Bachelor Thesis

Compact perturbations of operators on Hilbert spaces

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July 20, 2017

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Contents

1 Introduction

Initially I set out to find similarities between some of Weyl's theorems. The first of which is a duo of theorems that uses sequences to approximate elements of the spectrum, the second is a theorem that states: any two self-adjoint that differ a compact operator have a distinct part of the spectrum that align, and the third states that any self-adjoint operator differs a small compact operator from a diagonal operator. One can directly see the similarity of having two operators differ a compact operator. As time went on, this search for similarities evolved to a more broader search for compact perturbations, which can be used in the so called Calkin algebra, the algebra of bounded linear operators modulo compact operators.

The space of bounded linear operators ($\mathcal{B}(\mathcal{H})$) inherits a natural algebraic structure. There exists a natural addition and multiplication on operators and with the addition of the *adjoint* a algebraic structure natural to bounded linear operators arises, there are many variations of these algebra, we shall be looking at C^*-algebras. With this algebraic interpretation we shall prove a generalization of one of Weyl's theorems and give a interpretation for the Calkin algebra.

The term C^*-algebra was introduced by I. E. Segal in 1947 to describe norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$. In his paper Segal defines a C^*-algebra as a "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space".

A familiarity with the rudiments of functional analysis is required to read this thesis. To avoid any confusion in notation the first chapter will consist of notation standards and basic concepts of operator theory, specifically properties of compact operators and C^*-algebras.

2 Preliminaries

For this paper a basis in Hilbert spaces, operator theory and C^*-algebra is required. Therefore we start with some preliminaries regarding these subjects. A basic understanding of linear algebra and function analysis is required to fully understand the language used in this text, as know properties are used. These preliminaries will also clear up any misconceptions regarding notation.

Not every theorem in the preliminaries will be proven. Any of these proofs can be found in [1].

2.1 Hilbert spaces and Operator theory

We start of with one massive definition regarding Hilbert spaces, operators and the adjoint.

Definition 2.1. *A* ***Hilbert space*** *is a vector space with an inner product $\langle\cdot,\cdot\rangle$ such that any Cauchy sequence is convergent in the metric generated by said inner product. This metric is generated in following way:*

$$d(x, y) = \sqrt{\langle x - y, x - y\rangle}.$$

In this thesis the inner product will be linear in the first entry. In a Hilbert space ***unit vectors*** *are vectors with length 1, that is $\|x\| = 1$. Two elements $x, y \in \mathcal{H}$ are called* ***orthonormal*** *if*

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

$\mathcal{B}(\mathcal{H})$ is the space of all continuous linear operators[1] *on a Hilbert space, with an norm*

$$\|A\| = \sup\{\|Ax\| : x \text{ is a unit vector in } \mathcal{H}\},$$

where $\|Ax\| := \sqrt{\langle Ax, Ax\rangle}$ is the norm of Ax. The space $\mathcal{B}(\mathcal{H})$ can also be regarded as the space of bounded linear operators, in the sense that $\|Ax\| \leq C\|x\|$ for some $C > 0$ for all $x \in \mathcal{H}$.

Any Hilbert space has a orthonormal ***basis****; a net $(v_\lambda)_{\lambda\in\Lambda}$ of orthonormal elements such that any $x \in \mathcal{H}$ can be written as*

$$x = \sum_{\lambda\in\Lambda} \langle x, v_\lambda\rangle v_\lambda.$$

If such a base is countable we can use sequences instead and we say that a Hilbert space is ***separable*** *when the space has a countable base. We quickly note that for a Hilbert space $\mathcal{H}$ to be separable is equivalent to the topological definition; $\mathcal{H}$ contains a countable dense subset. Since any closed linear subspace of $\mathcal{H}$ is a Hilbert space, any closed linear subspace has a basis on it own.*

The ***spectrum*** *of an operator A is the set*

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\}$$

The complement of the spectrum of A is called the ***resolvent*** *of A written $\rho(A)$.*

The ***adjoint*** *of an operator A is the (unique) operator A^* that has the property:*

$$\forall x, y \in \mathcal{H} : \langle Ax, y\rangle = \langle x, A^*y\rangle.$$

[1] An operator is another term for function between vector spaces

Some interesting classes of operators are:

***Self-adjoint**: $A = A^*$;*

***Normal**: $NN^* = N^*N$;*

***Unitary**: $UU^* = U^*U = I$;*

***Projection**: $A = A^* = A^2$;*

***Diagonalizable**: there exists an orthonormal basis consisting of eigenvectors of T.*

Denote $\mathcal{B}(\mathcal{H})_{sa}$ by the space of all self-adjoint operators.

Lemma 2.2 (**Cauchy-Schwarz inequality**). *for any $x, y \in \mathcal{H}$ we have the inequality*

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

Lemma 2.3 (**Pythagorean theorem**). *Let $(v_i)_{i=1}^k$ be a orthonormal set; then for all $x \in \mathcal{H}$,*

$$\|x\|^2 = \sum_{i=1}^k |\langle x, v_i\rangle|^2 + \|x - \sum_{i=1}^k \langle x, v_i\rangle v_i\|^2.$$

Lemma 2.4 (**Bessel's inequality**). *For a separable Hilbert space $\mathcal{H}$ and a orthonormal sequence $(e_n)_{n=1}^\infty$ the inequality*

$$\sum_{n=1}^\infty |\langle x, e_n\rangle|^2 \leq \|x\|^2,$$

holds for all $x \in \mathcal{H}$.

Corollary 2.5. *For a separable Hilbert space $\mathcal{H}$, a orthonormal sequence $(e_n)_{n=1}^\infty$ in $\mathcal{H}$ and any $x \in \mathcal{H}$ we have*

$$\lim_{n\to\infty} \langle x, e_n\rangle = 0.$$

Proposition 2.6. *For a nonempty subset $Y \subseteq \mathcal{H}$ the set*

$$Y^\perp := \{x \in \mathcal{H} \mid \forall y \in Y : \langle x, y\rangle = 0\}$$

of all elements perpendicular to Y, is a closed linear subspace of $\mathcal{H}$.

As a result of this we have te following theorem.

Theorem 2.7. *If $Y \subseteq \mathcal{H}$ is a nonempty closed linear subset, then*

$$\mathcal{H} = Y \oplus Y^\perp,$$

in the sense that for any $x \in \mathcal{H}$ there exist unique points $x_1 \in A, x_2 \in Y^\perp$ such that $x = x_1 + x_2$.

Proposition 2.8. *For any $A \in \mathcal{B}(\mathcal{H})$ the kernel of A is a closed linear subspace of $\mathcal{H}$.*

This result will be used by combining it with the previous theorem, that is, for any $A \in \mathcal{B}(\mathcal{H})$ we have a decomposition

$$\mathcal{H} = \ker(A) \oplus \ker(A)^\perp$$

Much of the structure of a Hilbert space $\mathcal{H}$ and the space $\mathcal{B}(\mathcal{H})$ comes from the inner product, especially when in comes to properties of the adjoint. As such, there is a method to prove that two operators are the same, using the inner product. This is most useful to prove statements like $A = A^*$.

Lemma 2.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then $A = B$ if and only if $\langle Ax, y\rangle = \langle Bx, y\rangle$ for all $x, y \in \mathcal{H}$.*

Proof. From left to right the proof follows directly from the fact that $Ax = Bx$ for all x. For the converse, we let $x \in \mathcal{H}$ be arbitrary, we show that $Ax = Bx$. Choose $y := (A - B)x$, then we have the following line of implications.

$$\langle Ax, y\rangle = \langle Bx, y\rangle \Rightarrow \langle (A-B)x, y\rangle = 0 \Rightarrow \langle (A-B)x, (A-B)x\rangle = 0 \Rightarrow \|(A-B)x\| = 0 \Rightarrow Ax = Bx$$

And so $A = B$. □

To get a feeling for eigenvalues and diagonal operators, we prove the following theorem.

Theorem 2.10. *Suppose $\mathcal{H}$ is a separable Hilbert space. Then $A \in \mathcal{B}(\mathcal{H})$ is diagonalizable if and only if there is an orthonormal basis $(v_n)_{n=1}^\infty$ for $\mathcal{H}$ and a bounded sequence $(\lambda_n)_{n=1}^\infty$ such that*

$$Ax = \sum_{n=1}^\infty \lambda_n \langle x, v_n\rangle v_n \qquad \forall x \in \mathcal{H}, \tag{2.1}$$

in which case $\lambda_1, \lambda_2, \lambda_3, \ldots$ are exactly the eigenvalues of A.

Proof. Suppose A is diagonalizable. Choose an orthonormal basis $(v_n)_{n=1}^\infty$ of eigenvectors with corresponding eigenvalues $(\lambda_n)_{n=1}^\infty$. Then for $x \in \mathcal{H}$ we have

$$x = \sum_{n=1}^\infty \langle x, v_n\rangle v_n,$$

and so

$$Ax = \sum_{n=1}^\infty \langle x, v_n\rangle A v_n = \sum_{n=1}^\infty \langle x, v_n\rangle \lambda_n v_n,$$

since A is continuous. Now we need the sequence $(\lambda_n)_{n=1}^\infty$ to be bounded:

$$|\lambda_n| = \|\lambda_n v_n\| = \|Av_n\| \leq \|A\|\|v_n\| = \|A\|,$$

and so $\sup\{|\lambda_n, n \in \mathbb{N}\} \leq \|A\|$.

Conversely suppose A is of the form in (2.1), then for the eigenvalues we have

$$Av_j = \sum_{n=1}^\infty \langle v_j, v_n\rangle \lambda_n v_n = \lambda_j v_j,$$

since all eigenvectors are orthonormal. This means that $v_1, v_2, v_3, \dots$ are eigenvectors of A and $\lambda_1, \lambda_2, \lambda_3, \dots$ are eigenvalues of A. Now we only need to prove that A has no other eigenvalues, for in that case $B = \{v \mid v \text{ is a eigenvector of A}\}$ is not an orthonormal base.

Suppose $Ax = \lambda x$. Writing $x = \sum_{n=1}^{\infty} \langle x, v_n \rangle v_n$ and $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n$ we get the countable set of equations

$$\lambda_n \langle x, v_n \rangle v_n = \lambda \langle x, v_n \rangle v_n \qquad \forall n \in \mathbb{N},$$

$x \neq 0$, thus there exists a n_0 such that $\langle x, v_n \rangle \neq 0$. And so $\lambda_{n_0} = \lambda$. This means that $x \in B$ and $\lambda \in (\lambda_n)_{n=1}^{\infty}$, and so there are no further eigenvalues and eigenvectors.

□

Another decomposition we use is a decomposition into two self-adjoint operators. We shall use this for the Weyl-von Neumann-Berg theorem.

Proposition 2.11. *Any operator N can be written as $N = A + iB$ where A and B are self-adjoint operators. Moreover, if N is normal then A and B commute.*

This proposition can be proven directly by setting $A = \frac{N+N^*}{2}$ and $B = \frac{N-N^*}{2i}$.

Theorem 2.12 (Banach inverse mapping theorem). *If $T \in \mathcal{B}(\mathcal{H})$ is bijective then the inverse also lies in $\mathcal{B}(\mathcal{H})$.*

Theorem 2.13. *The spectrum of an operator is a non-empty compact set in $\mathbb{C}$. Moreover if $A = A^*$ then $\sigma(A) \subseteq \mathbb{R}^+$.*

2.2 Compact operators

There exist a unique norm-closed ideal in the space of bounded linear operators on a Hilbert space. This is the ideal of so called compact operators.

Definition 2.14. *An operator $K \in \mathcal{B}(\mathcal{H})$ is **compact** if for any sequence of unit vectors $(x_n)_{n=1}^{\infty}$ the sequence $(Kx_n)_{n=1}^{\infty}$ has a Cauchy subsequence.*

Although the word compact cannot be found directly in the definition, it is closely related to the compactness of sets. Indeed, for a Hilbert space a subset Y is compact if and only if it is sequentially compact (every sequence has a convergent/Cauchy subsequence). Therefore another equivalent definition of a compact operator is to say that the image of the unit sphere in $\mathcal{H}$ is a compact set.

Proposition 2.15. *For any compact operators K and K', bounded operator B, and scalar λ, the following properties hold:*

1) $K + K'$ is compact;

2) λK is compact;

3) BK and KB are compact.

This makes $\mathcal{K}(\mathcal{H})$, the set of all compact operators on $\mathcal{H}$, an ideal on $\mathcal{B}(\mathcal{H})$. This can be pushed further as $\mathcal{K}(\mathcal{H})$ is a closed ideal of $\mathcal{B}(\mathcal{H})$, which follows directly from the following proposition.

Proposition 2.16. *If $K \in \mathcal{B}(\mathcal{H})$ is the norm-limit of compact operators, then K is compact.*

Definition 2.17. *The **rank** of an operator T is the dimension of its range. Furthermore*

$$\mathcal{F}(\mathcal{H}) = \{F \in \mathcal{B}(\mathcal{H}) \mid F \text{ has finite rank } \}.$$

When a Hilbert space is finite dimensional, it always isomorphic (as a Hilbert space) to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Any operator on $\mathbb{C}^n$ can be represented as a matrix, and any matrix A is compact. Indeed if $(x_n)_{n=1}^{\infty}$ is a sequence of unit vectors then $(Ax_n)_{n=1}^{\infty}$ is a bounded sequence, which has a convergent subsequence by the Bolzano-Weierstrass theorem. Consequently if the rank of an operator is finite, it is compact.

Lemma 2.18. *Any finite rank operator $T \in \mathcal{B}(\mathcal{H})$ is compact.*

Proof. The rank of T is finite so for some $n \in \mathbb{N}$ we have $\text{Ran}(T) \overset{\Phi}{\cong} \mathbb{C}^n$. Now T is bounded so for a sequence of unit vectors $(x_n)_{n=1}^{\infty}$ the sequence $(\Phi(Tx_n))_{n=1}^{\infty}$ is contained in a closed ball $B_R(0)$ for some $R > 0$. Closed balls in $\mathbb{C}^n$ are compact and thus sequentially compact, so $(\Phi(Tx_n))_{n=1}^{\infty}$ has a convergent subsequence in $\mathbb{C}^n$. The function ψ is an isomorphism and thus $(Tx_n)_{n=1}^{\infty}$ has a convergent subsequence in $T(\mathcal{H})$. This makes T a compact operator. □

Theorem 2.19. *Any compact operators is the norm-limit of a sequence of finite rank operators. In other words, $\overline{\mathcal{F}(\mathcal{H})} = \mathcal{K}(\mathcal{H})$ with respect to the norm.*

A proof can be found in [1] (Theorem 4.11).

Theorem 2.20 (**Schauder's theorem**). *An operator is compact if and only if its adjoint is compact.*

Proof. We have $A^{**} = A$ as a consequence of lemma 2.9, so the implication from left to right is sufficient. Suppose A is a compact operator and $(x_n)_{n=1}^{\infty}$ is a sequence of unit vectors. The sequence $(AA^*x_n)_{n=1}^{\infty}$ must contain a Cauchy subsequence $(AA^*x_{n_k})_{k=1}^{\infty}$, as A is compact implies that AA^* is compact by proposition 2.15.(3). Comparing two terms in the subsequence $(A^*x_{n_k})_{k=1}^{\infty}$ of the sequence $(A^*x_n)_{n=1}^{\infty}$ we find, with the Cauchy–Schwarz inequality,

$$\begin{aligned}\|A^*(x_{n_k} - x_{n_m})\|^2 &= \langle AA^*(x_{n_k} - x_{n_m}), x_{n_k} - x_{n_m}\rangle \\ &\leq \|AA^*(x_{n_k} - x_{n_m})\| \cdot \|(x_{n_k} - x_{n_m})\| \\ &\leq 2\|AA^*(x_{n_k} - x_{n_m})\| \to 0,\end{aligned}$$

as $\|(x_{n_k} - x_{n_m})\| \leq 2$ since they are unit vectors. Consequently $(A^*x_{n_k})_{k=1}^{\infty}$ is a Cauchy subsequence of $(A^*x_n)_{n=1}^{\infty}$. □

Theorem 2.21. *The set of finite rank operators $\mathcal{F}(\mathcal{H})$ is contained in any non-trivial ideal $\mathfrak{J}$ of $\mathcal{B}(\mathcal{H})$.*

This theorem is proven in [5] (theorem 5.2.1). As $\overline{\mathcal{F}(\mathcal{H})} = \mathcal{K}(\mathcal{H})$ this theorem extends to the following corollary.

Corollary 2.22. *The set $\mathcal{K}(\mathcal{H})$ is contained in any non-trivial norm-closed ideal $\mathfrak{J}$ of $\mathcal{B}(\mathcal{H})$.*

It turns out that this can be pushed even further. The set of compact operators is unique in the sense that it is the only proper closed ideal in $\mathcal{B}(\mathcal{H})$.

Theorem 2.23. *IF $\mathfrak{J}$ is a proper closed ideal in $\mathcal{B}(\mathcal{H})$, then $\mathfrak{J} = \mathcal{K}(\mathcal{H})$.*

A proof can be found in [5] (lemma 5.4.20).

2.3 C*-algebras and the Calkin algebra

The space of bounded linear operators can be interpreted as an algebra. There already exist a addition and multiplication operation on operators in $\mathcal{B}(\mathcal{H})$, and the idea of an adjoint on operators can be generalized to a more general linear convolution. This gives the generalization of $\mathcal{B}(\mathcal{H})$ to so called C^*-algebra's. This generalization is useful to this thesis as it creates an interpretation for quotients such as the quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Definition 2.24. *A **Banach algebra** is a complex normed algebra $\mathfrak{A}$ that is complete (as a metric space with respect to the norm) and satisfies*

$$\|AB\| \leq \|A\|\|B\| \qquad \textit{for all} \qquad A, B \in \mathfrak{A}$$

*A **Banach $*$-algebra** is a Banach algebra $\mathfrak{A}$ with a conjugate linear convolution $*$(called the **adjoint**) that is anti-isomorphic. That is, for all A, B in $\mathfrak{A}$ and $\lambda \in \mathbb{C}$ we have,*

$$\begin{aligned}(A + B)^* &= A^* + B^* \\ (\lambda A)^* &= \overline{\lambda} A^* \\ A^{**} &= A \\ (AB)^* &= B^* A^*\end{aligned}$$

A Banach $*$-algebra is called a ***C*-algebra*** *if it has the following additional condition:*

$$\|A^*A\| = \|A\|^2 \qquad \textit{for all} \qquad A \in \mathfrak{A}.$$

For a C^*-algebra $\mathfrak{A}$ we denote by $\mathfrak{A}_{sa}$ the C^*-subalgebra of self-adjoint elements. A subset $\mathfrak{J} \subseteq \mathfrak{U}$ is called a ***subalgebra*** *of $\mathfrak{A}$ if it is closed under addition and multiplication.*

The term C^*-algebra was introduced in 1947 by I.E. Segal to describe norm-closed subalgebras of $\mathcal{B}(\mathcal{H})$. Consequently, it will be to no surprise that we have the following proposition.

Proposition 2.25. *$\mathcal{B}(\mathcal{H})$ is a C^*-algebra, with the adjoint and norm as in Definition 2.1. Moreover the ideal of compact operators is a subalgebra of $\mathcal{B}(\mathcal{H})$.*

Now we want the quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ to have the same structure as the original space $\mathcal{B}(\mathcal{H})$.

Proposition 2.26. *For a C^*-algebra $\mathfrak{A}$ with subalgebra $\mathfrak{J}$ the quotient $\mathfrak{A}/\mathfrak{J}$ is once again a C^*-algebra.*

A proof of this proposition can be found in [3] (Theorem I.5.4).

Definition 2.27. *The quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the* ***Calkin algebra****. When two cosets $T_1 + \mathcal{K}(\mathcal{H})$ and $T_2 + \mathcal{K}(\mathcal{H})$ are equal we denote $T_1 =_c T_2$.*

A large part of this thesis is focused on compact perturbations i.e. operators T_1 and T_2 that can be written as

$$T_1 = T_2 + K,$$

where K is a compact operator. This means that each time we find such a perturbation we have

$$T_1 =_c T_2.$$

i.e. $T_1 = T_2$ in the Calkin algebra.

3 Approximating the spectra of self-adjoint operators

3.1 The full spectrum

When a Hilbert space is finite dimensional the entire spectrum consists eigenvalues, as any injective matrix is also surjective, and vice versa. This property does not hold in the infinite dimensional case, and therefore for an element $\lambda \in \sigma(A)$ the operator $A - \lambda$ is *either* non-injective or non-surjective. This leads to the following decomposition of the spectrum:

Definition 3.1. $\sigma_p(A) = \{\lambda \in \sigma(A) \mid A - \lambda$ *is not injective*$\}$ *is called the* ***point spectrum****. Points in the point spectrum are called* ***eigenvalues****.*
$\sigma_c(A) = \{\lambda \in \sigma(A) \mid A - \lambda$ *is injective and* $\mathrm{Ran}(A - \lambda) \subsetneq \mathcal{H}$ *is dense*$\}$ *is called the* ***continuous spectrum****.*
$\sigma_r(A) = \{\lambda \in \sigma(A) \mid A - \lambda$ *is injective and* $\mathrm{Ran}(A - \lambda) \subsetneq \mathcal{H}$ *is not dense*$\}$ *is called the* ***residual spectrum****.*

We can interpret an element $\lambda \in \sigma_c(A)$ as an element where $A - \lambda$ is nearly invertable. The operator might not be surjective, but it comes pretty close. Now $\sigma(A)$ is the disjoint union of these three sets:

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

Proposition 3.2. *If A is self-adjoint, then $\sigma_r(A) = \emptyset$.*

This proposition also holds for normal operators. However, considering this thesis is focused on self-adjoint operators, we only proof the statement for $A = A^*$.

Proof. Suppose $\lambda \in \sigma(A)$ and that $A - \lambda$ is injective. We show that $\mathrm{Ran}(A - \lambda)$ must be dense. Firstly we note that $\lambda = \overline{\lambda}$ by theorem 2.13. Now the operator $A - \lambda$ is self-adjoint, as

$$(A - \lambda)^* = A^* - \overline{\lambda} = A - \lambda.$$

Note the identity

$$\overline{\mathrm{Ran}(B)} = \ker(B^*)^{\perp},$$

which holds for any bounded operator B. Then if $\ker((A - \lambda)^*) = \{0\}$ we have $\overline{\mathrm{Ran}(A - \lambda)} = \{0\}^{\perp} = \mathcal{H}$. Since $A - \lambda$ is self-adjoint and injective, we have

$$\ker((A - \lambda)^*) = \ker(A - \lambda) = \{0\}.$$

□

Example. In the finite dimensional case every element in the spectrum will always be an eigenvalue, i.e. $\sigma_p(A) = \sigma(A)$. This does not have to happen in the infinite case. Let us look at an example where the point spectrum is even empty.

Suppose $H = L^2([0, 1], \mu)$ is the space of quadratic integrable function on the interval $[0, 1]$ with respect to the Lebesgue measure μ. We look at the multiplication operator T where

$$(Tf)(x) := x \cdot f(x),$$

for all $x \in [0,1]$ and all integratable functions f. The operator T is bounded as the interval $[0,1]$ is bounded. As the interval is real, the operator is self-adjoint. Indeed, for $f, g \in \mathcal{H}$ we see

$$\begin{aligned} \int f(x)(T^*g)(x)\ d\mu(x) &= \langle f, T^*g \rangle \\ &= \langle Tf, g \rangle \\ &= \int xf(x)g(x)\ d\mu(x), \end{aligned}$$

which implies that

$$T^*g(x) = xg(x)\ \mu\text{-a.e.},$$

and so $T^*g = Tg$ in $\mathcal{H}$. We claim that T has no eigenvalues, since for $f \neq 0$ (which really means: $f(x) \neq 0\ \mu -$ a.e.), we have

$$(T - \lambda)f = 0 \Leftrightarrow (x - \lambda)f(x) = 0\ \forall x\ \mu - \text{a.e.}$$

As $x - \lambda \neq 0$ for all but one ($x = \lambda$), the operator $T - \lambda$ can not be injective, so $\sigma_p(T) = \emptyset$. Furthermore T is self-adjoint and so by proposition 3.2 we have $\sigma_c(A) = \sigma(A)$.

This operator is not diagonalizable, for one can not make a base of eigenvectors if there are no corresponding eigenvalues.

Now we prove that every element λ in the spectrum can be approximated with a sequence of element in $\mathcal{H}$ that are very close to being eigenvectors of λ.

Theorem 3.3. *Let A be self-adjoint. Then $\lambda \in \sigma(A)$ if and only if there exist a sequence $(u_n)_{n=1}^{\infty}$ such that $\|u_n\| = 1$ and $\|(A - \lambda)u_n\| \to 0$*

Proof. for $\lambda \in \sigma(A)$ the residual spectrum is empty so only two cases arise:

(a) $\ker(A - \lambda) \neq \{0\}$ (i.e. λ is an eigenvalue). Then $\forall n \in \mathbb{N}$ let $u_n = f$ for any fixed $f \in \ker(A - \lambda)$ with $\|f\| = 1$.

(b) $\ker(A - \lambda) = \{0\}$ and $\text{Ran}(A - \lambda)$ is dense but not equal to $\mathcal{H}$. We may choose a sequence $(v_n)_{n=1}^{\infty}$ in $\text{Ran}(A-\lambda)$ with $\|v_n\| = 1$ and $\|(A-\lambda)^{-1}v_n\| \to \infty$. Such a sequence can be found by taking a sequence with limit $x \in \mathcal{H}\backslash\text{Ran}(A - \lambda)$ which is nonempty by assumption such that $\|x\| = 1$. Using the sequence $(v_n)_{n=1}^{\infty}$ we can define $u_n = \frac{(A-\lambda)^{-1}v_n}{\|(A-\lambda)^{-1}v_n\|}$ for all n. Now $\|u_n\| = 1$ and

$$|(A - \lambda)u_n\| = \frac{\|v_n\|}{\|(A - \lambda)^{-1}v_n\|} \to 0$$

Conversely, let $\lambda \in \rho(A)$. Then the Banach inverse mapping theorem states: $(A - \lambda)^{-1}$ is bounded. Choose $M > 0$ such that

$$\forall v \in \mathcal{H} : \|(A - \lambda)^{-1}v\| \leq M\|v\|.$$

If we suppose that there is a sequence $(u_n)_{n=1}^{\infty}$ such that $\|u_n\| = 1$ and $\|(A - \lambda)u_n\| \to 0$ then in particular we have

$$1 = \|u_n\| = \|(A - \lambda)^{-1}(A - \lambda)u_n\| \leq M\|(A - \lambda)u_n\| \to 0,$$

this would mean $1 \leq 0$, which is a contradiction. □

The only consequence of $A = A^*$ that we used in the previous theorem is proposition 3.2, and so the theorem would also hold for normal operators, as remarked below proposition 3.2.

3.2 The essential spectrum and Weyl's criterion

We can push the result of theorem 3.3 a bit further for a specific part of the spectrum. Using the same argumentation in theorem 3.3, a classification arises for the so called *essential* spectrum.

Definition 3.4. *The* ***discrete spectrum*** *of an operator A is the set of all eigenvalues of A with finite multiplicity that are isolated points of $\sigma(A)$. More formally*

$$\sigma_d(A) = \{\lambda \in \sigma_p(A) \mid \dim\ker(A-\lambda) < \infty \text{ and } \exists\varepsilon > 0 \text{ such that } \forall z \in \sigma(A)\backslash\{\lambda\} \text{ we have } |z-\lambda| \geq \epsilon\}.$$

The ***essential spectrum*** *is defined as the complement of $\sigma_d(A)$ i.e.*

$$\sigma_{ess}(A) = \sigma(A)\backslash\sigma_d(A).$$

In this paragraph we shall prove a version of theorem 3.3 for the essential spectrum. To prove this we shall not be using the essential spectrum directly but instead we look at whether the operator $A - \lambda$ has an unbounded inverse. This brings along a problem. As when λ lies in the spectrum, the operator $A - \lambda$ does not have a proper inverse, so we must tweak the definition of inverse.

Remark. We say that the inverse of an operator A exists (as a possibly not bounded operator) if A is injective. We do so by restricting the domain of A^{-1} to the range of A. That is, there is a function A^{-1} such that

$$\begin{aligned} A^{-1}\colon \operatorname{Ran}(A) &\to \mathcal{H} \\ Ax &\mapsto x. \end{aligned}$$

It should be noted that for any self-adjoint (and normal) operator any element in the essential spectrum is either surjective of very close to being surjective, as any element in the continuous spectrum has produces an operator with dense range in $\mathcal{H}$.

Proposition 3.5. *Let λ be a isolated point of $\sigma(A)$ and A be self-adjoint. Then $(A - \lambda)$ restricted to $\ker(A - \lambda)^\perp$ has a bounded inverse.*

A proof of this proposition an the following theorem can be found in [2] (proposition 6.6 and theorem 6.7).

Theorem 3.6. *Let A be self-adjoint and $\lambda \in \sigma(A)$. Then $\lambda \in \sigma_d(A)$ if and only if $\ker(A - \lambda)$ is finite-dimensional and $A - \lambda$ restricted to $\ker(A - \lambda)^\perp$ has a bounded inverse.*

Looking at the negation of this statement we find the following corollary.

Corollary 3.7. *Let $\ker(A - \lambda)$ be finite-dimensional. Then $\lambda \in \sigma_{ess}(A)$ if and only if $A - \lambda$ restricted to $\ker(A - \lambda)^\perp$ has an unbounded inverse.*

Here *unbounded* means not bounded in the sense an operator B is unbounded if for any $n \in \mathbb{N}$ there exists a $v_n \in \mathcal{H}$ such that $\|Bv_n\| > n$.

Definition 3.8. *A sequence $(u_n)_{n=1}^\infty$ converges* ***weakly*** *to $u \in \mathcal{H}$ if for each $v \in \mathcal{H}$ we have $\langle u_i, v\rangle \to \langle u, v\rangle$. We write $u_n \xrightarrow{w} u$. A sequence $(u_n)_{n=1}^\infty$ converges* ***strongly*** *to u if $\|u_i - u\| \to 0$. Here we write $u_n \xrightarrow{s} u$.*

We note that strong convergence implies weak convergence and strong convergence is the same as regular convergence, now written differently as to not be confused with weak convergence.

Definition 3.9. *A sequence* $(u_n)_{n=1}^{\infty}$ *is called a* ***Weyl sequence*** *for* A *and* λ *if* $\|u_n\| = 1$, $u_n \xrightarrow{w} 0$ *and* $(A - \lambda)u_n \xrightarrow{s} 0$.

Note that $u_n \xrightarrow{w} 0$ if and only if $\langle u_n, y\rangle \to 0$ for all $y \in \mathcal{H}$, as $\langle 0, y\rangle = 0$ for all $y \in \mathcal{H}$. We shall show that an element $\lambda \in \sigma_{ess}(A)$ if and only if there exists a Weyl sequence for A and λ. To prove this, we need a couple of propositions.

Proposition 3.10. *Suppose* K *is a compact operator,* $u_n \xrightarrow{w} 0$, *and* $\|u_n\| = 1$ *for all* $n \in \mathbb{N}$. *Then* $Ku_n \xrightarrow{s} 0$.

Proof. Suppose Ku_n does not converge strongly to 0. Then we can choose an $\epsilon > 0$ and a subsequence $(u_{n_k})_{k=1}^{\infty}$ such that for k large enough we have

$$\|Ku_{n_k}\| \geq \epsilon. \tag{3.1}$$

K is compact thus the sequence $(Ku_{n_k})_{k=1}^{\infty}$ contains a convergent subsequence $(Ku_{n_{k_l}})_{l=1}^{\infty}$. First we note that $(Ku_n)_{n=1}^{\infty}$ converges weakly to 0 as $u_n \xrightarrow{w} 0$. We claim that this subsequence must converge to 0. If this claim holds we have a contradiction with (3.1). To prove the claim we show that if $v_n \xrightarrow{w} w$ and $v_n \xrightarrow{s} v$ then $w = v$. Indeed, by the continuity of the inner product we have

$$\langle w, y\rangle = \langle \lim_{n\to\infty} v_n, y\rangle = \lim_{n\to\infty} \langle v_n, y\rangle = \langle w, y\rangle,$$

and so for all $y \in \mathcal{H}$ we have

$$\langle v, y\rangle = \langle w, y\rangle.$$

Setting $y := w - v$ yields $w = v$. Now as $Ku_{n_{k_l}} \xrightarrow{w} 0$ and $Ku_{n_{k_l}}$ convergence strongly to some v, we must have $v = 0$. □

Proposition 3.11. *The property* $A = A^*$ *implies that* $\ker(A)^{\perp}$ *is* A*-invariant.*

Proof. For $x \in \ker(A)^{\perp}$ and any $y \in \ker(A)$ we have

$$\langle Ax, y\rangle = \langle x, A^*y\rangle = \langle x, Ay\rangle = \langle x, 0\rangle = 0,$$

and so $Ax \in \ker(A)^{\perp}$. □

To prove $u_n \xrightarrow{w} 0$, it is sufficient to prove the statement for a dense subset $Y \subseteq \mathcal{H}$.

Proposition 3.12. *Suppose* $\langle u_n, v\rangle \to 0$ *for all* v *in a dense subset of* $\mathcal{H}$. *Then* $u_n \xrightarrow{w} 0$.

Proof. Suppose $x \in \mathcal{H}$, then we can choose a sequence $(y_m)_{m=1}^{\infty}$ in Y, a dense subset in $\mathcal{H}$, such that

$$x = \lim_{m\to\infty} y_m.$$

Then, by interchanging limits, we find

$$\begin{aligned}
\lim_{n\to\infty} \langle u_n, x\rangle &= \lim_{n\to\infty} \langle u_n, \lim_{m\to\infty} y_m\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle u_n, y_m\rangle \\
&= \lim_{m\to\infty} \lim_{n\to\infty} \langle u_n, y_m\rangle = 0.
\end{aligned}$$

These limits can be interchanged due to following analytic proposition. □

Proposition 3.13. *Suppose $(a_{nm})_{n,m\in\mathbb{N}}$ is a sequence of complex numbers such that $\lim_{n\to\infty} a_{nm}$ exist for almost all m and $\lim_{m\to\infty} a_{nm}$ exist for almost all n. Then we can interchange limits i.e.*

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{nm} = \lim_{m\to\infty}\lim_{n\to\infty} a_{mn}.$$

Here *almost all* means all exact for a finite number.

Theorem 3.14 (Weyl's criterion). *Let $\mathcal{H}$ be a separable Hilbert space and let A be a self-adjoint operator. Then $\lambda \in \sigma_{ess}(A)$ if and only if there exists a Weyl sequence for A and λ.*

Proof. Fix an arbitrary $\lambda \in \sigma_{ess}(A)$, then $\ker(A-\lambda)$ is either infinite dimensional or finite dimensional. If the first case arises we can choose an orthonormal basis $(u_n)_{n=1}^{\infty}$ for $\ker(A-\lambda)$ since the kernel is a closed linear subspace of $\mathcal{H}$. Now we prove that this sequence is a Weyl sequence for λ and A.

By corollary 2.5 we have

$$\lim_{n\to\infty} \langle x, u_n\rangle = 0 \text{ for all } x \in \mathcal{H},$$

and so $u_n \xrightarrow{w} 0$. Naturally we have $(A-\lambda)u_n = 0$, and so in particular $(A-\lambda)u_n \xrightarrow{s} 0$.

The proof for the second case is quite similar to the proof of theorem 3.3. The spectrum of a self-adjoint operator is contained in $\mathbb{R}$ and so $\lambda = \overline{\lambda}$. From this, it follows that $A-\lambda$ is self-adjoint and furthermore $\ker(A-\lambda)^{\perp}$ is $(A-\lambda)$-invariant due to proposition 3.11. Using corollary 3.7 we find an unbounded inverse for $A-\lambda$ restricted to $\ker(A-\lambda)^{\perp}$. Now we can choose a sequence of unit vectors $(v_n)_{n=1}^{\infty}$ in $\ker(A-\lambda)^{\perp}$, which by assumption is infinite dimensional, so that

$$\|(A-\lambda)^{-1}v_n\| > n > 0 \text{ for all } n \geq 1.$$

Set $u_n = (A-\lambda)^{-1}v_n\|(A-\lambda)^{-1}v_n\|^{-1}$; then $\|u_n\| = 1$ and

$$\|(A-\lambda)u_n\| = \|v_n\| \cdot \|(A-\lambda)^{-1}v_n\|^{-1} \xrightarrow{s} 0.$$

It remains to show that $u_n \xrightarrow{w} 0$. Let $x \in \mathcal{H}$; then we have the decomposition

$$\mathcal{H} = \ker(A-\lambda) \oplus \ker(A-\lambda)^{\perp},$$

from which we can write $x = x_1 + x_2$ accordingly. We note that $\ker(A-\lambda)^{\perp}$ is $(A-\lambda)$-invariant thus $u_n = (A-\lambda)^{-1}v_n\|(A-\lambda)^{-1}v_n\|^{-1} \in \ker(A-\lambda)^{\perp}$ and

$$\langle u_n, x\rangle = \langle u_n, x_1\rangle + \langle u_n, x_2\rangle = \langle u_n, x_2\rangle.$$

This means that it is sufficient so show $\langle y, u_n\rangle \to 0$ for all $y \in \ker(A-\lambda)^{\perp}$. Moreover, it is sufficient to prove that $\langle u_n, y\rangle \to 0$ for y in a dense subset of $\ker(A-\lambda)^{\perp}$, as stated by proposition 3.12. Setting $A_\lambda = (A-\lambda)$ restricted to $\ker(A-\lambda)^{\perp}$, we claim that the domain $D\left((A_\lambda^{-1})^*\right)$ is dense in $\ker(A-\lambda)^{\perp}$. Temporarily assuming the claim, we compute,

$$\begin{aligned}
|\langle u_n, y\rangle| &= |\langle (A-\lambda)^{-1}v_n\|(A-\lambda)^{-1}v_n\|^{-1}, y\rangle| \\
&= \|(A-\lambda)^{-1}v_n\|^{-1} \cdot |\langle v_n, ((A-\lambda)^{-1})^* y\rangle| \\
&\leq \|(A-\lambda)^{-1}v_n\|^{-1} \cdot \|v_n\| \cdot \|((A-\lambda)^{-1})^* y\|,
\end{aligned}$$

which we can compute since $y \in D\left((A_\lambda^{-1})^*\right)$. The right hand side converges to zero as $n \to \infty$, hence $u_n \xrightarrow{w} 0$.

Now we only need to prove the claim. We note that any element of the form $(A-\lambda)x$, with $x \in D(A-\lambda)$, belongs to $D\left((A_\lambda^{-1})^*\right)$ since, for any $u \in D\left((A_\lambda^{-1})^*\right)$,

$$\langle A_\lambda x, A_\lambda^{-1} u\rangle = \langle x, u\rangle.$$

As $\operatorname{Ran}(A_\lambda)$ is dense in $\ker(A-\lambda)^\perp$, The domain of $(A_\lambda^{-1})^*$ is dense as well. This completes the proof from left to right.

Conversely, suppose $(u_n)_{n=1}^\infty$ is Weyl sequence for A and λ. Theorem 3.3 states that $\lambda \in \sigma(A)$, so we only need to show that λ is not an isolated eigenvalue with finite multiplicity (i.e. $\ker(A-\lambda)$ is finite dimensional). Again two options arise; the kernel of $A-\lambda$ is either infinite-or finite-dimensional. If $\ker(A-\lambda)$ is infinite, then we are done by the definition of the essential spectrum, so we can assume it is finite. By corollary 3.7 it suffices to show that $A-\lambda$ restricted to $\ker(A-\lambda)^\perp$ does not have a bounded inverse.

Let $(v_i)_{i=1}^k$ be a finite orthonormal basis for $\ker(A-\lambda)$ and let P_λ be the orthonormal projection onto $\ker(A-\lambda)$. Then, as these vectors form a orthonormal basis for $\operatorname{Ran} P_\lambda$, we obtain

$$\|P_\lambda u_n\|^2 = \|\sum_{i=1}^k \langle u_n, v_i\rangle v_i\|^2 = \sum_{i=1}^k \|\langle u_n, v_i\rangle v_i\|^2 = \sum_{i=1}^k |\langle u_n, v_i\rangle|^2,$$

by the Pythagorean theorem. And since $|\langle u_n, v_i\rangle| \to 0$ for any $i = 1, 2, \ldots, k$, we have

$$\lim_{n\to\infty} \|P_\lambda u_n\|^2 = \lim_{n\to\infty} \sum_{i=1}^k |\langle u_n, v_i\rangle|^2 = 0.$$

Set $\overline{P}_\lambda = \mathbb{1} - P_\lambda$. Now the Pythagorean theorem gives

$$\|\overline{P}_\lambda u_n\|^2 = 1 - \|P_\lambda u_n\|^2 \to 1. \tag{3.2}$$

Denote $v_n = \frac{\overline{P}_\lambda u_n}{\|\overline{P}_\lambda u_n\|}$, so that $\|v_n\| = 1$ and the denominator remains bounded by (3.2). Note that

$$(A-\lambda)\overline{P}_\lambda u_n = (A-\lambda)u_n - (A-\lambda)P_\lambda u_n = (A-\lambda)u_n,$$

because $P_\lambda u_n \in \ker(A-\lambda)$. Since $(u_n)_{n=1}^\infty$ is a Weyl sequence for A and λ and the denominator remained bounded, we have

$$\|(A-\lambda)v_n\| = \|(A-\lambda)u_n\| \cdot \|\overline{P}_\lambda u_n\|^{-1} \to 0.$$

Now the claimed result follows directly from the following lemma. □

Lemma 3.15. *Suppose B is a operator with an inverse. Then B^{-1} is unbounded if and only if there is a sequence of unit vectors $(v_n)_{n=1}^\infty$ such that $\|Bv_n\| \to 0$.*

Note that for Weyl's criterion we only need the implication from right to left.

Proof. If B^{-1} is unbounded then for any $n \in \mathbb{N}$ we can find a u_n such that $\|B^{-1}u_n\| \geq n$, consequently, $\|B^{-1}u_n\| \to \infty$ as $n \to \infty$. Setting $v_n = B^{-1}u_n\|B^{-1}u_n\|^{-1}$, we compute

$$\|Bv_n\| = \|u_n\| \cdot \|B^{-1}u_n\|^{-1} \to 0.$$

Conversely, suppose there is such a sequence $(v_n)_{n=1}^{\infty}$. To prove B^{-1} is unbounded, we must show that for any $n \in \mathbb{N}$ there exists a unit vector u_n with $\|B^{-1}u_n\| \geq n$. Take $n \in \mathbb{N}$ arbitrarily. $\|Bv_n\| \to 0$ so we can pick $m > n$ such that $\|Bv_m\| \leq \frac{1}{n}$. Now we pick $c \in \mathbb{R}_{\geq 1}$ such that

$$c \cdot n\|Bv_m\| = 1.$$

If we denote $u_n = B(c \cdot n \cdot v_m)$, then $\|u_n\| = 1$ and $\|B^{-1}u_n\| = c \cdot n\|v_m\| = c \cdot n \geq n$. □

Proposition 3.16. *Suppose $\mathcal{H}$ is a separable Hilbert space and D is a self-adjoint diagonal operator. Then the essential spectrum of D is equal to the set of all limit points of its eigenvalue sequence.*

To illustrate the uses of the previous, a (partial) proof shall be given using Weyl's criterion. With the criterion we prove

$$\{\lambda \in \mathbb{C} \mid \lambda \text{ is a limit point of } \sigma(D)\} \subseteq \sigma_{ess}(D).$$

A complete proof can be found in [4] (theorem 2.2.4) where they use the *Weyl-von Neumann-Berg theorem*, which we shall discuss in chapter 4.

Proof. Suppose λ is a limit point of the eigenvalues, say $\lambda = \lim_{k\to\infty} \lambda_{n_k}$. We look at the corresponding (sub)sequence of eigenvectors $(v_{n_k})_{k=1}^{\infty}$ and claim that this sequence is a Weyl sequence for λ and D. Then, by Weyl's criterion, λ lies in the essential spectrum of D. We must prove that $(A - \lambda)v_{n_k} \xrightarrow{s} 0$ and $v_{n_k} \xrightarrow{w} 0$. Suppose $\epsilon > 0$ and choose $N \in \mathbb{N}$ accordingly such that

$$|\lambda_{n_k} - \lambda| \leq \epsilon \text{ for any } k > N.$$

For any $k > N$ we can write

$$(D - \lambda)v_{n_k} = (\lambda_{n_k} - \lambda)v_{n_k},$$

as v_{n_k} is an eigenvector with corresponding eigenvalue λ_{n_k}. From this we estimate

$$\|(D - \lambda)v_{n_k}\| = |\lambda - \lambda_{n_k}| \cdot \|v_{n_k}\| \leq \epsilon,$$

and so $(D - \lambda)v_{n_k} \xrightarrow{s} 0$. Since $(v_n)_{n=1}^{\infty}$ is an orthonormal base for $\mathcal{H}$, corollary 2.5 states that for any $x \in \mathcal{H}$,

$$\lim_{n\to\infty} \langle v_n, x\rangle = 0,$$

which will also hold for a subsequence $(v_{n_k})_{k=1}^{\infty}$. This makes $(v_{n_k})_{k=1}^{\infty}$ a Weyl sequence for λ and D. □

3.3 Weyl's theorem

We stated earlier that in this chapter we shall prove that compact operators have no influence on the essential spectrum. However a stronger result can be proven. For this we need the following definition.

Definition 3.17. *Let $A \in \mathcal{B}(\mathcal{H})$ with a nonempty resolvent set $\rho(A)$. An operator B is called* ***relatively A-compact*** *if $BR_A(z) := B(A-z)^{-1}$ is compact for some $z \in \rho(A)$. Here*

$$R_A(z) := (A-z)^{-1},$$

is well-defined, as $z \in \rho(A)$ means $A - z$ is invertible.

The product of a compact operator and a bounded operator is always compact. $R_A(z)$ is bounded by the Banach inverse mapping theorem, so any compact operator B is relatively compact to any operator A (as $\rho(A) \neq \emptyset$). For two operators $A, B \in \mathcal{B}(\mathcal{H})$ we shall show that if the difference $A - B$ is either relatively A-or B-compact, then they have the same essential spectrum.

Lemma 3.18. *Let A and B be self-adjoint operators and let $V = A - B$. If V is A-compact, then V is B-compact.*

The proof of this lemma can be found in [2] (Theorem 14.2).

Theorem 3.19 (Weyl's theorem). *Let A and B be self-adjoint operators, and let $A - B$ be A-compact. Then,*

$$\sigma_{ess}(A) = \sigma_{ess}(B).$$

Proof. $A - B$ is A-compact, so we can choose $z \in \rho(A)$ such that $(A-B)(A-z)^{-1}$ is compact. Let $\lambda \in \sigma_{ess}(A)$. Then by Weyl's criterion there exists a Weyl sequence $(u_n)_{n=1}^{\infty}$ for A and λ, that is $\|u_n\| = 1$, $u_n \xrightarrow{w} 0$ and $(A-\lambda)u_n \xrightarrow{s} 0$. Now we have

$$(A-z)u_n = (A-\lambda)u_n + (\lambda - z)u_n \xrightarrow{w} 0, \tag{3.3}$$

as the first term converges strongly to zero and $u_n \xrightarrow{w} 0$ implies $(\lambda - z)u_n \xrightarrow{w} 0$. We claim that $(B-\lambda)u_n \xrightarrow{s} 0$, for we can write

$$(B-\lambda)u_n = (A-\lambda)u_n + (B-A)(A-z)^{-1}(A-z)u_n. \tag{3.4}$$

The first term on the right satisfies $(\lambda - A)u_n \xrightarrow{s} 0$. The operator $(A-B)(z-A)^{-1}$ is compact and so by (3.3) and proposition 3.10, the second term on the right in (3.4) converges strongly to zero. Now the conditions are met for $(u_n)_{n=1}^{\infty}$ to be a Weyl sequence for λ and B. This proves $\sigma_{ess}(A) \subseteq \sigma_{ess}(B)$. By Lemma 3.18, $(A - B)$ is B-compact. Thus, we can switch roles between A and B in the argument above. This gives the equality we were looking for. □

Corollary 3.20. *If K is a compact self-adjoint operator, then $\sigma_{ess}(K) = \emptyset$.*

For this proof we need three different zero-elements: the zero-operator, the zero-element in the Hilbert space, and the number zero. This should be noted beforehand, so there is no ambiguity.

Proof. We can write $K = 0 + K$, where

$$0 : \ x \mapsto 0_{\mathcal{H}}$$

is the operator that sends every element in $\mathcal{H}$ to the 0-element in $\mathcal{H}$. By Weyl's theorem we have $\sigma_{ess}(K) = \sigma_{ess}(0)$, so if $\sigma_d(0) = \sigma(0)$ we are done. Note $\sigma(0) = \{\lambda \in \mathbb{C} \mid -\lambda I \text{ is not invertible}\}$ and $-\lambda I$ is not invertible if and only if $\lambda = 0$. Therefore $\sigma(0) = \{0\}$. 0 is an eigenvalue because the operator $0 - 0 \cdot I = 0$ is not injective. Considering that 0 is the only point in the spectrum it must be isolated, and since $\dim(0) = 0 < \infty$, we have $0 \in \sigma_d(0)$. $\square$

4 The Weyl-von Neumann-Berg Theorem

The *Weyl-von Neumann-Berg theorem*, firstly proven by Weyl has since taken many names. It states that, after the addition of a compact operator (Weyl (1909)) or Hilbert-Schmidt operator (von Neumann (1935)) of arbitrarily small norm, a bounded self-adjoint operator on a Hilbert space is equal to a diagonal operator. The results are subsumed in later generalizations for bounded normal operators due to David Berg (1971, compact perturbation) and Dan-Virgil Voiculescu (1979, Hilbert-Schmidt perturbation). We shall prove the generalization of David Berg using C^*-algebras and consequently deduce the initial theorem of Weyl.

We prove that a self-adjoint operator can be decomposed as

$$A = D + K,$$

where D is a diagonal operator and K is a compact operator. To find such a decomposition, the operator A needs to be written in a such way that a diagonal operator will appear somewhere in the notation. This can be done by writing A as a (perhaps infinite) sum of projections. More precisely, as a sum of *spectral* projections.

4.1 Spectral projections

We wish to write any operator A as an infinite sum of projections. This is somewhat analogue to the idea that any number between 0 and 1 has an unique binary expansion. That is to say for any $\alpha \in [0,1]$ there is a sequence $(a_k)_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} 2^{-k} a_k = \alpha.$$

This can be achieved by setting $c_n = \lceil 2^n \cdot \alpha - 1 \rceil$ and $a_n = c_n - 2c_{n-1}$. Indeed we have

$$\lim_{n\to\infty} \frac{c_n}{2^n} = \alpha,$$

whilst simultaneously the identity

$$\frac{c_n}{2^n} = \sum_{k=1}^{n} 2^{-k} a_k$$

holds for any $n \in \mathbb{N}$. Now for an operator A with $\sigma(A) \subseteq [0,1]$ we shall show that we can write

$$A = \sum_{k=1}^{\infty} 2^{-k} E_k^A,$$

where E_k^A will be *spectral* projections of A.

Definition 4.1. *A **projection-valued measure** on a measurable space (X, Σ), is a mapping E from Σ to $\mathcal{B}(\mathcal{H})$ such that*

$$E(X) = id_H,$$

and for every $\psi, \phi \in \mathcal{H}$, the set function

$$Y \mapsto \langle E(Y)\psi, \phi \rangle,$$

is a complex measure on M.

In this thesis $(X, \Sigma) = (\sigma(A), B(\sigma(A))$ will be the spectrum of an operator A with the Borel measure on said spectrum. We shall not go into details of the actual measure, and only use it loosely. If we were to use the actual measure for proofs, then the difficulty of this thesis would skyrocket. For all intents and purposes we look at it as if it were just an ordinary measure.

Remark. It should be noted that the spectral theorem states that every self-adjoint operator A has an associated projection-valued measure E_A such that

$$A = \int_{\sigma(A)} x \; dE_A(x),$$

which we shall use extensively.

Definition 4.2. *With the previous remark we can define*

$$g(A) := \int_{\sigma(A)} g(x) \; dE_A(x),$$

for a measurable function g on $\sigma(A)$. In particular taking $g = \mathbb{1}_\Delta$ an indicator function for $\Delta \subseteq \sigma(A)$, we can define

$$E_A(\Delta) \equiv \mathbb{1}_\Delta(A) = \int_{\sigma(A)} \mathbb{1}_\Delta(x) \; dE_A(x)$$

called the ***spectral projection*** *of Δ.*

The name *spectral projection* has not been chosen arbitrarily: The operator $\mathbb{1}_\Delta(A)$ is a projection for any $A \in \mathcal{B}(\mathcal{H})_{sa}$ and $\Delta \subseteq \sigma(A)$. To prove this we need the following proposition.

Proposition 4.3. *Let A be a self-adjoint operator in $\mathcal{B}(\mathcal{H})$. Then for any Borel measurable function f on $\sigma(A)$ we have:*

1) $f \mapsto f(A)$ is a ring homomorphism from $B(\sigma(A))$ to $\mathcal{B}(\mathcal{H})$;

2) $f(A)^* = \overline{f}(A)$.

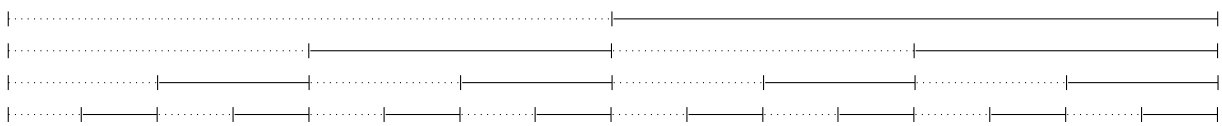
With this proposition we note $\mathbb{1}_\Delta(A)^* = \overline{\mathbb{1}}_\Delta(A) = \mathbb{1}_\Delta(A)$ and $\mathbb{1}_\Delta(A)^2 = \mathbb{1}_\Delta^2(A) = \mathbb{1}_\Delta(A)$.

Lemma 4.4. *For $\sigma(A) \subseteq [0,1]$ we have*

$$A = \sum_{k=1}^{\infty} 2^{-k} E_k^A \qquad \textit{where} \qquad E_k^A = E_A\left(\bigcup_{j=1}^{2^{k-1}} (2^{-k}(2j-1), 2^{-k} \cdot 2j] \right).$$

Here $E_A(\Delta)$ is the spectral projection of Δ, as in definition 4.2. These intervals are called dyadic intervals.

Proof. A formal proof will go beyond the scope of this thesis, so instead a sketch of the proof will be given. First a visualization of the dyadic intervals. Here the intervals are drawn for $k = 1, 2, 3, 4$.



A result from the spectral theorem is that a self-adjoint operator A can be written as

$$A = \int_{\sigma(A)} x \, dE_A.$$

Now we are looking for the equality

$$\int_{\sigma(A)} x \, dE_A = \sum_{k=1}^{\infty} 2^{-k} \int_{\sigma(A)} x \mathbb{1}_{\Delta_k}(x) \, dE_A,$$

where $\Delta_k = \bigcup_{j=1}^{2^{k-1}} (2^{-k}(2j-1), 2^{-k} \cdot 2j]$ is the k-th dyadic interval of [0,1]. With Beppo-Levi we can switch the integral and the sum, so if the functions x and $\sum_{k=1}^{\infty} 2^{-k} x \mathbb{1}_{\Delta_k}(x)$ are the same for every x the equality will hold. This is not the case however. The dyadic intervals create gaps, so the indicator functions will be 0 at times, and thus $\sum_{k=1}^{\infty} 2^{-k} \mathbb{1}_{\Delta_k}(x) < 1$. This means that for the equality to hold, the function must be equal E_A-almost everywhere. Proving this goes beyond this thesis. □

A more detailed look into spectral projections and Functional Calculus can be found in [3].

Lemma 4.5. *Let $\mathcal{H}$ be separable and let $\mathfrak{A}$ be a separable abelian C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then there is a countable commuting family $\mathcal{E} = \{E_n \mid n \geq 1\}$ of projections such that $\mathfrak{A}_{sa} \subseteq \mathfrak{E} = \overline{span(\mathcal{E})}$.*

Proof. For convenience, let us translate A by a multiple of the identity and scale it such that $0 \leq A \leq I$. We do this so that the spectrum of A is contained in $[0,1]$. Each self-adjoint operator is in the norm-closed span of its spectral projections corresponding to dyadic intervals. More precisely, for any $A \in \mathfrak{A}_{sa}$ we have

$$A = \sum_{k=1}^{\infty} 2^{-k} E_k^A, \qquad \text{where} \qquad E_k^A = E_A(\bigcup_{i=1}^{2^{k-1}} (2^{-k}(2i-1), 2^{-k}(2i)]).$$

Consequently, $A \in \overline{\text{span}\{E_k^A \mid k \in \mathbb{N}\}}$.

$\mathfrak{A}$ is separable and thus $\mathfrak{A}_{sa}$ is seperable too. Choose a countable dense subset $\{A_1, A_2, \dots\}$ for $\mathfrak{A}_{sa}$. Then the sequence $(A_n)_{n=1}^{\infty}$ in $\mathfrak{A}_{sa}$ has the following property: for any W open in $\mathfrak{A}_{sa}$ there is an $A_n \in W$. Consequently for $T \in \mathfrak{A}_{sa}$ we have

$$\forall m \in \mathbb{N}\ \exists A_{n_m}^T \qquad \text{such that} \qquad A_{n_m}^T \in B_{\frac{1}{m}}(T).$$

so the subsequence $(A_{n_m}^T)_{m=1}^{\infty}$ converges to T. Now $T \in \overline{\text{span}\{A_{n_m}^T \mid m \in \mathbb{N}\}}$ and hence

$$\begin{aligned}\mathfrak{A}_{sa} &\subseteq \overline{\text{span}\{A_n \mid n \in \mathbb{N}\}}\\ &\subseteq \overline{\text{span}\{E_k^{A_n} \mid n, k \in \mathbb{N}\}}.\end{aligned}$$

The set $\mathcal{E} = \{E_k^{A_n} \mid k, n \in \mathbb{N}\}$ is a countable family of projections. This family commutes as all operators A_n commute, and furthermore all $E_k^{A_n}$ commute for a given $n \in \mathbb{N}$.

□

4.2 Proving the Weyl-von Neumann-Berg theorem

Theorem 4.6. *Suppose that $\mathfrak{A}$ is a separable abelian C^*-subalgebra in $\mathcal{B}(\mathcal{H})$. Then there is an orthonormal basis $\{e_k \mid k \geq 1\}$ for $\mathcal{H}$ so that $\mathfrak{A}_{sa}$ is contained in $\mathfrak{D} + \mathcal{K}(\mathcal{H})$ where $\mathfrak{D}$ the algebra of diagonal operators with respect to this basis, and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators.*

By Lemma 4.5, $\mathfrak{A}_{sa}$ is contained in a C^*-algebra $\mathfrak{E} = \overline{span(\mathcal{E})}$ generated by a countable, commuting family $\mathcal{E} = \{E_n \mid n \geq 1\}$ of projections. If finitely many projections are given, we may assume that they are the first N on the list. We do so for the *Weyl-von Neumann-Berg Theorem*, which is a corollary of theorem 4.6. In this theorem N has no restriction, i.e. what we prove can be done for any chosen N.

Proof. Let N be arbitrary and choose $\mathfrak{E} = \overline{span(\mathcal{E})}$ generated by a countable, commuting family $\mathcal{E} = \{E_n \mid n \geq 1\}$ of projections such that $\mathfrak{A}_{sa}$ is contained in $\mathfrak{E}$. We show that a diagonal algebra $\mathfrak{D}$ can be constructed such that $\mathcal{E} \subseteq \mathfrak{D} + \mathcal{K}(\mathcal{H})$. Consequently we will have $\mathfrak{A}_{sa} \subseteq \mathfrak{E} \subseteq \mathfrak{D} + \mathcal{K}(\mathcal{H})$ as $\mathfrak{D}$ and $\mathcal{K}(\mathcal{H})$ are closed subsets of $\mathcal{B}(\mathcal{H})$. Fix an orthonormal basis $x_1, x_2, \dots$ for $\mathcal{H}$. For each projection $E_i \in \mathcal{E}$ let us denote $E_i^{(-1)} := I - E_i$ and $E_i^{(1)} := E_i$. Then for $k \geq N$, set

$$\mathcal{L}_k = \text{span}\left\{\prod_{i=1}^{k} E_i^{(\epsilon_i)} x_j \mid 1 \leq j \leq k, \epsilon_i = \pm 1\right\},$$

and let F_k be the projection on the subspace $\mathcal{L}_k$. Note that the product is taken over the functions $E_i^{(\epsilon_i)}$ in $\mathcal{B}(\mathcal{H})$ rather than the elements $E_i^{(\epsilon_i)} x_j$ in $\mathcal{H}$.

Also note that $(\mathcal{L}_k)_{k=N}^{\infty}$ is an increasing sequence of finite dimensional subspaces with dense union in $\mathcal{H}$. It is dense since $x_j \in \mathcal{L}_k$ for any $j = 1, 2, \dots, k$ and $(x_j)_{j=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. To prove $x_j \in \mathcal{L}_k\ \forall\ j < k$, we first note that for $k = 2$ we have

$$E_1 E_2 + E_1(I - E_2) + (I - E_1)E_2 + (I - E_1)(I - E_2) = I,$$

and so $x_j \in \mathcal{L}_2$. With an inductive argument this can be shown for all $\mathcal{L}_k$. Denote by σ_k the sum of all possible product combinations of $E_1^{(\epsilon_1)}, \dots, E_k^{(\epsilon_k)}$. Then by induction we have

$$\sigma_{k+1} = \sigma_k(I - E_{k+1}) + \sigma_k E_{k+1} = I(I - E_{k+1}) + I \cdot E_{k+1} = I.$$

Recall the initially set N and let

$$D_n = \begin{cases} E_n & \text{for } 1 \leq n \leq N \\ E_n(I - F_n) & \text{for } n > N. \end{cases}$$

The we let $\mathfrak{D}$ be the C^*-algebra generated by $\{D_n \colon n \geq 1\}$. Now we show that $\mathcal{E}$ is contained in $\mathfrak{D} + \mathcal{K}(\mathcal{H})$. If $n \leq N$ then $E_n = D_n$ and for $n > N$ we have

$$D_n = E_n(I - F_n) = E_n - E_nF_n,$$

and hence

$$E_n = D_n + E_nF_n.$$

Here E_nF_n is compact by lemma 2.18 as $\text{Ran}(E_nF_n) = \mathcal{L}_n$, which is finite dimensional.

It remains to show that $\mathfrak{D}$ is diagonalizable, meaning every element $D \in \mathfrak{D}$ is a diagonal operator with regards to a predetermined base of eigenvectors. As $\mathfrak{D}$ commutes with each F_k for $k \geq N$, the finite dimensional subspaces $\mathcal{H}_N := \mathcal{L}_N$ and $\mathcal{H}_k := \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$ for $k > N$ are all invariant for $D \in \mathfrak{D}$. This result requires 5 different case studies, as for varying k and n different scenarios appear. Suppose $D \in \mathfrak{D}$, we prove $D(\mathcal{H}_k) \subseteq \mathcal{H}_k$ for all $k \geq N$.

1) If $k = N, D = E_n$ for some $n \leq N$,
then $D(\mathcal{H}_k) = E_n(\mathcal{L}_N) \subseteq \mathcal{L}_N$, since $\mathcal{L}_N$ in invariant under E_n if $n \leq N$.

2) Suppose $k = N, D = E_n(I - F_n)$ for some $n > N$,
now $D(\mathcal{H}_k) = (E_n(I - F_n)(\mathcal{L}_N) = \{0\} \subseteq \mathcal{L}_N$ since $\mathcal{L}_N \subseteq \mathcal{L}_n \Rightarrow F_n(\mathcal{L}_N) = \mathcal{L}_N$.

3) If $k > N, D = E_n$ for some $n \leq N$,
then $D(\mathcal{H}_k) = E_n(\mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp})$. Suppose $x \in \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$ then $E_nx \in \mathcal{L}_k$ as $n \leq N < k$ and $\forall y \in \mathcal{L}_{k-1}$ we have $\langle E_nx, y\rangle = \langle x, E_ny\rangle = 0$. Consequently $E_nx \in \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$.

4) Now suppose $k > N, D = E_n(I - F_n)$ for some $n > N$ and $n > k$,
with the same argument used in 2) we find $D(\mathcal{H}_k) = E_n(I - F_n)(\mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}) = \{0\}$. As $\{0\} \subseteq \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$ we are done.

5) Lastly, we suppose $k > N, D = E_n(I - F_n)$ for some n with $k \geq n > N$, then E_n and F_n commute, and hence for $x \in \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$ we have $E_n(I - F_n)x = (E_n - F_nE_n)x = z - F_nz$ for $E_nx := z \in \mathcal{L}_k \cap \mathcal{L}_{k-1}^{\perp}$. Note that $\mathcal{L}_n \subseteq \mathcal{L}_k \Rightarrow F_nz = z$ So $z - F_nz = z - z = 0$. Once again we have $D(\mathcal{H}_k) = \{0\}$, which completes the final case study.

Now that we have checked operators of the form $D = D_n$, the conclusion extends to the entire C^*-algebra generated by $\{D_n : n \geq 1\}$. Consequently the finite dimensional subspaces $\mathcal{H}_N$ and $\mathcal{H}_k$ are all invariant under operators $D \in \mathfrak{D}$.

We note that the sequence $(\mathcal{H}_k)_{k=N}^{\infty}$ is a sequence of closed subspaces that are all perpendicular to each other and that the union is dense in $\mathcal{H}$. This is true as $(\mathcal{L}_k)_{k=N}^{\infty}$ is an increasing sequence of finite dimensional subspaces with dense union in $\mathcal{H}$.

The restriction of any $\mathfrak{D}$ to any $\mathcal{H}_k$ is a commuting family of normal matrices, which is diagonalizable by the finite dimensional spectral theorem. Choose $D \in \mathfrak{D}$ arbitrarily. As $\mathcal{H}_k$ is a closed subspace we can choose a basis $B_k = (v_1^k, \ldots, v_{n_k}^k)$ for $\mathcal{H}_k$. Since $k \neq k'$ implies that $\mathcal{H}_k$ and $\mathcal{H}_{k'}$ are perpendicular, we have $\langle v_i^k, v_j^{k'}\rangle = 0$ for all i, j if $k \neq k'$. And lastly, as the union is dense the set $B = \bigcup_{k=1}^{\infty} B_k$ is an orthonormal basis for $\mathcal{H}$, containing only eigenvectors of D. Vice versa, all eigenvectors of D are contained in B, resulting in a basis of eigenvectors of D, making any $D \in \mathfrak{D}$ diagonalizable with respect to the same basis. □

We can push this a bit further for normal and self-adjoint operators. The following corollary asserts that every normal operator is a *small* compact operator away from being a diagonalizable operator. Furthermore, in the Calkin algebra the coset of any self-adjoint operator is equal to the coset of a diagonal operator.

Theorem 4.7 (**Weyl-von Neumann-Berg Theorem**). *Every normal operator N on a separable Hilbert space can be expressed as a sum $N = D + K$ of a diagonal normal operator D and a compact operator K. Moreover for any n commuting self-adjoint operators $A_1, \dots, A_n$ and any $\epsilon > 0$, there are diagonal self-adjoint operators D_i and compact operators K_i such that $A_i = D_i + K_i$ and $\|K_i\| < \epsilon$ for any $i = 1, 2, \dots n$.*

If we drop the property $\|K_i\| < \epsilon$ we can deduce the result directly from the previous theorem by setting $\mathfrak{A}$ as the C-algebra generated by $A_1, A_2, \dots, A_n$. If we do want this property, we have to look at the series given in lemma 4.4.

Proof. Suppose $\epsilon > 0$ and $A_1, \dots, A_n$ are commuting self-adjoint operators. Lemma 4.4 states

$$A_i = \sum_{k=1}^{\infty} 2^{-k} E_k^{(i)} \qquad \text{where} \qquad E_k^{(i)} = E_{A_i}\left(\bigcup_{j=1}^{2^{k-1}} (2^{-k}(2j-1), 2^{-k} \cdot 2j] \right).$$

We choose N large enough such that $2^{-N} < \epsilon$. As mentioned at the beginning of theorem 4.6, the entire argument in the proof of theorem 4.6 works for any N. We did this, so it could be done for this particular N. By setting $\mathfrak{A}$ as the C^*-algebra generated by $A_1, A_2, \dots, A_n$ we can follow the same line of argumentation used in the previous theorem for $\mathcal{E} = \{E_k^{(i)} \mid k \geq 1 \;;\; 1 \leq i \leq n\}$, as $A_i, \dots, A_n \in \overline{\text{span}(\mathcal{E})}$ and $A_1, \dots, A_n$ all commute. From this, we find a diagonal algebra $\mathfrak{D}$ that contains the following elements:

$D_k^{(i)} = E_k^{(i)}$ for $1 \leq k \leq N$

$D_k^{(i)} = E_k^{(i)} - R_{i,k}$ for $k > N$,

where each $R_{i,k}$ is a projection of finite rank (and thus compact due to lemma 2.18) as seen in equation (4.2) of Theorem 4.6 and the arguments following said equation. Now we define

$$B_i = \sum_{k=1}^{\infty} 2^{-k} D_k^{(i)} \qquad \text{and} \qquad K_i = A_i - B_i = \sum_{k>N} 2^{-k} R_{i,k},$$

where all B_i are self-adjoint, as all $E_k^{(i)}$ and $R_{i.k}$ are self-adjoint. For $n > N$ we set

$$T_{i,n} = \sum_{n>k>N} 2^{-k} R_{i,k}.$$

Then K_i is the limit of the sequence $(T_{i,n})_{n=N}^{\infty}$. Any $T_{i,n}$ is a finite sum of compact operators, hence all $T_{i,n}$ are compact. Now proposition 2.16 states that K_i is compact. It remains to show that the compact operators in question are smaller then ϵ, indeed we have

$$\|K_i(x)\| = \left\| \sum_{k>N} 2^{-k} R_{i,k}(x) \right\| \leq \sum_{k>N} 2^{-k} \|R_{i,k}(x)\| \leq \sum_{k>N} 2^{-k} < 2^{-N}$$

for any unit vector $x \in \mathcal{H}$. As a result, $\|K_i\| < 2^{-N} < \epsilon$.

Now we have the decomposition $A_i = B_i + K_i$, where K_i is compact and $B_i \in \mathfrak{D}$ is self-adjoint. This proves the second part of the theorem. To obtain the result for a normal operator N, we apply this result to the real and imaginary part of N, as any normal operator N can be written as $N = A + iB$. Here A and B commute and are self-adjoint as seen in Proposition 2.11, so they meet the required conditions. □

Corollary 4.8. *For any $n \in \mathbb{N}$ we can write a self-adjoint operator as $A = D_n + K_n$ with $\|K_n\| < 1/n$ and D_n diagonalizable. Consequently we have the norm limit*

$$A = \lim_{n\to\infty} D_n.$$

Subalgebras are norm-closed, so one might think that therefore the above operator A is diagonal operator. However every D_n inherits a different algebra $\mathfrak{D}_n$, making this conclusion invalid. Indeed, not every self-adjoint operator is a diagonal operator, as seen in the example given in the beginning of chapter 3 (taking the multiplication operator $x \cdot f(x)$ on $[0, 1]$).

Eventhough the proof of the *Weyl-von Neumann-Berg* is a constructive one, it is not a simple one. If the construction would be used in an example, it would require a lot of difficult calculations using spectral projections and infnite sums. Therefore this last corollary is mostly usefull in the abstract.

5 Essential unitary equivalence

The results of chapter 3 and 4 can be combined in a result regarding the Calkin algebra, of two self-adjoint operators have the same essential spectrum then the are unitary equivalent in the Calkin algebra. Unitary equivalence is a very strong equivalence as it retains the structure of the underlying Hilbert space. That is to say for a unitary operator U we have

$$\langle Ux, Uy \rangle = \langle x, y \rangle \; \forall x, y \in \mathcal{H},$$

and consequently if two operators A and B are unitary equivalent (which means $A = U^*BU$ for some unitary operator U) we have

$$\|Bx\| = \|Ax\| \; \forall x \in \mathcal{H}.$$

Definition 5.1. *Two bounded operators T_1 and T_2 are **essentially unitarily equivalent** if there is a unitary isomorphism U such that the difference $T_1 - U^*T_2U$ is compact.*

Now if we consider the Calkin algebra, we see that if T_1 and T_2 are essentially unitary equivalent in $\mathcal{B}(\mathcal{H})$, they are unitary equivalent in the Calkin algebra.

Proposition 5.2. *Essential unitary equivalence is an equivalence relation. We shall denote the relation as $T_1 \sim_e T_2$.*

Proof. Any operator T is unitary equivalent to itself as $T - T = 0$ is compact and $I = I^*$ is a unitary isomorphism. Suppose $T_1 \sim_e T_2$ and choose U such that

$$T_1 - U^*T_2U = K,$$

is compact. Multiplying with U on the right and U^* on the left results in

$$UT_1U^* - UU^*T_2UU^* = UKU^*,$$

and therefore,

$$(U^*)^*T_1(U^*) - T_2 = UKU^*.$$

Hence UKU^* is compact by proposition 2.15, as U and U^* are bounded. This means that $T_2 \sim_e T_1$.

Now suppose $T_1 \sim_e T_2$ and $T_2 \sim_e T_3$. Then $T_3 \sim_e T_2$ and as such we can choose unitary isomorphisms U_1 and U_2 and compact operators K_1 and K_2 such that

$$T_2 - {U_1}^*T_1U_1 = K_1,$$
$$T_3 - {U_2}^*T_2U_2 = K_2.$$

Then we can compute

$$T_3 - (U_1U_2)^*T_1(U_1U_2) = K_2 + {U_2}^*K_1U_2,$$

where the right side is compact, and U_1U_2 a unitary isomorphism. □

Before we start with the main theorem for essential unitary equivalence we set aside an analytic lemma.

Lemma 5.3. *Let $(\lambda_i)_{i=1}^{\infty}$ and $(\mu_j)_{j=1}^{\infty}$ be two bounded sequences of real numbers, and suppose that they have the same limit points. Then there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $|\lambda_i - \mu_{\sigma(i)}| \to 0$ as $i \to \infty$.*

Proof. Finding such a bijection comes down to rearranging both sequences in such a way that the terms come $\epsilon-$close to each other as $n \to \infty$. As the sequences are bounded there must be at least one limit point. First we observe what happens when there are finitely many limit points, say $x_1, x_2, \ldots, x_n$. Then there are n subsequences in both $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{m=1}^{\infty}$ that converge to these points. For $i = 1, 2, \ldots, n$ we write

$$\lambda_{n_k}^{i} \to x_i \leftarrow \mu_{n_k}^{i},$$

for these subsequences. Without loss of generality we may assume the subsequences are pairwise disjoint, as to not have terms be part of two different subsequences. Now we rearrange $(\lambda_n)_{n=1}^{\infty}$. The first term in the new sequence becomes the first term in the first subsequence (to x_1), the second term becomes the first term in the second subsequence. We continue this listing for the first n terms, then repeat the process for terms $n+1$ trough $2n$ in the new sequence. Now the subsequences becomes

$$\lambda_{n_1}^{1}, \lambda_{n_1}^{2}, \ldots, \lambda_{n_1}^{n-1}, \lambda_{n_2}^{1}, \lambda_{n_2}^{2}, \ldots$$

Doing the same for $(\mu_m)_{m=1}^{\infty}$ results in two sequences that have the desired property.

If there are infinitely many limit points, then there will be countably many. We may paste $\mathbb{N}$ countably many times behind itself, which is as large as $\mathbb{N} \times \mathbb{N}$. This set is countable which results in a bijection Ψ to $\mathbb{N}$. Placing the first subsequence in the first iteration of $\mathbb{N}$, the second subsequence in the second iteration and the n-th in the n-th iteration gives a bijection trough Ψ using the same process as described above.

□

Now, using both Weyl's theorem and the Weyl-von Neumann-Berg theorem, we obtain a condition for two self-adjoint operators to be essentially unitarily equivalent and consequently unitarily equivalent in the Calkin algebra.

Theorem 5.4. *If T_1 and T_2 are self-adjoint operators on a separable Hilbert space, and have the same essential spectrum, then $T_1 \sim_e T_2$.*

Proof. Firstly we prove the statement for self-adjoint diagonal operators, then extend the argument using the Weyl-von Neumann-Berg theorem. Suppose D_1 and D_2 are diagonal operators with bases of eigenvectors $(v_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ respectively, and corresponding eigenvalues $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$. A unitary isomorphism U can be constructed by mapping these bases upon each other. That is, there is a unitary isomorphism U and a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that

$$Uv_n = w_{\sigma(n)} \text{ for all } n \in \mathbb{N}.$$

Choose U and σ. Since U is uniform we also have

$$U^* w_{\sigma(n)} = v_n \text{ for all } n \in \mathbb{N},$$

and thus for any element v_n of the base we have

$$\begin{aligned}(D_1 - U^* D_2 U)v_n &= \lambda_n v_n - U^* D_2 w_{\sigma(n)} \\ &= \lambda_n v_n - U^* \mu_{\sigma(n)} v_n \\ &= (\lambda_n - \mu_{\sigma(n)}) v_n.\end{aligned}$$

Since D_1 and D_2 have the same essential spectrum, proposition 3.16 states that the sequences $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ have the same limit points. Then, as they have the same limit points, lemma 5.3 states that there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $|\lambda_n - \mu_{\sigma(n)}| \to 0$ as $n \to \infty$. Since the sequence $(v_n)_{n=1}^{\infty}$ is a base for the Hilbert space, the operator $D_1 - U^*D_2U$ is a diagonal operator where the sequence $\left(\lambda_n - \mu_{\sigma(n)}\right)_{n=1}^{\infty}$ of eigenvalues converges to zero.

The converse of the spectral theorem for compact operators (proved in [6]) states that if T is an diagonal operator and the corresponding sequence of eigenvalues converges to zero, then T is compact. Hence the operator $D_1 - U^*D_2U$ is compact, which concludes the proof for diagonal self-adjoint operators.

Now, with the Weyl-von Neumann-Berg theorem we write $T_1 = D_1 + K_1$ and $T_2 = D_2 + K_2$, where $D_{1,2}$ are diagonal operators and $K_{1,2}$ are compact operators. Considering $\sigma_{ess}(T_1) = \sigma_{ess}(T_2)$, corollary 3.16 now states that D_1 and D_2 have the same essential spectrum. As we have already proved the theorem for diagonal operators, we can find an unitary isomorphism U and compact operator K such that

$$D_1 - U^*D_2U = K.$$

With the Weyl-von Neumann-Berg decompositions we compute

$$T_1 - K_1 - U^*(T_2 - K_2)U = K$$

and so

$$T_1 - U^*T_2U = K + K_1 - U^*K_2U,$$

where the right side is compact as a direct result of proposition 2.15. In conclusion, T_1 is essentially unitarily equivalent to T_2. □

Corollary 5.5. *If T_1 and T_2 are as in theorem 5.4 then their images in the Calkin algebra are unitarily equivalent. In other words, there exists a unitary isomorphism $U \in \mathcal{B}(\mathcal{H})$ such that*

$$T_1 =_c U^*T_2U.$$

An essential part proving theorem 5.4 is first comparing the essential spectrum of diagonal operators instead of the original self-adjoint (or normal) operators, and then revisiting the original operators with the Weyl-von Neumann-Berg theorem. I suspect that this method is applicable to more then just this theorem. Two diagonal operators with the same essential spectrum might have more in common then just having the same limit points. If so, then perhaps these similarities can lead to more results applicable to the Calkin algebra.

References

[1] Barbara D. MacCluer, *Elementary Functional Analysis*

[2] P.D. Hislop and I.M. Sigal, *Introduction to Spectral Theory*

[3] Kenneth R. Davidson, *C^*-Algebras by Example*

[4] Nigel Higson and John Roe, *Analytic K-Homology*

[5] S.R. Caradus, W.E. Pfaffenberger and Bertram Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*

[6] Shlomo Sternberg, *The spectral theorem for compact self-adjoint operators* http://isites.harvard.edu/fs/docs/icb.topic465490.files/212b0802handout.pdf