pacific Journal of Mathematics*, 272, 02 2013.

[7] J. Głowacki. Groupoid symmetry, constraints and quantization of General Relativity. *arXiv e-prints*, page arXiv:1905.03152, May 2019.

[8] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011.

[9] S. Hojman, K. Kuchar, and C. Teitelboim. Geometrodynamics Regained. *Annals Phys.*, 96:88–135, 1976.

[10] P. Iglesias-Zemmour. *Diffeology*. American Mathematical Society, 2013.

[11] P. Johnstone. *Sketches of an elephant: a Topos theory compendium*. Oxford logic guides. Oxford Univ. Press, New York, NY, 2002.

[12] N.P. Landsman. *Mathematical Topics between Classical and quantum mechanics*. Springer monographs in mathematics, 1998.

[13] K. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*. Cambridge University Press, 2005.

[14] S. MacLane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory*. Universitext. Springer, Berlin, 1992.

[15] nLab authors. diffeological space. `http://ncatlab.org/nlab/show/diffeological\%space`, December 2018.

[16] nLab authors. coverage. `http://ncatlab.org/nlab/show/coverage`, July 2019. Revision 39.

[17] nLab authors. smooth set. `http://ncatlab.org/nlab/show/smooth%20set`, July 2019. Revision 18.

[18] A. Schmeding and C. Wockel. The lie group of bisections of a lie groupoid. *Annals of Global Analysis and Geometry*, 09 2014.

[19] A. Schmeding and C. Wockel. (re)constructing lie groupoids from their bisections and applications to prequantisation. *Differential Geometry and its Applications*, 49:227–276, 08 2016.

[20] R. Wald. *General relativity*. Chicago Univ. Press, Chicago, IL, 1984.