Radboud University

Master's Thesis

Symmetries in Algebraic Quantum Theory

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August 2019

Abstract

There are various symmetries on subsets of the algebra of bounded operators on a Hilbert space $B(H)$, for example Jordan symmetries on the self-adjoint operators, Kadison symmetries on the density operators, and Wigner symmetries on the one-dimensional projections. We first prove the equivalences between these symmetries, after which we prove that all symmetries are induced by unitary or anti-unitary operators. In the main part of this thesis, we generalise Jordan, Kadison and Wigner symmetries to maps between state spaces of C*-algebras and prove that these equivalences still hold. In doing so we will fill in all details in previous work of notably F.W. Shultz (1982).

Preface

The last year of my master's was mostly dedicated to writing this thesis. Before starting this project, I had never even read a mathematical paper, let alone sifted through all the details. Suddenly, I was presented with material that I not only did not understand, but for which I also could find no other sources besides the paper itself. Although this was very frustrating at times, it allowed me to learn a great deal about mathematical physics and the mathematical skills required to write a long text like this.

Of course, I could not have accomplished this feat alone. First and foremost, I would like to thank my supervisor Klaas Landsman who helped me during the whole process through our weekly meetings, suggestions for ways of looking at the problem, and proofreading the thesis. I would also like to thank F.W. Shultz and C.A. Akemann who were kind enough to answer several of my questions. My fellow students, whom I spent countless hours with in the *masterkamer*, also deserve their fair share of praise for helping me figure out some details, and always having time for a cup of coffee or tea.

On the non-mathematical front I would like to thank my family and friends who were patient enough to listen to my struggles even though they had no idea what I was talking about. Special thanks goes out to Christa, who kept me motivated and believed in me even during the most difficult parts of this thesis.

Kitty Rang, 03.08.2019

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Introduction

In the second half of the 1920s quantum mechanics was still a new field of research. It was around this time that mathematicians and physicists started incorporating group theory into the mathematical description of quantum mechanics (Scholz 2006). In 1931 Eugene Wigner (1902-1995) published the book *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Wigner 1931), in which he argued for the use of group theory in quantum mechanics. In this book he gave the first proof of a theorem that he and John von Neumann first suggested in 1928 (Neumann and Wigner 1928), describing how quantum-mechanical symmetries are given by unitary or anti-unitary operator on a Hilbert space. This theorem is now known as Wigner's Theorem, and is regarded as fundamental.

In this text we will look at a few different symmetries in quantum mechanics, one of which is the symmetry described by von Neumann and Wigner. Roughly speaking, a symmetry is a bijective map that preserves the relevant mathematical structures. We will focus on the algebra of self-adjoint operators, viewed as a Jordan algebra, the convex set of normal states, and the set of pure states equipped with a transition probability.

In Chapter 1 we develop the language for the bounded operator case, including some convexity theory, which will play an important role throughout the thesis. We start by introducing some probability theory on a finite set, which we interpret as a classical configuration space, after which we introduce the quantum-mechanical analogues of their classical counterparts.

In Chapter 2 we equip each of the sets of self-adjoint operators, density operators and one-dimensional projections with a suitable structure. To this end we introduce the concept of a transition probability space in great generality, and define a transition probability on the set of one-dimensional projections. Once we have equipped the relevant sets with a mathematical structure, we define Wigner, Kadison, and Jordan symmetries to be bijections that preserve these structures. We prove that these symmetries are equivalent, and prove that they are all induced by either a unitary or an anti-unitary operator. The difficult part lies in proving that every Jordan symmetry is either a *-isomorphism or a *-anti-isomorphism. This is a consequence of the decomposition of a Jordan map given by Thomsen (Thomsen 1982), which we work out in great detail.

Chapters 3 and 4 are dedicated to generalising the results from Chapters 1 and 2 in the context of C*-algebras. The goal of Chapter 3 is finding the C*-algebra analogues of a Wigner, Kadison, and Jordan symmetry. These symmetries are still equivalent, as we prove in Chapter 4. In the first section of Chapter 4 we extend the equivalence between Kadison, and Jordan symmetries to include non-unital C*-algebras. After that we give the necessary framework to prove the main result from Shultz 1982, mainly the concept of an orientation of the state space. Once we have done this, we give a detailed proof of the equivalence between orientation-preserving Wigner symmetries and *-isomorphisms. In the last section we tweak this result slightly to prove the equivalence between Wigner symmetries and Jordan symmetries.

Chapter 1

Quantum mechanics on a Hilbert space

In this chapter we define key concepts we will study throughout this thesis. We start with a brief introduction into convexity theory and classical mechanics, after which we will define quantum-mechanical analogues to some classical notions. This Chapter is based on chapters 1 through 4 of Landsman (2017), with the exception of Section 1.1.

1.1 Convexity theory

We will see throughout this thesis that convexity plays an important role in the mathematical description of quantum mechanics, both in the bounded operator case and in the algebraic case. In this section we define some key definitions regarding convexity. This section is based on Chapter 1 of Alfsen and Shultz (2001). We use the letter V to denote a complex vector space.

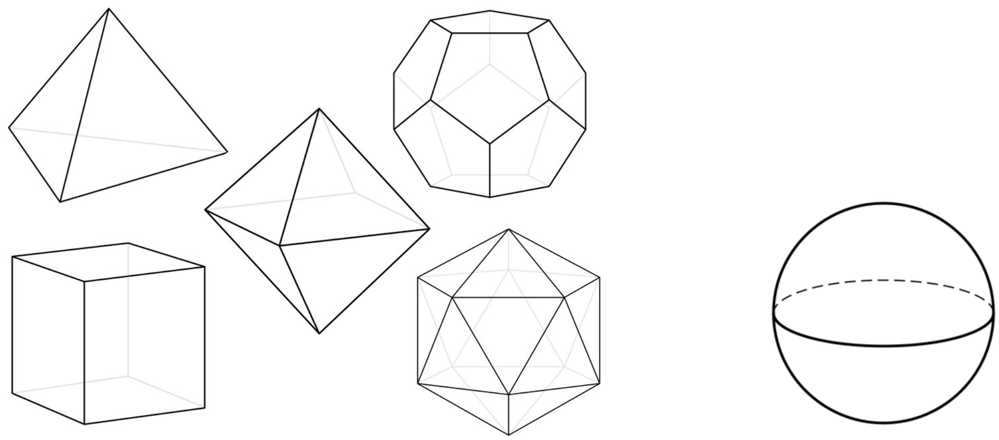
Definition 1.1. *A subset $M \subset V$ is said to be an **affine subspace** if for all $x, y \in M$ and $\lambda, \mu \in \mathbb{R}$ with $x + y = 1$ we have*

$$\lambda x + \mu y \in M.$$

*Similarly, a subset $K \subset M$ is said to be **convex** if for all $x, y \in M$ and $0 \leq \lambda < 1$ we have*

$$\lambda x + (1 - \lambda) y \in K.$$

*The **convex hull** of a subset E of V consists of all elements $x \in V$ of the form $\sum_{i=1}^{n} \lambda_i x_i$ where $x_i \in E$ and $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$. It will be denoted by $\mathrm{co}(E)$.*



(a) Platonic solids (b) Unit ball

Figure 1.1: Two examples of three-dimensional convex subsets.

Figure 1.1 shows some examples of three-dimensional convex subsets.

Definition 1.2. *A convex set C of a linear space V is said to be a **Euclidean ball** or an **n-ball** if it is affinely isomorphic to the n-ball*

$$B^n = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1^2 + \cdots + \lambda_n^2 \leq 1\}, \tag{1.1}$$

for some $n < \infty$.

Definition 1.3. *A convex subset F of a convex set $K \subset X$ is said to be a **face** of K if the following implication holds for $y, z \in K$ and $0 < \lambda < 1$:*

$$\lambda y + (1 - \lambda) z \in F \Rightarrow y, z \in F.$$

*For a given subset E of K the intersection of all faces containing E is said to be the **face generated by E**, denoted $\mathrm{face}_K(E)$.*

As the name suggests, a face (in the sense of a flat surface surrounded by edges) of the platonic solids shown in Figure 1.1a is also a face in the sense of the previous definition. However, also edges and vertices are faces in the sense of Definition 1.3.

Definition 1.4. *A point x in a convex set K is said to be an **extreme point** if there is no convex combination $x = \lambda y + (1 - \lambda) z$ with $y \neq x$, $z \neq x$ and $0 < \lambda < 1$, or, equivalently, if the singleton $\{x\}$ is a face of K. The set of all extreme points of K is called the **extreme boundary of K** and is denoted by $\partial_e K$.*

For example, the extreme points of a platonic solid are its vertices and the extreme points of the unit ball is the unit sphere.

Definition 1.5. *A function f from a convex set K to a linear space is said to be **affine** if f preserves convex combinations, i.e.,*

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y), \tag{1.2}$$

whenever $x, y \in K$ and $0 < \lambda < 1$.

Definition 1.6. *Two convex subsets F and G of a real vector space V are said to be **affinely independent** if every point z in their convex hull $\mathrm{co}(F \cup G)$ can be uniquely expressed as a convex combination*

$$z = \lambda x + (1-\lambda)y,$$

where $0 \leq \lambda \leq 1$, $x \in F$ and $y \in G$. Here uniqueness means uniqueness up to the indeterminancy of x when $\lambda = 0$ and y when $\lambda = 1$.

Definition 1.7. *We say that a convex set $K \subset V$ is the **free convex sum** of two convex subsets F and G, and we write $K = F \oplus_c G$, if $K = \mathrm{co}(F \cup G)$ and F, G are affinely independent. Observe that if $K = F \oplus_c G$, then the two sets must be faces of K. We say that a face F of K is a **split face** if there exists another face G such that $K = F \oplus_c G$. In this case G is unique; we call it the **complementary split face** of F, and we will use the notation $F' = G$. More specifically, F' consists of all points $x \in K$ whose generated face in K is disjoint from F, in symbols*

$$F' = \{x \in K \mid \mathrm{face}_K(x) \cap F = \emptyset\}. \tag{1.3}$$

Theorem 1.8 (Krein-Milman)**.** *Let V be a real normed vector space with dual V^*, and let K be a convex subset of V^* that is compact in the w^*-topology. Then $\partial_e K \neq \emptyset$, and each point of K lies in the w^*-closure of the convex hull of $\partial_e K$. In other words,*

$$K = \overline{\mathrm{co}(\partial_e K)}^{w^*}. \tag{1.4}$$

1.2 The classical case

In this section we give a short introduction to the mathematical description of classical mechanics. Although classical mechanics is *a priori* deterministic, we will develop a statistical approach to establish a close analogy between classical and quantum mechanics. Because we are only interested in the classical case to illustrate the analogies with the quantum-mechanical case, we can limit ourselves to *finite* systems. Let X be a *finite* set, playing the role of the ***configuration space*** of some physical system. By a configuration space we mean that every element $x \in X$ describes a certain configuration of the system. Take for instance a lattice $A \subset \mathbb{Z}^d$ for which each site is the home of some classical object that may assume N different configurations. For every site we may pick one of the N possible configurations, hence the total amount of configurations is given by $N^{|A|}$. In this case the configuration space is $X = \underline{N}^A$, in that $x\colon A \to \underline{N} = \{1, \ldots, N\}$ describes the configuration in which the site $a \in A$ has configuration $x(a) \in \underline{N}$. Once you know the configuration of the system, i.e., once you know which element $x \in X$ describes the system, you can determine all other characteristics of the system.

To develop the statistical approach, we need the following definitions.

Definition 1.9. *(a) A **probability distribution** on X is a function $p\colon X \to [0,1]$ such that $\sum_{x\in X} p(x) = 1$.*

*(b) A **probability measure** on X is a function $P\colon \mathscr{P}(X) \to [0,1]$ such that $P(X) = 1$ and $P(U \cup V) = P(U) + P(V)$ whenever $U, V \in \mathscr{P}(X)$ such that $U \cap V = \emptyset$.*

*(c) A **random variable** on X is a function $f\colon X \to \mathbb{R}$.*

*(d) The **spectrum** of a random variable f is the subset $\mathrm{sp}(f) = \{f(x) \mid x \in X\}$ of $\mathbb{R}$.*

Note that there is a bijection between the set of probability distribution p and probability measures P given by

- $P(U) = \sum_{x\in U} p(x)$, where $U \in \mathscr{P}(X)$, and;
- $p(x) = P(\{x\})$.

Furthermore, every $y \in X$ defines a probability distribution p_y by $p_y(x) = \delta_{xy}$. Then for the corresponding probability measure P_y one has $P_y(U) = 1$ if $y \in U$ and $P_y(U) = 0$ if $y \notin U$.

Proposition 1.10. *The set $\mathrm{Pr}(X)$ of all probability measures on X is convex. Furthermore, the set X is isomorphic to the extreme boundary $\partial_e \mathrm{Pr}(X)$ through $x \mapsto P_x$.*

Proof. The proof that $\mathrm{Pr}(X)$ is convex is trivial, so we will only prove the second part of the proposition. Because there is an affine bijection between the set of probability measures and the set of probability distributions, it is enough to prove that X is isomorphic to the set $\{p_x \mid x \in X\}$. Denote the set of probability distributions by $\mathrm{Pd}(X)$, so we want to prove that X is isomorphic to $\partial_e \mathrm{Pd}(X)$. First, $x \mapsto p_x$ is trivially bijective, so we only need to show that all p_x are the extreme points of $\mathrm{Pd}(X)$. Suppose that one has probability distributions p and q and $t \in (0,1)$ such that $p_x = tp + (1-t)q$. Taking $y \neq x$ yields $p(y) = q(y) = 0$, so that $p = 1 = p_x$. This shows that $\{p_x \mid x \in X\} \subseteq \partial_e \mathrm{Pd}(X)$. Conversely, suppose that $p \in \partial_e \mathrm{Pd}(x)$, but $p \neq p_x$ for all $x \in X$. Then there is some $x_0 \in X$ with $0 < p(x_0) < 1$. Now define $q \in \mathrm{Pd}(X)$ by $q(x_0) = 0$ and $q(x) = p(x)/(1-p(x_0))$ for $x \neq x_0$. Then

$$p = p(x_0)p_{x_0} + (1-p(x_0))q \tag{1.5}$$

is a nontrivial convex combination that yields p. This contradicts the fact that $p \in \mathrm{Pd}(X)$, so $\mathrm{Pd}(X) \subseteq \{p_x \mid x \in X\}$. □

The following theorem gives a very important construction.

Theorem 1.11. *A probability distribution p on X and a random variable $f\colon X \to \mathbb{R}$ jointly yield a probability distribution on p_f on the spectrum $\mathrm{sp}(f)$ by means of*

$$p_f(\lambda) = \sum_{x \in X \,|\, f(x) = \lambda} p(x). \tag{1.6}$$

In terms of the corresponding probability measure P on X, one has

$$p_f(\lambda) = P(f = \lambda), \tag{1.7}$$

where $f = \lambda$ denotes the closed set $\{x \in X \mid f(x) = \lambda\}$ in X. Similarly, the probability measure P_f on $\mathrm{sp}(f)$ corresponding to the probability distribution p_f is given by

$$P_f(\Delta) = P(f \in \Delta), \tag{1.8}$$

where $\Delta \subseteq \mathrm{sp}(f)$ and $f \in \Delta$ denotes the set $\{x \in X \mid f(x) \in \Delta\}$ in X.

Proof. The proof is trivial and just a matter of writing out definitions. □

Because every random variable gives us some information about the system, we also call them ***observables***. Suppose that the system is mixed, in the sense that there is probability distribution p such that the probability of the system being in the configuration $x \in X$ is given by $p(x)$. Then we are interested in the ***expectation value*** of a random variable:

$$E_P(f) = \sum_{x \in X} f(x)p(x). \tag{1.9}$$

Using the previous theorem, this is the same as

$$E_P(f) = \sum_{\lambda \in \mathrm{sp}(f)} \lambda p_f(\lambda). \tag{1.10}$$

We now develop another language to describe the statistical approach to classical mechanics. Let X still denote a finite set.

Definition 1.12. *The **algebra of observables** is the space $C(X)$ of all complex-valued functions on X, viewed as a commutative C^*-algebra under the pointwise operations:*

$$(\lambda \cdot f)(x) = \lambda f(x) \quad (\lambda \in \mathbb{C}); \tag{1.11}$$
$$(f + g)(x) = f(x) + g(x); \tag{1.12}$$
$$(f \cdot g)(x) = f(x)g(x), \tag{1.13}$$

and with involution and norm given by:

$$f^*(x) = \overline{f(x)}; \tag{1.14}$$
$$\|f\|_\infty = \sup_{x \in X}\{|f(x)|\}. \tag{1.15}$$

Although $C(X)$ is called the algebra of *observables*, the term observables should only be applied to the self-adjoint elements, i.e., to the random variables. We denote the set of random variables by $R(X)$, so $R(X) = C(X)_{\mathrm{sa}}$.

Definition 1.13. *A **state** on $C(X)$ is a complex-linear map $\omega\colon C(X) \to \mathbb{C}$ satisfying:*

*(a) **positivity**: $\omega(f^*f) \geq 0$ for each $f \in C(X)$;*

*(b) **normalisation**: $\omega(1_X) = 1$.*

*The set of all states is called the **state space** and is denoted by $S(C(X))$.*

Equivalently, we may define a state on $R(X)$ as a real-linear map $\omega_{\mathbb{R}}\colon R(X) \to \mathbb{R}$ that satisfies the same conditions. Indeed, we may restrict any state $\omega \in S(C(X))$ to the $R(X)$, which gives us a state on $R(X)$. Conversely, given a state $\omega_{\mathbb{R}}$ on $R(X)$ we may extend it to a state ω on $C(X)$ by setting $\omega(f + ig) = \omega_{\mathbb{R}}(f) + i\omega_{\mathbb{R}}(g)$, where $f, g \in R(X)$. Hence we can use states on $C(X)$ and on $R(X)$ interchangeably.

It is clear from the definition of the state space that it is a compact convex subset of $C(X)^*$. We call elements of the extreme boundary of $S(C(X))$ ***pure states*** and the set of all pure states the ***pure state space***, which we denote by $P(C(X))$. The following theorem establishes the link between states on $C(X)$ and probability measure (and hence probability distributions) on X.

Theorem 1.14. *The relation $\omega(f) = E_P(f)$ induces an isomorphism*

$$S(C(X)) \cong \Pr(X) \tag{1.16}$$

of convex sets. Furthermore, for the corresponding pure states we have

$$P(C(X)) \cong X, \tag{1.17}$$

i.e., any pure state on $C(X)$ is given by $\omega(f) = f(x)$, for some $x \in X$.

Proof. Given a state ω, define a function $p\colon X \to \mathbb{R}$ by $p(x) = \omega(\delta_x)$. Since $\delta_x > 0$ pointwise, positivity of ω yields $p(x) \geq 0$. Because $1_X = \sum_{x\in X} \delta_x$, normalisation then forces $\sum_{x\in X} p(x) = 1$, so that p is a probability distribution on X. Hence ω corresponds to a probability measure $P \in \Pr(X)$, where P is the probability measure corresponding to p. Conversely, $P \in \Pr(X)$ defines a map $E_P\colon R(X) \to \mathbb{R}$, which is positive and normalised.

It is easy to see that the map $\varphi\colon \Pr(X) \to S(C(X))$ given by $P \mapsto E_P$ is convex. Hence is maps the extreme boundary of $\Pr(X)$, which according to Proposition 1.10 is isomorphic to X, onto that of $S(C(X))$. □

1.3 From classical mechanics to quantum mechanics

We now give the quantum analogues to some important classical notions. In this section H denotes a (possibly infinite-dimensional) Hilbert space.

The quantum analogue of the configuration space X is a finite-dimensional Hilbert space H. Infinite-dimensional Hilbert spaces are the quantum analogues of infinite *phase spaces*. For more information on infinite phase spaces we refer to Landsman (2017), Chapter 3. The quantum analogue of a random variable is a bounded self-adjoint operator on H, which we call an **observable**. As the name suggests, an observable is a physical quantity that can be measured, for instance spin or position. Given an observable $a \in B(H)_{\mathrm{sa}}$, supposing that the system is described by $x \in H$, the expectation value of the observable a is given by

$$E_x(a) = \langle x, ax\rangle. \tag{1.18}$$

This shows that the element $x \in H$ does not *uniquely* describe the state of the system. Indeed, suppose that $z \in \mathbb{T}$ (where $\mathbb{T} = \{x \in \mathbb{C} \mid |x| = 1\}$). Then for every $a \in B(H)_{\mathrm{sa}}$:

$$E_{zx}(a) = \langle zx, a(zx)\rangle = \overline{z}z\langle x, ax\rangle = |z|^2 \langle x, ax\rangle = \langle x, ax\rangle = E_x(a). \tag{1.19}$$

Hence the unit vector that describes the quantum-mechanical system is unique up to a phase $z \in \mathbb{T}$. So instead of using unit vectors to describe the system, we should use ***one-dimensional projections***. Denote the set of projections on H by $\mathcal{P}(H)$, so

$$\mathcal{P}(H) = \{e \in B(H) \mid e^2 = e^* = e\}. \tag{1.20}$$

Note that for all $e \in \mathcal{P}(H)$:

$$\mathrm{Tr}(e) = \dim(eH). \tag{1.21}$$

The set of one-dimensional projections $\mathcal{P}_1(H)$ is then given by

$$\mathcal{P}_1(H) = \{e \in B(H) \mid e^2 = e^* = e,\ \dim(eH) = 1\} \tag{1.22}$$

$$= \{e \in B(H) \mid e^2 = e^* = e,\ \mathrm{Tr}(e) = 1\} \tag{1.23}$$

$$= \{e_x \mid x \in H,\ \|x\| = 1\}, \tag{1.24}$$

where e_x is the projection onto xH given by

$$e_x \colon H \to H; \quad y \mapsto \langle x, y\rangle x. \tag{1.25}$$

The one-dimensional projection e_x is also sometimes denoted by $|x\rangle\langle x|$. The expectation value of the observable $a \in B(H)_{\mathrm{sa}}$ is then given by

$$\mathrm{Tr}(e_x a) = \langle x, ax\rangle. \tag{1.26}$$

The quantum analogue of a probability distribution is a ***density operator*** (cf. Appendix A.4).

Definition 1.15. *A **density operator** is a positive trace-class operator ρ on H such that*

$$\mathrm{Tr}(\rho) = 1. \tag{1.27}$$

Equivalently[1]*, an operator ρ is a density operator if and only if it has a norm-convergent expansion*

$$\rho = \sum_{\lambda \in \mathrm{sp_p}(\rho)} \lambda \cdot e_\lambda, \tag{1.28}$$

where $\mathrm{sp_p}(\rho)$ is some countable subset of $\mathbb{R}^+$ with 0 as its only possible accumulation point, the multiplicity $m_\lambda = \dim H_\lambda$ of each eigenvalue $\lambda > 0$ is finite, and

$$\sum_{\lambda \in \mathrm{sp_p}(\rho)} \lambda \cdot m_\lambda = 1. \tag{1.29}$$

We denote the set of density operators on H by $\mathcal{D}(H)$.

The set of density operators forms a convex subset of $B(H)$. We interpret $\mathrm{Tr}(\rho a)$ as the expectation value of the observable $a \in B(H)_{\mathrm{sa}}$, i.e.,

$$E_\rho(a) = \mathrm{Tr}(\rho a). \tag{1.30}$$

Of course, the one-dimensional projections are in particular density operators, more precisely, they are the extreme points of the convex set of density operators. Denote the density operator induced by the unit vector x by ρ_x, i.e., $\rho_x = e_x = |x\rangle\langle x|$.

Lemma 1.16. *A density operator ρ is an extreme point of the convex set $\mathcal{D}(H)$ of density operators if and only if $\rho = \rho_x$ for some unit vector $x \in H$.*

Proof. By definition of the density operator we have $\partial_e \mathcal{D}(H) \subseteq \mathcal{P}_1(H)$, so it only remains to be shown that $\mathcal{P}_1(H) \subseteq \partial_e \mathcal{D}(H)$. If H is one-dimensional, the claim is trivial, so suppose that $\dim H > 1$. Let $x_0 \in H$ be a unit vector and assume that $\rho_{x_0} = \lambda\rho_1 + (1-\lambda)\rho_2$ for some $\lambda \in (0,1)$ and $\rho_1, \rho_2 \in \mathcal{D}(H)$. Let $y \in H$ be a unit vector that is perpendicular to x_0. Then

$$\langle y, \rho_{x_0} y\rangle = \langle y, \langle x_0, y\rangle x_0\rangle = \big|\langle x_0, y\rangle\big|^2 = 0, \tag{1.31}$$

which implies that

$$0 = \langle y, \rho_{x_0} y\rangle = \lambda\langle y, \rho_1 y\rangle + (1-\lambda)\langle y, \rho_2 y\rangle. \tag{1.32}$$

Because ρ_1 and ρ_2 are positive we have $\langle y, \rho_i y\rangle \geq 0$ for $i = 1, 2$, which gives us

$$\langle y, \rho_1 y\rangle = \langle y, \rho_2 y\rangle = 0. \tag{1.33}$$

Now let $\{v_i\}$ be an orthonormal basis for H with $v_1 = x_0$. Then

$$1 = \mathrm{Tr}(\rho_1) = \sum_i \langle v_i, \rho_1 v_i\rangle = \langle x_0, \rho_1 x_0\rangle, \tag{1.34}$$

and similarly $\langle x_0, \rho_2 x_0\rangle = 1$. This shows that

$$\langle v_i, \rho_1 v_i\rangle = \langle v_i, \rho_2 v_i\rangle = \langle v_i, \rho_{x_0} v_i\rangle, \tag{1.35}$$

for every v_i in the orthonormal basis. It now follows from Theorem A.13 that $\rho_1 = \rho_2 = \rho_{x_0}$, so ρ_{x_0} is an extreme point of $\mathcal{D}(H)$. □

[1]This equivalence follows from the spectral theorem for self-adjoint compact operators.

The following result is a consequence of the symbolic calculus for normal operators, which is explained in Section A.3. It explains how a density operator induces a unique probability measure on the spectrum of an observable. This is the quantum-mechanical analogue of Theorem 1.11.

Corollary 1.17. *Let H be a Hilbert space, let $a^* = a \in B(H)$ and let $\rho \in \mathcal{D}(H)$ be a density operator. There exists a unique probability measure μ_ρ on the spectrum* $\mathrm{sp}(a)$ *for which*

$$\mathrm{Tr}(\rho f(a)) = \int_{\mathrm{sp}(a)} d\mu_\rho f, \quad f \in C(\mathrm{sp}(a)); \tag{1.36}$$

$$\mu_\rho(\Delta) = \mathrm{Tr}(\rho e_\Delta). \tag{1.37}$$

This measure on $\mathrm{sp}(a)$ *is called the* ***Born measure*** *(defined by a and ρ).*

Specifically, a unit vector $x \in H$ induces a unique probability measure μ_x on $\mathrm{sp}(a)$ *such that*

$$\langle x, f(a)x\rangle = \int_{\mathrm{sp}(a)} f d\mu_x, \quad f \in C(\mathrm{sp}(a)). \tag{1.38}$$

In terms of spectral projections $e_\Delta = 1_\Delta(a)$ (defined for Borel sets $\Delta \subseteq \mathrm{sp}(a)$), the Born measure is given by

$$\mu_x(\Delta) = \|e_\Delta x\|^2 . \tag{1.39}$$

Proof. See Landsman (2017), Corollary 4.4. □

1.4 The state space of the algebra of bounded operators

Just as in the classical case, we can use ***states*** to develop the statistical approach to quantum mechanics.

Definition 1.18. *A* ***state*** *on $B(H)$ is a complex-linear map $\omega\colon B(H) \to \mathbb{C}$ satisfying:*

(a) ***positivity****: $\omega(a^*a) \geq 0$, $a \in B(H)$;*

(b) ***normalisation****: $\omega(1_H) = 1$.*

The set of all states is called the ***state space*** *and is denoted by $S(B(H))$.*

Proposition 1.19. *Let H be a Hilbert space. The state space is a w^*-compact convex subset of $B(H)^*$.*

Proof. It is clear from the definition of a state that the state space is convex. Note further that it is a w^*-closed subset of $B(H)^*$. Indeed, suppose that $\{\omega_n\}$ is a sequence in $S(B(H))$ that converges in the w^*-topology to $\omega \in B(H)^*$. Then in particular

$$1 = \omega_n(1_H) \to \omega(1_H), \tag{1.40}$$

which proves that ω is normalised. In the same way it follows that ω is positive, so $\omega \in S(B(H))$. By Banach-Alaoglu (Theorem A.53) the unit ball is w^*-compact. Hence the state space is a w^*-closed subset of a w^*-compact subset. This shows that $S(B(H))$ is w^*-compact. □

By 1.8 the extreme boundary of $S(B(H))$ is non-empty. We define ***pure states*** to be extreme points of $S(B(H))$ and the ***pure state space*** to be the extreme boundary of $S(B(H))$. We denote the pure state space by $P(B(H))$, i.e., $P(B(H)) \equiv \partial_e S(B(H))$.

Proposition 1.20. *Let H be a Hilbert space. Then the pure state space $P(B(H))$ is non-empty and*

$$S(B(H)) = \overline{(\mathrm{co}(P(B(H)))}^{w^*}. \tag{1.41}$$

Proof. This follows immediately from the fact that $S(B(H)))$ is a w^*-compact subset of $B(H)^*$ and Krein-Milman (Theorem 1.8). □

The following theorem relates the state space $S(B(H))$ to the set of density operators $\mathcal{D}(H)$, in the case that H is finite-dimensional.

Theorem 1.21. *Let H be a finite-dimensional Hilbert space. Then there is a bijective correspondence between states ω on $B(H)$ and density operators ρ on H, given by*

$$\omega(a) = \mathrm{Tr}(\rho a). \tag{1.42}$$

Proof. Because H is finite-dimensional, every operator is trace-class, i.e., $B_1(H) = B(H)$. Then

$$\langle a, b \rangle = \mathrm{Tr}(a^* b) \tag{1.43}$$

defines an inner product on $B(H)$. By elementary linear algebra for every $\omega \in B(H)^*$ there is a unique $a \in B(H)$ such that $\omega = \mathrm{Tr}(a\cdot)$. It follows easily that ω is positive if and only if the corresponding $a \in B(H)$ is positive. Similarly, ω is normalised if and only if the corresponding bounded operator has trace 1. □

Let us give an example of a state space. Take $H = \mathbb{C}^2$, then the bounded operators of H are the 2×2 complex matrices, i.e., $B(H) = M_2(\mathbb{C})$. The following proposition gives a description of the state space of $M_2(\mathbb{C})$.

Proposition 1.22. *The state space $S(M_2(\mathbb{C}))$ of the 2×2 matrices is isomorphic (as a compact convex set) to the closed unit ball $B^3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\}$. The isomorphism $\varphi\colon B^3 \to S(M_2(\mathbb{C}))$ is explicitly given by*

$$\begin{aligned} \varphi\colon B^3 &\to S(M_2(\mathbb{C})); \\ \mathbf{x} &\mapsto \mathrm{Tr}(\rho_{\mathbf{x}}\cdot), \end{aligned} \tag{1.44}$$

where $\mathbf{x} = (x,y,z)$ and $\rho_{\mathbf{x}}$ is the density matrix given by

$$\rho_{\mathbf{x}} = \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}. \tag{1.45}$$

On this isomorphism, the extreme boundary

$$\partial_e B^3 = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\} \tag{1.46}$$

corresponds to the set of all density matrices $\rho = \rho_{\mathbf{x}}$, where $\mathbf{x}$ is a unit vector in $\mathbb{C}^2$.

Proof. We follow the proof of Proposition 2.9 from Landsman (2017). The adjoint of a matrix $A = \{A_{ij}\}$ is given by $A^* = \{\overline{A_{ji}}\}$, hence any self-adjoint 2×2 matrix may be parametrized by $(t,x,y,z) \in \mathbb{R}^4$ as

$$\rho(t,x,y,z) = \tfrac{1}{2}\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}. \tag{1.47}$$

The eigenvalues λ_i of $\rho(t,x,y,z)$ are

$$\lambda_\pm = \tfrac{1}{2}(t \pm \sqrt{x^2+y^2+z^2}). \tag{1.48}$$

Now let $\rho = \rho(t,x,y,z)$ be a density operator. Condition (1.27) yields $t = 1$. Positivity of ρ is equivalent to positivity of its eigenvalues, which gives us $x^2+y^2+z^2 \geq 1$.

For the second claim, note that the $\rho_{\mathbf{x}}$ are just the one-dimensional projections, which in turn are the density matrices satisfying $\rho^2 = \rho$, so $x^2+y^2+z^2 = 1$. Finally, since convex sums $t\mathbf{v} + (1-t)\mathbf{w}$ in B^3 are given by straight line segments connecting $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, it immediately follows geometrically that $\partial_e B^3 = S^2$. □

By Corollary A.101 the set $B(H)_{\mathrm{sa}}$ is monotone complete, i.e., each increasing net of self-adjoint operators that is bounded above converges strongly (and weakly) to its supremum. For a bounded increasing net $\{a_\lambda\}$ in $B(H)_{\mathrm{sa}}$ with supremum $a \in B(H)_{\mathrm{sa}}$ we write $a_\lambda \nearrow a$ to describe this situation.

Lemma 1.23. *The following conditions on a state $\omega \in S(B(H))$ are equivalent:*

(i) For each orthogonal family $\{e_i\}$ of projections (i.e. $e_i^* = e_i$ and $e_i e_j = \delta_{ij}$) one has

$$\omega\Big(\sum_i e_i\Big) = \sum_i \omega(e_i). \tag{1.49}$$

Here $\sum_i e_i$ is defined as the projection on the smallest closed subspace K of H that contains each $e_i H$ (that is, $\sum_i e_i = \vee_i e_i$ in the sense of Theorem A.100). Furthermore, the sum over i on the right-hand side is defined as the supremum (in $\mathbb{R}$) of the set of all sums $\sum_{i\in F}\omega(e_i)$ over finite subsets $F \subset I$ of the index set I in which i takes values. It is finite because $\sum_{i\in F} e_i \leq 1_H$ and hence, since ω is positive,

$$\sum_{i\in F}\omega(e_i) \leq \omega(1_H) = 1. \tag{1.50}$$

(ii) $\omega(a) = \lim_\lambda \omega(a_\lambda)$ whenever $a_\lambda \nearrow a$.

(iii) $\omega(a) = \mathrm{Tr}(\rho a)$ for some density operator $\rho \in \mathcal{D}(H)$.

(iv) ω is σ-weakly continuous.

Proof. See Landsman (2017), Corollary 4.13. □

Definition 1.24. *A state $\omega \in S(B(H))$ is called **normal** if it satisfies one, and hence all, of the equivalent conditions in Lemma 1.23. We call the set of normal states on $B(H)$ the **normal state space** and denote it by $S_n(B(H))$. We denote the set of normal pure states by $P_n(B(H))$.*

Using Lemma 1.23, it is easy to see that $S_n(B(H))$ is a convex subset of $S(B(H))$. In fact, it is a split face of the state space, as we will see in Proposition 3.24. This implies that the extreme boundary of $S_n(B(H))$ consists of pure states of $B(H)$, i.e., $P_n(B(H)) = \partial_e S_n(B(H))$. The following theorem gives an isomorphism between the normal state space and the set of density operators.

Theorem 1.25. *The relation $\omega(a) = \mathrm{Tr}(\rho a)$ induces an isomorphism*

$$S_n(B(H)) \cong \mathcal{D}(H) \tag{1.51}$$

of convex sets. Furthermore, for the corresponding pure states we have

$$P_n(B(H)) \cong \mathcal{P}_1(H), \tag{1.52}$$

i.e., any normal pure state on $B(H)$ *is given by* $\omega = \omega_x$ *for some unit vector* $x \in H$*, where* $\omega_x(a) = \langle x, ax \rangle$*.*

Proof. See Landsman (2017), Corollary 4.14. □

Because every pure normal state is induced by a unit vector in H, we call these states ***vector states***. Note that for finite-dimensional Hilbert spaces every state is normal. This follows directly from the first characterisation of a normal functional, because every orthogonal family of projections consists of at most $\dim H < \infty$ elements. In that case, Theorem 1.25 simplifies to Theorem 1.21. If, however, H is infinite-dimensional, then $S(B(H))$ is strictly larger than $S_n(B(H))$.

Chapter 2

Symmetries in quantum mechanics

In the previous chapter we introduced the following concepts:

1. ***Observables***, which are the self-adjoint operators $B(H)_{\text{sa}}$.

2. ***One-dimensional projections*** $\mathcal{P}_1(H)$, which according to Theorem 1.25 are in 1-1 correspondence with the normal pure states on $B(H)$.

3. ***Density operators*** $\mathcal{D}(H)$, which according to Theorem 1.25 are affinely isomorphic to the normal state space $S_n(H)$.

In this chapter we will impose more structure on $B(H)_{\text{sa}}$ and $\mathcal{P}_1(H)$ and define ***symmetries*** on $B(H)_{\text{sa}}$, $\mathcal{P}_1(H)$ and $\mathcal{D}(H)$. By a symmetry we mean an invertible bijection that preserves all relevant structure. In Section 2.3 we will prove that symmetries on $B(H)_{\text{sa}}$, $\mathcal{P}_1(H)$ and $\mathcal{D}(H)$ are in fact equivalent. In the last sections of this chapter we will prove that the symmetries correspond in all cases to either unitary or anti-unitary operators. A key ingredient in proving this is a theorem by Thomsen (1982), which we will explain in great detail in Section 2.4.

This chapter is largely based on Chapter 5 of Landsman (2017), with the exception of Section 2.1 and Section 2.4.

2.1 The transition probability

Transition probabilities are a key concept in this thesis and we will use transition probabilities on different spaces, such as the set one-dimensional projections on a Hilbert space in this chapter and the pure state space of a C*-algebra in section 3.6.

Definition 2.1. *A **transition probability** on a set X is a function*

$$\tau \colon X \times X \to [0,1] \tag{2.1}$$

that satisfies $\tau(x,y) = 1$ if and only if $x = y$, and $\tau(y,x) = \tau(x,y)$

*A **transition probability space** is a set P with a transition probability*

$$\tau \colon P \times P \to [0,1]. \tag{2.2}$$

Definition 2.2. *Let (P,τ) be a transition probability space and let $X \subseteq P$. Define the **orthocomplement in P** of X by*

$$X^{\perp} = \{x \in P \mid \tau(x,y) = 0 \text{ for all } y \in X\}. \tag{2.3}$$

*We denote the **double orthocomplement** by $X^{\perp\perp} = (X^{\perp})^{\perp}$ and similarly $X^{\perp\perp\perp} = (X^{\perp\perp})^{\perp}$, etc.*

Compare this definition to that of orthogonal complement using inner product given in Definition A.10: the transition probability plays the role of the inner product on a vector space. This leads to similar results, as stated in the following lemma.

Lemma 2.3. *Let $X, Y \subseteq P$ be subsets of a transition probability space such that $X \subseteq Y$. Then $Y^{\perp} \subseteq X^{\perp}$ and $X^{\perp\perp\perp} = X^{\perp}$.*

Proof. It follows immediately from the definition of the orthocomplement that $Y^{\perp} \subseteq X^{\perp}$ and that $X \subseteq X^{\perp\perp}$. Combining these relations shows that

$$X^{\perp\perp\perp} = (X^{\perp\perp})^{\perp} \subseteq X^{\perp} \quad \text{and} \quad X^{\perp} \subseteq (X^{\perp})^{\perp\perp} = X^{\perp\perp\perp}. \tag{2.4}$$

□

Definition 2.4. *Let X be a subset of a transition probability space P. The **orthoclosure** of X is defined as $X^{\perp\perp}$. We call the subset X **orthoclosed** if $X^{\perp\perp} = X$.*

Note that it follows from the previous lemma that $X^{\perp}$ is always orthoclosed.

Definition 2.5. *A family of subsets of a transition probability space P is called **orthogonal** if $\tau(x,y) = 0$ whenever x and y do not lie in the same subset. A **basis** of P is an orthogonal family B of points of P with the property that*

$$\sum_{x \in B} \tau(x,y) = 1 \quad \forall y \in P, \tag{2.5}$$

where the sum over x on the left-hand side is defined as the supremum (in $\mathbb{R}$) of all sums $\sum_{i \in F} \tau(x,y)$ over finite subsets $F \subseteq B$ of the index set B.

Definition 2.6. *A transition probability space is* ***well-behaved*** *if every orthoclosed subset X of P has the property that any maximal orthogonal subset of X is a basis of X.*

The notion of a basis is due to Mielnik (1968). Mielnik defines transition probabilities to be transition probabilities in our sense, with the additional requirement that they are well-behaved. The following theorem proves that every basis of a well-behaved transition probability space has the same cardinality.

Theorem 2.7. *Let (P, τ) be a well-behaved transition probability space and let B_1 and B_2 be two bases of P. Then B_1 and B_2 have the same cardinality.*

Proof. See Mielnik (1968), Theorem 1. □

Lemma 2.8. *Let (P, τ) be a well-behaved transition probability space and X be an orthogonal family of points. Then*

$$X^{\perp\perp} = \{y \in P \mid \sum_{x \in X} \tau(x, y) = 1\}. \tag{2.6}$$

Proof. Let $B \subseteq X^{\perp}$ be a maximal orthogonal subset of $X^{\perp}$. Because $X^{\perp}$ is orthoclosed and P is well-behaved, the set B is a basis of $X^{\perp}$.

Claim: The set $X \cup B$ forms a basis of P.

Proof: Because P is orthoclosed and well-behaved, it is enough to show that $X \cup B$ is a maximal orthogonal subset of P. Suppose for the sake of contradiction that it is not a basis. Because $X \cup B$ is clearly a family or orthogonal points, this implies that there exists a $y \in P$ such that $\tau(x, y) = 0$ for all $x \in X \cup B$ and $y \notin X \cup B$. Then in particular $\tau(x, y) = 0$ for every $x \in X$, and hence $y \in X^{\perp}$. But because B is a basis for $X^{\perp}$ we have $\sum_{x \in B} \tau(x, y) = 1$, which contradicts our assumption that $\tau(x, y) = 0$ for every $x \in X \cup B$. Thus $X \cup B$ indeed is a maximal orthogonal set, and hence a basis for P. ■

Now suppose that $y \in X^{\perp\perp}$. Then $\tau(x, y) = 0$ for all $x \in B \subseteq X^{\perp}$, and hence

$$1 = \sum_{x \in X \cup B} \tau(x, y) = \sum_{x \in X} \tau(x, y). \tag{2.7}$$

This shows that $X^{\perp\perp} \subseteq \{y \in P \mid \sum_{x \in X} \tau(x, y) = 1\}$. Finally, suppose that $y \notin X^{\perp\perp}$. Then there exists a $x \in X^{\perp}$ such that $\tau(x, y) > 0$. Now choose B to be a maximal orthogonal subset of $X^{\perp}$ that contains x. Then $1 = \sum_{x \in X \cup B} \tau(x, y) > \sum_{x \in X} \tau(x, y)$, which contradicts the fact that $X \cup B$ is a basis. This implies that $\{y \in P \mid \sum_{x \in X} \tau(x, y) = 1\} \subseteq X^{\perp\perp}$. □

2.2 Basic mathematical structures of quantum mechanics

We view $B(H)_{\mathrm{sa}}$, $\mathcal{D}(H)$ and $\mathcal{P}_1(H)$, not just as sets, but as mathematical structures. Recall that

$$\mathcal{P}_1(H) = \{e \in B(H) \mid e^2 = e^* = 2,\ \mathrm{Tr}(e) = \dim(eH) = 1\}; \tag{2.8}$$
$$\mathcal{D}(H) = \{\rho \in B(H) \mid \rho \geq 0,\ \mathrm{Tr}(\rho) = 1\}; \tag{2.9}$$
$$B(H)_{\mathrm{sa}} = \{a \in B(H) \mid a^* = a\}. \tag{2.10}$$

The additional structure on $\mathcal{P}_1(H)$ is a transition probability $\tau^{\mathcal{P}_1(H)}$ given by

$$\tau^{\mathcal{P}_1(H)}\colon\ \mathcal{P}_1(H) \to P_1(H); \quad (e, f) \mapsto \mathrm{Tr}(ef). \tag{2.11}$$

It is easy to see that this is indeed a transition probability, because for one-dimensional projections e_x and e_y in $\mathcal{P}_1(H)$ we have

$$\tau^{\mathcal{P}_1(H)}(e_x, e_y) = \mathrm{Tr}(e_x e_y) = \left|\langle x, y\rangle\right|^2. \tag{2.12}$$

We then define a ***Wigner map*** to be a map that preserves this transition probability.

Definition 2.9. *Let H_1 and H_2 be Hilbert spaces. A* ***Wigner symmetry*** *is a bijection*

$$W\colon\ \mathcal{P}_1(H_1) \to \mathcal{P}_1(H_2) \tag{2.13}$$

that satisfies

$$\mathrm{Tr}(W(e)W(f)) = \mathrm{Tr}(ef), \quad e, f \in \mathcal{P}_1(H). \tag{2.14}$$

We saw in the previous chapter that $\mathcal{D}(H)$ is a convex set. This is the relevant structure on $\mathcal{D}(H)$ that we want to preserve. This leads to the definition of a ***Kadison symmetry***.

Definition 2.10. *Let H_1 and H_2 be Hilbert spaces. A* ***Kadison symmetry*** *is an affine bijection*

$$K\colon\ \mathcal{D}(H_1) \to \mathcal{D}(H_2), \tag{2.15}$$

i.e., a bijection that preserves convex sums.

Lastly, we add structure to the set $B(H)_{\mathrm{sa}}$ by viewing it as a ***Jordan algebra***. We will discuss Jordan algebras in more detail in Section 3.5, but for now it is enough to equip $B(H)_{\mathrm{sa}}$ with the *Jordan product* "$\circ$" given by

$$a \circ b = \tfrac{1}{2}(ab + ba). \tag{2.16}$$

This turns $(B(H)_{\mathrm{sa}}, \circ)$ into a real associative vector space. A map that preserves the Jordan product is called a ***Jordan map***.

Definition 2.11. *Let H_1 and H_2 be Hilbert spaces. A **Jordan symmetry** is an invertible Jordan map*

$$J\colon B(H_1)_{sa} \to B(H_2)_{sa}, \tag{2.17}$$

i.e., an $\mathbb{R}$-linear bijection that satisfies the equivalent conditions

$$J(a \circ b) = J(a) \circ J(b); \tag{2.18}$$
$$J(a^2) = J(a)^2. \tag{2.19}$$

Note that every Jordan map $J\colon B(H_1)_{\mathrm{sa}} \to B(H_2)_{\mathrm{sa}}$ has a unique extension to a $\mathbb{C}$-linear map

$$J_{\mathbb{C}}\colon B(H_1) \to B(H_2); \tag{2.20}$$
$$J_{\mathbb{C}}(a^*) = J_{\mathbb{C}}(a)^*, \tag{2.21}$$

which preserves the Jordan product for all a, b, as well as

$$J_{\mathbb{C}}(a + ib) = J(a) + iJ(b), \tag{2.22}$$

for all $a, b \in A_{\mathrm{sa}}$. We call such a map a ***complex Jordan map***. Conversely, such a complex Jordan map defines a real Jordan map by restricting it to $B(H_1)_{\mathrm{sa}}$.

Recall from section A.2 the definition of a unitary and anti-unitary operator:

Definition 2.12. *Let H_1 and H_2 be Hilbert spaces and $u \in B(H_1, H_2)$. We call u a **unitary operator** or a **unitary** if it satisfies one and hence all of the following equivalent conditions*

(a) $u^*u = 1_{H_1}$ and $uu^* = 1_{H_2}$;

(b) u is surjective and $\langle ux, uy\rangle_{H_2} = \langle x, y\rangle$ for all $x, y \in H_1$.

Definition 2.13. *Let H_1 and H_2 be Hilbert spaces.*

(a) A real-linear operator $u\colon H_1 \to H_2$ *is* ***anti-linear*** *if*

$$u(\lambda x) = \overline{\lambda} u(x) \ \ (\lambda \in \mathbb{C}). \tag{2.23}$$

(b) An anti-linear operator $u\colon H_1 \to H_2$ *is* ***anti-unitary*** *if it is surjective, and*

$$\langle ux, uy\rangle_{H_2} = \overline{\langle x, y\rangle}_{H_1} \ \ (x, y \in H_1). \tag{2.24}$$

The adjoint $u^*\colon H_2 \to H_1$ *of a bounded anti-linear operator* u *is defined by the property*

$$\langle u^* x, y\rangle_{H_1} = \overline{\langle x, uy\rangle}_{H_2} \ \ (x \in H_2, \ y \in H_1), \tag{2.25}$$

in which case $u^*\colon H_2 \to H_1$ *is anti-linear too. Hence we may equally say that an anti-linear operator is anti-unitary if* $u^*u = 1_{H_1}$ *and* $uu^* = 1_{H_2}$.

The goal for the rest of this chapter is to prove that any Wigner, Jordan or Kadison symmetry is induced by a unitary or anti-unitary in the following sense:

$$W(e) = ueu^*; \tag{2.26}$$

$$K(\rho) = u\rho u^*; \tag{2.27}$$

$$J(a) = uau^*, \tag{2.28}$$

where $u \in B(H_1, H_2)$ is either a unitary or anti-unitary or a *-anti-isomorphism. Moreover, u is uniquely determined by the symmetry in question up to a phase. The theorem involving Wigner symmetries is known as ***Wigner's Theorem***. It was first stated by von Neumann and Wigner in 1928, and proven by Wigner in 1931. Mirroring the name "Wigner's Theorem", we call the statements involving Kadison and Jordan symmetries ***Kadison's Theorem*** and ***Jordan's Theorem*** respectively. A direct proof of Wigner's Theorem is given in Landsman (2017), who follows Simon (1976). Wigner's Theorem is first proven for two-dimensional Hilbert spaces, after which it is generalised to arbitrary Hilbert spaces. We will take a different route, using the equivalence between Jordan symmetries and Wigner symmetries. Once we have proven Jordan's Theorem, Wigner's Theorem and Kadison's Theorem will follow immediately.

2.3 Equivalences between symmetries

We will prove the equivalence between the Wigner symmetries, Kadison symmetries and Jordan symmetries in the sequence Wigner ↔ Kadison ↔ Jordan. We follow the proofs from Landsman 2017 almost *verbatim*. We start by proving Wigner ↔ Kadison.

Proposition 2.14. *There is an isomorphism of groups between:*

- *The group of Kadison symmetries* $K\colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$*;*
- *The group of Wigner symmetries* $\mathcal{P}_1(H_1) \to P_2(H_2)$*,*

given by

$$W = K_{|\mathcal{P}_1(H_1)}; \tag{2.29}$$

$$K\Big(\sum_i \lambda_i e_{v_i}\Big) = \sum_i \lambda_i W(v_{v_i}), \tag{2.30}$$

where $\rho = \sum_i \lambda_i e_{v_i}$ *is some (not necessarily unique) expansion of* $\rho \in \mathcal{D}(H)$ *in terms of a basis of eigenvectors* v_i *with eigenvalues* λ_i*, where* $\lambda_i \geq 0$ *and* $\sum_i \lambda_i = 1$*. In particular,* (2.29) *and* (2.30) *are well defined.*

Proof. We start with 2.29. Let K be a Kadison symmetry. Because it is affine, it preserves the extreme boundary. By Lemma 1.16 this means that K maps $\mathcal{P}_1(H_1)$ bijectively onto $\mathcal{P}_1(H_2)$. So it remains to be shown that $K_{|\mathcal{P}_1(H_1)}$ preserves transition probabilities, i.e.,

$$\mathrm{Tr}(K(e)K(f)) = \mathrm{Tr}(ef), \tag{2.31}$$

for all $e, f \in H_1$. To prove this, we need the following two claims.

<u>Claim:</u> An affine bijection $K\colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$ extends to an isomorphism

$$K_1\colon B_1(H_1)_{\mathrm{sa}} \to B_1(H_2)_{\mathrm{sa}} \tag{2.32}$$

that is isometric in the trace norm.

<u>Proof:</u> Put $K_1(0) = 0$ and for $a > 0$, $a \in B_1(H_1)^+$ define

$$K_1(a) = \|a\|_1 \, K\Big(\frac{a}{\|a\|_1}\Big). \tag{2.33}$$

For $a \in B_1(H_1)^+$, $a \neq 0$ we have $\mathrm{Tr}(a) = \|a\|_1$, hence $a/\|a\|_1 \in \mathcal{D}(H_1)$, on which K is defined. Hence K_1 is isometric and preserves positivity. Let $\lambda > 0$. Then for $a \in B_1(H_1)^+$, $a > 0$ we have $\lambda a \in B_1(H_1)^+$, and

$$K_1(\lambda a) = \|\lambda a\|_1 \, K\Big(\frac{\lambda a}{\|\lambda a\|_1}\Big) = \lambda \|a\|_1 \, K\Big(\frac{a}{\|a\|_1}\Big). \tag{2.34}$$

Note that if $a, b \in B_1(H_1)^+$, then $a + b \in B_1(H_1)^+$. Now use

$$a + b = \left(\|a\|_1 + \|b\|_1\right) \cdot \left(t\frac{a}{\|a\|_1} + (1-t)\frac{b}{\|b\|_1}\right), \tag{2.35}$$

with $t = \|a\|_1 / (\|a\|_1 + \|b\|_1)$, and equation (2.34) to see that

$$\begin{aligned} K_1(a+b) &= K_1\Big(\left(\|a\|_1 + \|b\|_1\right)\big(t\frac{a}{\|a\|_1} + (1-t)\frac{b}{\|b\|_1}\big)\Big) \\ &= \left(\|a\|_1 + \|b\|_1\right) K_1\Big(t\frac{a}{\|a\|_1} + (1-t)\frac{b}{\|b\|_1}\Big) \\ &= \left(\|a\|_1 + \|b\|_1\right)(tK\Big(\frac{a}{\|a\|_1}\Big) + (1-t)K\Big(\frac{b}{\|b\|_1}\Big)) \\ &= K_1(a) + K_1(b) \end{aligned} \tag{2.36}$$

This shows linearity of K_1 with positive coefficients on $B_1(H_1)^+$. Now let $a \in B_1(H_1)_{\mathrm{sa}}$ and ue Corollary A.41 to decompose $a = a_+ - a_-$, where $a_\pm \geq 0$. We then define

$$K_1(a) = K_1(a_+) - K_1(a_-). \tag{2.37}$$

This is independent of the decomposition of a, for suppose that $a_+ - a_- = a = a'_+ - a'_-$. Then by (2.36):

$$K_1(a_+) + K_1(a'_-) = K_1(a_+ + a'_-) = K_1(a'_+ + a_-) = K_1(a'_+) + K_1(a_-). \tag{2.38}$$

Hence for $a, b \in B_1(H_1)_{\mathrm{sa}}$ we may compute

$$\begin{aligned} K_1(a+b) &= K_1(a_+ + b_+ - (a_- + b_-)) = K_1(a_+ + b_+) - K_1(a_- + b_-) \\ &= K_1(a_+) + K_1(b_+) - K_1(a_-) - K_1(b_-) = K_1(a) + K_1(b), \end{aligned} \tag{2.39}$$

since $a_+ + b_+$ and $a_- + b_-$ are both positive. Because the construction is invertible, we have now shown that $K_1 \colon B(H_1)_{\mathrm{sa}} \to B(H_2)_{\mathrm{sa}}$ is an isomorphism. It only remains to be shown that it is isometric in the trace norm. By Corollary A.41 we have $|a| = a_1 + a_2$. Using this property, we have

$$\begin{aligned} \left\|K_1(a)\right\|_1 &= \mathrm{Tr}(\left|K_1(a)\right|) = \mathrm{Tr}(\left|K_1(a_+) - K_1(a_-)\right|) = \mathrm{Tr}(K_1(a_+) + K_1(a_-)) \\ &= \mathrm{Tr}(\|a_+\|_1 K(a_+/\|a_+\|_1) + \mathrm{Tr}(\|a_-\|_1 K(a_-/\|a_-\|_1))) \\ &= \|a_+\|_1 + \|a_-\|_1 = \mathrm{Tr}(a_+) + \mathrm{Tr}(a_-) = \mathrm{Tr}(a_+ + a_-) \\ &= \mathrm{Tr}(|a_+ - a_-|) = \mathrm{Tr}(|a|) = \|a\|_1, \end{aligned} \tag{2.40}$$

i.e., K_1 is a trace norm isometry. ■

Claim: For any two unit vectors $x, y \in H$ we have the formula

$$\left\|e_x - e_y\right\|_1 = 2\sqrt{1 - \mathrm{Tr}(e_x e_y)}. \tag{2.41}$$

Proof: If $y = zx$ for some $z \in \mathbb{T}$ we have $e_x = e_y$, in which case equation (2.41) reads $0 = 0$. So assume that x and y are linearly independent. Then everything takes place in the two-dimensional subspace of H spanned by $x = (1, 0)$ and $y = (c_1, c_2)$, with $|c_1|^2 + |c_2|^2$. So we can prove (2.41) using 2×2 matrices. Then

$$e_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} |c_1|^2 & c_1\overline{c_2} \\ \overline{c_1}c_2 & |c_2|^2 \end{pmatrix}, \tag{2.42}$$

from which it follows that

$$(e_x - e_y)^2 = |c_2|^2 \cdot 1_2; \tag{2.43}$$

$$\left|e_x - e_y\right| = \sqrt{(e_x - e_y)^2} = |c_2| \cdot 1_2; \tag{2.44}$$

$$\left\|e_x - e_y\right\|_1 = \mathrm{Tr}(\left|e_x - e_y\right|) = 2|c_2|. \tag{2.45}$$

Hence

$$\left\|e_x - e_y\right\|_1 = 2|c_2| = 2\sqrt{|c_2|^2} = 2\sqrt{1 - |c_1|^2} = 2\sqrt{1 - \mathrm{Tr}(e_x e_y)}. \tag{2.46}$$

■

Now let $e_x, e_y \in \mathcal{P}_1(H_1)$. By the first claim we have

$$\left\|K(e_x) - K(e_y)\right\|_1 = \left\|K_1(e_x - e_y)\right\|_1 = \left\|e_x - e_y\right\|_1. \tag{2.47}$$

Using the second claim, this implies that

$$2\sqrt{1 - \mathrm{Tr}(K(e_x)K(e_y))} = 2\sqrt{1 - \mathrm{Tr}(e_x e_y)}, \tag{2.48}$$

and hence

$$\mathrm{Tr}(K(e_x)K(e_y)) = \mathrm{Tr}(e_x e_y), \tag{2.49}$$

i.e., $K_{|\mathcal{P}_1(H_1)}$ preserves transition probabilities. We conclude that $K_{|\mathcal{P}_1}: \mathcal{P}_1(H_1) \to \mathcal{P}_1(H_2)$ is a Wigner symmetry.

We move on to equation (2.30). Our main concern is that this expression is well defined, because the norm convergent expansion

$$\rho = \sum_{\lambda \in \mathrm{sp_p}(\rho)} \lambda \cdot e_\lambda \tag{2.50}$$

is in general far from unique. We settle this as follows.

Claim: Let $W\colon \mathcal{P}_1(H_1) \to P_1(H_2)$ be a bijection that preserves transition probabilities. Let $L \subset H$ be a finite-dimensional subspace, and let $\{v_i\}$ and $\{v'_j\}$ be bases of L. Then

$$\sum_i W(e_{v_i}) = \sum_j W(e_{v'_j}). \tag{2.51}$$

Proof: From Bessel's inequality (Lemma A.12) we obtain that for every unit vector $x \in H_1$:

$$\sum_i \big|\langle v_i, x\rangle\big|^2 \leq \|x\| = 1. \tag{2.52}$$

The equality holds if and only if $x \in L$. In other words, $e_x \leq e_L$ if and only if $x \in L$, and hence $e_x \leq e_L$ if and only if $\sum_i \mathrm{Tr}(e_{v_i} e_x) = 1$. Furthermore, because W preserves transition probabilities the images $W(e_{v_i})$ remain orthogonal. Hence $\sum_i W(e_{v_i})$ is a projection, and for every projection $e \in P_1(H_2)$ we have $e \leq \sum_i W(e_{v_i})$ if and only if $\mathrm{Tr}(\sum_i W(e_{v_i})e) = \sum_i \mathrm{Tr}(W(e_{v_i})e) = 1$. Because W preserves transition probabilities, this is satisfied for $e = W(e_{v'_j})$. Because all $W(e_{v'_j})$ are orthogonal, we conclude that

$$\sum_j W(e_{v'_j}) \leq \sum_i W(e_{v_i}). \tag{2.53}$$

Interchanging the roles of the two bases gives the converse, yielding the claim. ■

Finally, to prove bijectivity of the correspondence $K \leftrightarrow W$, we need the property

$$K\Big(\sum_i \lambda_i e_{v_i}\Big) = \sum_i \lambda_i K(e_{v_i}), \tag{2.54}$$

since this implies that K is determined by its action on $\mathcal{P}_1(H_1) \subset \mathcal{D}(H_1)$. In finite dimension this follows from convexity of K, and we are done. For infinite dimensions we need the extension of K (see the first claim) to be continuous in the trace norm, as well as convergence of $\sum_i \lambda_i e_{v_i}$ in the trace norm. To prove the latter, note that for finite n, m,

$$\left\| \sum_{i=n}^{m} \lambda_i e_{v_i} \right\|_1 \leq \sum_{i=n}^{m} |\lambda_i| \left\| e_{v_i} \right\|_1 = \sum_{i=n}^{m} \lambda_i, \tag{2.55}$$

where we use the triangle inequality and the fact that $e_{v_i} \in \mathcal{D}(H_1)$. Because $\sum_i \lambda_i = 1$, the above expression vanishes as $n, m \to \infty$.

Lastly, continuity of $K_1 \colon B_1(H_1) \to B_1(H_2)$ in the trace norm follows from the fact that K is isometric in the trace norm and hence bounded. □

To prove the equivalence between Kadison symmetries and Jordan symmetries we take a short detour to the real vector space of bounded affine function. In what follows, for any convex set C, the notation $A_b(C))$ stands for the real vector spaces of bounded affine functions $f \colon C \to \mathbb{R}$, that is, bounded functions satisfying

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y), \ x, y \in C, \ t \in (0,1). \tag{2.56}$$

Proposition 2.15. *For any Hilbert space H we have an isometric isomorphism*

$$A_b(\mathcal{D}(H)) \cong B(H)_{sa}, \tag{2.57}$$
$$f \leftrightarrow a, \tag{2.58}$$
$$f(\rho) = \mathrm{Tr}(\rho a), \tag{2.59}$$

which preserves the unit as well as the order.

Proof. See Landsman (2017), Proposition 5.17. □

The equivalence between Kadison symmetries and Jordan symmetries follows directly from the following lemma.

Lemma 2.16. *1. There is a bijective correspondence between:*

- *Kadison symmetries $K \colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$;*
- *unital positive linear bijections $\alpha \colon B(H_2)_{sa} \to B(H_1)_{sa}$,*

such that for any $a \in B(H)_{sa}$ one has

$$\mathrm{Tr}(K(\rho)a) = \mathrm{Tr}(\rho\alpha(a)). \tag{2.60}$$

2. *A map* $\alpha\colon B(H_1) \to B(H_2)$ *is a unital positive linear bijection if and only if it is a Jordan symmetry.*

Proof. 1. A Kadison symmetry $K\colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$ induces an isomorphism

$$K^*\colon A_b(\mathcal{D}(H_2)) \to A_b(\mathcal{D}(H_1)); \tag{2.61}$$
$$f \mapsto f \circ K, \tag{2.62}$$

which is unital, positive and isometric. Let

$$\Phi_i\colon A_b(\mathcal{D}(H_i)) \to B(H_i)_{\text{sa}} \;\; (i = 1, 2) \tag{2.63}$$

be the isometric isomorphism from Proposition 2.15. Then

$$\alpha \equiv \Phi_1 \circ K^* \circ \Phi_2^{-1}\colon B(H_2)_{\text{sa}} \to B(H_1)_{\text{sa}} \tag{2.64}$$

is an isomorphism that is also unital, positive and isometric.

Conversely, such a map $\alpha\colon B(H_2)_{\text{sa}} \to B(H_1)_{\text{sa}}$ yields a map K directly by (2.60); to see this, we identify $\mathcal{D}(H_i)$ with the normal state space of $B(H_i)$ through $\rho \leftrightarrow \omega$ (see Theorem 1.25). For $\omega \in S_n(B(H_1))$ we then define $K\omega \in S_n(B(H_2))$ to be the state given by

$$(K\omega)(a) = \omega(\alpha(a)), \tag{2.65}$$

or briefly $K\omega = \omega \circ \alpha$.

2. The trivial direction of the proof is from Jordan symmetries to unital positive linear bijections. Indeed, let $J\colon B(H_1) \to B(H_2)$ be a Jordan symmetry. Then it is clearly a unital linear bijection. Now let $a \in B(H_1)^+$. Then $a = b^2$ for certain $b \in B(H_1)_{\text{sa}}$, and hence $J(a) = J(b^2) = J(b)^2 \geq 0$, which proves that J is positive.

The nontrivial direction is based on a number of facts from operator theory:

(i) Unital positive linear maps $B(H_1)_{\text{sa}} \to B(H_2)_{\text{sa}}$ map $\mathcal{P}(H_1)$ onto $\mathcal{P}(H_2)$.

(ii) Any two projections e and f are orthogonal ($ef = 0$) if and only if $e + f \leq 1_H$.

(iii) Any $a \in B(H)_{\text{sa}}$ is a norm-limit of finite sums of the kind $\sum_i \lambda_i e_i$, where $\lambda_i \in \mathbb{R}$ and the e_i are mutually orthogonal projections.

(iv) Any unital positive linear map $\alpha\colon B(H_1)_{\text{sa}} \to B(H_2)_{\text{sa}}$ is continuous and of norm 1.

For details we refer the reader to the proof of Lemma 5.20 in Landsman 2017. Therefore, by (i), (ii) and (iii), any unital positive linear map α preserves orthogonality of projections, so if $a = \sum_i \lambda_i e_i$ (finite sum), then

$$\alpha(a^2) = \alpha\Big(\sum_i \lambda_i^2 e_i\Big) = \sum_i \lambda_i^2 \alpha(e_i) = \sum_{i,j} \lambda_i \lambda_j \alpha(e_i)\alpha(e_j) = \alpha(a)^2. \quad (2.66)$$

By continuity of α, this property extends to arbitrary $a \in B(H_1)_{\mathrm{sa}}$. Finally, since

$$a \circ b = \tfrac{1}{2}((a+b)^2 - a^2 - b^2), \quad (2.67)$$

preserving squares implies preserving the Jordan product $\circ$.

□

Corollary 2.17. *There is an isomorphism of groups between:*

- *The group of Kadison symmetries* $K\colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$*;*
- *The group of Jordan symmetries* $J\colon B(H_2)_{sa} \to B(H_1)_{sa}$*,*

such that for any $a \in B(H)_{sa}$ *one has*

$$\mathrm{Tr}(K(\rho)a) = \mathrm{Tr}(\rho J(a)) \ \ (\rho \in \mathcal{D}(H)). \quad (2.68)$$

We now move on to Jordan's Theorem, see Section 2.5.

2.4 Decomposing a Jordan map

The proof of Jordan's Theorem consists of two main steps: firstly proving that every Jordan symmetry $J_{\mathbb{C}}\colon B(H_1) \to B(H_2)$ is either a *-isomorphism or a *-anti-isomorphism, and secondly proving that every *-isomorphism and *-anti-isomorphism are of the desired form. In his 1982 paper *Jordan-morphisms in *-algebras* (Thomsen 1982) Klaus Thomsen proved a theorem which allows us to decompose a Jordan map into three parts: a part that is a *-homomorphism and not a *-anti-homomorphism, a part that is a *-anti-homomorphism and not a *-homomorphism, and a part that is both a *-homomorphism as well as a *-anti-morphism. This result is a generalisation of Størmer (1965), which in turn is a generalisation of Jacobson and Rickart (1950). Using the special properties of the algebra of bounded operators, we will use this theorem to prove the first step of the proof of Jordan's Theorem.

Thomsen's theorem takes place in a more general setting than Jordan's Theorem. For this, we need a more general definition of a Jordan map.

Definition 2.18. *Let A and B be associative algebras over $\mathbb{R}$. A* ***Jordan map*** *is an $\mathbb{R}$-linear map that satisfies the equivalent conditions*

$$J(a \circ b) = J(a) \circ J(b) \tag{2.69}$$
$$J(a^2) = J(a)^2, \tag{2.70}$$

where '$\circ$' denotes the Jordan product given by

$$a \circ b = \tfrac{1}{2}(ab + ba). \tag{2.71}$$

As you can see, the only difference between this definition of a Jordan map and the definition given in Section 2.2 is that we now work with general algebras over $\mathbb{R}$, and not just with $B(H)_{\mathrm{sa}}$. Now let A be a *-algebra, then A_{sa} is an associative algebra over $\mathbb{R}$. Just as before, we can uniquely extend a Jordan map $J\colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ to a $\mathbb{C}$-linear map

$$J_{\mathbb{C}}\colon A \to B; \quad J_{\mathbb{C}}(a^*) = J_{\mathbb{C}}(a)^*, \tag{2.72}$$

which satisfies

$$J_{\mathbb{C}}(a \circ b) = J_{\mathbb{C}}(a) \circ J_{\mathbb{C}}(b). \tag{2.73}$$

Let A and B again be *-algebras. We will call a $\mathbb{C}$-linear map $\varphi\colon A \to B$ a Jordan map if $\varphi_{|_{A_{sa}}} : A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ is a Jordan map, in which case

$$\varphi = \big(\varphi_{|_{A_{\mathrm{sa}}}}\big)_{\mathbb{C}}. \tag{2.74}$$

Using this terminology, we can state Thomsen's theorem.

Theorem 2.19. *Let A be a *-algebra, and let $\varphi : A \to B(H)$ be a Jordan map. Assume that $\varphi(A)$ is again a *-algebra. Then there exist three mutually orthogonal projections p_i, $i = 1, 2, 3$, in the center of the von Neumann algebra generated by $\varphi(A)$, such that:*

- *(i) $\varphi(\cdot)p_1$ is a *-homomorphism, and not a *-anti-homomorphism;*
- *(ii) $\varphi(\cdot)p_2$ is an *-anti-homomorphism, and not a *-homomorphism;*
- *(iii) p_3 is the largest central projection such that $\varphi(\cdot)p_3$ is a *-homomorphism, as well an a *-anti-homomorphism;*
- *(iv) $1_H = p_1 \oplus p_2 \oplus p_3$.*

Conversely, the above conditions (1)-(4), determine the central projections p_i uniquely.

We will prove the theorem by explicitly constructing the projections p_1, p_2 and p_3. To do this, we need to do some preparatory work in the form of Lemma 2.22 and Lemma 2.23. To ease notation in these lemmas, we use the following definition.

Definition 2.20. *Let A and B be *-algebras and let $\varphi\colon A \to B$ be a Jordan map. Then we define for $a, b \in A$:*

$$a^b = i[\varphi(ab) - \varphi(a)\varphi(b)]; \tag{2.75}$$

$$a_b = i[\varphi(ab) - \varphi(b)\varphi(a)]. \tag{2.76}$$

Remark 2.21. It is easy to see that if $a, b \in A_{\text{sa}}$, then $a^b, a_b \in B_{\text{sa}}$. Indeed:

$$\begin{aligned}(a^b)^* &= \big(i[\varphi(ab) - \varphi(a)\varphi(b)]\big)^* \\ &= -i[\varphi(b^*a^*) - \varphi(b^*)\varphi(a^*)] \\ &= -i[\varphi(ba + ab) - \varphi(ab) - \varphi(b)\varphi(a)] \\ &= -i[\varphi(b)\varphi(a) + \varphi(a)\varphi(b) - \varphi(ab) - \varphi(b)\varphi(a)] \\ &= i[\varphi(ab) - \varphi(a)\varphi(b)] \\ &= a^b, \end{aligned} \tag{2.77}$$

and similarly for a_b.

Using elementary linear algebra, we can deduce several useful relations concerning a^b and a_b. These relations will be useful in the proof of Lemma 2.23. Lemma 2.22 is due to Herstein (1956).

Lemma 2.22. *Let A and B be *-algebras and $\varphi\colon A \to B$ a Jordan map. Then for all $a, b, c \in A$:*

(i) $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$;

(ii) $\varphi(abc + cba) = \varphi(a)\varphi(b)\varphi(c) + \varphi(c)\varphi(b)\varphi(a)$;

(iii) $a^b a_b = 0$;

(iv) $a^b\varphi(c)a^b = ia^b\varphi((ab - ba)c)$;

(v) $a^b\varphi((ab - ba)c)a_b = 0$;

(vi) $a^b\varphi(ab - ba)\varphi(c)\varphi(ab - ba)a_b = 0$;

(vii) $a^b + a^c = a^{b+c}$, *and* $a_b + a_c = a_{b+c}$;

(viii) $a^c + b^c = (a+b)^c$, *and* $a_c + b_c = (a+b)_c$.

Proof. (i) For all $a, b \in A$

$$aba = 2(a \circ (a \circ b)) - b \circ (a \circ a), \tag{2.78}$$

which shows that

$$\begin{aligned}\varphi(aba) &= 2\varphi\big((a \circ (a \circ b))\big) - \varphi\big(b \circ (a \circ a)\big)\\ &= 2\big(\varphi(a) \circ (\varphi(a) \circ \varphi(b))\big) - \varphi(b) \circ \big(\varphi(a) \circ \varphi(a)\big)\\ &= \varphi(a)\varphi(b)\varphi(a).\end{aligned} \tag{2.79}$$

(ii) Because for all $a, b, c \in A$

$$abc + cba = (a+c)b(a+c) - aba - cbc, \tag{2.80}$$

we can use (2.79) to show that

$$\begin{aligned}\varphi(abc + cba) &= \varphi\big((a+c)b(a+c)\big) - \varphi(aba) - \varphi(cbc)\\ &= \varphi(a+c)\varphi(b)\varphi(a+c) - \varphi(a)\varphi(b)\varphi(a) - \varphi(c)\varphi(b)\varphi(c)\\ &= \varphi(a)\varphi(b)\varphi(c) + \varphi(c)\varphi(b)\varphi(a).\end{aligned} \tag{2.81}$$

(iii) It now follows from (2.79) and (2.81) that $a^b a_b = 0$ for all $a, b \in A$:

$$\begin{aligned}a^b a_b &= -\big[\varphi(ab) - \varphi(a)\varphi(b)\big]\big[\varphi(ab) - \varphi(b)\varphi(a)\big]\\ &= -\varphi(ab)^2 + \big[\varphi(ab)\varphi(b)\varphi(a) + \varphi(a)\varphi(b)\varphi(ab)\big] - \varphi(a)\varphi(b)^2\varphi(a)\\ &= -\varphi(ab) \circ \varphi(ab) + \varphi(ab^2a + abab) - \varphi(ab^2a)\\ &= -\varphi(abab) + \varphi(ab^2a + abab) - \varphi(ab^2a) = 0.\end{aligned} \tag{2.82}$$

(iv) We can now use (2.81) and the Jordan property to see that

$$\varphi(c)a^b \tag{2.83}$$

$$\begin{aligned}&= i\varphi(c)\varphi(ab) - i\varphi(c)\varphi(a)\varphi(b)\\ &= i\varphi(c)\varphi(ab) + i\varphi(b)\varphi(a)\varphi(c) - i\varphi(cab + bac)\\ &= i\varphi(c)\varphi(ab) + i\varphi(b)\varphi(a)\varphi(c) - i\varphi\big(c(ab) + (ab)c\big) + i\varphi\big((ab - ba)c\big)\\ &= i\varphi(c)\varphi(ab) + i\varphi(b)\varphi(a)\varphi(c) - i\varphi(c)\varphi(ab) - i\varphi(ab)\varphi(c) + i\varphi\big((ab - ba)c\big)\\ &= i\big[\varphi(ab) - \varphi(a)\varphi(b)\big]\varphi(c) + i\varphi\big((ab - ba)c\big)\\ &= a_b\varphi(c) + i\varphi((ab - ba)c).\end{aligned} \tag{2.84}$$

Now multiply both sides of the equation from the left with a^b and use (2.82) to conclude that

$$\begin{aligned} a^b\varphi(c)a^b &= a^b a_b\varphi(c) + ia^b\varphi\big((ab-ba)c\big) \\ &= ia^b\varphi\big((ab-ba)c\big). \end{aligned} \tag{2.85}$$

(v) Multiply equation (2.85) from the right with a_b and use equation (2.82) to conclude that

$$a^b\varphi\big((ab-ba)c\big)a_b = -ia^b\varphi(c)a^b a_b = 0. \tag{2.86}$$

(vi) Use equation (2.79) and (2.86) and replace c with $c(ab-ba)$ to conclude that

$$0 = a^b\varphi\big((ab-ba)c(ab-ba)\big)a_b = a^b\varphi(ab-ba)\varphi(c)\varphi(ab-ba)a_b. \tag{2.87}$$

(vii) This follows immediately from the linearity of φ.

(viii) This follows immediately from the linearity of φ.

□

Lemma 2.23. *Let A be a *-algebra, and let $\varphi\colon A \to B(H)$ be a Jordan map such that $\varphi(A)$ is a *-algebra. Then*

$$\big(\varphi(ab) - \varphi(a)\varphi(b)\big)\big(\varphi(cd) - \varphi(d)\varphi(c)\big) = 0, \tag{2.88}$$

for all quadruples $a, b, c, d \in A$.

Proof. • We first reduce to the case that (A, φ) is unital. For if A is not unital, then we use the unitisation $\widetilde{A}$ and define $\widetilde{\varphi}\colon \widetilde{A} \to B(H)$ by $\widetilde{\varphi}\big((a,\lambda)\big) = \varphi(a) + \lambda 1_H$. The map $\widetilde{\varphi}$ is a Jordan map. Indeed, it preserves the adjoint:

$$\begin{aligned} \widetilde{\varphi}\big((a,\lambda)\big)^* &= \big(\varphi(a) + \lambda 1_H\big)^* \\ &= \varphi(a^*) + \overline{\lambda}1_H \\ &= \widetilde{\varphi}\big((a^*,\overline{\lambda})\big) \\ &= \widetilde{\varphi}\big((a,\lambda)^*\big), \end{aligned} \tag{2.89}$$

and it preserves the Jordan product:

$$
\begin{aligned}
\widetilde{\varphi}\big((a,\lambda)\circ(b,\mu)\big) &= \widetilde{\varphi}\big(\tfrac{1}{2}(a,\lambda)(b,\mu)+\tfrac{1}{2}(b,\mu)(a,\lambda)\big)\\
&= \widetilde{\varphi}\big(\tfrac{1}{2}(\mu a+\lambda b+ab,\lambda\mu)+\tfrac{1}{2}(\mu a+\lambda b+ba,\lambda\mu)\big)\\
&= \widetilde{\varphi}\big(\mu a+\lambda b+\tfrac{1}{2}(ab+ba),\lambda\mu\big)\\
&= \varphi(\mu a+\lambda b+\{a,b\})+\lambda\mu 1_H\\
&= \mu\varphi(a)+\lambda\varphi(b)+\{\varphi(a),\varphi(b)\}+\lambda\mu 1_H\\
&= \tfrac{1}{2}\big(\varphi(a)+\lambda 1_H\big)\big(\varphi(b)+\mu 1_H\big)+\tfrac{1}{2}\big(\varphi(b)+\mu 1_H\big)\big(\varphi(a)+\lambda 1_H\big)\\
&= \tfrac{1}{2}\widetilde{\varphi}\big((a,\lambda)\big)\widetilde{\varphi}\big((b,\mu)\big)+\tfrac{1}{2}\widetilde{\varphi}\big((b,\mu)\big)\widetilde{\varphi}\big((a,\lambda)\big)\\
&= \widetilde{\varphi}\big((a,\lambda)\big)\circ\widetilde{\varphi}\big((b,\mu)\big). \qquad (2.90)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big(\widetilde{\varphi}\big((a,\alpha)(b,\beta)\big)-\widetilde{\varphi}\big((a,\alpha)\big)\widetilde{\varphi}\big((b,\beta)\big)\Big)\Big(\widetilde{\varphi}\big((c,\gamma)(d,\delta)\big)-\widetilde{\varphi}\big((d,\delta)\big)\widetilde{\varphi}\big((c,\gamma)\big)\Big)\\
&\quad= \Big(\widetilde{\varphi}\big((\beta a+\alpha b+ab,\alpha\beta)\big)-\widetilde{\varphi}\big((a,\alpha)\big)\widetilde{\varphi}\big((b,\beta)\big)\Big)\cdot\\
&\qquad\qquad \Big(\widetilde{\varphi}\big((\delta c+\gamma d+cd,\gamma\delta)\big)-\widetilde{\varphi}\big((d,\delta)\big)\widetilde{\varphi}\big((c,\gamma)\big)\Big)\\
&\quad= \Big(\varphi(\beta a+\alpha b+ab)+\alpha\beta 1_H-\big(\varphi(a)+\alpha 1_H\big)\big(\varphi(b)+\beta 1_H\big)\Big)\cdot\\
&\qquad\qquad \Big(\varphi(\delta c+\gamma d+cd)+\gamma\delta 1_H-\big(\varphi(d)+\delta 1_H\big)\big(\varphi(c)+\gamma 1_H\big)\Big)\\
&\quad= \big(\varphi(ab)-\varphi(a)\varphi(b)\big)\big(\varphi(cd)-\varphi(d)\varphi(c)\big). \qquad (2.91)
\end{aligned}
$$

This shows that if the statement holds for unital (A,φ), it holds for all (A,φ).

- Secondly, we will reduce to the case where $\varphi(A)''$ is a factor, i.e., where the center of $\varphi(A)''$ only consists of scalar multiples of the identity.

  Because we have reduced to the case that (A,φ) is unital, the von Neumann bicommutant theorem states that $\varphi(A)''$ is a von Neumann algebra. Because the norm topology is stronger than the strong topology, every strongly closed set is also norm-closed. Hence $\varphi(A)''$ is a C*-algebra. Let P be the set of pure states on $\varphi(A)''$ and let $\pi_a=\oplus_{\omega\in P}\pi_\omega$ be its atomic representation. According to Proposition A.90 this representation is faithful, so it is enough to show that

$$
\begin{aligned}
&\big[(\pi_\omega\circ\varphi)(ab)-(\pi_\omega\circ\varphi)(a)(\pi_\omega\circ\varphi)(b)\big]\cdot\\
&\qquad\qquad \big[(\pi_\omega\circ\varphi)(cd)-(\pi_\omega\circ\varphi)(d)(\pi_\omega\circ\varphi)(c)\big]=0, \qquad (2.92)
\end{aligned}
$$

for all $a, b, c, d \in A$ and $\omega \in P$. So let $\omega \in P$ be a pure state. Note that

$$\pi_\omega \circ \varphi\colon A \to B(H_\omega) \tag{2.93}$$

is a Jordan map because φ is a Jordan map and π_ω is a *-homomorphism. Further note that and $(\pi_\omega \circ \varphi)(A)$ is a *-algebra, because $\varphi(A)$ is a *-algebra and π_ω is a *-homomorphism. We now show that $\pi_\omega(\varphi(A)'')$ is a factor. By Corollary A.73 we know that $\varphi(A)$ is σ-weakly dense in $\varphi(A)''$. We want to conclude that $\pi_\omega(\varphi(A))$ is σ-weakly dense in $\pi_\omega(\varphi(A)'')$, but we cannot do so directly, because we do not know if π_ω is σ-weakly continuous. We can work around this problem by using some results from Chapter 3. Corollary 3.32 shows that π_ω has an extension to a map from A^{**} to $B(H_\omega)$ that is σ-weakly continuous. Indeed, let $\widetilde{\omega}$ be the normal state on the enveloping von Neumann algebra A^{**} that corresponds to ω. Then Lemma 3.15 and 3.16 show that

$$\pi_{\widetilde{\omega}}\colon (\varphi(A)'')^{**} \to B(H_\omega) \tag{2.94}$$

is σ-weakly continuous, which implies that $\pi_{\widetilde{\omega}}(\varphi(A))$ is σ-weakly dense in $\pi_{\widetilde{\omega}}(\varphi(A)'')$. Because $\pi_{\widetilde{\omega}}$ is an extension of $\pi_\omega\colon \varphi(A)'' \to B(H_\omega)$ we have

$$\pi_{\widetilde{\omega}}(\varphi(A)) = \pi_\omega(\varphi(A)), \tag{2.95}$$

and

$$\pi_{\widetilde{\omega}}(\varphi(A)'') = \pi_\omega(\varphi(A)''). \tag{2.96}$$

This shows that $\pi_\omega(\varphi(A))$ is σ-weakly dense in $\pi_\omega(\varphi(A)'')$, and hence

$$\pi_\omega(\varphi(A))'' = \pi_\omega(\varphi(A)'')''. \tag{2.97}$$

Since ω is in particular a pure state on $\varphi(A)''$, we have by Theorem A.95 that

$$\pi_\omega(\varphi(A))'' = \pi_\omega(\varphi(A)'')'' = B(H_\omega). \tag{2.98}$$

This implies that

$$Z\big(\pi_\omega(\varphi(A))''\big) = B(H_\omega)' = \mathbb{C} \cdot 1_H, \tag{2.99}$$

i.e., $\big((\pi_\omega \circ \varphi)(A)\big)''$ is a factor. We have now shown that in general $\varphi(A)''$ is a direct sum of von Neumann algebras that are factors. Because we can consider all of the summands independently, we can reduce to the case that $\varphi(A)''$ is a factor.

For the rest of the proof we will assume that (A, φ) is unital and that $\varphi(A)''$ is a factor. Because $\varphi(A)$ is dense in $\varphi(A)''$, Lemma 2.22 part 6 implies that

$$a^b \varphi(ab - ba)\varphi(A)''\varphi(ab - ba)a_b = 0. \tag{2.100}$$

So we can apply Corollary A.81 to conclude that for all $a, b \in A$,

$$a^b \varphi(ab - ba) = 0 \text{ or } \varphi(ab - ba)a_b = 0. \tag{2.101}$$

We use the Jordan property to rewrite

$$\begin{aligned} \varphi(ab - ba) &= 2\varphi(ab) - \varphi(ab + ba) \\ &= 2\varphi(ab) - \varphi(a)\varphi(b) - \varphi(b)\varphi(a) \\ &= -i(a^b + a_b). \end{aligned} \tag{2.102}$$

First assume that $a^b \varphi(ab - ba) = 0$ in equation (2.101). Using (2.102) and equation (2.82), we find

$$0 = a^b \varphi(ab - ba) = -ia^b(a^b + a_b) = -i(a^b)^2, \tag{2.103}$$

hence $(a^b)^2 = 0$. Similarly, the other possibility in equation (2.101) implies $(a_b)^2 = 0$. If $a, b \in A_{\text{sa}}$, we know that a^b and a_b are self-adjoint. In that case $(a^b)^2 = \left|a^b\right|^2 = 0$ or $(a_b)^2 = |a_b|^2 = 0$. Therefore, we can conclude that for all $a, b \in A_{\text{sa}}$:

$$a^b = 0 \text{ or } a_b = 0. \tag{2.104}$$

This shows that for all $a, b \in A_{\text{sa}}$ and $c \in A$:

$$a_b \varphi(c) a^b = 0. \tag{2.105}$$

Using this relation and Lemma 2.22 part 7 we see that for all $a, b, d \in A_{\text{sa}}$ and $c \in A$:

$$\begin{aligned} 0 &= a_{b+d}\varphi(c)a^{b+d} \\ &= a_b\varphi(c)a^b + a_b\varphi(c)a^d + a_d\varphi(c)a^b + a_d\varphi(c)a^d \\ &= a_b\varphi(c)a^d + a_d\varphi(c)a^b. \end{aligned} \tag{2.106}$$

Combining this with equation (2.104) implies that for all $a, b, d \in A_{\text{sa}}$ we have $a_d = 0$ or $a^b = 0$, hence:

$$a_d a^b = 0. \tag{2.107}$$

Replacing a by $a+c$ and using Lemma 2.22 part 8 we find that for all $a,b,c,d \in A_{\mathrm{sa}}$:

$$\begin{aligned}0 &= (a+c)_d + (a+c)^b \\ &= a_d a^b + a_d c^b + c_d a^b + c_d c^b \\ &= a_d c^b + c_d a^b. \end{aligned} \tag{2.108}$$

We know that $a_d = 0$ or $a^b = 0$, and $c^b = 0$ or $c_d = 0$, hence (2.108) implies that $a^b = 0$ or $c_b = 0$. So for all $a,b,c,d \in A_{\mathrm{sa}}$:

$$a^b c_d = 0. \tag{2.109}$$

Now let $a,b,c,d \in A$ and write these as $a = a_1 + ia_2, \ldots, d = d_1 + id_2$ with $a_1, a_2, \ldots, d_2, d_2 \in A_{\mathrm{sa}}$. Then it follows from Lemma 2.22 part 7 and 8 and the previous relation that

$$\begin{aligned}&\big(\phi(ab - \varphi(a)\varphi(b)\big)\big(\varphi(cd) - \varphi(d)\varphi(c)\big) \\ &\quad = -a^b c_d \\ &\quad = (a_1 + ia_2)^{b_1 + ib_2}(c_1 + ic_2)_{d_1 + id_2} \\ &\quad = (a_1^{b_1} + ia_1^{b_2} + ia_2^{b_1} - a_2^{b_2})(c_{1d_1} + ic_{1d_2} + ic_{2d_1} - c_{2d_2}) \\ &\quad = 0. \end{aligned} \tag{2.110}$$

□

Corollary 2.24. *Let A be a *-algebra, and let $\varphi\colon A \to B(H)$ be a Jordan map such that $\varphi(A)$ is a *-algebra. Then*

$$\big(\varphi(ab) - \varphi(b)\varphi(a)\big)\big(\varphi(cd) - \varphi(c)\varphi(d)\big) = 0, \tag{2.111}$$

for all quadruples $a,b,c,d \in A$.

Proof. Let $a,b,c,d \in A$. Then by Lemma 2.23:

$$\begin{aligned}&\big(\varphi(ab) - \varphi(b)\varphi(a)\big)\big(\varphi(cd) - \varphi(c)\varphi(d)\big) \\ &\qquad = \Big[\big(\varphi(d^*c^*) - \varphi(d^*)\varphi(c^*)\big)\big(\varphi(b^*a^*) - \varphi(a^*)\varphi(b^*)\big)\Big]^* = 0. \end{aligned} \tag{2.112}$$

□

Using Lemma 2.23 and Corollary 2.24, we are finally able to prove Theorem 2.19.

Proof of Theorem 2.19. Define

$$A_1 = \{\varphi(ab) - \varphi(a)\varphi(b) \mid a, b \in A\}, \tag{2.113}$$
$$A_2 = \{\varphi(ab) - \varphi(b)\varphi(a) \mid a, b \in A\}. \tag{2.114}$$

Also define

$$q_1 = \Big[\bigcap_{\alpha\in A_1} \ker\alpha\Big], \quad q_2 = \Big[\bigcap_{\alpha\in A_2} \ker\alpha\Big], \quad q_3 = \Big[\bigcap_{\alpha\in A_1\cup A_2} \ker\alpha\Big], \tag{2.115}$$

where $[\dots]$ means that we take the orthogonal projection onto the closed subspace of $B(H)$ between the brackets. Now define

$$A_1' = \{\varphi(ab) - \varphi(a)\varphi(b) \mid a, b \in (A_1)_{\mathrm{sa}}\}; \tag{2.116}$$
$$A_2' = \{\varphi(ab) - \varphi(a)\varphi(b) \mid a, b \in (A_2)_{\mathrm{sa}}\}, \tag{2.117}$$

and

$$q_1' = \Big[\bigcap_{\alpha\in A_1'} \ker\alpha\Big], \quad q_2' = \Big[\bigcap_{\alpha\in A_2'} \ker\alpha\Big], \quad q_3' = \Big[\bigcap_{\alpha\in A_1'\cup A_2'} \ker\alpha\Big]. \tag{2.118}$$

<u>Claim:</u> We have $q_1 = q_1'$, $q_2 = q_2'$ and $q_3 = q_3'$.

<u>Proof:</u> We prove that $q_1 = q_1'$; the other two identities are proven similarly. First note that $A_1' \subset A_1$, and hence

$$\bigcap_{\alpha\in A_1'} \ker\alpha \supseteq \bigcap_{\alpha\in A_1} \ker\alpha. \tag{2.119}$$

Using Proposition A.30 this shows that $q_1 \le q_1'$, so it only remains to be shown that

$$\bigcap_{\alpha\in A_1'} \ker\alpha \subseteq \bigcap_{\alpha\in A_1} \ker\alpha. \tag{2.120}$$

Let $x \in \bigcap_{\alpha\in A_1'} \ker\alpha$. We want to show that $\alpha x = 0$ for all $\alpha \in A_1$. So let α be any element of A_1, i.e., $\alpha = \varphi(ab) - \varphi(a)\varphi(b)$ for certain $a, b \in A$. Write $a = a_1 + ia_2$ and $b = b_1 + ib_2$ with $a_1, a_2, b_1, b_2 \in A$. Then

$$\begin{aligned}\alpha &= \varphi\big((a_1+ia_2)(b_1+ib_2)\big) - \varphi(a_1+ia_2)\varphi(b_1+ib_2)\\ &= \big[\varphi(a_1b_1) - \varphi(a_1)\varphi(b_1)\big] + i\big[\varphi(a_1b_2) - \varphi(a_1)\varphi(b_2)\big]\\ &\qquad + i\big[\varphi(a_2b_1) - \varphi(a_2)\varphi(b_1)\big] - i\big[\varphi(a_2b_2) - \varphi(a_2)\varphi(b_2)\big]\\ &= \alpha_1 + i\alpha_2 + i\alpha_3 - \alpha_4,\end{aligned} \tag{2.121}$$

where

$$\begin{aligned}\alpha_1 &= \big[\varphi(a_1b_1) - \varphi(a_1)\varphi(b_1)\big], \quad \alpha_2 = \big[\varphi(a_1b_2) - \varphi(a_1)\varphi(b_2)\big],\\ \alpha_3 &= \big[\varphi(a_2b_1) - \varphi(a_2)\varphi(b_1)\big], \quad \alpha_4 = \big[\varphi(a_2b_2) - \varphi(a_2)\varphi(b_2)\big].\end{aligned} \tag{2.122}$$

Because $x \in \bigcap_{\alpha\in A_1'} \ker\alpha$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in A_1'$ we can now conclude that

$$\alpha x = (\alpha_1 + i\alpha_2 + i\alpha_3 - \alpha_4)x = 0. \tag{2.123}$$

This shows that $\bigcap_{\alpha\in A_1'} \ker\alpha = \bigcap_{\alpha\in A_1} \ker\alpha$ and hence $q_1 = q_1'$. ■

Now define $p_\alpha = \left[\overline{\operatorname{Ran}\alpha}\right]$, i.e., the range projection of α, and define

$$\begin{aligned}e_1 &= \inf\{1_H - p_\alpha \mid \alpha \in A_1\};\\ e_2 &= \inf\{1_H - p_\alpha \mid \alpha \in A_2\};\\ e_3 &= \inf\{1_H - p_\alpha \mid \alpha \in A_1 \cup A_2\}.\end{aligned} \tag{2.124}$$

Claim: We have $q_1 = e_1$, $q_2 = e_2$ and $q_3 = e_3$.

Proof: We prove that $q_1 = e_1$; the other two identities are proven similarly. We will show that $q_1 = e_1$ by proving that $q_1 \leq e_1$ and $e_1 \leq q_1$.

We first show that $q_1 \leq e_1$. Let $\alpha = \varphi(ab) - \varphi(a)\varphi(b) \in A_1$. Then according to A.14 we can decompose H as

$$H = \overline{\operatorname{Ran}\alpha} \oplus \overline{\operatorname{Ran}\alpha}^{\perp} = \operatorname{Ran} p_\alpha \oplus \big(\operatorname{Ran} p_\alpha\big)^{\perp}. \tag{2.125}$$

Using Theorem A.25 we can rewrite this decomposition further as

$$H = \operatorname{Ran} p_\alpha \oplus \ker p_\alpha^* = \operatorname{Ran} p_\alpha \oplus \ker p_\alpha. \tag{2.126}$$

Let $x \in H$ be any element of H and decompose it as $x = y + z$ with $y \in \operatorname{Ran} p_\alpha$ and $z \in \ker p_\alpha$. Note that $\alpha^* = \varphi(b^*a^*) - \varphi(b^*)\varphi(a^*) \in A_1$ and

$$\operatorname{Ran} p_\alpha \cap \ker\alpha^* = \overline{\operatorname{Ran}\alpha} \cap \big(\operatorname{Ran}\alpha\big)^{\perp} = \big(\operatorname{Ran}\alpha\big)^{\perp\perp} \cap \big(\operatorname{Ran}\alpha\big)^{\perp} = \{0\}. \tag{2.127}$$

So for $0 \neq y \in \operatorname{Ran} p_\alpha$ we have $y \notin \ker\alpha^*$ and thus $q_1 y = 0$. Then

$$\begin{aligned}q_1 x &= q_1(y + z) = q_1 y + q_1 z = 0 + q_1 z\\ &= q_1(y - y) + q_1(z - 0) = q_1(1_H - p_\alpha)y + q_1(1_H - p_\alpha)z\\ &= q_1(1_H - p_\alpha)(y + z) = q_1(1_H - p_\alpha)x.\end{aligned} \tag{2.128}$$

This shows that $q_1 \leq (1_H - p_\alpha)$ for all $\alpha \in A_1$, and hence

$$q_1 \leq \inf\{1_H - p_\alpha \mid \alpha \in A_1\} = e_1. \tag{2.129}$$

So it remains to be shown that $e_1 \leq q_1$. To this end, note that for all

$$\alpha = \big[\varphi(ab) - \varphi(a)\varphi(b)\big] \in A_1' \tag{2.130}$$

we have:

$$\begin{aligned} \alpha^* &= \varphi(ba) - \varphi(b)\varphi(a) = \varphi(ab + ba) - \varphi(ab) - \varphi(b)\varphi(a) \\ &= -\big[\varphi(ab) - \varphi(a)\varphi(b)\big] = -\alpha. \end{aligned} \tag{2.131}$$

Using this relation and Theorem A.25 we see that

$$\ker\alpha = \ker -\alpha = \ker\alpha^* = \big(\operatorname{Ran}\alpha\big)^\perp. \tag{2.132}$$

Now decompose H, using the relation above and the previous claim:

$$\begin{aligned} H &= \operatorname{Ran} q_1 \oplus \big(\operatorname{Ran} q_1\big)^\perp \\ &= \bigcap_{\alpha\in A_1'} \ker\alpha \oplus \Big(\bigcap_{\alpha\in A_1'} \ker\alpha\Big)^\perp \\ &= \bigcap_{\alpha\in A_1'} \big(\operatorname{Ran}\alpha\big)^\perp \oplus \Big(\bigcap_{\alpha\in A_1'} \big(\operatorname{Ran}\alpha\big)^\perp\Big)^\perp \\ &= \bigcap_{\alpha\in A_1'} \big(\operatorname{Ran}\alpha\big)^\perp \oplus \Big(\bigcup_{\alpha\in A_1'} \operatorname{Ran}\alpha\Big)^{\perp\perp} \\ &= \bigcap_{\alpha\in A_1'} \big(\operatorname{Ran}\alpha\big)^\perp \oplus \overline{\bigcup_{\alpha\in A_1'} \operatorname{Ran}\alpha}. \end{aligned} \tag{2.133}$$

Let $x \in H$ be any element of H and decompose it as $x = y + z$ with

$$y \in \operatorname{Ran} q_1 = \bigcap_{\alpha\in A_1'} \big((\operatorname{Ran}\alpha)^\perp\big); \tag{2.134}$$

$$z \in \big(\operatorname{Ran} q_1\big)^\perp = \overline{\bigcup_{\alpha\in A_1'} \operatorname{Ran}\alpha}. \tag{2.135}$$

Then it follows immediately that

$$q_1 x = q_1(y + z) = y. \tag{2.136}$$

Since $z \in \overline{\bigcup_{\alpha \in A_1'} \operatorname{Ran} \alpha}$ there are $z_n \in \bigcup_{\alpha \in A_1'} \operatorname{Ran} \alpha$ such that z_n converges to z in H. Now let $\alpha_n \in A_1'$ be such that $z_n \in \operatorname{Ran} \alpha_n$. Because $e_1 \leq 1_H - p_\alpha$ for all $\alpha \in A_1'$, we have $e_1 = e_1(1 - p_\alpha)$ for all $\alpha \in A_1'$, and hence:

$$\begin{aligned} e_1 x &= e_1(y + z) \\ &= e_1 y + \lim_{n \to \infty} e_1 z_n \\ &= e_1 y + \lim_{n \to \infty} e_1(1_H - p_{\alpha_n}) z_n \\ &= e_1 y + \lim_{n \to \infty} e_1(z_n - z_n) \\ &= e_1 y \\ &= e_1 q_1 x. \end{aligned} \tag{2.137}$$

This shows that $e_1 q_1 = e_1$ and hence $e_1 \leq q_1$. We can therefore conclude that $e_1 = q_1$. ■

Claim: The projections q_1, q_2 and q_3 are central in the von Neumann algebra generated by $\varphi(A)$.

Proof: We prove the statement for q_1; the other statements are proven analogously. We first show that $q_1 \in \varphi(A)''$. Because $q_1 = \inf\{1_H - p_\alpha \mid \alpha \in A_1\}$ it is enough to show that $1_H - p_\alpha \in \varphi(A)''$ for all $\alpha \in \varphi(A)$. Since $1_H \in \varphi(A)''$ it is even enough to show that $p_\alpha \in \varphi(A)''$. By definition, p_α is the range projection of $\alpha \in A_1 \subseteq \varphi(A)$. Because $\varphi(A)''$ is a von Neumann algebra and von Neumann algebras contain range projections, we have $p_\alpha \in \varphi(A)''$.

Secondly, we show that $q_1 \in \varphi(A)'$; the proofs for q_2 and q_3 go analogously. We want to show that $q_1 \varphi(a) = \varphi(a) q_1$ for all $a \in A$. Decompose H as follows:

$$H = \bigcap_{\alpha \in A_1} \ker \alpha \oplus \Big(\bigcap_{\alpha \in A_1} \ker \alpha \Big)^{\perp}. \tag{2.138}$$

Note that the range space of q_1 is invariant under $\varphi(A)$, i.e.,

$$\varphi(a) \bigcap_{\alpha \in A_1} \ker \alpha \subseteq \bigcap_{\alpha \in A_1} \ker \alpha, \tag{2.139}$$

for all $a \in A$. Indeed, let $a \in A$, $x \in \bigcap_{\alpha \in A_1} \ker \alpha$, and $\alpha = \varphi(bc) - \varphi(b)\varphi(c) \in A_1$. Then

$$\begin{aligned} \alpha \varphi(a) &= \big[\varphi(bc) - \varphi(b)\varphi(c)\big]\varphi(a) \\ &= -\big[\varphi(bca) - \varphi(bc)\varphi(a)\big] + \big[\varphi(bca) - \varphi(b)\varphi(ca)\big] + \varphi(b)\big[\varphi(ca) - \varphi(c)\varphi(a)\big] \\ &= -\alpha_1 + \alpha_2 + \varphi(b)\alpha_3, \end{aligned} \tag{2.140}$$

where

$$\alpha_1 = \varphi(bca) - \varphi(bc)\varphi(a); \tag{2.141}$$
$$\alpha_2 = \varphi(bca) - \varphi(b)\varphi(ca); \tag{2.142}$$
$$\alpha_3 = \varphi(ca) - \varphi(c)\varphi(a), \tag{2.143}$$

and $\alpha_1, \alpha_2, \alpha_3 \in A_1$. Because $x \in \bigcap_{\alpha \in A_1} \ker \alpha$, it follows immediately that

$$\alpha\varphi(a)x = (-\alpha_1 + \alpha_2 + \varphi(b)\alpha_3)x = 0, \tag{2.144}$$

so $\varphi(a)x \in \ker \alpha$, and hence

$$\varphi(a)x \in \bigcap_{\alpha \in A_1} \ker \alpha. \tag{2.145}$$

Because $\bigcap_{\alpha \in A_1} \ker \alpha$ is invariant under $\varphi(A)$ we also know that $\left(\bigcap_{\alpha \in A_1} \ker \alpha\right)^{\perp}$ is invariant under $\varphi(A)$. Now let $x \in H$ and write it as $x = y + z$ with $y \in \bigcap_{\alpha \in A_1} \ker \alpha$ and $z \in \left(\bigcap_{\alpha \in A_1} \ker \alpha\right)^{\perp}$. Then

$$\begin{aligned} q_1\varphi(a)x &= q_1\varphi(a)(y+z) \\ &= \varphi(a)y + 0 \\ &= \varphi(a)q_1 y + \varphi(a)q_1 z \\ &= \varphi(a)q_1 x, \end{aligned} \tag{2.146}$$

which shows that $q_1 \in \varphi(A)'$. So now we know that $q_1 \in \varphi(A)'' \cap \varphi(A)'$, which proves that q_1 is central in the von Neumann algebra generated by $\varphi(A)$. ■

The second claim showed that

$$\begin{aligned} q_1 &= \inf\{1_H - p_\alpha \mid \alpha \in A_1\}, \\ q_2 &= \inf\{1_H - p_\alpha \mid \alpha \in A_2\}, \\ q_3 &= \inf\{1_H - p_\alpha \mid \alpha \in A_1 \cup A_2\}, \end{aligned} \tag{2.147}$$

which in turn means that

$$\begin{aligned} 1_H - q_1 &= \sup\{p_\alpha \mid \alpha \in A_1\}, \\ 1_H - q_2 &= \sup\{p_\alpha \mid \alpha \in A_2\}, \\ 1_H - q_3 &= \sup\{p_\alpha \mid \alpha \in A_1 \cup A_2\}. \end{aligned} \tag{2.148}$$

Now define

$$p_1 = 1_H - q_2, \quad p_2 = 1_H - q_1, \quad p_3 = q_1. \tag{2.149}$$

We now show that p_1, p_2 and p_3 satisfy the requirements of the theorem. The third claim showed that p_1, p_2 and p_3 are central projections. We now show that they are mutually orthogonal and that $p_1+p_2+p_3 = 1_H$. Let $\alpha \in A_1$ and $\beta \in A_2$. By Lemma 2.23 we have $\alpha^*\beta = 0$, so $\operatorname{Ran}\beta \subseteq \ker\alpha^* = (\operatorname{Ran}\alpha)^\perp$, and hence

$$\overline{\operatorname{Ran}\beta} \subseteq \overline{(\operatorname{Ran}\alpha)^\perp}. \tag{2.150}$$

Because $(\operatorname{Ran}\alpha)^\perp$ is closed we have $\overline{(\operatorname{Ran}\alpha)^\perp} = (\operatorname{Ran}\alpha)^\perp$, and

$$(\operatorname{Ran}\alpha)^\perp = \big((\operatorname{Ran}\alpha)^{\perp\perp}\big)^\perp = \overline{\operatorname{Ran}\alpha}^\perp, \tag{2.151}$$

which implies $p_\alpha p_\beta = 0$ for all $\alpha \in A_1$ and $\beta \in A_2$. Because $\{p_\alpha \mid \alpha \in A_1\}$ and $\{p_\beta \mid \beta \in A_2\}$ are bounded sets in $B(H)$, it follows from Lemma A.49 that

$$p_2 p_1 = \big(\sup_{\alpha\in A_1} p_\alpha\big)\big(\sup_{\beta\in A_2} p_\beta\big) = \sup_{\alpha\in A_1}\sup_{\beta\in A_2}\big(p_\alpha p_\beta\big) = 0. \tag{2.152}$$

This shows that p_1 and p_2 are mutually orthogonal and hence that $p_1 + p_2$ is a projection. Because $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$, we can immediately conclude from (2.148) that

$$p_1 \leq 1_H - p_3 \quad \text{and} \quad p_2 \leq 1_H - p_3. \tag{2.153}$$

This shows that

$$(p_1 + p_2)(1_H - p_3) = p_1(1_H - p_3) + p_2(1_H - p_3) = p_1 + p_2, \tag{2.154}$$

which implies

$$p_1 + p_2 \leq 1_H - p_3. \tag{2.155}$$

Again, it follows from Lemma A.49 that for every $\alpha \in A_1$ and $\beta \in A_2$

$$p_1\alpha = 0, \quad p_2\beta = 0, \tag{2.156}$$

and hence

$$p_1 p_\alpha = 0, \quad p_2 p_\beta = 0. \tag{2.157}$$

Because $p_\alpha \leq p_2$ for all $\alpha \in A_1$ and $p_\alpha \leq p_1$ for all $\alpha \in A_2$, the above relation shows that

$$\big[p_1 + p_2\big] p_\alpha = p_\alpha, \tag{2.158}$$

for all $\alpha \in A_1 \cup A_2$. Hence
$$p_\alpha \leq p_1 + p_2, \tag{2.159}$$
which shows that
$$1_H - p_3 = \sup_{\alpha \in A_1 \cup A_2} p_\alpha \leq p_1 + p_2. \tag{2.160}$$
Because $p_1 + p_2 \leq 1_H - p_3$ and $1_H - p_3 \leq p_1 + p_2$ we have
$$p_1 + p_2 = 1_H - p_3, \tag{2.161}$$
and hence
$$p_1 + p_2 + p_3 = 1_H. \tag{2.162}$$
Now use this relation to see that
$$p_1 p_3 = p_1(1_H - p_1 - p_2) = p_1 - p_1 - p_1 p_2 = 0, \tag{2.163}$$
and similarly $p_2 p_3 = 0$. So the projections p_1, p_2 and p_3 are mutually orthogonal. It is clear from equation (2.148) and equation (2.156) that for all $a, b \in A$,
$$\varphi(ab)p_1 - \varphi(a)p_1\varphi(b)p_1 = [\varphi(ab) - \varphi(a)\varphi(b)]p_1 = 0, \tag{2.164}$$
which shows that $\varphi(\cdot)p_1$ is a *-homomorphism. Now suppose that $p_1 \neq 0$. By equation (2.148) this means that there exists an $\alpha = \varphi(ab) - \varphi(b)\varphi(a) \in A_2$ such that $\alpha \neq 0$. Then
$$\varphi(ab)p_1 - \varphi(b)p_1\varphi(a)p_1 = \alpha p_1 = p_1 p_\alpha \alpha = p_\alpha \alpha = \alpha \neq 0, \tag{2.165}$$
which shows that $\varphi(\cdot)p_1$ is not an anti-morphism. The proof that $\varphi(\cdot)p_2$ is a *-anti-homomorphism and not a *-homomorphism and the proof that $\varphi(\cdot)p_3$ is a *-homomorphism as well as a *-anti-homomorphism are similar. Lastly, suppose that there is a central projection p such that $\varphi(\cdot)p$ is a *-homomorphism as well as a *-anti-homomorphism. Then
$$p[\varphi(ab) - \varphi(a)\varphi(b)] = 0 = p[\varphi(ab) - \varphi(b)\varphi(a)], \tag{2.166}$$
for all $a, b \in A$. This shows that $\operatorname{Ran} p \subseteq \ker \alpha$ for all $\alpha \in A_1 \cup A_2$ and hence:
$$\operatorname{Ran} p \subseteq \bigcap_{\alpha \in A_1 \cup A_2} \ker \alpha = \operatorname{Ran} p_3. \tag{2.167}$$
This implies that $p \leq p_3$, which proves that p_3 is the largest central projection such that $\varphi(\cdot)p_3$ is a *-homomorphism, as well as a *-anti-homomorphism.

Finally, we show that the conditions in the theorem determine p_1, p_2 and p_3 uniquely. So assume that p_1, p_2 and p_3 satisfy the conditions of the theorem and let q_1, q_2 and q_3 be as in equation (2.148). We show that $p_1 = 1_H - q_2$, $p_2 = 1_H - q_1$ and $p_3 = q_3$. By the previous argument, we have $p_3 \leq q_3$. Because p_3 is the largest central projection such that $\varphi(\cdot)p_3$ that is a *-homomorphism, as well as a *-anti-homomorphism, we also have $q_3 \leq p_3$. This proves that $p_3 = q_3$. Furthermore, $p_1 + p_2 + p_3 = 1_H$, which implies

$$p_1 \leq 1_H - p_3; \tag{2.168}$$

$$p_2 \leq 1_H - p_3; \tag{2.169}$$

$$p_1 + p_2 = 1_H - p_3. \tag{2.170}$$

Let $a, b \in A$. Because p_1 is a central projection such that $\varphi(\cdot)p_1$ is a *-homomorphism, we have

$$\varphi(ab)p_1 - \varphi(a)p_1\varphi(b)p_1 = [\varphi(ab) - \varphi(a)\varphi(b)]p_1 = 0, \tag{2.171}$$

and hence

$$p_1 p_\alpha = 0, \tag{2.172}$$

where $\alpha = \varphi(ab) - \varphi(a)\varphi(b) \in A_1$. This shows that

$$p_1(1_H - p_\alpha) = p_1, \tag{2.173}$$

for all $\alpha \in A_2$, and therefore,

$$p_1 \leq \inf\{1_H - p_\alpha \mid \alpha \in A_1\} = q_1. \tag{2.174}$$

Similarly, $p_2 \leq q_2$. This in turn implies that $1_H - q_1 \leq 1_H - p_1$ and $1_H - q_2 \leq 1_H - p_2$. Therefore,

$$1_H - q_2 = (1_H - q_2)(1_H - p_3) \leq (1_H - p_2)(1_H - p_3) = 1_H - p_2 - p_3 = p_1, \tag{2.175}$$

and similarly,

$$1_H - q_1 \leq p_2, \tag{2.176}$$

where we used the fact that $(1_H - q_2)p_3 = (1_H - q_1)p_3 = 0$. When we combine these inequalities we find that

$$1_H - p_3 = (1_H - q_1) + (1_H - q_2) \leq p_1 + p_2 = 1_H - p_3, \tag{2.177}$$

so the equalities must hold, i.e.,

$$1_H - q_2 = p_1, \tag{2.178}$$

$$1_H - q_1 = p_2. \tag{2.179}$$

□

In the next section we study the case where the *-algebra A is the algebra $B(H)$ of bounded operators on a Hilbert space. In this case the possibilities for central projections are limited, because the center of $B(H)$ only consists of scalar multiples of the identity. We will use this fact to prove that every Jordan symmetry is either a *-isomorphism or a *-anti-isomorphism.

2.5 Proof of Jordan's Theorem

We are now able to prove Jordan's Theorem. Throughout this sections we use the following notation: let $u \in B(H_1, H_2)$ be a unitary operator. We denote by α_u the *-isomorphism from $B(H_1)$ to $B(H_2)$ given by

$$\alpha_u(a) = uau^*, \tag{2.180}$$

for all $a \in B(H_1)$. Similarly, let $u \in B(H_1, H_2)$ be anti-unitary. We denote by α'_u the *-anti-isomorphism from $B(H_1)$ to $B(H_2)$ given by

$$\alpha'_u(a) = ua^*u^*. \tag{2.181}$$

Theorem 2.25 (Jordan's Theorem)**.** *Any Jordan symmetry $J_{\mathbb{C}}\colon B(H_1) \to B(H_2)$ is given by either*

$$J_{\mathbb{C}}(a) = \alpha_u(a) \equiv uau^*, \tag{2.182}$$

where $u \in B(H_1, H_2)$ is unitary and is determined by $J_{\mathbb{C}}$ up to a phase, or by

$$J_{\mathbb{C}}(a) = \alpha'_u(a) \equiv ua^*u^*, \tag{2.183}$$

where $u \in B(H_1, H_2)$ is anti-unitary and is determined by $J_{\mathbb{C}}$ up to a phase.

The first step of the proof is a consequence of Theorem 2.19.

Proposition 2.26. *A Jordan symmetry $J_{\mathbb{C}}\colon B(H_1) \to B(H_2)$ is either a *-isomorphism or a *-anti-isomorphism.*

Proof. Let $J\colon B(H_1) \to B(H_2)$ be a Jordan symmetry. Then in particular $B(H_2)$ is a *-algebra, J is a Jordan map and $J(B(H_1)) = B(H_2)$ is again a *-algebra. The von Neumann algebra generated by $J(B(H_1))$ is just $B(H_2)$, the center of which only consists of scalar multiples of the identity. Now let p_1, p_2 and p_3 be the projections as in Theorem 2.19. Since these are central projections, we have $p_1, p_2, p_3 \in \{0, 1_{H_2}\}$. Because $p_1 + p_2 + p_3 = 1_{H_2}$, exactly one of p_1, p_2 and p_3 equals 1_{H_2}, and both other projections equal 0. Suppose that $p_3 = 1_{H_2}$, which would imply that J is a *-isomorphism as well as a *-anti-isomorphism. Then for all $a, b \in B(H_1)$:

$$J(a)J(b) = J(ab) = J(b)J(a). \tag{2.184}$$

This implies that $J(B(H_1)) \subseteq Z(B(H_2)) = \mathbb{C} \cdot 1_{H_2} \subsetneq B(H_2)$, which contradicts the assumption that J is bijective. So we have $p_3 \neq 1_{H_2}$ and hence $p_3 = 0$. We can now conclude that $p_1 = 1_{H_2}$ and $p_2 = 0$, i.e., J is a *-isomorphism, or $p_1 = 0$ and $p_2 = 1_{H_2}$, i.e., J is a *-anti-isomorphism. □

It remains to be proven that every *-isomorphism of $B(H)$ is given by α_u for a certain unitary operator $u \in B(H_1, H_2)$, and every *-anti-isomorphism is given by α'_u for a certain anti-unitary operator $u \in B(H_1, H_2)$. We work towards this result using the following two lemma's.

Lemma 2.27. *Let $\alpha\colon B(H_1) \to B(H_2)$ be an *-isomorphism and $a \in B(H)$, then*

$$\left\|\alpha(a)\right\| = \|a\|. \tag{2.185}$$

Proof. Since α is injective, this follows from part (*iii*) of Theorem A.67. □

Lemma 2.28. *Let $\alpha\colon B(H_1) \to B(H_2)$ be a *-isomorphism and $e \in B(H_1)$ a one-dimensional projection, then so is $\alpha(e)$.*

Proof. First note that α preserves projections: $\alpha(e)^2 = \alpha(e^2) = \alpha(e)$ and $\alpha(e)^* = \alpha(e^*) = \alpha(e)$, which shows that $\alpha(e)$ is a projection. Further note that for all $a \in B(H)^+$ we have $a = b^*b$ for certain $b \in B(H)$ and hence $\alpha(a) = \alpha(b^*b) = \alpha(b)^*\alpha(b) \in B(H)^+$. This shows that α preserves positivity and hence order. The one-dimensional projections are minimal non-zero projections in the lattice of projections, i.e, they are ***atoms*** (see Definition A.99). Because α preserves projections and order, it restricts to an isomorphism of the lattice of projections in $B(H_1)$ to the lattice of projections in $B(H_2)$. Hence it preserves atoms, because atoms are intrinsically defined by the partial order. □

We are now ready to give a constructive proof of the following proposition.

Proposition 2.29. *Any isomorphism $\alpha\colon B(H_1) \to B(H_2)$ takes the form $\alpha = \alpha_u$, where $u\colon H_1 \to H_2$ is unitary and uniquely determined by α up to a phase.*

Proof. For some fixed unit vector $x \in H_2$, take the corresponding one-dimensional projection e_x and define a new unit vector $y \in H_1$ (up to a phase) by

$$e_y = \alpha^{-1}(e_x). \tag{2.186}$$

Now any $z \in H_1$ may be written as $z = ay$, for some $a \in B(H_1)$. We attempt to define an operator $u\colon H_1 \to H_2$ by $uz = \alpha(a)x$, i.e.,

$$uay = \alpha(a)x. \tag{2.187}$$

It is not immediately clear that this is well defined, because many different operators a may give rise to the same z. Fortunately, we may compute

$$\begin{aligned}\|ay\|_{H_1} &= \|ae_y y\|_{H_1} = \|ae_y\|_{B(H_1)} = \|\alpha(ae_y)\|_{B(H_1)}\\ &= \|\alpha(a)\alpha(e_y)\|_{B(H_2)} = \|\alpha(a)e_x\|_{B(H_2)} = \|\alpha(a)x\|_{H_2}\\ &= \|uay\|_{H_2},\end{aligned} \tag{2.188}$$

so that if $ay = by$, then $\alpha(a)x = \alpha(b)x$ and hence u is well defined. By this computation u is also isometric. It is also surjective: for suppose that $x' \in H_2$, then there is a $a \in B(H_2)$ such that $ax = x'$ and hence

$$u\alpha^{-1}(a)y = \alpha(\alpha^{-1}(a))x = ax = x'. \tag{2.189}$$

Because u is an isometric surjection, it is unitary. The property $\alpha(a) = uau^*$ is equivalent to $ua = \alpha(a)u$, which in turn is equivalent to $uaby = \alpha(a)uby$ for any $b \in B(H_1)$, which by definition of u is the same as

$$\alpha(ab)x = \alpha(a)\alpha(b)x. \tag{2.190}$$

But this holds by virtue of α being a *-isomorphism. Finally, all arbitrariness in u lies in the lack of uniqueness of y. □

We use the previous proposition to prove that every *-anti-isomorphism is of the form α'_u.

Proposition 2.30. *Any *-anti-isomorphism* $\alpha\colon B(H_1) \to B(H_2)$ *takes the form* $\alpha = \alpha'_u$, *where* $u\colon H_1 \to H_2$ *is anti-unitary, and uniquely determined by* α *up to a phase.*

Proof. Let $v \in B(H_1)$ be an anti-unitary operator and define

$$\beta\colon B(H_1) \to B(H_1) \tag{2.191}$$

$$a \mapsto va^*v^*. \tag{2.192}$$

Then $\alpha \circ \beta$ is a *-isomorphism, to which Proposition 2.29 applies, so that

$$\alpha \circ \beta = \alpha_{\tilde{u}}, \tag{2.193}$$

for some unitary $\widetilde{u} \in B(H_1, H_2)$. Hence

$$\alpha(a) = \alpha(\beta \circ \beta^{-1}(a)) = \alpha \circ \beta(v^* a^* v) = \widetilde{u} v^* a^* v \widetilde{u}^*, \tag{2.194}$$

so that $\alpha(a) = ua^*u^*$ with $u = \widetilde{u}v^*$, which is an anti-unitary operator. The lack of uniqueness of u is inherited from the unitary case. □

The proof of Jordan's theorem now just consists of assembling the different parts of the proof.

Proof of Jordan's Theorem. Let $J_{\mathbb{C}} \colon B(H_1) \to B(H_2)$ be a Jordan symmetry. By Proposition 2.26 it is either a *-isomorphism or a *-anti-isomorphism. If it is a *-isomorphism, Proposition 2.29 shows that it is of the form $J_{\mathbb{C}} = \alpha_u$, where $u \in B(H_1, H_2)$ is a unitary operator and determined by $J_{\mathbb{C}}$ up to a phase. If, on the other hand, $J_{\mathbb{C}}$ is a *-anti-isomorphism, then Proposition 2.30 shows that $J_{\mathbb{C}} = \alpha'_u$, where $u \in B(H_1, H_2)$ is an anti-unitary operator and determined by $J_{\mathbb{C}}$ up to a phase. □

Using the equivalences of the previous section, we can now prove Kadison's Theorem and Wigner's Theorem.

Theorem 2.31 (Kadison's Theorem)**.** *Any Kadison symmetry $K \colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$ is given by either*

$$K(\rho) = \alpha_u(\rho) \equiv u\rho u^*, \tag{2.195}$$

where $u \in B(H_1, H_2)$ is unitary and is determined by K up to a phase, or by

$$K(\rho) = \alpha_u(\rho) \equiv u\rho^* u^*, \tag{2.196}$$

where $u \in B(H_1, H_2)$ is anti-unitary and is determined by K up to a phase.

Proof. Let K be a Kadison symmetry. By Corollary 2.17 there is a bijective equivalence between Kadison symmetries from $\mathcal{D}(H_1)$ to $\mathcal{D}(H_2)$ and Jordan symmetries from $B(H_2)_{\mathrm{sa}}$ to $B(H_1)_{\mathrm{sa}}$. Let J be the corresponding Jordan symmetry. By Theorem 2.25 either $J_{\mathbb{C}} = \alpha_u$ for some unitary operator $u \in B(H_2, H_1)$ determined by J (and hence K) up to a phase, or $J_{\mathbb{C}} = \alpha'_u$ for some anti-unitary operator $u \in B(H_2, H_1)$ determined by J (and hence K) up to a phase. Then for all $\rho \in \mathcal{D}(H)$ and $a \in B(H_2)_{\mathrm{sa}}$

$$\mathrm{Tr}(K(\rho)a) = \mathrm{Tr}(\rho J(a)) = \mathrm{Tr}(\rho u a u^*) = \mathrm{Tr}(u^* \rho u a). \tag{2.197}$$

By Theorem 1.25, equation (2.197) implies that $K(\rho)a = u^*\rho u a$, i.e., $K = \alpha_{u^*}$, with $u^* \in B(H_1, H_2)$ unitary or anti-unitary and determined by K up to a phase. □

Finally, we use the equivalence between Kadison symmetries and Wigner symmetries to prove Wigner's Theorem.

Theorem 2.32 (Wigner's Theorem)**.** *Any Wigner symmetry* $W\colon B(H_1) \to B(H_2)$ *is given by either*

$$W(e) = \alpha_u(e) \equiv u\alpha u^*, \tag{2.198}$$

where $u \in B(H_1, H_2)$ *is unitary and is determined by* W *up to a phase, or by*

$$W(e) = \alpha_u(e) \equiv ueu^*, \tag{2.199}$$

where $u \in B(H_1, H_2)$ *is anti-unitary and is determined by* W *up to a phase.*

Proof. Let W be a Wigner symmetry. By Proposition 2.14 there is a unique Kadison symmetry $K\colon \mathcal{D}(H_1) \to \mathcal{D}(H_2)$ such that $W = K_{|\mathcal{P}_1(H_1)}$. By Kadison's Theorem, $K = \alpha_u$, where u is either a unitary or an anti-unitary operator and is determined by K up to a phase. Hence

$$W(e) = K_{|\mathcal{P}_1(H_1)}(e) = \alpha_{u|\mathcal{P}_1}(e) = ueu^*, \tag{2.200}$$

where $u \in B(H_1, H_2)$ is unitary or anti-unitary, which is determined by W up to a phase. □

Chapter 3

Algebraic quantum theory

In the previous chapter we introduced some important concepts in quantum mechanics, namely one-dimensional projections, density operators and self-adjoint operators (observables). Our starting point was a Hilbert space, but the important symmetries (Wigner, Kadison, and Jordan symmetry respectively) act only on the operators on that Hilbert space, rather than on the elements itself. So the question arises if we could develop a theory that does not have a Hilbert space as a starting point, but a generalisation of the set of bounded operators thereon. It turns out that we can indeed develop such a theory, where we generalise the algebra of bounded operators to a C*-algebra. This is called algebraic quantum theory. In the next section we introduce suitable replacements for Wigner, Kadison, and Jordan symmetries.

3.1 The state space of a C*-algebra

The first step to generalising to algebraic quantum physics is finding suitable replacements for one-dimensional projections, density operators and self-adjoint operators. The easiest way to do this is to use ***states***, because by definition states only act on the operators and do not rely on the underlying Hilbert space. Recall that a state on $B(H)$ is a normalised positive complex linear map $\omega\colon B(H) \to \mathbb{C}$ (cf. Definition 1.18).

Definition 3.1. *A **state** on a C*-algebra A is a functional $\omega\colon A \to \mathbb{C}$ that satisfies*

*(a) **positivity**: $\omega(a^*a) \geq 0$, $a \in A$;*

*(b) **normalisation**: $\|\omega\| = 1$.*

Note that if the C*-algebra A is unital, the condition that a state is normalised simplifies to the condition that $\omega(1_A) = 1$.

The following Proposition is a (partial) generalisation of Proposition 1.19.

Proposition 3.2. *Let A be a C*-algebra. The state space is a convex subset of A^*. If A is unital, then the state space $S(A)$ is w^*-compact.*

Proof. Let $\omega, \omega' \in S(A)$ and $t \in [0,1]$. It is clear that $t\omega + (1-t)\omega'$ is still positive, so we only need to show that $t\omega + (1-t)\omega'$ is normalised. If A is unital, then for any positive functional $\sigma \in (A^*)^+$ it follows from Theorem A.65 part (ii) that

$$\|\sigma\| = \sigma(1_A). \tag{3.1}$$

So for unital A we have

$$\big\|t\omega + (1-t)\omega'\big\| = t\omega(1_A) + (1-t)\omega'(1_A) = 1, \tag{3.2}$$

which proves that $t\omega + (1-t)\omega'$ is a state. If A is not unital, we can use the canonical approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ and Theorem A.65 to conclude:

$$\big\|t\omega + (1-t)\omega'\big\| = t\lim_\lambda \omega(e_\lambda) + (1-t)\lim_\lambda \omega'(e_\lambda) = 1. \tag{3.3}$$

The proof that $S(A)$ is w^*-compact for unital A is exactly the same as for Proposition 1.19, namely by noting that $S(A)$ is a w^*-closed subspace of the w^*-compact unit ball $A^*_{\leq 1}$. □

Remark 3.3. If A is a non-unital C*-algebra, the state space may not be w^*-closed in A^*, and hence not w^*-compact. For instance, take $A = C_0(\mathbb{R})$, which is a C*-algebra under pointwise addition and multiplication, involution

$$f^*(x) = \overline{f(x)}, \tag{3.4}$$

and

$$\|f\|_\infty = \sup_{x\in\mathbb{R}}\{\big|f(x)\big|\}. \tag{3.5}$$

For all $n \in \mathbb{N}$, let the functional $\omega_n \colon C_0(\mathbb{R}) \to \mathbb{C}$ be given by

$$\omega_n(f) = f(n). \tag{3.6}$$

Then ω_n is a state on $C_0(\mathbb{R})$, but the $\{\omega_n\}_{n\in\mathbb{N}}$ converges pointwise to 0, i.e., for all $f \in C_0(\mathbb{R})$

$$\lim_{n\to\infty} \omega_n(f) = \lim_{n\to\infty} f(n) = 0. \tag{3.7}$$

This shows that $C_0(\mathbb{R})$ is not w^*-closed.

Just as in the case $A = B(H)$ we define ***pure states*** to be extreme points of the pure state space and the ***pure state space*** to be the extreme boundary of $S(A)$. We denote the pure state space by $P(A)$, i.e., $P(A) = \partial_e S(A)$. For unital C*-algebras, we can generalise Proposition (1.20).

Proposition 3.4. *Let A be a C*-algebra. Then the pure state space $P(A)$ is non-empty and*

$$S(A) = \overline{(\mathrm{co}(P(A))}^{w^*}. \tag{3.8}$$

Proof. This follows immediately from the fact that $S(A)$ is a w^*-compact subset of of A^*, and Krein-Milman (Theorem 1.8). □

This proof does not work for non-unital C*-algebras, because in that case the state space is not w^*-compact. However, we can tweak the proof slightly to get the following result, which in particular shows that the pure state space is non-empty.

Proposition 3.5. *Let A be a C*-algebra and let B be the set of positive functionals with norm smaller or equal to 1, i.e., $B = \left(A^*\right)^+_{\leq 1}$ (the **quasi-state space**).*

(i) B is convex and w^*-compact subset of A^*.

(ii) The extreme points of B consist of 0 and the pure states of A.

(iii) B is the w^*-closed convex hull of 0 and the set of pure states.

Proof. We follow the proof of Dixmier (1977), Proposition 2.5.5.

(i) The proof that B is convex follows from the fact that the convex sum of two positive functionals is positive and Theorem A.65.

(ii) Let $\omega, \omega' \in B$ and $t \in (0, 1)$ be such that

$$t\omega + (1-t)\omega' = 0. \tag{3.9}$$

Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be the canonical approximate unit of A. Then

$$0 = t\lim_\lambda \omega(e_\lambda) + (1-t)\lim_\lambda \omega(e_\lambda) = t\|\omega\| + (1-t)\left\|\omega'\right\|. \tag{3.10}$$

It follows that $\|\omega\| = \|\omega'\| = 0$ and hence $\omega = \omega' = 0$. This shows that 0 is an extreme point of B. Now let $\omega \neq 0$ be an extreme point of B. By Krein-Milman (Theorem 1.8) such an extreme point exists. To conclude that $\omega \in P(A)$ we only need to show that $\|\omega\| = 1$. Suppose for the sake of contradiction that $\|\omega\| < 1$. Then $\frac{1}{\|\omega\|}\omega \in B$, and we can write ω as a non-trivial convex combination of 0 and $\frac{1}{\|\omega\|}\omega$:

$$\omega = \|\omega\| \left(\frac{1}{\|\omega\|}\omega\right) + (1 - \|\omega\|)0. \tag{3.11}$$

But this contradicts the fact that ω is an extreme point of B. Hence $\|\omega\| = 1$, i.e., $\omega \in P(A)$.

(iii) By the first part and Krein-Milman we have

$$B = \overline{\mathrm{co}(\partial_e B)}^{w^*}. \tag{3.12}$$

By the second part, this is the same as

$$B = \overline{\mathrm{co}(P(A) \cup \{0\})}^{w^*}. \tag{3.13}$$

□

In view of Theorem 1.25, it would be logical to generalise normal states and normal pure states of A and view these as generalisations of density operators and one-dimensional projections. However, although we can indeed generalise the notion of a normal state (which we do in the following section), not all C*-algebras admit pure normal states, as is the case for $L^\infty(0,1)$. Hence we cannot develop algebraic quantum theory using normal states and pure normal states. Instead, we use states and pure states, which always exist by Proposition 3.4 and Proposition 3.5. However, normal states are still of great importance, so we will study them in the next section.

3.2 Normal maps and the normal state space

In this section we focus on the concept of normal functionals on C*-algebras and normal maps between C*-algebras. We are especially interested in normal functionals on von Neumann algebras, which we study in detail. For the proofs of the results, we refer to Alfsen and Shultz (2001), but other books like Pedersen (1979) and Kadison and Ringrose (1986) also give a detailed description of normal functionals and normal maps.

The following Lemma is the von Neumann algebra version of Proposition A.50.

Lemma 3.6. *The self-adjoint part of a von Neumann algebra is monotone complete. More specifically, if M is a von Neumann algebra acting on a Hilbert space H and $\{a_\lambda\}$ is an increasing family bounded above in M_{sa}, then $\{a_\lambda\}$ has a supremum in M_{sa}, and a is also the limit of $\{a_\lambda\}$ in each of the weak, strong σ-weak, and σ-strong topologies. Moreover, a is also the supremum of $\{a_\lambda\}$ in $B(H)$. Similarly for decreasing nets.*

Proof. See Alfsen and Shultz (2001), Lemma 2.81. □

Definition 3.7. *A positive linear functional φ on a C^*-algebra A is said to be **normal** if $\varphi(a) = \lim_\lambda \varphi(a_\lambda)$ for each increasing net $\{a_\lambda\}$ with supremum a in A. More generally, a linear functional on A is said to be **normal** if it is a linear combination of normal positive linear functionals on A.*

Definition 3.8. *Let A be a C^*-algebra. The set of all normal states on A is called the **normal state space** of A and we denote it by $S_n(A)$.*

The previous definition clearly generalises Definition 1.24. What follows is a short overview of some characteristics of normal functionals.

Theorem 3.9. *A positive linear functional φ on a von Neumann algebra M acting on a Hilbert space H is normal if and only if is is σ-weakly (or σ-strongly) continuous.*

Proof. See Alfsen and Shultz (2001), Theorem 2.86. □

Corollary 3.10. *The normal functionals on a von Neumann algebra M form a norm-closed subspace of M^*, and the normal states separate the points of M, i.e., for each non-zero $a \in M$ there is a normal state ω such that $\omega(a) \neq 0$.*

Proof. See Alfsen and Shultz (2001), Corollary 2.88. □

Definition 3.11. *The Banach space of all normal linear functionals on a von Neumann algebra M is called the **predual** of M and is denoted by M_*.*

The name "predual" suggests that if we take its dual, we end up with the original space. The following theorem proves that this is indeed the case.

Theorem 3.12. *If M is a von Neumann algebra, then the map $\Psi\colon M \to (M_*)^*$ defined by $(\Psi a)(\omega) = \omega(a)$ for $a \in M$ and $\varphi \in M_*$ is a surjective isometric isomorphism and a homeomorphism from the σ-weak topology on M to the w^*-topology on $(M_*)^*$. Moreover $\Psi a \geq 0$ if and only if $a \geq 0$.*

Proof. See Alfsen and Shultz (2001), Theorem 2.92. □

An interesting side note is that Kadison proved that 3.6 and 3.10 are necessary and sufficient conditions to describe von Neumann algebras. The following theorem states this result.

Theorem 3.13. *A C*-algebra A acting on a Hilbert space H is a von Neumann algebra if and only if A_{sa} is monotone complete and the normal states separate the points of A.*

Proof. See Alfsen and Shultz (2001), Theorem 2.93. □

We now turn our attention to normal maps between C*-algebras, and especially normal maps between von Neumann algebras.

Definition 3.14. *Let $\varphi\colon A \to B$ be a positive linear map from a C*-algebra to another. We say φ is* ***normal*** *if $a_\lambda \nearrow a$ implies $\varphi(a_\lambda) \nearrow \varphi(a)$ for increasing nets $\{a_\lambda\}$ in A_{sa}.*

Lemma 3.15. *A positive linear map φ from a von Neumann algebra M into a von Neumann algebra N is normal iff it is σ-weakly continuous.*

Proof. See Alfsen and Shultz (2001), Lemma 2.99. □

Lemma 3.16. *If ω is a normal state with GNS-representation π_ω on a von Neumann algebra M, then π_ω is a normal positive map from M into $B(H_\omega)$.*

Proof. See Alfsen and Shultz (2001), Lemma 2.101. □

To each positive normal functional of a von Neumann algebra M we assign a unique projection $p \in M$ in the following way.

Lemma 3.17. *If M is a von Neumann algebra and $\omega \in M_*^+$, then there is a (necessarily unique) least projection $p \in M$ such that $\omega(p) = \|\omega\|$.*

Proof. See Alfsen and Shultz (2001), Lemma 2.132. □

For example, if $M = B(H)$, and $\omega = \omega_x$ for certain $x \in H$, i.e., $\omega_x(a) = \langle x, ax\rangle$ for every $a \in M$, then $p = e_x = |x\rangle\langle x|$.

Definition 3.18. *Let M be a von Neumann algebra and $\omega \in M_*^+$. Then the least projection $p \in M$ such that $\omega(p) = \|\omega\|$ is called the* ***carrier projection*** *(or just* ***carrier****) of ω, denoted by* carrier(ω).

We will use the carrier projection in Section 3.6 to define transition probabilities on $P(A)$.

We have now defined the carrier of a single positive normal functional, but we can generalise this to the carrier of a subset of M_*^+.

Lemma 3.19. *If M is a von Neumann algebra and F is a subset of M_*^+, then there is a least projection $p \in M$ such that $\omega(p) = \|\omega\|$ for all $\omega \in F$, namely $p = \inf\{\text{carrier}(\omega) \mid \omega \in F\}$.*

Proof. See Alfsen and Shultz (2001), Lemma 3.19. □

Definition 3.20. *If M is a von Neumann algebra and F is a subset of M_*^+, then the least projection $p \in M$ such that $\omega(p) = \|\omega\|$ for all $\omega \in F$ is called the* ***carrier projection*** *of F and is denoted by* carrier(F).

If F is not just a set, but a face of the normal state space of M, it has additional properties, which we explain in the following proposition and theorem.

Proposition 3.21. *Let F be a face of the normal state space $S_n(M)$ of a von Neumann algebra M and set $p =$ carrier(F). Then the norm closure $\overline{F}$ of F consists of all $\sigma \in S_n(M)$ such that $\sigma(p) = 1$.*

Proof. See Alfsen and Shultz (2001), Proposition 3.30. □

Corollary 3.22. *The norm closure of a face F of the normal state space $S_n(M)$ of a von Neumann algebra M is a face.*

Proof. See Alfsen and Shultz (2001), Corollary 3.32. □

Theorem 3.23. *Let M be a von Neumann algebra with normal state space $S_n(M)$, and denote by $\mathcal{F}$ the set of norm-closed faces of $S_n(M)$, and by $\mathcal{P}$ the set of all projections in M, each equipped with the natural ordening. Then there is an order-preserving bijection $\Phi\colon \mathcal{P} \to \mathcal{F}$. The map Φ and its inverse are explicitly given by the bijection*

$$F = \{\sigma \in S_n(M) \mid \sigma(p) = 1\}; \tag{3.14}$$

$$p = \text{carrier}(F). \tag{3.15}$$

Proof. See Alfsen and Shultz (2001), Theorem 3.35. □

Proposition 3.24. *The normal state space of a von Neumann algebra M is a split face of the state space of M (cf. Definition 1.7).*

Proof. See Alfsen and Shultz (2001), Corollary 3.42. □

Proposition 3.25. *Let $S_n(M)$ be the normal state space of a von Neumann algebra M. Then the intersection of any family of split faces of $S_n(M)$ is a split face.*

Proof. See Alfsen and Shultz (2001), Corollary 3.43. □

The bounded operators on a Hilbert space form a von Neumann algebra, so we can apply Theorem 3.23 to $B(H)$. This even gives us more detail on the dimension of a norm-closed face.

Theorem 3.26. *Let H be a Hilbert space, let $S_n(B(H))$ be the normal state space of $B(H)$ and denote by F_p the norm-closed face of $S_n(B(H))$ associated with a projection $p \in B(H)$. Then the map $p \mapsto F_p$ determines an isomorphism from the lattice of closed subspaces of H to the lattice of norm-closed faces of $S_n(B(H))$. This map carries subspaces of dimension $k < \infty$ to faces of dimension $k^2 - 1$. Each such face can be inscribed in a Euclidean ball, and it is a full Euclidean ball if and only if $k = 2$.*

Remark 3.27. We say that a convex set K is ***inscribed*** in a Euclidean ball B if there is an affine isomorphism φ of K into B which takes every extreme point of K to a point on the surface of B.

Proof. See Alfsen and Shultz (2001), Theorem 4.6. □

The following corollary looks specifically at projections on linear subspaces spanned by a family of unit vectors.

Corollary 3.28. *Let H be a Hilbert space and let $S_n\big(B(H)\big)$ be the normal state space of $B(H)$. If p is the projection onto the closed subspace spanned by a family of unit vectors x_λ in H, then the norm-closed face F associated with p is the smallest norm-closed face of $S_n\big(B(H)\big)$ that contains the vector states ω_{x_λ} $(\lambda \in \Lambda)$.*

Proof. See Alfsen and Shultz (2001), Corollary 4.7. □

3.3 The enveloping von Neumann algebra

Frequently, we want to work with von Neumann algebras instead of general C*-algebras. Using the universal representation (given by Definition A.88) we can embed every C*-algebra into its ***enveloping von Neumann algebra***. This construction has the additional feature that it extends states on the C*-algebra to normal states on its enveloping von Neumann algebra.

Definition 3.29. *Let A be a C*-algebra with universal representation π_u on the Hilbert space H_u. Then the von Neumann algebra $\overline{\pi_u(A)}^w$ on H is called the* ***enveloping von Neumann algebra*** *of A.*

Because A is in general not unital, the C*-algebra $\pi_u(A) \subseteq B(H_u)$ is in general not unital either. So we cannot apply von Neumann's bicommutant theorem directly to conclude that $\overline{\pi_u(A)}^w$ is a von Neumann algebra. However, according to Theorem A.87 the GNS-representation π_ω is cyclic for every state $\omega \in S(A)$. Because every cyclic representation is in particular non-degenerate, it follows that the universal representation is a direct sum of non-degenerate representations and hence is also non-degenerate. Now, by Proposition A.75 we have

$$\overline{\pi_u(A)}^w = \overline{\pi_u(A)}^s = \pi_u(A)'', \tag{3.16}$$

which shows that $\overline{\pi_u(A)}^w$ is indeed a von Neumann algebra.

The following results show that we can identify the enveloping von Neumann algebra with the double dual A^{**}, and that we can identify the state space of A with the normal state space of A^{**}.

Proposition 3.30. *Let A be a C*-algebra with state space $S(A)$ and denote the enveloping von Neumann algebra by $\widetilde{A}$. Each $\rho \in S(A)$ has a unique extension to a normal state $\widetilde{\rho}$ on $\widetilde{A}$, i.e., $\rho = \widetilde{\rho} \circ \pi_u$ where $\pi_u \colon A \to B(H_u)$ is the universal representation of A. Specifically, $\widetilde{\rho}$ is the restriction to $\widetilde{A}$ of the vector state ω_{x_ρ} on $B(H_u)$. Each normal state on $\widetilde{A}$ arises in this way, i.e., it is the restriction to $\widetilde{A}$ of a vector state ω_x where x can be chosen to be the distinguished cyclic vector x_ρ for some $\rho \in S(A)$.*

Proof. See Alfsen and Shultz (2001), Proposition 2.124. □

Corollary 3.31. *The weak and σ-weak topologies coincide on the enveloping von Neumann algebra of a C*-algebra, and so do the the strong and σ-strong topologies.*

Proof. See Alfsen and Shultz (2001), Corollary 2.125. □

Corollary 3.32. *Let A be a C*-algebra with state space $S(A)$ and and let $\widetilde{A}$ be its enveloping von Neumann algebra with normal state space $S_n(\widetilde{A})$. The map $\omega \mapsto \widetilde{\omega}$ that assigns to each state ω on A the corresponding normal state $\widetilde{\omega}$ on $\widetilde{A}$ is an affine isomorphism from $S(A)$ onto $S_n(\widetilde{A})$, which extends uniquely to an isometric linear isomorphism from the dual space A^* of A onto the predual $\widetilde{A}_*$ of $\widetilde{A}$.*

Proof. See Alfsen and Shultz (2001), Corollary 2.126. □

Corollary 3.33. *If A is a C*-algebra with enveloping von Neumann algebra $\widetilde{A}$, then there is an isometric linear isomorphism Φ of $\widetilde{A}$ onto A^{**} defined by the equation*

$$(\Phi b)(\omega) = \widetilde{\omega}(b), \tag{3.17}$$

where $b \in \widetilde{A}$ and $\omega \mapsto \widetilde{\omega}$ is the isomorphism of A^* onto $\widetilde{A}_*$ established in Corollary 3.32.

Proof. See Alfsen and Shultz (2001), Corollary 2.127. □

Because we can embed any C*-algebra A into its double dual A^{**} (see Proposition A.51), we identify A with its image under this embedding. Thus we write $A \subset A^{**}$, and for $a \in A$ and $\omega \in A^*$ we write $a(\omega) = \omega(a)$. Using the affine isomorphism described in Corollary 3.32, we identify the state space of A with the normal state space of the enveloping von Neumann algebra. Lastly, using the isometric linear isomorphism described in Corollary 3.33, we identify $\widetilde{A}$ with A^{**} equipped with the induced involution and product. Thus from now on we denote the enveloping von Neumann algebra by A^{**}.

Theorem 3.34. *A unital *-homomorphism $\varphi\colon A \to M$ from a C*-algebra A into a von Neumann algebra M has a unique extension to a normal *-homomorphism*

$$\widetilde{\varphi}\colon A^{**} \to M. \tag{3.18}$$

Proof. See Alfsen and Shultz 2001, Theorem 2.129. □

Using the enveloping von Neumann algebra, we can extend Theorem A.95.

Theorem 3.35. *If π is a representation of a C*-algebra A on a Hilbert space H, then the following are equivalent:*

(i) *π is irreducible.*

(ii) *$\pi(A)' = \mathbb{C}\,1_H$.*

(iii) *$\pi(A)$ is strongly dense in $B(H)$.*

(iv) *$\pi(A)$ is weakly (or σ-weakly) dense in $B(H)$.*

(v) *The normal extension $\widetilde{\pi}$ of π maps A^{**} onto $B(H)$.*

(vi) *For any two vectors $x, y \in H$ with $x \neq 0$ there is an $a \in A$ such that $\pi(a)x = y$.*

(vii) Each non-zero vector in H is cyclic for $\pi(A)$.

(viii) (H, π) is unitarily equivalent to the GNS-representation associated with a pure state on A.

Proof. See Alfsen and Shultz (2001), Proposition 5.15. □

Definition 3.36. *Let H be a Hilbert space. For each subset A of $B(H)_{sa}$ denote the set of operators in $B(H)_{sa}$, that can be obtained as strong limits of monotone increasing (respectively decreasing) nets from A by A^m (respectively A_m).*

Let M be a von Neumann algebra and let $A \subset M_{\text{sa}}$. Because M is by definition strongly closed, it contains A^m and A_m. In the special case that M is the double dual of a C*-algebra A, we can look at A^m and A_m. This gives us the following definition.

Definition 3.37. *Let A be a C*-algebra. A projection $p \in A^{**}$ is called* ***open*** *if $p \in (A^+)^m$ and is called* ***closed*** *if $p \in (A^+)_m$.*

Using closed projections, we can give the C*-analogue of Theorem 3.23.

Theorem 3.38. *If A is a C*-algebra with state space $S(A)$, and denote by $\mathcal{F}$ the set of w^*-closed faces of $S(A)$, and by $\mathcal{P}_c$ the set of closed projections in A^{**}. Then there are canonical bijective correspondences between $\mathcal{F}$ and $\mathcal{P}_c$ given by*

$$F = \{\sigma \in S(A) \mid \sigma(p) = 1\}; \tag{3.19}$$

$$p = \text{carrier}(F). \tag{3.20}$$

Proof. See Alfsen and Shultz (2001), Theorem 3.61. □

Note that for projections $p, q \in A^{**}$ we have $\{\sigma \in S(A) \mid \sigma(p) = 1\} = \{\sigma \in S(A) \mid \sigma(q) = 1\}$ if and only if $p = q$, because the normal states on A^{**}, i.e., the states on $S(A)$ separate points. So the bijective correspondence from the previous theorem implies not only that there is no other closed projection in A^{**} such that $F = F_p$, but also that carrier(F) is the *unique* projection such that $F = F_{\text{carrier}(F)}$.

3.4 Carriers, covers and the equivalence of states

The ***center*** of a C*-algebra A consists of those elements z that commute with all elements of A. The center is a norm-closed *-subalgebra, i.e., a C*-subalgebra of A. If M is a von Neumann algebra, then its center is σ-weakly closed, and hence is a von Neumann subalgebra. A projection in the center of a C*-algebra is said to be a ***central projection***.

Proposition 3.39. *If σ is a normal state on a von Neumann algebra M, there is a smallest central projection $c \in M$ such that $\sigma(c) = 1$.*

Proof. Suppose that c_1 and c_2 are central projections such that $\sigma(c_1) = \sigma(c_2) = 1$. This implies that $1 - c_2$ is a central projection and that $\sigma(1 - c_2) = 0$. Note that

$$\langle\, ,\, \rangle_\sigma \colon M \times M \to \mathbb{C}; \quad (a, b) \mapsto \sigma(a^* b) \tag{3.21}$$

is a positive semidefinite sesquilinear form, so that by Cauchy-Schwarz (Theorem A.2)

$$\left|\sigma(a^* b)\right|^2 \leq \sigma(a^* a)\sigma(b^* b). \tag{3.22}$$

Using equation (3.22) we find that

$$0 \leq \sigma(c_1(1 - c_2)) \leq \sigma(c_1^* c_1)\sigma((1 - c_2)^*(1 - c_2)) = \sigma(c_1)\sigma(1 - c_2) = 0, \tag{3.23}$$

and hence

$$\sigma(c_1 c_2) = \sigma(c_1(1 - c_2) + c_1) = \sigma(c_1(1 - c_2)) + \sigma(c_1) = 0 + 1 = 1. \tag{3.24}$$

Furthermore, $c_1 c_2$ is a central projection for which $c_1 c_2 \leq c_1$ and $c_1 c_2 \leq c_2$. Thus the collection of central projections on which σ takes the value 1 is directed downwards. Because the lattice of projections on a von Neumann algebra is complete (see Corollary A.101) the decreasing net of such projections converges σ-weakly to its infimum, which is a projection $c \in M$. This projection is central, because the center M is σ-weakly closed. Because normal states are σ-weakly continuous (see Theorem 3.9) it follows that $\sigma(c) = 1$. □

Definition 3.40. *Let σ be a normal state on a von Neumann algebra M. The projection c in Proposition 3.39 is called the* ***central carrier*** *of σ, and is denoted $c(\sigma)$.*

Compare the definition of the central carrier of σ to that of the carrier of σ given in Definition 3.18. The only difference is that the central carrier is *central*, whereas the carrier does not need to be. It follows that for a normal state σ on a von Neumann algebra M:

$$\text{carrier}(\sigma) \leq c(\sigma). \tag{3.25}$$

Let σ be a normal state on a von Neumann algebra M. Denote the intersection of all split faces of $S_n(M)$ containing σ by F_σ, so

$$F_\sigma = \bigcap\{F \mid F \text{ is a split face of } S_n(M) \text{ and } \sigma \in F\}. \tag{3.26}$$

By Proposition 3.24 the normal state space of M is a split face and by Proposition 3.25 the intersection of split faces of $S_n(M)$ is a split face. This implies that F_σ is a split face of $S_n(M)$. We call it the **split face generated by** σ. Now let σ be a state on a C*-algebra A and let $\widetilde{\sigma}$ be its normal extension. By the previous argument $\widetilde{\sigma}$ generates a split face of $S_n(A^{**})$. We can identify $S(A)$ with the normal state space of the enveloping von Neumann algebra. Hence σ generates a split face of $S(A)$, which we also denote by F_σ.

Recall from Theorem 3.23 that there is a bijection from the set of projections in M to the set of norm-closed faces of $S_n(M)$. Denote by F_p the norm-closed face associated to a projection $p \in M$, i.e.,

$$F_p = \{\sigma \in S_n(M) \mid \sigma(p) = 1\}. \tag{3.27}$$

The following lemma relates the split face generated by a state to the face generated by its central carrier.

Lemma 3.41. *If σ is a state on a C*-algebra or a normal state on a von Neumann algebra, then*

$$F_\sigma = F_{c(\sigma)}. \tag{3.28}$$

Proof. See Alfsen and Shultz (2001), Lemma 5.1. □

Definition 3.42. *Let A be a C*-algebra with state space $S(A)$ and let $\pi \colon A \to B(H)$ be a representation with normal extension $\widetilde{\pi}$. The* ***central cover*** *of π is the central projection $c(\pi)$ in A^{**} such that $\ker(\widetilde{\pi}) = \big(1 - c(\pi)\big)A^{**}$.*

To complicate things further, let us define the central cover of a *self-adjoint element* instead of a state. If $p \in M$ is a projection in M and $c \in M$ is a central projection, we say that c **dominates** p if $c \geq p$, or, equivalently, if $cp = p$. Note that if c_1 and c_2 are two central projections that dominate p, then $c_1 c_2$ dominates p. Indeed,

$$c_1 c_2 p = c_1(c_2 p) = c_1 p = p. \tag{3.29}$$

Hence the infimum of all central projections dominating p also dominates p. Because the center of M is σ-weakly closed, the infimum is a central projection. Recall from Definition A.78 that the range projection of a self-adjoint element $a \in M$ is the smallest projection p such that $pa = a$. We denote the range projection by $r(a)$.

Definition 3.43. *If a is a self-adjoint element in a von Neumann algebra M, then the least central projection dominating $r(a)$ is called the* ***central cover*** *of a and is denoted $c(a)$.*

Note that for every projection $p \in M$ we have $r(p) = p$, in which case the central cover of p is just the smallest central projection dominating p.

Definition 3.44. *Let M be a von Neumann algebra. A projection $p \in M$ is said to be* ***abelian*** *if pMp is an abelian subalgebra of M*

Definition 3.45. *Let M be a von Neumann algebra. M is said to be of* ***type I*** *if it has an abelian projection p such that $c(p) = 1$*

We associate to each state σ a projection, namely carrier(σ), to which in turn we associate a central projection, namely the central cover of carrier(σ). We also directly associate a central projection with the state σ, namely the central carrier $c(\sigma)$. Each state also has a corresponding representation, namely its GNS-representation to which we associate a third the central projection $c(\pi_\sigma)$. The following lemma relates these three central projections.

Lemma 3.46. *Let σ be a state on a C*-algebra A and let π_σ be its GNS-representation. Then*

$$c(\pi_\sigma) = c(\mathrm{carrier}(\sigma)) = c(\sigma). \tag{3.30}$$

Proof. For the proof that $c(\pi_\sigma) = c(\sigma)$, see Alfsen and Shultz (2001), Corollary 3.41, Lemma 5.1 and Corollary 5.5. We now prove that $c(\mathrm{carrier}(\sigma)) = c(\sigma)$. By definition of the carrier projection and the central carrier we have

$$\mathrm{carrier}(\sigma) = \inf\{p \in A^{**} \mid p \text{ is a projection and } \widetilde{\sigma}(p) = 1\}; \tag{3.31}$$
$$c(\sigma) = \inf\{c \in A^{**} \mid c \text{ is a central projection and } \widetilde{\sigma}(c) = 1\}. \tag{3.32}$$

Hence it follows immediately from these definitions that $c(\sigma)$ is the smallest central projection that dominates carrier(σ), i.e., $c(\mathrm{carrier}(\sigma)) = c(\sigma)$. □

Proposition 3.47. *If A is a C*-algebra and $\pi\colon A \to B(H)$ is an irreducible representation, then π^* is an affine isomorphism of the normal state space of $B(H)$ onto $F_{c(\pi)}$. In particular, if σ is a pure state on A, then π_σ^* is an affine isomorphism of the normal state space of $B(H_\sigma)$ onto F_σ.*

Proof. See Alfsen and Shultz (2001), Corollary 5.16. □

Recall from Definition A.92 that two representations (H_1, π_1) and (H_2, π_2) are unitarily equivalent if there exists a unitary $u \in B(H_1, H_2)$ such that $u\pi_1(a)u^* = \pi_2(a)$ for all $a \in A$. The following definition gives a similar description of unitary equivalence of states.

Definition 3.48. *Two states σ and ρ on a C^*-algebra A are* ***unitarily equivalent*** *if there is a unitary $u \in A$ such that*

$$\rho(a) = \sigma(uau^*) \quad \text{for all } a \in A. \tag{3.33}$$

It turns out that for pure states these notions of unitary equivalence coincide via the GNS-construction. Moreover, we can relate it to the split face generated by the states and their central carriers.

Theorem 3.49. *Let σ and ρ be pure states on a C^*-algebra A with state space $S(A)$. The following are equivalent.*

(i) *σ is unitarily equivalent to ρ.*

(ii) *π_σ is unitarily equivalent to π_ρ.*

(iii) *σ and ρ generate the same split face of $S(A)$.*

(iv) $c(\sigma) = c(\rho)$.

(v) *There is a unit vector $x \in H_\sigma$ such that $\rho = \omega_x \circ \pi_\sigma = \langle x, \pi_\sigma(\cdot)x\rangle$.*

Proof. See Alfsen and Shultz (2001), Theorem 5.19. □

Definition 3.50. *Let A be a C^*-algebra and let $P \subseteq P(A)$ be a maximal set of pairwise inequivalent (i.e. pairwise not unitarily equivalent) pure states. Then*

$$\pi_{ra} = \bigoplus_{\sigma \in P} \pi_\sigma \tag{3.34}$$

is called the ***reduced atomic representation*** *of A.*

Strictly speaking, we cannot speak of *the* reduced atomic representation of A, because we choose the set P and this choice is not unique. However, it follows from Theorem 3.49 that if we choose a different maximal set of pairwise inequivalent pure states, say P', then $\bigoplus_{\sigma\in P'} \pi_\sigma$ is unitarily equivalent to $\bigoplus_{\sigma\in P} \pi_\sigma$. So up to unitary equivalence, the reduced atomic representation is unique.

Theorem 3.51. *Let A be a C^*-algebra and let π_{ra} be its reduced atomic representation. Then π_{ra} is faithful and*

$$\pi_{ra}(A)'' = \bigoplus_{\sigma \in P} B(H_\sigma). \tag{3.35}$$

Proof. See Kadison and Ringrose (1986), Proposition 10.3.10. The proof follows from the fact that the atomic representation is faithful (Proposition A.90) and the fact that for pure states σ its image $\pi_\sigma(A)$ is weakly dense in $B(H_\sigma)$ (Theorem 3.35). □

3.5 Jordan algebras

We have already seen several Jordan maps, which preserve a certain Jordan product. However, we have not given the general definition of a Jordan algebra. In this section we give an introduction to Jordan algebras, JB-algebras (which are the Jordan analogues of C*-algebras) and JBW-algbras (which are the Jordan analogues of von Neumann algebras). In the next section we will use several results about JBW-algebras to define a transition probability on $P(A)$.

Definition 3.52. *A* ***Jordan algebra*** *over* $\mathbb{R}$ *is a vector space* A *over* $\mathbb{R}$ *equipped with a commutative bilinear product* $\circ$ *that satisfies the identity*

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a). \tag{3.36}$$

Lemma 3.53. *If* A *is an associative algebra over* $\mathbb{R}$*, then* A *equipped with the product*

$$a \circ b = \tfrac{1}{2}(ab + ba) \tag{3.37}$$

is a Jordan algebra.

Proof. It is clear that $\circ$ is a commutative, bilinear product. So we only have to check if the relation (3.36) holds. Note that $a \circ a = a^2$, so that

$$\begin{aligned}
(a \circ a \circ b) \circ a &= \tfrac{1}{2}(a^2 b + ba^2) \circ a && (3.38)\\
&= \frac{1}{4}(a^2 ba + ba^3 + a^4 b + aba^2) && (3.39)\\
&= \tfrac{1}{2}\big(a^2[\tfrac{1}{2}(ba + ab)] + [\tfrac{1}{2}(ba + ab)]a^2\big) && (3.40)\\
&= a^2 \circ (b \circ a). && (3.41)
\end{aligned}$$

□

In any Jordan algebra we define the ***triple product*** $\{abc\}$ by

$$\{abc\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b. \tag{3.42}$$

Note that for the Jordan algebra from Lemma 3.53, the triple product is given by

$$\begin{aligned}
\{abc\} &= \frac{1}{4}(abc + bac + cab + cba) + \frac{1}{4}(bca + cba + abc + acb)\\
&\qquad - \frac{1}{4}(acb + cab + bac + bca)\\
&= \tfrac{1}{2}(abc + cba). && (3.43)
\end{aligned}$$

Definition 3.54. *A* ***JB-algebra*** *is a Jordan algebra A over $\mathbb{R}$ with identity element 1_A equipped with a complete norm satisfying for all $a, b \in A$:*

$$\|a \circ b\| \leq \|a\| \|b\| \,, \tag{3.44}$$

$$\left\|a^2\right\| = \|a\|^2 \,, \tag{3.45}$$

$$\left\|a^2\right\| \leq \left\|a^2 + b^2\right\| . \tag{3.46}$$

Lemma 3.55. *The self-adjoint part of a unital C*-algebra A is a JB-algebra with respect to the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.*

Proof. Let $a, b \in A_{\mathrm{sa}}$. Because the self-adjoint part of a unital C*-algebra is an algebra over $\mathbb{R}$, Lemma 3.53 shows that $\frac{1}{2}(ab + ba)$ indeed defines a Jordan product. Because a C*-algebra is in particular a normed algebra, we can conclude that (3.44) holds:

$$\|a \circ b\| = \left\|\tfrac{1}{2}(ab + ba)\right\| \leq \tfrac{1}{2}(\|ab\| + \|ba\|) \leq \|a\| \|b\| \,. \tag{3.47}$$

Property (3.45) follows directly from the fact that $\|aa^*\| = \|a\|^2$ for all $a \in A$. Because a and b are self-adjoint we have $a^2, b^2 \geq 0$ and hence $a^2 + b^2 \geq a^2$. This implies that $\left\|a^2\right\| \leq \left\|a^2 + b^2\right\|$, so (3.46) holds. □

Because a von Neumann algebra is in particular a unital C*-algebra, the previous lemma implies that M_{sa} is a JB-algebra for every von Neumann algebra M.

Definition 3.56. *An ordered Banach space A is* ***monotone complete*** *if each increasing net a_α which is bounded above has a supremum a in A. We denote such a net by $a_\alpha \nearrow a$. A bounded linear functional σ on a monotone complete space A is* ***normal*** *if whenever $a_\alpha \nearrow a$, then $\sigma(a_\alpha) \to \sigma(a)$*

Definition 3.57. *A* ***JBW-algebra*** *is a JB-algebra that is monotone complete and admits a separating set of normal states.*

JB-algebras are the Jordan analogs of a C*-algebras and JBW-algebras are the Jordan analogs of von Neumann algebras, as you can see by looking at Theorem 3.13. The only JBW-algebra we will be working with in this thesis is the set of self-adjoint elements of a von Neumann algebra. However, note that the definition of a normal functional given by Definition 3.56 differs slightly from that of a normal functional on a C*-algebra given by Definition 3.7. The following proposition proves that these definitions are in fact compatible.

Proposition 3.58. *Let M be a JBW-algebra. Every normal linear functional on M is the difference of positive normal linear functionals, and the normal state space of M is a split face of the state space of M.*

Proof. See Alfsen and Shultz (2003), Proposition 2.52. □

Corollary 3.59. *Let M be a von Neumann algebra. Then M_{sa} equipped with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ is a JBW-algebra.*

Proof. It follows from Lemma 3.55 that M_{sa} is a JB-algebra. Proposition 3.58 shows that the normal states on M_{sa} are precisely the normal states on M_{sa} viewed as a JB-algebra. Theorem 3.13 proves that M_{sa} is complete and that the normal states on M_{sa} separate points. □

Definition 3.60. *A projection p is an JBW-algebra M is* ***abelian*** *if the algebra*

$$M_p = \{pMp\} \tag{3.48}$$

is associative.

Let M be a von Neumann algebra and p a projection in M. Then by equation (3.43) the algebra $(M_{\mathrm{sa}})_p$ is just $pM_{\mathrm{sa}}p$. By Alfsen and Shultz (2003), Proposition 1.49, it follows that $p \in M_{\mathrm{sa}}$ is abelian in the sense of Definition 3.60 if and only if it is abelian in the sense of Definition 3.44. So the definition of an abelian projection in a JBW-algebra generalises the definition of an abelian projection in a von Neumann algebra given in Definition 3.44. Hence the following definition is a generalisation of Definition 3.45.

Definition 3.61. *A JBW-algebra M is* ***type I*** *if it contains an abelian projection with central cover 1_M.*

Recall that a minimal non-zero element of a lattice is called an **atom**. Note that the lattice of projections for the von Neumann algebra M and the JBW-algebra M_{sa} are the same. Therefore, the set of atoms in M is identical to the set of atoms in M_{sa} and hence the following proposition also holds for von Neumann algebras.

Proposition 3.62. *A JBW-factor M is type I if and only if it contains an atom.*

Proof. See Alfsen and Shultz (2003), Proposition 3.44. □

3.6 Transition probability on the pure state space

The goal of this section is to define a transition probability on the pure states of a C*-algebra that in some way generalises the transition function on the one-dimensional projections of a Hilbert space. Let H be a Hilbert space. Recall that the transition function on $\mathcal{P}_1(H)$ was given by

$$(e, f) \mapsto \mathrm{Tr}(ef) \ \ (e, f \in \mathcal{P}_1(H)). \tag{3.49}$$

Denote this transition function by $\tau^{\mathcal{P}_1(H)}$. We want to give $\tau^{\mathcal{P}_1(H)}$ in such a way that we can use Theorem 1.25 to define a transition function on $P_n(B(H))$.

Lemma 3.63. *Let H be a Hilbert space and $x, y \in H$ unit vectors. Then*

$$\mathrm{Tr}(e_x e_y) = \inf\{\langle x, ax\rangle \mid a \in B(H)_{sa},\ 0 \leq a \leq 1_H,\ \langle y, ay\rangle = 1\}. \tag{3.50}$$

Proof. The proof is due to Landsman (2017), Proposition 5.18.

Since $\mathrm{Tr}(e_x e_y) = \langle x, e_y x\rangle$, we are ready if we can show that the infimum is reached at $a = e_y$. Therefore, we prove that for any a as specified we must have $\langle x, ax\rangle \geq \mathrm{Tr}(e_x e_y) = \left|\langle x, y\rangle\right|^2$. To do so, we are going to find a contradiction if

$$\langle x, ax\rangle < \mathrm{Tr}(e_x e_y), \tag{3.51}$$

for some such a. Indeed, $\langle y, ay\rangle$ with $\|a\|_{B(H)} \leq 1$ and $\|y\| = 1$ imply, by Cauchy-Schwarz (A.2, that $ay = y$ Since $a^* = a$ (by positivity of a), we also have $a\colon (\mathbb{C}\cdot y)^\perp \to (\mathbb{C}\cdot y)^\perp$, so we may write $a = e_y + a'$, with $a'y = 0$ and a' mapping $(\mathbb{C}\cdot y)^\perp$ to itself. Then $a \geq 0$ implies $a' \geq 0$. If (3.51) holds, then $\langle x, a'x\rangle < 0$, which contradicts positivity of a' (and hence of a). □

Definition 3.64. *Let H be a Hilbert space and $P_n(B(H))$ be its normal pure state space. Then $\tau^{B(H)}\colon P_n(B(H)) \times P_n(B(H))$, given by*

$$\tau^{B(H)}(\omega_x, \omega_y) = \inf\{\omega_x(a) \mid a \in B(H), 0 \leq a \leq 1_H, \omega_y(a) = 1\}, \tag{3.52}$$

for all $\omega_x, \omega_y \in P_n(B(H))$ defines a transition probability on $P_n(B(H))$.

Remark 3.65. It is not immediately clear that $\tau^{B(H)}$ defines a transition probability. However, this follows directly from the previous lemma and Theorem 1.25. Indeed, let $\omega_x, \omega_y \in P_n(B(H))$. Then

$$\begin{aligned}\tau^{B(H)}(\omega_x, \omega_y) &= \inf\{\omega_x(a) \mid a \in B(H), 0 \leq a \leq 1_H, \omega_y(a) = 1\}\\ &= \inf\{\langle x, ax\rangle \mid a \in B(H)_{\mathrm{sa}},\ 0 \leq a \leq 1,\ \langle y, ay\rangle = 1\}\\ &= \mathrm{Tr}(e_x e_y) = \mathrm{Tr}(e_y e_x) = \tau^{B(H)}(\omega_y, \omega_x).\end{aligned} \tag{3.53}$$

Furthermore, $\tau^{B(H)}(\omega_x, \omega_y) = \mathrm{Tr}(e_x e_y) = 1$ if and only if $e_x = e_y$, in which case $y = zx$ for certain $z \in \mathbb{T}$. This implies that $\tau^{B(H)}(\omega_x, \omega_y) = 1$ if and only if $\omega_x = \omega_y$.

Hence the pairs $(\mathcal{P}_1(H), \tau^{\mathcal{P}_1(H)})$ and $(P_n(B(H)), \tau^{B(H)})$ are isomorphic as sets with a transition probability. We now aim to generalise this transition function to one on the pure state on a general C*-algebra. For unital C*-algebras, we can mirror Definition 3.64:

Definition 3.66. *Let A be a unital C*-algebra with pure state space $P(A)$. Then the function $\tau\colon P(A) \times P(A) \to [0,1]$, given by*

$$\tau(\sigma, \rho) = \inf\{\sigma(a) \mid a \in A, 0 \leq a \leq 1_A, \rho(a) = 1\}, \tag{3.54}$$

for all $\sigma, \rho \in P(A)$ defines a transition probability on $P(A)$.

The following proposition gives a more intuitive way to interpret the transition probability on $P(A)$. In fact, it *proves* that equation (3.54) defines a transition probability in the first place. We will see later on in Proposition 3.80 that the dichotomy is valid even if A has no unit.

Proposition 3.67. *Let A be a unital C*-algebra and define τ by equation* (3.54). *Then*

$$\tau(\sigma, \rho) = 1 - \frac{1}{4}\|\sigma - \rho\|^2, \tag{3.55}$$

and the following dichotomy applies:

(i) If σ and ρ are unitarily equivalent, so that we may assume that the associated cyclic vectors x_σ and x_ρ lie in the same Hilbert space, we have

$$\tau(\sigma, \rho) = \mathrm{Tr}(e_{x_\sigma}, e_{x_\rho}) = \left|\langle x_\sigma, x_\rho\rangle\right|^2. \tag{3.56}$$

(ii) If σ and ρ are unitarily inequivalent, then

$$\tau(\sigma, \rho) = 0. \tag{3.57}$$

Proof. See Landsman (2017), Proposition C.177. □

We still need to define a transition probabiliy on $P(A)$ that also holds for non-unital C*-algebras. This will be accomplished in Theorem 3.79, which requires considerable preparation. The key to defining the transition probability lies in the fact that we can identify the pure states of A with the pure normal states of A^{**}. We start by embedding A into its enveloping von Neumann algebra A^{**}, as described in section 3.3. According to Corollary 3.59 the self-adjoint elements of A^{**} form a JBW-algebra. Corollary 3.32 gives an affine isomorphism from the convex set $S(A)$ onto the convex set $S_n(A^{**})$. Because the map is affine, it maps the extreme boundary of $S(A)$, i.e., the pure states $P(A)$, onto the extreme boundary of $S_n(A)$, i.e., the pure normal states $P_n(A^{**})$.

Define a state on a JB-algebra as usual as a positive linear functional of norm 1. Let A be a C*-algebra. Note that we can uniquely extend a state, or indeed any functional, on the JB-algebra A_{sa} to a state on A by setting

$$\sigma(a + ib) = \sigma(a) + i\sigma(b), \tag{3.58}$$

for every $\sigma \in S(A_{\mathrm{sa}})$ and $a, b \in A_{\mathrm{sa}}$. Conversely, every state on a C*-algebra A defines a state on the JB-algebra A_{sa}. Therefore, we can use $S(A)$ and $S(A_{\mathrm{sa}})$ interchangeably and the same holds for normal states and pure states.

Lemma 3.68. *Let M be a JBW-algebra, and F a non-empty set of positive normal functionals on M. Then there is a smallest projection p such that $\omega(p) = \|p\|$ for all $\omega \in F$.*

Proof. See Alfsen and Shultz (2003), Lemma 5.1. □

Definition 3.69. *Let M be a JBW-algebra, and F a non-empty set of positive normal functionals on M. The smallest projection p such that $\omega(p) = \|p\|$ for all $\omega \in F$ is called the **support projection** or **carrier projection**[1] of F, and is denoted* carrier(F). *For a single positive normal functional ω we refer to the carrier projection of the set $\{\omega\}$ as the carrier projection of ω.*

Definition 3.70. *Let M be a JBW-algebra with normal state space $S_n(M)$, and let p be a projection in M. Then we define the norm-closed face F_p of $S_n(M)$ by*

$$F_p = \{\omega \in S_n(M) \mid \omega(p) = 1\}. \tag{3.59}$$

*We say F_p is the **face of $\mathbf{S_n(M)}$ associated with $\mathbf{p}$**. We call a face of the form F_p a **projective face**.*

Given a projective face F we can find the projection $p \in M$ such that $F_p = F$ by using the following proposition.

Proposition 3.71. *If p is a projection in a JBW-algebra M, then*

$$p = \inf\{a \in M \mid 0 \leq a \leq 1_M \text{ and } \omega(a) = 1 \text{ for every } \omega \in F_p\}. \tag{3.60}$$

[1]Note that for the JBW-algebra M_{sa} this definition of the carrier projection coincides with Definition 3.20.

Proof. See Alfsen and Shultz (2003), Proposition 5.8. □

Note that Proposition 3.71 implies that

$$\text{carrier}(F_p) = p, \tag{3.61}$$

which implies that there is a bijective correspondence between projections in a JBW-algebra and projective faces. This is the JBW-algebra version of Theorem 3.23. We now want to find a bijective correspondence between certain projections, namely atoms, and certain projective faces, namely faces containing only a single extreme point. Recall from Corollary A.101 that the lattice of projections of a von Neumann algebra M is complete. Recall from Definition A.99 that an atom is a minimal non-zero projection, i.e., p is an atom if for all projections q such that $q \leq p$ we have $q = 0$ or $q = p$.

Proposition 3.72. *Let M be a JBW-algebra with normal state space $S_n(M)$. The map $p \mapsto F_p$ is a 1-1 correspondence of atoms of M and faces of $S_n(M)$ consisting of a single extreme point.*

Proof. See Alfsen and Shultz (2003), Proposition 5.39. □

Definition 3.73. *Let p be an atom in a JBW-algebra M with normal state space $S_n(M)$. We denote by ω_p the unique extreme point of $S_n(M)$ satisfying $F_p = \{\omega_p\}$.*

Proposition 3.74. *Let p and q be atoms in a JBW-algebra M. Then*

$$\omega_p(q) = \omega_q(p). \tag{3.62}$$

Proof. See Alfsen and Shultz (2003), Corollary 5.57. □

If A is a C*-algebra with enveloping von Neumann algebra A^{**} the extreme points of $S_n(A^{**})$ are precisely the normal extensions of the pure states of A. Hence Proposition 3.72 gives a 1-1 correspondence between pure states of A and their carrier projections, which are atoms in A^{**}.

Lemma 3.75. *Let p be a projection in a JBW-algebra M. Then there is a smallest central projection $c(p) \geq p$.*

Proof. See Alfsen and Shultz (2003), Lemma 2.37. □

Definition 3.76. *Let p be a projection in a JBW-algebra M. The smallest central projection $c(p) \geq p$ is called the **central cover** of p.*[2]

Lemma 3.77. *Let M be a JBW-algebra, and p and q distinct atoms in M. Then either $c(p) = c(q)$ or $c(p) \perp c(q)$.*

Proof. See Alfsen and Shultz (2003), Lemma 5.53. □

Corollary 3.78. *Let A be a C^*-algebra and let σ and ρ be pure states of A. Then σ and ρ are unitarily inequivalent if and only if $c(\sigma) \perp c(\rho)$.*

Proof. Because σ and ρ are pure states their central carriers are atoms of A^{**}. By Lemma 3.46,

$$c(\text{carrier}(\sigma)) = c(\sigma), \quad c(\text{carrier}(\rho)) = c(\rho). \tag{3.63}$$

By Theorem 3.49 we have $c(\sigma) \neq c(\rho)$. Then by Lemma 3.77 it follows that $c(\sigma) \perp c(\rho)$. □

As we explained earlier, every extreme point of $S_n(A^{**})$ corresponds to a unique pure state $\omega \in P(A)$. Thus Proposition 3.72 gives a 1-1 correspondence of atoms of A^{**} and pure states of A. We denote by ω_p the pure state on A corresponding to the atom $p \in A^{**}$. As in Section 3.3 we denote the normal state on A^{**} corresponding to a state $\omega \in S(A)$ by $\widetilde{\omega}$. To ease notation we denote carrier(ω) by p_ω.

Theorem 3.79. *Let A be a C^*-algebra. The map*

$$\tau \colon P(A) \times P(A) \to [0,1]; \quad (\rho, \sigma) \mapsto \widetilde{\sigma}(p_\rho), \tag{3.64}$$

defines a transition probability on $P(A)$.

Proof. Because $\widetilde{\sigma}$ is a state, it is by definition a positive linear functional of norm 1. Because p_ρ is a projection it is in particular positive and $\|p_\rho\| = 1$. This implies that $\tau(\rho, \sigma) \geq 0$, and

$$\tau(\rho, \sigma) = \widetilde{\sigma}(p_\rho) = \left|\widetilde{\sigma}(p_\rho)\right| \leq \|\widetilde{\sigma}\| \|p_\rho\| = 1 \cdot 1 = 1. \tag{3.65}$$

So τ is indeed a map from $P(A) \times P(A)$ to $[0,1]$. It follows from Proposition 3.74 that $\tau(\rho, \sigma) = \tau(\sigma, \rho)$, hence τ is symmetric. Lastly, we have to show that $\tau(\rho, \sigma) = 1$ if and only if $\rho = \sigma$. This follows immediately from the fact that

$$\{\widetilde{\rho}\} = F_{p_\rho} \equiv \{\omega \in S_n(M) \mid \omega(p_\rho) = 1\}. \tag{3.66}$$

□

[2]Note that for the JBW-algebra M_{sa}, where M is a von Neumann algebra, this definition of the central cover coincides with that given in Definition 3.43.

We have now defined a transition probability on $P(A)$, but it is not immediately clear why this is the transition probability we want to work with. The following proposition shows that it is in fact a generalisation of the transition probability given in equation (3.54), because the dichotomy from Proposition 3.67 still applies.

Proposition 3.80. *Proposition 3.67 also holds verbatim for non-unital C^*-algebras.*

Proof. We first prove the dichotomy. To do this, we describe $\widetilde{\sigma}$ and p_ρ, after which the proof follows easily. View the enveloping von Neumann algebra $\widetilde{A}$ as $\pi_u(A)'' \subseteq B(H_u)$. Let $x_\sigma \in H_\sigma \subseteq H_u$ be the cyclic vector associated with the GNS representation of σ. Note that according to Proposition 3.30 the normal extension of σ is the restriction of the vector state ω_{x_σ} to $\widetilde{A}$, i.e., $\widetilde{\sigma} = \omega_{x_\sigma|_{\widetilde{A}}}$. Similarly for ρ: $\widetilde{\rho} = \omega_{x_\rho|_{\widetilde{A}}}$.

Claim: For a pure state $\rho \in P(A)$ we have $p_\rho \equiv \text{carrier}(\widetilde{\rho}) = \langle x_\rho, \cdot \rangle x_\rho = e_{x_\rho}$, i.e., the projection onto the unit vector $x_\rho \in H_u$.

Proof: By definition

$$\text{carrier}(\widetilde{\rho}) = \inf\{p \in \widetilde{A} \mid p \text{ is a projection and } \widetilde{\rho}(p) = 1\}. \tag{3.67}$$

First note that e_{x_ρ} is a projection and

$$\widetilde{\rho}(e_{x_\rho}) = \omega_{x_\rho}(e_{x_\rho}) = \langle x_\rho, e_{x_\rho}(x_\rho)\rangle = \langle x_\rho, \langle x_\rho, x_\rho\rangle x_\rho\rangle = \langle x_\rho, x_\rho\rangle^2 = 1^2 = 1. \tag{3.68}$$

Now suppose that $q \in \widetilde{A}$ is a projection in $\widetilde{A}$ such that $\widetilde{\rho}(q) = 1$. Then

$$1 = \widetilde{\rho}(q) = \langle x_\rho, q(x_\rho)\rangle, \tag{3.69}$$

which implies that $q(x_\rho) = x_\rho$. The smallest projection $q \in \widetilde{A}$ such that $q(x_\rho) = x_\rho$ is precisely the one-dimensional projection onto the unit vector x_ρ, i.e., $e_{x_\rho} \leq q$ and hence $\text{carrier}(\widetilde{\rho}) = e_{x_\rho}$. ∎

Now suppose that ρ and σ are unitarily equivalent. Let $(H_\rho, \pi_\rho, x_\rho)$ be the GNS representation of ρ. Using Theorem 3.49 we may assume that cyclic vector of σ lies in the same Hilbert space. Then

$$\begin{aligned}\tau(\rho,\sigma) = \widetilde{\sigma}(p_\rho) &= \omega_{x_\sigma}(e_{x_\rho}) = \langle x_\sigma, e_{x_\rho}(x_\sigma)\rangle \\ &= \langle x_\sigma, \langle x_\rho, x_\sigma\rangle x_\rho\rangle = \langle x_\rho, x_\sigma\rangle\langle x_\sigma, x_\rho\rangle = \left|\langle x_\rho, x_\sigma\rangle\right|^2.\end{aligned} \tag{3.70}$$

Finally, suppose that ρ and σ are not unitarily equivalent. By Corollary 3.78 we have $c(\sigma) \perp c(\rho)$, and hence by Lemma 3.46, $c(p_\sigma) \perp c(p_\rho)$. This implies

$$1_{A^{**}} - c(p_\rho) \geq c(p_\sigma), \tag{3.71}$$

indeed

$$(1_{A^{**}} - c(p_\rho))c(p_\sigma) = c(p_\sigma) - c(p_\rho)c(p_\sigma) = c(p_\sigma). \tag{3.72}$$

Note further that $1_{A^{**}} - c(p_\rho)$ is a central projection, so in particular

$$0 \leq \widetilde{\sigma}(1_{A^{**}} - c(p_\rho)) = \|\widetilde{\sigma}\| \left\|1_{A^{**}} - c(p_\rho)\right\| = 1^2 = 1. \tag{3.73}$$

Because $\widetilde{\sigma}$ is positive it follows that

$$1 = \widetilde{\sigma}(c(p_\sigma)) \leq \widetilde{\sigma}(1_{A^{**}} - c(p_\rho)) \leq 1, \tag{3.74}$$

so that

$$1 - \widetilde{\sigma}(c(p_\rho)) = \widetilde{\sigma}(1_{A^{**}} - c(p_\rho)) = 1. \tag{3.75}$$

This shows that $\widetilde{\sigma}(c(p_\rho)) = 0$. Because $0 \leq p_\rho \leq c(p_\rho)$ we have

$$0 \leq \widetilde{\sigma}(p_\rho) \leq \widetilde{\sigma}(c(p_\rho)) = 0, \tag{3.76}$$

so we can conclude that $\tau(\sigma, \rho) = \widetilde{\sigma}(p_\rho) = 0$.

It only remains to be shown that $\tau(\rho, \sigma) = 1 - \frac{1}{4}\|\rho - \sigma\|^2$. First suppose that ρ and σ are unitarily equivalent. This proof is identical to that in the unital case, given in Landsman (2017), Proposition C.177. If $\rho = \sigma$, the identity is trivial, so suppose that ρ and σ distinct pure states. Let π_ρ be the GNS-representation of ρ and let x_ρ and x_σ be the unit vectors in H_σ such that $\rho = \omega_{x_\rho} \circ \pi_\rho$ and $\sigma = \omega_{x_\sigma} \circ \pi_\rho$. Then

$$\begin{aligned} \|\rho - \sigma\| &= \sup\{\left|\rho(a) - \sigma(a)\right| \mid a \in A, \|a\| = 1\} \\ &= \sup\{\left|\langle x_\rho, \pi_\rho(a)x_\rho\rangle - \langle x_\sigma, \pi_\rho(a)x_\sigma\rangle\right| \mid a \in A, \|a\| = 1\} \\ &= \sup\{\mathrm{Tr}((e_{x_\rho} - e_{x_\sigma})\pi_\rho(a)) \mid a \in A, \|a\| = 1\} \\ &= \sup\{\mathrm{Tr}((e_{x_\rho} - e_{x_\sigma})a) \mid a \in \pi_\sigma(A), \|a\| = 1\} \\ &= \sup\{\mathrm{Tr}((e_{x_\rho} - e_{x_\sigma})a) \mid a \in B(H_\rho), \|a\| = 1\} \\ &= \left\|e_{x_\rho} - e_{x_\sigma}\right\|_1, \end{aligned} \tag{3.77}$$

where $\|\cdot\|_1$ is the trace norm on $B_1(H_\rho)$. In the fifth step we used the fact that the map $a \mapsto \mathrm{Tr}(ba)$ is σ-weakly continuous for any $b \in B_1(H_\rho)$. Indeed, that is how σ-weak continuity was defined in Definition A.46. Hence instead of taking the supremum over all $a \in \pi_\rho(A)$, we can take the supremum over its σ-weak closure. Because σ is a pure state, its GNS-representation is irreducible and hence it follows from Theorem 3.35 that $\pi_\sigma(A)$ is σ-weakly dense in $B(H_\sigma)$. So we can take the supremum over all $a \in B(H_\sigma)$. The last step then follows from Theorem A.54.

Because ρ and σ are distinct, the unit vectors x_ρ and x_σ are not proportional, so we may work in the 2-dimensional Hilbert space spanned by $x_\rho \equiv (1,0)$ and $x_\sigma \equiv (c_1, c_2)$, with $|c_1|^2 + |c_2|^2 = 1$, just as we did in the proof of Proposition 2.14. In that case,

$$(e_{x_\rho} - e_{x_\sigma})^2 = |c_2|^2 \cdot 1_2; \tag{3.78}$$

$$\left|e_{x_\rho} - e_{x_\sigma}\right| = \sqrt{(e_{x_\rho} - e_{x_\sigma})^2} = |c_2| \cdot 1_2; \tag{3.79}$$

$$\left\|e_{x_\rho} - e_{x_\sigma}\right\|_1 = \mathrm{Tr}(\left|e_{x_\rho} - e_{x_\sigma}\right|) = 2|c_2|\,. \tag{3.80}$$

Using equation (3.77), this gives

$$1 - \frac{1}{4}\|\rho - \sigma\|^2 = 1 - \frac{1}{4}\left\|e_{x_\rho} - e_{x_\sigma}\right\|_1^2 = 1 - |c_2|^2 = |c_1|^2 = \left|\langle x_\rho, x_\sigma\rangle\right|^2 = \tau(\rho, \sigma). \tag{3.81}$$

Now suppose that ρ and σ are not unitarily equivalent. Then $\tau(\rho, \sigma) = 0$, so we want to show that $\|\rho - \sigma\| = 2$. It follows from the triangle inequality that

$$\|\rho - \sigma\| \leq \|\rho\| - \|\sigma\| = 2, \tag{3.82}$$

so we only have to show that $\|\rho - \sigma\| \geq 2$. Because the central carriers of ρ and σ are perpendicular (Corollary 3.78), it follows that their carrier projections are also perpendicular. Indeed:

$$p_\rho p_\sigma = c(p_\rho) p_\rho c(p_\sigma) p_\sigma = c(p_\rho) c(p_\sigma) p_\rho p_\sigma = 0. \tag{3.83}$$

By a similar argument as before, we have

$$0 \leq \rho(p_\sigma) \leq \rho(1_{A^{**}} - p_\rho) = 0, \tag{3.84}$$

which shows that $\rho(p_\sigma) = 0$. Similarly, we have $\sigma(p_\rho) = 0$. Hence

$$(\rho - \sigma)(p_\rho - p_\sigma) = \rho(p_\rho) + \sigma(p_\sigma) = 2. \tag{3.85}$$

Because p_ρ and p_σ are perpendicular projections, $\left\|p_\rho - p_\sigma\right\| \leq 1$. Hence

$$\|\rho - \sigma\| \leq (\rho - \sigma)(p_\rho - p_\sigma) = 2. \tag{3.86}$$

□

3.7 Uniform structure

As we will see in Sections 4.7, to ensure that the equivalence between Wigner symmetries, Kadison symmetries and Jordan symmetries still holds, we have to add the additional requirements that a Wigner symmetry be ***uniformly continuous***. For more information on uniform structures, see Bourbaki (1989), Chapter II. Recall the definition of uniform continuity of functions between metric spaces:

Definition 3.81. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is called* ***uniformly continuous*** *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x, y \in X$ with $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.*

General topological spaces do not have a metric, so we cannot at once generalise the notion of uniform continuity to topological spaces. We need some structure to replace the metric, which we call the **uniform structure**. To define it, we first need to define a **filter**.

Definition 3.82. *Let X be a set. A* ***filter*** *$\mathcal{U}$ on X is a collection $\mathcal{U} \subseteq \mathscr{P}(X)$ of subsets of X such that*

(a) $U \in \mathcal{U}$ and $U \subseteq V$ imply $V \in \mathcal{U}$;

(b) $U \in \mathcal{U}$ and $V \in \mathcal{U}$ imply $U \cap V \in \mathcal{U}$, and;

(c) $\emptyset \notin \mathcal{U}$.

This brings us the the definition of uniform structure and the new definition of uniform continuity.

Definition 3.83. *A* ***uniform structure*** *or* ***uniformity*** *on a set X is a structure given by a set $\mathcal{U}$ of subsets of $X \times X$ that satisfies conditions (1) and (2) of Definition 3.82, and that also satisfies the following conditions:*

(a) Each $U \in \mathcal{U}$ contains the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$;

(b) If $U \in \mathcal{U}$, then $U^T \in \mathcal{U}$, where $U^T = \{(y, x) \mid (x, y) \in U\}$;

(c) If $U \in \mathcal{U}$, then there is some $V \in U$ such that $V^2 \subseteq U$, where

$$V^2 = \{(x, z) \mid \exists y \in X\colon (x, y) \in V \ (y, z) \in V\}. \tag{3.87}$$

The elements of $\mathcal{U}$ are called ***entourages of the uniformity defined on*** **X**. *A set endowed with a uniform structure is called a* ***uniform space***.

Definition 3.84. *A **fundamental system of entourages of a uniformity** $\mathcal{U}$ is any set $\mathcal{B}$ of entourages such that for every $U \in \mathcal{U}$ there exists a $V \in \mathcal{B}$ so that $V \subseteq U$.*

Definition 3.85. *If X and Y are uniform spaces with uniform structures $\mathcal{U}_X$, respectively $\mathcal{U}_Y$, a function $f\colon X \to Y$ is **uniformly continuous** if $f^{-1}(V) \in \mathcal{U}_X$ whenever $V \in \mathcal{U}_Y$. By $f^{-1}(V)$ we mean the set*

$$\{(x, y) \in X \times X \mid (f(x), f(y)) \in V\}. \tag{3.88}$$

Proposition 3.86. *Let X, Y and Z be uniform spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are two uniformly continuous maps, then $g \circ f$ is uniformly continuous*

Proof. This follows immediately from the definition of uniform continuity. □

It is not immediately clear how uniform structures generalise the notion of uniform continuity. We will show that every metric induces a uniformity and that a function between metric spaces is uniformly continuous in the sense of Definition 3.81 if and only if it is uniformly continuous in the sense of Definition 3.83.

Definition 3.87. *Let (X, d) be a metric space. Then the **$\mathcal{U}_{\mathbf{d}}$-uniformity** is the smallest uniformity on $X \times X$ containing all*

$$U_\epsilon \equiv \{(x, y) \in X \times X \mid d(x, y) < \epsilon\}, \tag{3.89}$$

where $\epsilon > 0$.

By definition of a uniform structure, this implies that $U \in \mathcal{U}_X$ if and only if U contains U_ϵ for certain $\epsilon > 0$, i.e. $\{U_\epsilon \mid \epsilon > 0\}$ is a fundamental system of entourages for $\mathcal{U}_d$.

Lemma 3.88. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$ be a function. Then f is uniformly continuous in the sense of Definition 3.81 if and only if it is uniformly continuous in the sense of Definition 3.83.*

Proof. First assume that f is uniformly continuous in the sense of Definition 3.81 and let $V \in \mathcal{U}_{d_Y}$. Then there is an $\epsilon > 0$ such that $U_\epsilon^Y \in V$. Let $\delta > 0$ be such that $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \epsilon$. Then:

$$\begin{aligned} f^{-1}(V) &\supseteq f^{-1}(U_\epsilon^Y) \\ &= f^{-1}(\{(x, y) \in Y \times Y \mid d_Y(x, y) < \epsilon\} \\ &\supseteq \{(x, y) \in X \times X \mid d_X(x, y) < \epsilon\} = U_\delta^X, \end{aligned} \tag{3.90}$$

which implies that $f^{-1}(V) \in \mathcal{U}_{d_X}$, and hence that f is uniformly continuous in the sense of Definition 3.83. Conversely, assume that f is uniformly continuous in the sense of Definition 3.83 and let $\epsilon > 0$. Then $f^{-1}(U_\epsilon^Y) \in \mathcal{U}_{d_X}$, so there exists a $\delta > 0$ such that $U_\delta^X \subseteq f^{-1}(U_\epsilon^Y)$. Now let $x, y \in X$ with $d_X(x,y) < \delta$. Then $(x,y) \in U_\delta^X \subseteq f^{-1}(U_\epsilon^Y)$ and hence $(f(x), f(y)) \in U_\epsilon^Y$, i.e., $d_Y(f(x), f(y)) < \epsilon$. This shows that f is uniformly continuous in the sense of Definition 3.81. □

The uniformity we will use is the **w*-uniformity** on A^*.

Definition 3.89. *Let A be a Banach space. The **w*-uniformity** $\mathcal{U}_{\mathbf{w}^*}$ on A^* is the uniformity given by the fundamental system of entourages consisting of all subsets of the type*

$$U_\epsilon^a \equiv \{(\sigma, \rho) \in A^* \times A^* \mid \left|\sigma(a) - \rho(a)\right| < \epsilon\}, \tag{3.91}$$

where $a \in A$ and $\epsilon > 0$.

The w^*-uniformity on $P(A) \cup \{0\} \subset A^*$ is just $\{U \cap (P(A) \cup \{0\}) \mid U \in \mathcal{U}_{w^*}\}$.

3.8 Mathematical structures in algebraic quantum theory

We now have all the ingredients to define Wigner, Kadison and Jordan symmetries in algebraic quantum theory. In the following definition the continuity of the maps is defined with respect to the w^*-topology.

Definition 3.90. *Let A and B be C*-algebras. A **Kadison symmetry** is a homeomorphism $K\colon S(A) \cup \{0\} \to S(B) \cup \{0\}$ that maps 0 to 0 and is affine on $S(A)$.*

Because for *unital* C*-algebras the state space is a w^*-closed subset of their dual space, the previous definition in that case simplifies to the following.

Definition 3.91. *Let A and B be unital C*-algebras. A **Kadison symmetry** is an affine homeomorphism $K\colon S(A) \to S(B)$.*

Definition 3.92. *Let A and B be C*-algebras. A **Wigner symmetry** is a uniformly continuous bijection $W\colon P(A) \cup \{0\} \to P(B) \cup \{0\}$ with uniformly continuous inverse that maps 0 to 0 and that preserves transition probabilities, i.e.,*

$$\tau(W(\sigma), W(\rho)) = \tau(\sigma, \rho), \tag{3.92}$$

for all $\sigma, \rho \in P(A)$.

Similarly, for *unital* C*-algebras we can drop the requirement that a Wigner symmetry has to map 0 to 0.

Definition 3.93. *Let A and B be unital C*-algebras. A* ***Wigner symmetry*** *is a uniformly continuous bijection $W: P(A) \to P(B)$ with uniformly continuous inverse that preserves transition probabilities, i.e.,*

$$\tau(W(\sigma), W(\rho)) = \tau(\sigma, \rho), \tag{3.93}$$

for all $\sigma, \rho \in P(A)$.

For both Kadison symmetries and Wigner symmetries we had to add continuity requirements. Jordan symmetries, however, remain almost unchanged.

Definition 3.94. *Let A and B be C*-algebras. A* ***Jordan symmetry*** *is an invertible Jordan map*

$$J: A_{sa} \to B_{sa}, \tag{3.94}$$

i.e., an $\mathbb{R}$-linear bijection $J: A_{sa} \to B_{sa}$ that preserves the Jordan product

$$a \circ b = \tfrac{1}{2}(ab + ba). \tag{3.95}$$

As in the previous chapter, every Jordan map J has a unique extension to a $\mathbb{C}$-linear map

$$J_{\mathbb{C}}: A \to B; \tag{3.96}$$

$$J_{\mathbb{C}}(a^*) = J_{\mathbb{C}}(a)^*, \tag{3.97}$$

which preserves the Jordan product, as well as

$$J_{\mathbb{C}}(a + ib) = J(a) + iJ(b), \tag{3.98}$$

for all $a, b \in A_{\mathrm{sa}}$. We call such map a ***complex Jordan map***. Conversely, such a complex Jordan map defines a real Jordan map by restricting to A_{sa}.

Now that we have generalised the notion of Wigner, Kadison and Jordan symmetries, we can further develop algebraic quantum theory. Although there does not exist a generalised version of Wigner's Theorem, the equivalences between Wigner, Kadison and Jordan symmetries still hold. The next chapter is dedicated to proving those equivalences.

Chapter 4

Symmetries in algebraic quantum theory

The goal of this chapter is to prove that Wigner, Kadison and Jordan symmetries are equivalent. We will do this by first showing that $K \leftrightarrow J$ and then that $W \leftrightarrow J$. The first equivalence follows from the unital case that is proven in Alfsen and Shultz (2001). The equivalence between Wigner symmetries and Jordan symmetries follows from a result by Shultz (1982), which we will explain in great detail. To be able to do this, we need to define an *orientation* of the state space (Section 4.2) and we need to look at the "atomic part" of the enveloping von Neumann algebra (Section 4.4). Then we have the tools to prove the result by Shultz (1982), which we do in Section 4.5 and Section 4.6. Finally, we will tweak the result from Section 4.5 slightly to prove the equivalence between Wigner symmetries and Jordan symmetries.

4.1 Equivalence between Kadison symmetries and Jordan symmetries

The correspondence between Kadison symmetries and Jordan symmetries for *unital* C*-algebras is proven in Alfsen and Shultz (2001) through the equivalence with unital order isomorphisms, just as we did with Lemma 2.16.

Proposition 4.1. *Let A and B be unital C*-algebras. Then $\Phi \mapsto \Phi^*$ is a bijective correspondence between Jordan symmetries Φ from A onto B, and affine homeomorphisms from $S(B)$ onto $S(A)$.*

Proof. See Alfsen and Shultz (2001), Corollary 4.20. □

The goal for the rest of this section is to generalise this result to *all* C*-algebras.

Let $\widetilde{A}$ be the unitisation of A, as in Definition A.57. Then A is an ideal in $\widetilde{A}$ and $\widetilde{A}/A$ is one-dimensional. Let ω_0 be the state given by:

$$\omega((a,\lambda)) = \lambda. \tag{4.1}$$

Then ω_0 is the unique state on $\widetilde{A}$ that annihilates A, in the sense that $\omega((a,0)) = 0$ for every $a \in A$.

Lemma 4.2. *Let A be a C*-algebra with unitisation $\widetilde{A}$ and let ω_0 be the unique state on $S(\widetilde{A})$ that annihilates A. Then $\{\omega_0\}$ is a split face of the state space of $\widetilde{A}$ and the complementary split face with the relative w^*-topology can be identified with the state space of A.*

Proof. First note that ω_0 is a pure state. For suppose that there exist $\sigma, \rho \in S(\widetilde{A})$ and $t > 0$ such that $\omega_0 = t\omega_1 + (1-t)\omega_2$. Then for all $(a,\mu) \in \widetilde{A}$:

$$\begin{aligned}\mu = \omega_0((a,\mu)) &= \omega_0((a,0)) + \omega((0,\mu))\\ &= [t\omega_1((a,0)) + (1-t)\omega_2((a,0))] + [t\omega_1((0,\mu)) + (1-t)\omega_2((0,\mu))]\\ &= [t\omega_1((a,0)) + (1-t)\omega_2((a,0))] + \mu[t\omega_1((0,1)) + (1-t)\omega_2((0,1))]\\ &= [t\omega_1((a,0)) + (1-t)\omega_2((a,0))] + \mu,\end{aligned} \tag{4.2}$$

so we must have:

$$t\omega_1((a,0)) + (1-t)\omega_2((a,0)) = 0, \tag{4.3}$$

for every $a \in A$. Because ω_1 and ω_2 are positive, it follows that

$$\omega_1((a,0)) = \omega_2((a,0)) = 0 \tag{4.4}$$

for every $a \in A$. But then ω_1 and ω_2 are states on $\widetilde{A}$ that annihilate A and ω_0 was the unique state to do that. So we conclude that $\omega_1 = \omega_2 = \omega_0$, i.e., ω_0 is a pure state. Note further that ω_0 is ***abelian***, i.e., $\pi_{\omega_0}(A)$ is an abelian subset of $B(H_{\omega_0})$. Then by Alfsen and Shultz (2001), Proposition 5.63(ii), the split face generated by ω_0 only consists of ω_0, i.e.,

$$F_{\omega_0} = \{\omega_0\}. \tag{4.5}$$

Let F be the complementary split face of $\{\omega_0\}$. Recall from Definition 1.7 that this means that $S(\widetilde{A}) = \mathrm{co}(\{\omega_0\} \cup F)$ and that every $\omega \in S(A)$ can be uniquely expressed as a convex combination

$$t\sigma + (1-t)\omega_0, \tag{4.6}$$

where $0 \leq t \leq 1$ and $\sigma \in F$. Let $\omega \in S(A)$. Identify ω with $\widetilde{\omega} \in S(\widetilde{A})$, given by

$$\widetilde{\omega}((a, \lambda)) = \omega(a) + \lambda. \tag{4.7}$$

We want to show that $S(A) = F$ under this identification. By Definition 1.7, F is given by

$$F = \{\omega \in S(\widetilde{A}) \mid \mathrm{face}_{S(\widetilde{A})}(\omega) \cap \{\omega_0\} = \emptyset\}. \tag{4.8}$$

Let $\omega \in S(\widetilde{A})$. Then

$$\omega((a, \lambda)) = \omega((a, 0)) + \omega((0, \lambda)) = \omega((a, 0)) + \lambda\omega((0, 1)) = \omega((a, 0)) + \lambda. \tag{4.9}$$

Define $\alpha \in [0, 1]$ by

$$\alpha = \sup\{\omega((a, 0)) \mid \|a\| \leq 1\}. \tag{4.10}$$

If $\alpha = 0$ we have $\omega((a, \lambda)) = \lambda$, i.e., $\omega = \omega_0$. If $0 < \alpha < 1$, then the map σ given by

$$\sigma(a) = \frac{1}{\alpha}\omega((a, 0)) \tag{4.11}$$

is a state on A and

$$\omega = \alpha\sigma + (1 - \alpha)\omega_0, \tag{4.12}$$

which implies that $\omega_0 \in \mathrm{face}_{S(\widetilde{A})}(\omega)$. Thus for all $\omega \in F$ we have $\alpha = 1$, in which case ω is a state on A, i.e., $F \subseteq S(A)$. It remains to be shown that $S(A) \subseteq F$. Suppose that this is not the case. Then there exists $\omega \in S(A)$, $\sigma \in S(\widetilde{A})$ and $t \in (0, 1)$ such that

$$\omega = t\sigma + (1 - t)\omega_0. \tag{4.13}$$

But then

$$\begin{aligned} 1 &= \sup\{\omega((a, 0)) \mid \|a\| \leq 1\} = \sup\{t\sigma((a, 0)) \mid \|a\| \leq 1\} \\ &= t \sup\{\sigma((a, 0)) \mid \|a\| \leq 1\} \leq t < 1, \end{aligned} \tag{4.14}$$

which leads to a contradiction. Hence $S(A) \subseteq F$, from which it follows that $F = S(A)$.

□

So we can identify $S(A)$ with a face of $S(\widetilde{A})$. Let σ and ρ be pure states of A. We are interested in the face that σ and ρ generate in $S(A)$, so we want to make sure that

$$\mathrm{face}_{S(\widetilde{A})}(\sigma, \rho) = \mathrm{face}_{S(A)}(\sigma, \rho). \tag{4.15}$$

The following lemma proves that this in indeed the case.

Lemma 4.3. *Let K be a convex set and F a face of K. If $E \subseteq F$, then*

$$\mathrm{face}_K(E) = \mathrm{face}_F(E). \tag{4.16}$$

Proof. The face F contains the set E, so $\mathrm{face}_K(E) \subseteq F$. Note that $G \subseteq F$ is a face of F if and only if G is a face of K. Indeed, suppose G is a face of F and that $x, y \in K$ and $t \in (0,1)$, so that $tx + (1-t)y \in G$. Because F is a face of K and $G \subseteq F$ it follows that $x, y \in F$. Because G is a face of F, we can conclude that $x, y \in G$, which shows that G is a face of K. Conversely, suppose that G is a face of K and let $x, y \in F$ and $t \in (0,1)$, so that $tx + (1-t)y \in G$. Because G is a face of K it follows that $x, y \in G$, which implies that G is a face of F. Hence

$$\begin{aligned}
\mathrm{face}_K(E) &= \bigcap \{G \mid G \text{ is a face of } K \text{ that contains } E\} \\
&= \bigcap \{G \cap F \mid G \text{ is a face of } K \text{ that contains } E\} \\
&= \bigcap \{G \mid G \text{ is a face of } F \text{ that contains } E\} = \mathrm{face}_F(E).
\end{aligned} \tag{4.17}$$

□

Proposition 4.4. *Let A and B be C^*-algebras. Then $\Phi \mapsto \Phi^*$ is a bijective correspondence of Jordan symmetries Φ from A onto B, and Kadison symmetries from $S(B) \cup \{0\}$ onto $S(A) \cup \{0\}$.*

Proof. Let $\widetilde{A}$ be the unitisation of A and $\widetilde{B}$ be the unitisation of B. Define $\widetilde{\Phi} \colon \widetilde{A} \to \widetilde{B}$ by

$$\widetilde{\Phi}\big((a, \lambda)\big) = (\Phi(a), \lambda). \tag{4.18}$$

Then $\widetilde{\Phi}$ is a Jordan symmetry, because it is clearly a $\mathbb{C}$-linear bijective map and it preserves the Jordan product:

$$\begin{aligned}
\widetilde{\Phi}\big((a, \lambda) \circ (b, \mu)\big) &= \widetilde{\Phi}\big(\tfrac{1}{2}(ab + \mu a + \lambda b, \lambda\mu) + \tfrac{1}{2}(ba + \mu a + \lambda b, \lambda\mu)\big) \\
&= \widetilde{\Phi}\big((a \circ b + \mu a + \lambda b, \lambda\mu)\big) \\
&= (\Phi(a) \circ \Phi(b) + \mu\Phi(a) + \lambda\Phi(b), \lambda\mu) \\
&= \tfrac{1}{2}(\Phi(a)\Phi(b) + \mu\Phi(a) + \lambda\Phi(b), \lambda\mu) + \tfrac{1}{2}(\Phi(b)\Phi(a) + \mu\Phi(a) + \lambda\Phi(b), \lambda\mu) \\
&= \tfrac{1}{2}(\Phi(a), \lambda)(\Phi(b), \mu) + \tfrac{1}{2}(\Phi(b), \mu)(\Phi(a), \lambda) \\
&= \widetilde{\Phi}\big((a, \lambda)\big) \circ \widetilde{\Phi}\big((b, \mu)\big).
\end{aligned} \tag{4.19}$$

Conversely, every Jordan symmetry $\widetilde{\Phi} \colon \widetilde{A} \to \widetilde{B}$ induces a unique Jordan symmetry $\Phi \colon A \to B$ by restricting to the first coordinate. So there is a bijective correspondence of Jordan symmetries $\Phi\colon A \to B$ and Jordan symmetries $\widetilde{\Phi}\colon \widetilde{A} \to \widetilde{B}$. Therefore, by Proposition 4.1, $\Phi \mapsto \widetilde{\Phi}^*$ is a bijective correspondence of Jordan symmetries $\Phi\colon A \to B$, and affine homeomorphisms from $S(\widetilde{B})$ onto $S(\widetilde{A})$. Lemma 4.2 gives us

$$S(\widetilde{A}) = S(A) \oplus_c \{\omega_0^A\}; \tag{4.20}$$

$$S(\widetilde{B}) = S(B) \oplus_c \{\omega_0^B\}. \tag{4.21}$$

Now let φ^A be the map from $S(\widetilde{A}) = \mathrm{co}(S(A) \cup \{\omega_0^A\})$ onto $\mathrm{co}(S(A) \cup \{0\})$ that takes ω_0^A onto $\{0\}$, and similarly for φ^B. Then φ^A and φ^B are affine homeomorphisms. Hence $\Phi \mapsto \varphi^A \circ \Phi^* \circ (\varphi^B)^{-1}$ is a bijective correspondence of Jordan symmetries between A and B and affine homeomorphisms from the convex hull $\mathrm{co}(S(B) \cup \{0\})$ onto the convex hull $\mathrm{co}(S(A) \cup \{0\})$ taking 0 to 0. Finally, because this map is affine and takes 0 to 0, we get a bijective correspondence of Jordan symmetries between A and B and Kadison symmetries from $S(B) \cup \{0\}$ onto $S(A) \cup \{0\}$. This proves the proposition. □

4.2 Orientation of state spaces

Let A be a C*-algebra with state space $S(A)$ and denote the set of all faces of $S(A)$ by $F(A)$. Let σ and ρ be pure states of A. Recall that the face of $S(A)$ generated by σ and ρ is the intersection of all faces of $S(A)$ containing σ and ρ:

$$\mathrm{face}_{S(A)}(\sigma, \rho) = \bigcap \{F \mid F \in F(A),\ \sigma, \rho \in F\}. \tag{4.22}$$

In this section we describe the faces of $S(A)$ generated by pure states and give them something called an **orientation**. The following theorem gives an description of the faces generated by pure states on a *unital* C*-algebra. Later on, we generalise this to *all* C*-algebras. Recall from Definition 1.2 that a 3-ball is a convex set that is affinely isomorphic to the unit ball $B^3 \subset \mathbb{R}^3$.

Theorem 4.5. *Let σ and ρ be distinct pure states on a unital C*-algebra A. If the GNS-representation π_σ and π_ρ are unitarily equivalent, then the face generated by σ and ρ is a 3-ball. If these representations are not unitarily equivalent, then the face they generate is the line segment*

$$[\sigma, \rho] = \{\lambda\sigma + (1-\lambda)\rho \mid \lambda \in [0,1]\} \tag{4.23}$$

in $S(A)$.

Proof. We follow the proof given by Alfsen and Shultz (2001), Theorem 5.36. First suppose that σ and ρ are unitarily equivalent. Then by Theorem 3.49, $c(\sigma) = c(\rho)$, hence by Lemma 3.41 they generate the same split face of $S_n(A^{**})$, i.e., $F_\sigma = F_\rho$. According to Proposition 3.47, the map

$$\pi_\sigma^*\colon S_n(B(H_\sigma)) \to F_\sigma \tag{4.24}$$

is an affine isomorphism. Because σ and ρ are extreme points of F_σ and π_σ^* is an affine isomorphism, there are extreme points $\sigma', \rho' \in S_n(B(H_\sigma))$ such that $\sigma = \pi_\sigma^*(\sigma')$ and $\rho = \pi_\sigma^*(\rho')$. It is now enough to show that the face generated by σ' and ρ' in $S_n(B(H_\sigma))$ is affinely isomorphic to a 3-ball. The extreme points of $S_n(B(H_\sigma))$ are the vector states, so there are linearly independent unit vectors $x, y \in H_\sigma$ such that $\sigma' = \omega_x$ and $\rho' = \omega_y$. Let p be the projection onto the closed subspace of H_σ spanned by x and y. Then by Corollary 3.28 the face generated by σ' and ρ' is equal to F_p, which, by Theorem 3.26 is a 3-ball.

Now suppose that σ and ρ are not unitarily equivalent. Then the split faces they generate are not equal, so we may assume without loss of generality that $\tau \notin F_\sigma$. It then follows from Alfsen and Shultz (2001), Proposition 1.30 that the face generated by σ and τ is $[\sigma, \tau]$. □

Let us look at two examples to illustrate Theorem 4.5. We saw in Chapter 1 that the state space of the 2×2 complex matrices is isomorphic as a compact convex set to the closed unit ball (Proposition 1.22). The pure state space $P(M_2(\mathbb{C}))$ is then isomorphic to the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}, \tag{4.25}$$

which shows that all pure states are unitarily equivalent. Note further that the only faces of B^3 is are single pure states or B^3 itself. This shows that the face generated by two distinct states σ and ρ is affinely isomorphic to B^3, i.e., it is a facial 3-ball.

Now take as a C*-algebra $A = \mathbb{C} \oplus \mathbb{C}$ with pointwise addition and multiplication, and norm given by

$$\|(a, b)\| = \max\{|a|, |b|\}. \tag{4.26}$$

The unit in A is the element $(1, 1)$, the adjoint of an element $(a, b) \in \mathbb{C} \oplus \mathbb{C}$ is given by $(\overline{a}, \overline{b})$, and the positive elements of A are $\{(a, b) \in (\mathbb{R}^+)^2\}$. Let $t \in [0, 1]$. Then the map $\omega_t\colon A \to \mathbb{C}$ given by

$$\omega_t(a, b) = ta + (1 - t)b \tag{4.27}$$

is a state on A, and in fact all states arise this way. So $S(A) = \{\omega_t \mid t \in [0, 1]\} \cong [0, 1]$ and $P(A) = \{\omega_0, \omega_1\} \cong \{0, 1\}$. The states ω_0 and ω_1 are not unitarily equivalent and the face generated by ω_0 and ω_1 is the entire state space (and line segment) $S(A) = [\omega_0, \omega_1]$, as predicted by Theorem 4.5.

Definition 4.6. *Let K be a convex set. A* ***facial 3-ball*** *is a 3-ball that is a face of K.*

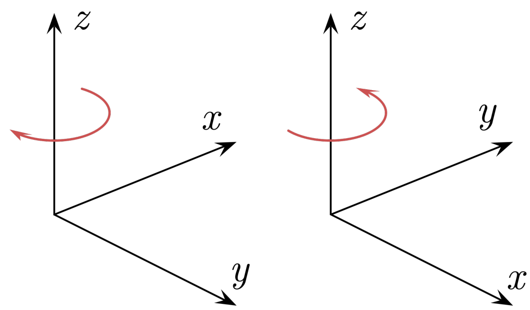


Figure 4.1: The left-handed orientation is shown on the left and the right-handed orientation is shown on the right.

In the next section we define orientations of facial 3-balls, and moreover we want our choice or orientation for each facial 3-ball to be continuous. First, recall that there are two orientations of $\mathbb{R}^3$, namely the left-handed and the right-handed orientation, as shown in figure 4.1. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be an ordered orthonormal basis of $\mathbb{R}^3$. The basis is called ***right-handed*** if, when you move the palm of your right hand from the basis vector $\mathbf{x}$ to the basis vector $\mathbf{y}$, your thumb points towards the direction of $\mathbf{z}$, and analogously for the left-handed orientation. Recall that the group $O(3)$ of real orthogonal 3×3 matrices M is defined by either one of these four equivalent conditions:

(i) $MM^T = M^TM = 1_3$;

(ii) M is invertible and $M^{-1} = M^T$;

(iii) M is an isometry, i.e., $\|M\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^3$;

(iv) M preserves the inner product, i.e., $\langle M\mathbf{x}, M\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

It follows from the definition of $O(3)$ that $\det(M) = \pm 1$ for every $M \in O(3)$. Thus $O(3)$ consists of two parts, namely

$$O_- = \{M \in O(3) \mid \det(M) = -1\}, \tag{4.28}$$
$$O_+ = \{M \in O(3) \mid \det(M) = 1\} \equiv SO(3). \tag{4.29}$$

Orthogonal maps ***preserve orientation*** if whenever $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a right-handed (respectively left-handed) basis of $\mathbb{R}^3$, the basis $\{M\mathbf{x}, M\mathbf{y}, M\mathbf{z}\}$ is also right-handed (respectively left-handed). Of course, that means that a map ***reverses orientation*** if whenever $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a right-handed (respectively left-handed) basis of $\mathbb{R}^3$, the basis $\{M\mathbf{x}, M\mathbf{y}, M\mathbf{z}\}$ is left-handed (respectively right-handed). The orientation-preserving orthogonal maps are precisely the elements of $SO(3)$, and the orientation-reversing orthogonal maps are precisely the elements of O_-.

Lemma 4.7. *Any affine bijection $\varphi\colon B^3 \to B^3$ in $\mathbb{R}^3$ is given by an orthogonal map $R \in O(3)$.*

Proof. We give a sketch of the proof. For a detailed proof, we refer the reader to Landsman (2017) Lemma 5.11. First, because φ is affine it maps the boundary $\partial_e B^3 = S^2$ bijectively onto itself. Second, because $\mathbf{0}$ is intrinsic to the convex structure (it is the unique point with the property that for any $\mathbf{x} \in S^2$ there exists a unique $\mathbf{x}'$ such that $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}' = \mathbf{0}$, namely $\mathbf{x}' = -\mathbf{x}$), φ maps $\mathbf{0}$ onto itself. Third, for $\mathbf{x} \in B^3$ and $t \in [0, 1]$ the second point implies that

$$\varphi(t\mathbf{x}) = \varphi(t\mathbf{x} + (1-t)\mathbf{0}) = t\varphi(\mathbf{x}) + (1-t)\varphi(\mathbf{0}) = t\varphi(\mathbf{x}). \tag{4.30}$$

The same then holds for $\mathbf{x} \in B^3$ and all $t \geq 0$ as long as $t\mathbf{x} \in B^3$. This shows that for all $\mathbf{x}, \mathbf{y} \in B^3$ for which $\mathbf{x} + \mathbf{y} \in B^3$, we have

$$\varphi(\mathbf{x} + \mathbf{y}) = 2\varphi(\tfrac{1}{2}\mathbf{x} + \tfrac{1}{2}\mathbf{y}) = 2 \cdot (\tfrac{1}{2}\varphi(\mathbf{x}) + \tfrac{1}{2}\varphi(\mathbf{y})) = \varphi(\mathbf{x}) + \varphi(\mathbf{y}). \tag{4.31}$$

In particular, because $\varphi(\mathbf{0}) = 0$, equation (4.31) implies that

$$\varphi(-\mathbf{x}) = -\varphi(\mathbf{x}). \tag{4.32}$$

Now, for some nonzero $\mathbf{x} \in \mathbb{R}^3$, take $s \geq \|\mathbf{x}\|$ and $t \geq \|\mathbf{x}\|$. Then equation (4.30) implies that

$$s\varphi(\frac{\mathbf{x}}{s}) = s\varphi(\frac{t}{s}\frac{\mathbf{x}}{t}) = t\varphi(\frac{\mathbf{x}}{t}). \tag{4.33}$$

We may therefore define a map $R\colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$R(\mathbf{0}) = \mathbf{0}; \tag{4.34}$$

$$R(\mathbf{x}) = s \cdot \varphi(\frac{\mathbf{x}}{s}) \;\; (\mathbf{x} \neq \mathbf{0}), \tag{4.35}$$

for any choice $s \geq \|x\|$. For $\mathbf{x} \in B^3$ we may take $s = 1$, so that R extends φ. Linearity of R follows from equation (4.31), (4.32) and (4.33). Finally, R is an isometry by (4.35) and step one of the proof. Being also linear and invertible, R must therefore be an orthogonal transformation. □

Using this lemma, we can define the concept of orientation on a facial 3-ball.

Definition 4.8. *Let K be a convex set and F a facial 3-ball. A **parametrisation** of F is an affine bijection $\varphi\colon B^3 \to F$. Let Φ be the set of all affine bijections. We define an equivalence relation $\sim$ on Φ by*

$$\varphi_1 \sim \varphi_2 \quad \Leftrightarrow \quad \det(\varphi_2^{-1} \circ \varphi_1) = 1, \tag{4.36}$$

*where we view $\varphi_2^{-1} \circ \varphi_1$ as an element of $O(3)$. Because $\det(\varphi_2^{-1} \circ \varphi_1) = \pm 1$ there are exactly two equivalence classes, which we refer to as being **opposite** to each other. Each equivalence class of affine bijections is called an **orientation** of F. If $\varphi\colon B^3 \to F$ is a parametrisation, we refer to $[\varphi]$ as the orientation of φ. If F and G are 3-balls equiped with orientations $[\varphi_1]$ and $[\varphi_2]$ respectively, and $\psi\colon F \to G$ is an affine bijection, we say ψ **preserves orientation** if $[\psi \circ \varphi_1] = [\varphi_2]$, and **reverses** orientation if the orientation $[\psi \circ \varphi_1]$ is the opposite of $[\varphi_2]$.*

Definition 4.9. *Let $S(A)$ be the state space of a C^*-algebra A. The notation $\mathrm{Param}(S(A))$ denotes the set of all parametrisations of facial 3-balls of $S(A)$. We equip $\mathrm{Param}(S(A))$ with the topology of pointwise convergence of maps from B^3 into the space $S(A)$ with the w^*-topology.*

Let us unpack this definition. First write

$$\begin{aligned} \mathrm{Param}(S(A)) = \{\varphi\colon B^3 \to S(A) \mid \varphi(B^3) &\text{ is a facial 3-ball of } S(A) \text{ and} \\ \varphi\colon B^3 \to \varphi(B^3) &\text{ is an affine isomorphism}\}. \end{aligned} \tag{4.37}$$

Now let $\{\varphi_n\}$ be a sequence in $\mathrm{Param}(S(A))$, and let $\varphi \in \mathrm{Param}(S(A))$. We say that $\{\varphi_n\}$ converges to φ (and write $\varphi_n \to \varphi$) if $\varphi_n(x) \to \varphi(x)$ for every $x \in B^3$ in the w^*-topology, i.e., $\varphi_n \to \varphi$ if and only if

$$\varphi_n(x)(a) \to \varphi(x)(a), \tag{4.38}$$

for all $x \in B^3$ and $a \in A$.

Definition 4.10. *We let $B_{S(A)}$ denote the set of facial 3-balls equipped with the quotient topology from the map of $\mathrm{Param}(S(A))$ onto $B_{S(A)}$ given by $\varphi \mapsto \varphi(B^3)$.*

The set $B_{S(A)}$ consists of all facial 3-balls, so by definition of a facial 3-ball and the set $\mathrm{Param}(S(A))$ we have

$$B_{S(A)} = \{\varphi(B^3) \mid \varphi \in \mathrm{Param}(S(A))\}. \tag{4.39}$$

Let F be a facial 3-ball of $S(A)$, so $F \in B_{S(A)}$, and let $\varphi\colon B^3 \to B_{S(A)}$ be an affine isomorphism. Then for every affine isomorphism $\psi\colon B^3 \to F$ there is a unique $M \in O(3)$ such that $\psi = \varphi \circ M$, namely $M = \varphi^{-1} \circ \psi$. Hence we can describe every facial 3-ball as an element of $\mathrm{Param}(S(A))/O(3)$, and $B_{S(A)}$ is homeomorphic to $\mathrm{Param}(S(A))/O(3)$.

Definition 4.11. *We call $\mathcal{O}B_{S(A)} = \mathrm{Param}(S(A))/SO(3)$ the **space of oriented facial 3-balls of S(A)**. We equip it with the quotient topology. If $\varphi \in \mathrm{Param}(S(A))$, then we denote its equivalence class by $[\varphi]$, which is an orientation of the 3-ball $\varphi(B^3)$.*

Let us also unpack this definition. First, write

$$\mathcal{O}B_{S(A)} = \{[\varphi] \mid \varphi \in \mathrm{Param}(S(A))\}. \tag{4.40}$$

A sequence $[\varphi_n]$ in $\mathcal{O}B_{S(A)}$ converges to $[\varphi] \in \mathcal{O}B_{S(A)}$ if and only if there are $M_n \in SO(3)$ such that $M_n\varphi_n \to \varphi$ in $\mathrm{Param}(S(A))$. Because $B_{S(A)}$ is the space of all facial 3-balls and $\mathcal{O}B_{S(A)}$ is the space of all *oriented* facial 3-balls, there is a canonical map π from $\mathcal{O}B_{S(A)}$ to $B_{S(A)}$, given by

$$\pi\colon \mathcal{O}B_{S(A)} \to B_{S(A)}; \quad [\varphi] \mapsto \varphi(B^3). \tag{4.41}$$

Proposition 4.12. *Let $S(A)$ be the state space of the C*-algebra A. The spaces $\mathcal{O}B_{S(A)}$ and $B_{S(A)}$ are Hausdorff, the canonical map from $\mathcal{O}B_{S(A)}$ onto $B_{S(A)}$ is continuous and open, and $\mathcal{O}B_{S(A)} \to B_{S(A)}$ is a $\mathbb{Z}_2$ bundle.*

Proof. See Alfsen and Shultz (2001), Proposition 5.40. □

By a $\mathbb{Z}_2$ bundle we mean a surjection such that the preimage of every element contains exactly two elements. This is the case because every facial 3-ball admits precisely two different orientations.

Definition 4.13. *Given a bundle $p\colon X \to Y$, a **cross-section** or **section** of p is a map $s\colon Y \to X$ such that $p(s(y)) = y$ for every $y \in Y$, i.e., $s(y) \in p^{-1}(y)$ for every $y \in Y$.*

Definition 4.14. *Let $S(A)$ be the state space of the C*-algebra A. A continuous cross-section of the bundle $\mathcal{O}B_{S(A)} \to B_{S(A)}$ is called a **global orientation**, or simply an **orientation**, of $S(A)$.*

What does it mean for a cross-section to be continuous? Let $\pi\colon \mathcal{O}B_{S(A)} \to B_{S(A)}$ be the canonical map and $s\colon B_{S(A)} \to \mathcal{O}B_{S(A)}$ be a continuous cross-section. So s maps a facial 3-ball F to one of its two possible orientations. Now let $\{F_n\}$ be a sequence in $B_{S(A)}$ that converges to a facial 3-ball $F \in B_{S(A)}$. Let $s(F_n) = [\varphi_n]$ and $s(F) = [\varphi]$. Note that this implies that $F_n = \varphi_n(B^3)$ and $F = \varphi(B^3)$. Because s is continuous, the sequence $\{[\varphi_n]\}$ converges to $[\varphi]$ in $\mathcal{O}B_{S(A)}$, i.e., there exists $M_n \in SO(3)$ such that $M_n\varphi_n$ converges to φ in the sense of equation (4.38). Because $F_n \to F$, $F_n = \varphi_n(B^3)$ and $F = \varphi(B^3)$ we know that there exist $M_n \in O(3)$ such that $M_n\varphi_n \to \varphi$, but the point is that because s is continuous, we can choose the M_n to be in $SO(3)$.

Now the question remains: does such a continuous cross-section exists? It turns out it does, but to prove this, we first need to define the orientation induced by A. To do this, recall from Proposition 1.22 that the state space of the 2×2 matrices $M_2(\mathbb{C})$ is isomorphic to B^3. For the next lemma and definition, we identify $S(M_2(\mathbb{C}))$ with B^3.

Lemma 4.15. *If F is a facial 3-ball in the state space of a unital C*-algebra A, and p is the carrier projection of F in A^{**}, then $pA^{**}p$ is *-isomorphic to $M_2(\mathbb{C})$. If π is any *-isomorphism from $pA^{**}p$ onto $M_2(\mathbb{C})$, then π^* is an affine isomorphism from the state space B^3 onto F, i.e., π^* is a parametrisation of F.*

Proof. See Alfsen and Shultz (2001), Lemma 5.43. □

Concretely, the map π^* is given by

$$\pi^*\colon B^3 \to F; \quad \mathbf{x} \mapsto \big(\omega\colon a \mapsto \mathrm{Tr}(\rho_{\mathbf{x}}\pi(a))\big), \tag{4.42}$$

where $\rho_{\mathbf{x}}$ is the density matrix given by

$$\rho_{\mathbf{x}} = \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}. \tag{4.43}$$

Definition 4.16. *Let A be a unital C*-algebra with state space $S(A)$, and F a facial 3-ball of $S(A)$ with carrier projection p. Let π be any *-isomorphism from $pA^{**}p$ onto $M_2(\mathbb{C})$. The* ***orientation on F induced by A*** *is the equivalence class of the map π^* from B^3 onto F.*

Theorem 4.17. *The state space $S(A)$ of a unital C*-algebra A is orientable. Specifically, the orientation of each facial 3-ball induced by A gives a global orientation of $S(A)$.*

Proof. See Alfsen and Shultz (2001), Theorem 5.54. □

The attentive reader has noticed that the last lemma and theorem were stated for *unital* C*-algebras. We will now generalise this to *all* C*-algebras.

Theorem 4.18. *Let σ and ρ be distinct pure states on a C*-algebra A. If the GNS-representation π_σ and π_ρ are unitarily equivalent, then the face generated by σ and ρ is a 3-ball. If these representations are not unitarily equivalent, then the face they generate is the line segment $[\sigma, \rho]$.*

Proof. Let $\widetilde{A}$ be the unitisation of A and let ω_0 be the state given in equation (4.1), i.e., let ω_0 be the unique state on $\widetilde{A}$ that annihilates A. Denote the complementary split face of ω_0 by F. By Lemma 4.2 we can identify F with the state space of A. Now let σ and ρ be pure states on A, hence $\sigma, \rho \in F$. By Theorem 4.5 the face they generate is a 3-ball of $S(\widetilde{A})$. By Lemma 4.2 and Lemma 4.3 we have:

$$\text{face}_{S(\widetilde{A})}(\sigma, \rho) = \text{face}_F(\sigma, \rho) = \text{face}_{S(A)}(\sigma, \rho). \tag{4.44}$$

Hence $\text{face}_{S(A)}(\sigma, \rho)$ is a facial 3-ball of $S(A)$. Now let σ and ρ be unitarily inequivalent pure states of $S(A)$. Then by Theorem 4.5 the face they generate in $S(\widetilde{A})$ is equal to the line segment $[\sigma, \rho]$. Again, using Lemma 4.2 and Lemma 4.3 we can conclude that this is equal to the line segment $[\sigma, \rho]$ in $S(A)$. □

The orientability of $\widetilde{A}$ now follows from the orientability of A, as stated in the following theorem.

Theorem 4.19. *The state space $S(A)$ of a C*-algebra A is orientable. Specifically, the orientation of each facial 3-ball induced by A gives a global orientation of $S(A)$.*

Proof. Because the split face generated by ω_0 is just $\{\omega_0\}$, it follows from Theorem 3.49 that ω_0 is not unitarily equivalent to any other pure state on $\widetilde{A}$. By Theorem 4.18 this implies that ω_0 is not an element of any facial 3-ball of $S(\widetilde{A})$. This implies that every facial 3-ball of $S(\widetilde{A})$ lies in the complementary split face of ω_0, which we identified with $S(A)$ in Lemma 4.2. So every facial 3-ball of $S(\widetilde{A})$ lies in $S(A)$. Therefore the bundles $\mathcal{O}B_{S(A)} \to B_{S(A)}$ and $\mathcal{O}B_{S(\widetilde{A})} \to B_{S(\widetilde{A})}$ are identical, so orientations of $S(A)$ coincide with orientations of $S(\widetilde{A})$. Hence by Theorem 4.17 the orientation of each facial 3-ball induced by A gives a global orientation of $S(A)$. □

Every *-isomorphism and *-anti-isomorphism between C*-algebras induces a bijection between the corresponding state spaces. The following proposition states when the induced bijection preserves orientation and when it reverses orientation.

Proposition 4.20. *Let A and B be C*-algebras with state spaces $S(A)$ and $S(B)$. If $\Phi\colon A \to B$ is a *-isomorphism, then Φ^* preserves orientation, and if Φ is a *-anti-isomorphism, then Φ^* reverses orientation.*

Proof. See Alfsen and Shultz (2001), Proposition 5.55. □

Theorem 4.21. *If $\Phi\colon A \to B$ is a Jordan symmetry between C*-algebras, then Φ is a *-isomorphism if and only if Φ^* preserves orientation, and is a *-anti-isomorphism if and only if Φ^* reverses orientation.*

Proof. See Alfsen and Shultz (2001), Theorem 5.71 for the proof of the unital case. Now suppose that A and/or B are non-unital C*-algebras. Then the Jordan product on $\widetilde{A}$ is given by

$$\begin{aligned}(a,\lambda)\circ(b,\mu) &= \tfrac{1}{2}(a,\lambda)(b,\mu)+\tfrac{1}{2}(b,\mu)(a,\lambda)\\ &= \tfrac{1}{2}(ab+\mu a+\lambda b,\lambda\mu)+\tfrac{1}{2}(ba+\mu a+\lambda b,\lambda\mu)\\ &= (a\circ b+\lambda a+\mu b,\lambda\mu),\end{aligned}\tag{4.45}$$

for all $(a,\lambda),(b,\mu)\in\widetilde{A}$, and similarly for $\widetilde{B}$. Let Φ be a Jordan symmetry on A, then $\widetilde{\Phi}$ given by

$$\widetilde{\Phi}((a,\lambda)) = (\Phi(a),\lambda)\tag{4.46}$$

is a Jordan symmetry between $\widetilde{A}$ and $\widetilde{B}$. Indeed, it is clearly $\mathbb{C}$-linear and bijective, and

$$\begin{aligned}\widetilde{\Phi}((a,\lambda)\circ(b,\mu)) &= \widetilde{\Phi}(\widetilde{a}\circ\widetilde{b}+\mu a+\lambda b,\lambda\mu)\\ &= (\Phi(a)\circ\Phi(b)+\mu\Phi(a)+\lambda\Phi(b),\lambda\mu)\\ &= \widetilde{\Phi}((a,\lambda))\circ\widetilde{\Phi}((b,\mu)).\end{aligned}\tag{4.47}$$

Hence $\widetilde{\Phi}$ is a *-isomorphism if and only if $\widetilde{\Phi}$ preserves orientation, and is a *-anti-isomorphism if and only if $\widetilde{\Phi}^*$ reverses orientation. Now note that $\widetilde{\Phi}$ is a *-isomorphism if and only if Φ is a *-isomorphism, and $\widetilde{\Phi}$ is a *-anti-isomorphism if and only if Φ is a *-isomorphism. This concludes the proof. □

In the next corollaries the continuity of the maps φ and φ^{-1} is defined with respect to the w^*-topology.

Corollary 4.22. *Let A and B be unital C*-algebras with state spaces $S(A)$ and $S(B)$. Let $\varphi\colon S(B)\to S(A)$ be an affine homeomorphism, and $\Phi\colon A\to B$ the map such that $\Phi^*=\varphi$ on $S(B)$. Then Φ is a *-isomorphism if and only if φ preserves orientation.*

Proof. First note that by Proposition 4.1 there exists a Jordan symmetry $\Phi\colon A\to B$ such that $\varphi=\Phi^*$. By Theorem 4.21, Φ is a *-isomorphism if and only if $\Phi^*=\varphi$ preserves orientation. □

Using Lemma 4.2 we can extend Corollary 4.22 to include the non-unital case, which proves the equivalence between orientation-preserving Kadison symmetries and *-isomorphisms.

Corollary 4.23. *Let A and B be C*-algebras with state spaces $S(A)$ and $S(B)$. Let $\varphi\colon S(B)\cup\{0\}\to S(A)\cup\{0\}$ be a Kadison symmetry, and $\Phi\colon A\to B$ the map such that $\Phi^*=\varphi$ on $S(B)\cup\{0\}$. Then Φ is a *-isomorphism if and only if φ preserves orientation.*

Proof. Let $\varphi\colon S(B)\cup\{0\}\to S(A)\cup\{0\}$ be a Kadison symmetry, i.e. a homeomorphism that maps 0 to 0 and is affine on $S(B)$. Extend φ to

$$\varphi\colon\ \mathrm{co}(S(B)\cup\{0\})\to\mathrm{co}(S(A)\cup\{0\}),\tag{4.48}$$

by setting

$$\varphi(\lambda\omega+(1-\lambda)0)=\lambda\varphi(\omega),\tag{4.49}$$

which is an affine homeomorphism between $\mathrm{co}(S(B)\cup\{0\})$ and $\mathrm{co}(S(A)\cup\{0\})$. Let $\widetilde{A}$ be the unitisation of A and let $S(\widetilde{A})$ be its state space. Let ω_0^A be the unique state on $\widetilde{A}$ that annihilates A. By Lemma 4.2 we have

$$S(\widetilde{A})=S(A)\oplus_c\{\omega_0^A\},\tag{4.50}$$

so the map

$$\psi^A\colon S(\widetilde{A})=\mathrm{co}(S(A)\cup\{\omega_0^A\})\to\mathrm{co}(S(A)\cup\{0\})\tag{4.51}$$

that takes ω_0^A to 0 is an affine homeomorphism, and similarly for

$$\psi^B\colon S(\widetilde{B})=\mathrm{co}(S(B)\cup\{\omega_0^B\})\to\mathrm{co}(S(B)\cup\{0\}).\tag{4.52}$$

Then

$$(\psi^A)^{-1}\circ\varphi\circ\psi^B\colon S(\widetilde{B})\to S(\widetilde{A})\tag{4.53}$$

is an affine homeomorphism between state spaces of unital C*-algebras. Define $\Psi\colon\widetilde{A}\to\widetilde{B}$ by

$$\Psi((a,\lambda))=(\Phi(a),\lambda).\tag{4.54}$$

Then $\Psi^*=(\psi^A)^{-1}\circ\varphi\circ\psi^B$. By Corollary 4.22 Ψ is a *-isomorphism if and only if $(\psi^A)^{-1}\circ\varphi\circ\psi^B$ preserves orientation. Note that Ψ is a *-isomophism if and only if Φ is a *-isomorphism and $(\psi^A)^{-1}\circ\varphi\circ\psi^B$ preserves orientation if and only if φ preserves orientation. We conclude that Φ is a *-isomorphism if and only if φ preserves orientation. □

In the case that $A = B(H)$ we prefer to work with the normal state space instead of the entire state space, as we did in the first two chapters. The following result is an application of Theorem 4.21 to $B(H)$ using the normal state space instead of the entire state space.

Proposition 4.24. *Let H_1 and H_2 be Hilbert spaces and $S_n(H_1)$, respectively $S_n(H_2)$ their normal state spaces. Let $\varphi\colon B(H_1) \to B(H_2)$ be a Jordan symmetry. If φ is a *-isomorphism (*-anti-isomorphism), then φ^* maps each facial 3-ball of $S_n(H_2)$ orientation-preservingly (orientation-reversingly) onto a facial 3-ball of $S_n(H_1)$.*

Proof. See Alfsen, Hanche-Olsen, and Shultz (1980), Proposition 6.2. □

4.3 Double orthocomplement

In this section, we describe the pure states in the face generated by $\sigma, \rho \in P(A)$, using the double orthocomplement from Definition 2.2. This is done in Proposition 4.27. Using this description, it follows easily that a bijective map $P(B) \to P(A)$ that preserves transition probabilities, also preserves equivalence of pure states (Corollary 4.28). Let A be a C*-algebra and let σ, ρ be unitarily equivalent pure states. Recall from Theorem 4.18 that the face generated by σ and ρ, which we denote by $\mathrm{face}(\sigma, \rho)$, is affinely isomorphic to the closed unit ball $B^3 \subset R^3$. Denote by $S^2(\sigma, \rho)$ the extreme boundary of $\mathrm{face}(\sigma, \rho)$.

Lemma 4.25. *Let A be a C*-algebra, let σ and ρ be distinct unitarily equivalent pure states, and let $\varphi\colon B^3 \to \mathrm{face}(\sigma, \rho)$ be an affine isomorphism. Then φ bijectively maps S^2 onto $S^2(\sigma, \rho)$ and $S^2(\sigma, \rho)$ consists of all pure states in $\mathrm{face}(\sigma, \rho)$.*

Proof. Let $\varphi\colon B^3 \to \mathrm{face}(\sigma, \rho)$ be an affine isomorphism. Because φ is affine, it maps the extreme boundary of B^3, i.e., the unit sphere S^2, bijectively onto the extreme boundary of $\mathrm{face}(\sigma, \rho)$, i.e., $S^2(\sigma, \rho)$. Let $\omega \in S^2(\sigma, \rho)$ and let $\omega_1, \omega_2 \in S(A)$ and $\lambda \in (0, 1)$ be such that $\lambda\omega_1 + (1 - \lambda)\omega_2 = \omega$. Because $\omega \in \mathrm{face}(\sigma, \rho)$ it follows from the definition of a face that $\omega_1, \omega_2 \in \mathrm{face}(\sigma, \rho)$. Since ω is an element of the extreme boundary of $\mathrm{face}(\sigma, \rho)$ this implies that $\omega_1 = \omega_2 = \omega$, which proves that $\omega \in P(A)$. Finally, note that these are all the pure states in $\mathrm{face}(\sigma, \rho)$, because every $\omega \in \mathrm{face}(\sigma, \rho)$ that is not on the extreme boundary can be written as a non-trivial combination of states on the extreme boundary and hence is not pure. □

We now want to describe $S^2(\sigma, \rho)$ using the transition probability from Theorem 3.79. We will use the terminology from Section 2.1.

Theorem 4.26. *Let A be a C^*-algebra and $P(A)$ its pure state space. Then $P(A)$ equipped with the transition probability defined in Theorem 3.79 is a well-behaved transition probability space.*

Proof. This follows from Landsman (1998), Theorem 2.8.2 and Proposition 3.80. □

This brings us to our next proposition, in which we describe the pure states in the face generated by σ and ρ using the double orthocomplement. This result is stated, but not proved in Shultz (1982).

Proposition 4.27. *Let $\sigma, \rho \in P(A)$. Then the set of pure states in the face generated by σ and ρ is given by $\{\sigma, \rho\}^{\perp\perp}$.*

Proof. First suppose that σ and ρ are not unitarily equivalent. In that case, Theorem 4.18 states that

$$\text{face}(\sigma, \rho) = [\sigma, \rho] = \{\lambda\sigma + (1-\lambda)\rho \mid \lambda \in [0,1]\}, \tag{4.55}$$

which implies that the pure states in face(σ, ρ) are precisely σ and ρ. According to Proposition 3.80, we have $\tau(\sigma, \rho) = 0$, so $\{\sigma, \rho\}$ is a family of orthogonal points. Then by Theorem 4.26 and Lemma 2.8,

$$\{\sigma, \rho\}^{\perp\perp} = \{\omega \in P(A) \mid \tau(\omega, \sigma) + \tau(\omega, \rho) = 1\}. \tag{4.56}$$

Now suppose that $\omega \in \{\sigma, \rho\}^{\perp\perp}$. Using Proposition 3.80 and the fact that σ and ρ are not unitarily equivalent, we know that ω is equivalent to either σ or to ρ. If ω is equivalent to σ, we have

$$1 = \tau(\omega, \sigma) + \tau(\omega, \rho) = \tau(\omega, \sigma), \tag{4.57}$$

and because τ is a transition probability, this implies that $\omega = \sigma$. Analogously, if we assume that ω is equivalent to ρ it follows that $\omega = \rho$. This proves that

$$\{\sigma, \rho\}^{\perp\perp} = \{\sigma, \rho\} = \text{face}(\sigma, \rho) \cap P(A). \tag{4.58}$$

Now suppose that σ and ρ are unitarily equivalent, so that according to Theorem 4.18 the face generated by σ and ρ is a 3-ball. As before, we denote the pure states of that 3-ball by $S^2(\sigma, \rho)$. So we want to prove that

$$\{\sigma, \rho\}^{\perp\perp} = S^2(\sigma, \rho). \tag{4.59}$$

We first give a precise description of $S^2(\sigma, \rho)$. Because σ and ρ are unitarily equivalent, we may assume that the associated cyclic vectors x_σ and x_ρ lie in the same

Hilbert space, which we denote by H. As usual, we denote the GNS-representation of σ by π_σ. Then according to Proposition 3.24 the split face generated by σ, which we denote by F_σ, is affinely isomorphic to the normal state space of $B(H)$. By Theorem 3.49 the split face generated by σ is the same as the split face generated by ρ. So we have $\sigma, \rho \in F_\sigma = F_\rho$, and F_σ is a split face. In particular, F_σ is a face that contains both σ and ρ, hence $\mathrm{face}(\sigma, \rho) \subseteq F_\sigma$.

It now follows from Lemma 4.3 that we can identify $\mathrm{face}(\sigma, \rho)$ with a face of the normal state space of $B(H)$, namely the face generated by the vector states ω_{x_σ} and ω_{x_ρ}. By Corollary 3.28 this is the norm-closed face $F_p = \{\omega \in S_n\big(B(H)\big) \mid \omega(p) = 1\}$ where p is the projection onto $\mathrm{span}\{x_\sigma, x_\rho\}$. Because the normal pure state in $B(H)$ are precisely the vector states, we see that

$$\begin{aligned} \mathrm{face}_{S(B(H))}(\omega_{x_\sigma}, \omega_{x_\rho}) \cap P(B(H)) &= \mathrm{face}_{S_n(B(H))}(\omega_{x_\sigma}, \omega_{x_\rho}) \cap P(B(H)) \\ &= F_p \cap P(B(H)) \\ &= \{\omega_y \mid y \in H, \|y\| = 1, \, \omega_y(p) = 1\}. \end{aligned} \tag{4.60}$$

Let ω_y be such that $\omega_y(p) = 1$. Then

$$1 = \omega_y(p) = \langle y, py \rangle, \tag{4.61}$$

which implies that $py = y$, i.e., $y \in \mathrm{span}\{x_\sigma, x_\rho\}$. Conversely, if y is a unit vector in $\mathrm{span}\{x_\sigma, x_\rho\}$, then $\omega_y(p) = 1$ and $\omega_y \in F_p \cap P(B(H))$. This shows that $S^2(\sigma, \rho)$ is isomorphic to $\{\omega_y \mid y \in \mathrm{span}\{x_\sigma, x_\rho\}, \|y\| = 1\}$.

Denote the pure states in a subset X of $S(A)$ by $P(X)$.

Claim: Let F be a split face of $S(A)$ and let F' be its complementary split face. Then $P(F)^\perp = P(F')$.

Proof: By definition of the complementary split face we have $S(A) = F \oplus_c F'$. By definition of the orthocomplement

$$P(F)^\perp = \{\sigma \in P(A) \mid \tau(\sigma, \rho) = 0 \text{ for all } \rho \in P(F)\}. \tag{4.62}$$

Let ω be a pure state. It follows that either $\omega \in F$ or $\omega \in F'$. If $\omega \in F$, then in particular $\omega \in P(F)$ and $\tau(\omega, \omega) = 1 \neq 0$. Hence $\omega \notin P(F)^\perp$. This shows that $P(F') \subseteq P(F)^\perp$. Conversely, suppose that $\omega \in P(F)^\perp$, so $\tau(\omega, \rho) = 0$ for all $\rho \in P(F)$. Then we must have $\omega \notin P(F)$, hence $\omega \in P(F')$. This shows that $P(F)^\perp \subseteq P(F')$ and hence $P(F)^\perp = P(F')$. ∎

To ease notation, denote the set $\{\sigma, \rho\}$ by S. Clearly, $S \subseteq F_\sigma$. Then $S \subseteq P(F_\sigma)$, which implies that

$$S^\perp \supseteq P(F_\sigma)^\perp = P(F'_\sigma). \tag{4.63}$$

Therefore,

$$S^\perp = P(F'_\sigma) \cup (S^\perp \cap P(F_\sigma)), \tag{4.64}$$

and

$$\begin{aligned} S^{\perp\perp} &= \big(P(F'_\sigma) \cup (S^\perp \cap P(F_\sigma))\big)^\perp \\ &= P(F'_\sigma)^\perp \cap (S^\perp \cap P(F_\sigma))^\perp \\ &= P(F_\sigma) \cap (S^\perp \cap P(F_\sigma))^\perp. \end{aligned} \tag{4.65}$$

We can conclude that the double orthocomplement of S calculated in $P(A)$ is the same as the double orthocomplement calculated solely within $P(F_\sigma)$. Because F_σ is affinely isomorphic to $S_n(B(H))$, the double orthocomplement of S is isomorphic to the double orthocomplement of $\{\omega_{x_\sigma}, \omega_{x_\rho}\}$ in $S_n(B(H))$. Recall that for vector states $\omega_x, \omega_y \in S_n(B(H))$ we have

$$\tau(\omega_x, \omega_y) = \left|\langle x, y\rangle\right|^2, \tag{4.66}$$

so $\tau(\omega_x, \omega_y) = 0$ if and only if the unit vectors x and y are orthogonal. Then

$$\begin{aligned} \{\omega_{x_\sigma}, \omega_{x_\rho}\}^{\perp\perp} &= \{\omega_y \mid y \in \{x_\sigma, x_\rho\}^{\perp\perp}, \|y\| = 1\} \\ &= \{\omega_y \mid y \in \operatorname{span}\{x_\sigma, x_\rho\}^{\perp\perp}, \|y\| = 1\} \\ &= \{\omega_y \mid y \in \operatorname{span}\{x_\sigma, x_\rho\}, \|y\| = 1\}. \end{aligned} \tag{4.67}$$

As we deduced in equation (4.60), these are precisely the vector states in the face generated by ω_{x_σ} and ω_{x_ρ}. Hence both $\{\sigma, \rho\}^{\perp\perp}$ and $S^2(\sigma, \rho) \cap P(A)$ are isomorphic to

$$\{\omega_y \mid y \in \operatorname{span}\{x_\sigma, x_\rho\}, \|y\| = 1\} \tag{4.68}$$

under the same isomorphism, which implies that $S^2(\sigma, \rho) = \{\sigma, \rho\}^{\perp\perp}$. □

Corollary 4.28. *Let A and B be C^*-algebras, and $\psi\colon P(B) \to P(A)$ a bijection that preserves transition probabilities. Then ψ preserves equivalence of pure states, and maps $S^2(\sigma, \rho)$ onto $S^2(\psi(\sigma), \psi(\rho))$.*

Proof. Let $X \subseteq P(B)$. Then:

$$\begin{aligned}\big(\psi(X)\big)^{\perp} &= \{\omega \in P(A) \mid \tau(\omega,\rho) = 0 \ \forall \rho \in \psi(X)\}\\ &= \{\omega \in P(A) \mid \tau(\psi^{-1}(\rho),\psi^{-1}(\omega)) = 0 \ \forall \rho \in \psi(X)\}\\ &= \{\omega \in P(A) \mid \tau(\sigma,\psi^{-1}(\omega)) = 0 \ \forall \sigma \in X\}\\ &= \{\psi(\nu) \mid \nu \in P(B),\ \tau(\sigma,\nu) = 0 \ \forall \sigma \in X\}\\ &= \psi(X^{\perp}). \end{aligned} \tag{4.69}$$

Let $\sigma, \rho \in P(B)$. Using Proposition 4.27 and the fact that $\psi(X)^{\perp} = \psi(X^{\perp})$, we find:

$$\begin{aligned}\sigma,\rho \text{ equivalent} &\Leftrightarrow \{\sigma,\rho\}^{\perp\perp} \text{ properly contains } \{\sigma,\rho\}\\ &\Leftrightarrow \psi(\{\sigma,\rho\}^{\perp\perp} = \{\psi(\sigma),\psi(\rho)\}^{\perp\perp} \text{ properly contains } \{\psi(\sigma),\psi(\rho)\}\\ &\Leftrightarrow \psi(\sigma),\psi(\rho) \text{ equivalent.}\end{aligned} \tag{4.70}$$

Now let σ, ρ be unitarily equivalent pure states of B. Again, using Proposition 4.27 and the fact that $\psi(X)^{\perp} = \psi(X^{\perp})$, we find:

$$\begin{aligned}\psi\big(S^2(\sigma,\rho)\big) &= \psi\big(\{\sigma,\rho\}^{\perp\perp}\big) = \psi(\{\sigma,\rho\})^{\perp\perp}\\ &= \{\psi(\sigma),\psi(\rho)\}^{\perp\perp} = S^2(\psi(\sigma),\psi(\rho)),\end{aligned} \tag{4.71}$$

so ψ maps $S^2(\sigma,\rho)$ onto $S^2(\psi(\sigma,\psi(\rho))$. □

4.4 Atomic algebras

In this section we will decompose A^{**} into two part, namely a part that is "atomic" and a part that contains no atoms. Let $[\sigma]$ denote the equivalence class of all states that are unitarily equivalent to $\sigma \in P(A)$. The aim of this section is to prove that the atomic part of A^{**} is given by

$$\bigoplus_{[\sigma]\in P(A)/\sim} c([\sigma])A^{**} \cong \bigoplus_{[\sigma]\in P(A)/\sim} B(H_{\sigma}). \tag{4.72}$$

Once we have proven this, we can identify the normal pure states of the atomic part of A^{**} with the normal pure states of A^{**} itself. We will also show that A is isomorphic to a certain subset of the atomic part of A^{**}.

Definition 4.29. *A von Neumann algebra M is **atomic** if every non-zero projection dominates an atom.*

Definition 4.30. *A JBW-algebra M is* ***atomic*** *if every non-zero projection dominates an atom.*

Note that the von Neumann algebra M is atomic if and only if the JBW-algebra M_{sa} is atomic.

Lemma 4.31. *Let M be a JBW-algebra, and $\mathcal{P}$ the lattice of projections in M. The supremum in $\mathcal{P}$ of the atoms in M is a central projection z such that zM is atomic and $(1-z)M$ contains no atoms. This decomposition of M into atomic and non-atomic parts is unique.*

Proof. See Alfsen and Shultz (2003), Lemma 3.42. □

Let A be a C*-algebra, let $A**_{\mathrm{sa}}$ be the associated JBW-algebra, and let z be the central projection from Lemma 4.31. Then z is also the central projection such that zA^{**} is atomic and $(1-z)A^{**}$ contains no atoms. We now give a description of the atomic part of A^{**}. Denote by "$\sim$" the equivalence relation on $P(A)$ given by unitary equivalence of pure states and denote by $[\sigma]$ the equivalence class of $\sigma \in P(A)$ in $P(A)/\sim$.

Corollary 4.32. *Let A be a C*-algebra and let z be the central projection such that zA^{**} is atomic and $(1-z)A^{**}$ contains no atoms. Then*

$$zA^{**} = \bigoplus_{[\sigma]\in P(A)/\sim} c(\sigma)A^{**}. \tag{4.73}$$

Proof. By Theorem 3.49 $c(\sigma) = c(\rho)$ for pure states σ and ρ if and only if σ and ρ are unitarily equivalent. Hence $c([\sigma])$ is well defined, and so is

$$\bigoplus_{[\sigma]\in P(A)/\sim} c([\sigma])A^{**}. \tag{4.74}$$

We first show that

$$z = \sum_{[\sigma]\in P(A)/\sim} c([\sigma]). \tag{4.75}$$

By Proposition 3.72 and equation (3.61) the atoms in A^{**} are precisely the carriers of the pure states $\sigma \in P(A)$, so by Lemma 4.31,

$$z = \sup\{\mathrm{carrier}(\sigma) \mid \sigma \in P(A)\}. \tag{4.76}$$

By Lemma 3.46 the smallest central projection dominating carrier(σ) is precisely $c(\sigma)$. Let w be the supremum of $\{c(\sigma) \mid \sigma \in P(A)\}$, i.e.,

$$w = \sup\{c(\sigma) \mid \sigma \in P(A)\}, \tag{4.77}$$

which is a central projection in A^{**}. Because carrier(σ) $\leq c(\sigma)$ for all $\sigma \in P(A)$, we have $z \leq w$. On the other hand, for every $\sigma \in P(A)$, z is a central projection, which dominates carrier(σ). By definition $c(\sigma)$ is the smallest central projection that dominates σ, hence $c(\sigma) \leq z$ for every $\sigma \in P(A)$. It follows that

$$w = \sup\{c(\sigma) \mid \sigma \in P(A)\} \leq z, \tag{4.78}$$

from which we conclude that $w = z$. Because unitarily equivalent pure states have the same central carrier (see Theorem 3.49 part (iv)), it follows that

$$z = \sup\{c([\sigma] \mid [\sigma] \in P(A)/\sim\}. \tag{4.79}$$

Let σ and ρ be unitarily inequivalent pure states, so that $c([\sigma]) \neq c([\rho])$. By Corollary 3.78 we have $c([\sigma]) \perp c([\rho])$ and hence by Theorem A.100:

$$c([\sigma]) \vee c([\rho]) = c([\sigma]) + c([\rho]). \tag{4.80}$$

We can now conclude that

$$z = \sup\{c([\sigma] \mid [\sigma] \in P(A)/\sim\} = \sum_{[\sigma]\in P(A)/\sim} c([\sigma]). \tag{4.81}$$

Because the elements of $\{c([\sigma] \mid [\sigma] \in P(A)/\sim\}$ are orthogonal, this shows that

$$zA^{**} = \bigoplus_{[\sigma]\in P(A)/\sim} c(\sigma)A^{**}. \tag{4.82}$$

□

Note that by equation (4.73) and Lemma 4.31, we have

$$A^{**} = \bigoplus_{[\sigma]\in P(A)/\sim} c([\sigma])A^{**} \oplus (1-z)A^{**}. \tag{4.83}$$

Corollary 4.33. *Let A be a C^*-algebra and let z be the central projection from Lemma 4.31. Then there is a bijective correspondence between the following sets:*

- $P_n(A^{**})$*;*
- $P_n(zA^{**})$*;*
- $\bigcup_{[\sigma]\in P(A)/\sim} P_n(c([\sigma])A^{**})$*.*

These bijective correspondences are explicitly given by extending the pure normal states with 0 to the larger space, and of restricting the pure normal states to the smaller space.

Proof. Let $\rho \in P_n(c[\sigma]A^{**})$. Extend ρ by 0 on all other summands of equation (4.73). This gives a pure normal state on zA^{**}. Extend ρ by 0 on all other summands of equation (4.83). This gives a pure normal state on A^{**}.

Let $\rho \in P_n(A^{**})$. Then $\rho(c[\rho]) = 1$, so ρ restricted to $c([\rho])A^{**}$ gives a pure normal state on $c([\rho])A^{**}$. Furthermore, by 3.78 we have $\rho(c[\sigma]) = 0$ for every $\sigma \in P(A)$ that is unitarily inequivalent to ρ. By equation (4.73) we have

$$\rho(z) = \sum_{[\sigma]\in P(A)/\sim} \rho(c[\sigma]) = \rho(c[\rho]) = 1, \tag{4.84}$$

so $\rho\big|_{zA^{**}} \in P_n(zA^{**})$. Let $\rho \in P_n(zA^{**})$. Extend ρ by 0 on $(1-z)A^{**}$. Then

$$\rho(1_{A^{**}}) = \rho(z) + \rho(1-z) = 1, \tag{4.85}$$

so ρ is a pure normal state on A^{**}. Furthermore,

$$1 = \rho(z) = \sum_{[\sigma]\in P(A)/\sim} \rho(c[\sigma]). \tag{4.86}$$

Because ρ is pure, ρ must be equal to 0 on all but one of the summands. Hence there is a $\sigma \in P(A)$ such that $\rho(c[\sigma]) = 1$. Hence ρ restricts to a pure normal state on $c([\sigma])A^{**}$.

It is easy to see that the constructions given above are inverses of each other, i.e., first restricting a state to a smaller space and then extending by 0 to the original space gives the original state, and *vice versa*. □

Corollary 4.34. *Let z be the central projection in A^{**} such that zA^{**} is atomic and $(1-z)A^{**}$ contains no atoms. Then A is *-isomorphic to zA.*

Proof. Define $\varphi_z\colon A \to zA$ by $a \mapsto za$. Because z is a central projection, it follows easily that φ_z is a *-homomorphism. Indeed, for all $a, b \in A$:

$$\varphi_z(a)^* = (za)^* = a^*z^* = za^* = \varphi_z(a^*); \tag{4.87}$$

$$\varphi_z(ab) = zab = zazb = \varphi_z(a)\varphi_z(b). \tag{4.88}$$

So we only need to show that φ_z is an isomorphism, i.e., that it is bijective. Surjectivity of φ_z is obvious, so we will only show injectivity. Let $a, b \in A$ be distinct elements of A. By equation (4.73) it is enough to find a pure state σ such that $c(\sigma)a \neq c(\sigma)b$. Because the atomic representation is faithful (see Proposition A.90), there exists a pure state $\sigma \in P(A)$ such that $\pi_\sigma(a) \neq \pi_\sigma(b)$. Because $\widetilde{\pi_\sigma}\colon c(\sigma)A^{**} \to B(H_\sigma)$ is a *-isomorphism we have:

$$\begin{aligned}\widetilde{\pi_\sigma}(c(\sigma)(a-b)) &= \widetilde{\pi_\sigma}(c(\sigma))\widetilde{\pi_\sigma}(a-b)\\ &= 1_{H_\sigma}(\widetilde{\pi_\sigma}(a) - \widetilde{\pi_\sigma}(b))\\ &= \pi_\sigma(a) - \pi_\sigma(b) \neq 0,\end{aligned} \tag{4.89}$$

from which we conclude that $c(\sigma)(a-b) \neq 0$, and hence $c(\sigma)a \neq c(\sigma)b$. □

We now want to prove that each summand in equation (4.73) is a type I factor.

Lemma 4.35. *Let A be a C*-algebra and σ a pure state on A. Then*

$$c(\sigma)A^{**} \cong B(H_\sigma), \tag{4.90}$$

and $c(\sigma)A^{**}$ is a type I factor.

Proof. By Lemma 3.46 we have $c(\sigma) = c(\pi_\sigma)$. By definition of the central cover (Definition 3.42), $c(\pi_\sigma)$ is the central projection in A^{**} such that

$$\ker(\widetilde{\pi_\sigma}) = \big(1 - c(\pi_\sigma)\big)A^{**}, \tag{4.91}$$

where $\widetilde{\pi_\sigma}$ is the normal extension of π_σ. Because the GNS-representation of a pure state is irreducible, Theorem 3.35 part 5 shows that $\widetilde{\pi_\sigma}$ maps A^{**} onto $B(H_\sigma)$. Hence $\widetilde{\pi_\sigma}|_{c(\sigma)A^{**}}$ is a *-isomorphism from $c(\sigma)A^{**}$ onto $B(H_\sigma)$. Because $B(H_\sigma)$ is a factor and $B(H_\sigma)$ is isomorphic to $c(\sigma)A^{**}$, we can conclude that $c(\sigma)A^{**}$ is a factor. Lastly, we have to show that $c(\sigma)A^{**}$ is of type I. This follows directly from Proposition 3.62 and the fact that $\mathrm{carrier}(\sigma) \in c(\sigma)A^{**}$ is an atom in $c(\sigma)A^{**}$. □

Corollary 4.36. *Let A be a C*-algebra, let π_{ra} be the reduced atomic representation and let z be the central projection such that zA^{**} is the atomic part of A^{**}. Then*

$$zA^{**} \cong \pi_{ra}(A)''. \tag{4.92}$$

Proof. This follows directly from Theorem 3.51 and Lemma 4.35. □

4.5 Wigner symmetries that preserve orientation

In this section we establish an equivalence between Wigner symmetries that preserve orientation and *-isomorphisms. The key results, namely Proposition 4.38, Theorem 4.41 and Theorem 4.42 are due to Shultz (1982). However, we greatly increased the details in his proofs of Proposition 4.38 and Theorem 4.42 in order to complete the argument.

We start with the following Theorem, which just paraphrases results from Chapters 1 and 2.

Theorem 4.37. *A bijective map ψ from $P_n(B(H_2))$ onto $P_n(B(H_1))$ that preserves transition probabilities extends to a unique affine isomorphism of the normal state spaces and is induced by a unique *-isomorphism or *-anti-isomorphism Ψ from $B(H_1)$ onto $B(H_2)$.*

Proof. By Theorem 1.25 a bijective map from $P_n(B(H_2))$ onto $P_n(B(H_2))$ that preserves transition probabilities is equivalent to a Wigner symmetry ψ' from $\mathcal{P}_1(H_2)$ onto $\mathcal{P}_2(H_1)$. Because Wigner symmetries and Kadison symmetries are equivalent, ψ' extends to a unique affine isomorphism from $\mathcal{D}(H_2)$ onto $\mathcal{D}(H_1)$. Again, using Theorem 1.25, this means that ψ extends to a unique affine isomorphism of the normal state spaces. By Corollary 2.17 every Kadison symmetry is induced by a Jordan symmetry and by Proposition 2.26 every Jordan symmetry is either a *-isomorphism or a *-anti-isomorphism. □

Corollary 4.23 showed the equivalence between orientation-preserving (reversing) Kadison symmetries and *-isomorphisms (*-anti-isomorphisms). We now want to look at bijections between the pure state spaces, rather than the entire state spaces. We still need the notion of orientation, but now on the pure states. Recall from Theorem 4.19 that A induces a global orientation of $S(A)$. By restricting each induced orientation of a 3-ball face(σ, ρ) to its extreme boundary $S^2(\sigma, \rho)$ we get a collection of orientations of all 2-spheres $S^2(\sigma, \rho)$. We refer to this collection of orientations as the **canonical orientation of $\mathbf{P}(\mathbf{A})$**. If we ignore topology and only look at the pure states, we get the following result.

Proposition 4.38. *Let A and B be C*-algebras. A bijective map $\psi\colon P(B) \to P(A)$ is induced by a *-isomorphism (*-anti-isomorphism) of the atomic part of A^{**} onto the atomic part of B^{**} if and only if ψ preserves transition probabilities and preserves (reverses) orientation.*

Remark 4.39. What do we mean if we say that a *-isomorphism Ψ from the atomic part of A^{**} onto the atomic part of B^{**} induces ψ? Using Lemma 4.31, denote the atomic part of A^{**} by $z_A A^{**}$ and the atomic part of B^{**} by $z_B B^{**}$. The pure states of A correspond to the pure normal states of A^{**}. Corollary 4.33 gives a correspondence between the pure normal states of A^{**} and the pure normal states of zA^{**}. So identify $P(A)$ with $P_n(z_A A^{**})$ and $P(B)$ with $P_n(z_B B^{**})$. Then ψ is a bijective map

$$\psi\colon P_n(z_B B^{**}) \to P_n(z_A A^{**}). \tag{4.93}$$

We say that $\Psi\colon z_A A^{**} \to z_B B^{**}$ induces $\psi\colon P(B) \to P(A)$ if

$$\Psi^* = \psi\colon P_n(z_B B^{**}) \to P_n(z_A A^{**}). \tag{4.94}$$

Proof of Proposition 4.38. We will only prove the correspondence between *-isomorphisms of the atomic parts and maps that preserve transition probabilities and orientation. The proof for *-anti-isomorphisms and orientation-reversing bijections is analogous.

First suppose that ψ is induced by a *-isomorphism Ψ from the atomic part of A^{**} onto the atomic part of B^{**}, in the sense of Remark 4.39. By Proposition 4.20 this map preserves orientation. We still need to show that ψ preserves transition probabilities.

Claim: Let $\sigma \in P_n(z_B B^{**})$. Then $\Psi^{-1}\colon z_B B^{**} \to z_A A^{**}$ maps the carrier projection of σ onto the carrier projection of $\psi(\sigma) = \Psi^*(\sigma)$.

Proof: Let p_σ be the carrier projection of σ and $p_{\psi(\sigma)}$ be the carrier projection of $\psi(\sigma)$. First note that $p_\sigma \leq c(\sigma)$, so that $p_\sigma \in z_B B^{**}$. Because Ψ is a *-isomorphism, $\Psi^{-1}(p_\sigma)$ is a projection. Then

$$\psi(\sigma)(\Psi^{-1}(p_\sigma)) = \sigma\big((\Psi \circ \Psi^{-1})p_\sigma\big) = \sigma(p_\sigma) = 1, \tag{4.95}$$

which shows that $p_{\psi(\sigma)} \leq \Psi^{-1}(p_\sigma)$. On the other hand $\Psi(p_{\psi(\sigma)})$ is a projection and

$$\sigma(\Psi(p_{\psi(\sigma)})) = \big(\sigma \circ \Psi \circ \Psi^{-1}\big)\big(\Psi(p_{\psi(\sigma)})\big) = (\Psi^* \circ \sigma)(p_{\psi(\sigma)}) = \psi(\sigma)(p_{\psi(\sigma)}) = 1, \tag{4.96}$$

which shows that $p_\sigma \leq \Psi(p_{\psi(\sigma)})$ and hence $\Psi^{-1}(p_\sigma) \leq \Psi^{-1}\big(\Psi(p_{\psi(\sigma)})\big) = p_{\psi(\sigma)}$. We conclude that $p_{\psi(\sigma)} = \Psi^{-1}(p_\sigma)$. ■

Let $\sigma, \rho \in P_n(z_B B^{**})$. Then by Theorem 3.79 and the claim we have

$$\tau(\psi(\sigma), \psi(\rho)) = \psi(\sigma)(p_{\psi(\rho)}) = (\sigma \circ \Psi)(\Psi^{-1}(p_\rho)) = \sigma(p_\rho) = \tau(\sigma, \rho), \tag{4.97}$$

i.e., ψ preserves transition probabilities.

Conversely, suppose that $\psi\colon P(B) \to P(A)$ preserves transition probabilities and orientation. Because ψ preserves transition probabilities, Corollary 4.28 implies that it preserves equivalence of pure states. Using Corollary 3.32, identify the state space of A (respectively B), with the normal state space of A^{**} (respectively B^{**}). Hence ψ is a bijection from $\{\rho \in P_n(B^{**}) \mid \rho \sim \sigma\}$ onto $\{\rho \in P_n(A^{**}) \mid \rho \sim \psi(\sigma)\}$. By Lemma 4.35 and equation (4.73), the atomic part of B^{**} is the direct sum of type I factors $c([\sigma])B^{**} \cong B(H_\sigma)$, where $[\sigma]$ denotes the equivalence class of $\sigma \in P(B)$. Using Corollary 4.33, identify the states in the equivalence class of $\sigma \in P(A)$ with the pure normal states of $c([\sigma]A^{**})$. Because ψ preserves the equivalence pure states, ψ bijectively maps the pure normal states of $c([\sigma])B^{**} \cong B(H_\sigma)$ onto those of $c([\psi(\sigma)])A^{**} \cong B(H_{\psi(\sigma)})$. So we have a bijection

$$W\colon P_n(B(H_\sigma)) \to P_n(B(H_{\psi(\sigma)})), \tag{4.98}$$

which preserves transition probabilities and preserves orientation. Then by Theorem 4.37 this bijection extends to a unique affine isomorphism

$$K\colon S_n(B(H_\sigma)) \to S_n(B(H_{\psi(\sigma)})), \tag{4.99}$$

which preserves orientation. Furthermore, K is induced by a unique *-isomorphism or *-anti-isomorphism

$$J\colon B(H_{\psi(\sigma)}) \to B(H_\sigma). \tag{4.100}$$

In particular, J is a Jordan symmetry that preserves the orientation of facial 3-balls in the normal state space. By Proposition 4.24, this implies that J is a *-isomorphism. Let

$$\Psi_{[\sigma]}\colon c([\psi(\sigma)])A^{**} \to c([\sigma])B^{**} \tag{4.101}$$

be the corresponding *-isomorphism between $c([\psi(\sigma)])A^{**}$ and $c([\sigma])B^{**}$. Then $\Psi_{[\sigma]}$ is the unique *-isomorphism that induces

$$\psi\colon P_n(c[\psi(\sigma)])A^{**}) \to P_n(c([\sigma])B^{**}). \tag{4.102}$$

Note that because ψ preserves the equivalence of pure states we have

$$\bigoplus_{[\sigma]\in P(B)/\sim} c([\psi(\sigma)])A^{**} = \bigoplus_{[\sigma]\in P(A)/\sim} c([\sigma])A^{**}. \tag{4.103}$$

Let Ψ be the direct sum of all *-isomorphisms $\Psi_{[\sigma]}$, i.e.,

$$\Psi = \bigoplus_{[\sigma]\in P(B)/\sim} \Psi_{[\sigma]}. \tag{4.104}$$

By equation (4.73) this is a *-isomorphism that maps the atomic part of A^{**} onto the atomic part of B^{**}, and induces ψ. $\square$

The *-isomorphism from the previous proposition maps $z_A A^{**}$ onto $z_B B^{**}$, but this does not necessarily mean that it maps $z_A A$ onto $z_B B$. We would like this to be the case, because according to Corollary 4.34 the algebras A and $z_A A$ are *-isomorphic, as are B and $z_B B$. If the map Ψ maps z_A onto z_B, then using the *-isomorphism from Corollary 4.34, we find a *-isomorphism from A onto B that induces ψ. The key to proving that Ψ maps $z_A A$ onto $z_B B$ is the fact that every C*-algebra is ***weakly perfect***.

Definition 4.40. *Let A be a C*-algebra. We denote by A_u the set of elements $a \in zA^{**}$ such that a, a^*a and aa^* are uniformly continuous on $P(A) \cup \{0\}$. We say A is **weakly perfect** if $zA = A_u$.*

Theorem 4.41. *Every C*-algebra A is weakly perfect.*

Proof. See Shultz (1982), Theorem 17. □

The following theorem proves that there is a bijective correspondence $\Phi \mapsto \Phi^*$ between *-isomorphisms (*-anti-isomorphisms) $\Phi\colon A \to B$ and orientation-preserving (reversing) Wigner symmetries Φ^*.

Theorem 4.42. *Let A and B be C*-algebras and $\psi\colon P(B) \cup \{0\} \to P(A) \cup \{0\}$ a bijection with $\psi(0) = 0$. Then ψ is induced by a *-isomorphism (*-anti-isomorphism) of A onto B if and only if ψ and ψ^{-1} are uniformly continuous and ψ preserves transition probabilities and preserves (reverses) orientation.*

Proof. We will only prove the correspondence between *-isomorphisms and uniformly continuous bijections with uniformly continuous inverse that preserve transition probabilities and orientation. The proof for *-anti-isomorphisms and orientation-reversing bijections is analogous.

First, suppose that ψ is induced by a *-isomorphism $\Psi\colon A \to B$. As in the proof of Proposition 4.38, this implies that ψ preserves orientation and transition probabilities. So we only need to show that ψ and ψ^{-1} are uniformly continuous. We will prove that ψ is uniformly continuous. The proof that ψ^{-1} is uniformly continuous is analogous, by noting that $\psi^{-1} = (\Psi^{-1})^*$.

<u>Claim:</u> let $\Psi\colon A \to B$ be a *-isomorphism. Then $\Psi^*\colon B^* \to A^*$ is uniformly continuous.

<u>Proof:</u> Denote the w^*-uniformity of A by $\mathcal{U}_{w^*}$ and the w^*-uniformity of B by $\mathcal{V}_{w^*}$. By definition of the w^*-uniformity, it is enough to show that for every $a \in A$ and

$\epsilon > 0$ there is a $b \in B$ and a $\delta > 0$ such that

$$V_\delta^b \subseteq (\Psi^*)^{-1}(U_\epsilon^a). \tag{4.105}$$

So let $a \in A$, $\epsilon > 0$, and let $b = \Psi(a)$. Then

$$\begin{aligned}(\Psi^*)^{-1}(U_\epsilon^a) &= \{(\sigma,\rho) \in B^* \times B^* \mid (\sigma\circ\Psi, \rho\circ\Psi) \in U_\epsilon^a\}\\ &= \{(\sigma,\rho) \in B^* \times B^* \mid \big|\sigma(\Psi(a)) - \rho(\Psi(a))\big| < \epsilon\}\\ &= \{(\sigma,\rho) \in B^* \times B^* \mid \big|\sigma(b) - \rho(b)\big| < \epsilon\} = V_\epsilon^b.\end{aligned} \tag{4.106}$$

Hence $(\Psi^*)^{-1}(U_\epsilon^a) = V_\epsilon^b$, which implies that $\Psi^*\colon B^* \to A^*$ is uniformly continuous. ■

Again, denote the w^*-uniformity of A by $\mathcal{U}_{w^*}$ and the w^*-uniformity of B by $\mathcal{V}_{w^*}$. Furthermore write $\mathcal{U}_{w^*}^{P(A)}$ for the w^*-uniformity of A^* restricted to $P(A)\cup\{0\}$, and $\mathcal{V}_{w^*}^{P(B)}$ for the w^*-uniformity of B^* restricted to $P(B)\cup\{0\}$. Let $U \in \mathcal{U}_{w^*}^{P(A)}$. Then there exists a $\widetilde{U} \in \mathcal{U}_{w^*}$ such that

$$U = \widetilde{U} \cap (P(A)\cup\{0\}). \tag{4.107}$$

By the above claim, $\Psi^*\colon B^* \to A^*$ is uniformly continuous, so $(\Psi^*)^{-1}(\widetilde{U}) \in \mathcal{V}_{w^*}$. Then, using the fact that $\Psi^*\colon B^* \to A^*$ is a bijection that maps $P(B)\cup\{0\}$ onto $P(A)\cup\{0\}$, we find:

$$\begin{aligned}(\Psi^*)^{-1}(U) &= (\Psi^*)^{-1}\big(\widetilde{U} \cap (P(A)\cup\{0\})\big)\\ &= (\Psi^*)^{-1}(\widetilde{U}) \cap (\Psi^*)^{-1}(P(A)\cup\{0\})\\ &= (\Psi^*)^{-1}(\widetilde{U}) \cap \big(P(B)\cup\{0\}\big).\end{aligned} \tag{4.108}$$

Because $(\Psi^*)^{-1}(\widetilde{U}) \in \mathcal{V}_{w^*}$, this means that $(\Psi^*)^{-1}(U) \in \mathcal{V}_{w^*}^{P(B)}$, i.e.,

$$\psi = \Psi^*\colon P(B)\cup\{0\} \to P(A)\cup\{0\} \tag{4.109}$$

is uniformly continuous.

Now suppose that $\psi\colon P(B)\cup\{0\} \to P(A)\cup\{0\}$ is a bijection with $\psi(0) = 0$ that is uniformly continuous, has uniformly continuous inverse, and preserves orientation as well as transition probabilities. Let z_A be the central projection such that $z_A A^{**}$ is the atomic part of A^{**} and let z_B be the central projection such that $z_B B^{**}$ is the atomic part of B^{**}. By Proposition 4.38, ψ is induced by a *-isomorphism

$\Psi\colon z_A A^{**} \to z_B B^{**}$. It remains to be shown that Ψ maps $z_A A$ onto $z_B B$.

Claim: Let $\Psi\colon z_A A^{**} \to z_B B^{**}$ be the *-isomorphism that induces $\psi\colon P(A) \cup \{0\} \to P(B) \cup \{0\}$ in the sense of Remark 4.39. Then

$$\Psi(a)(\sigma) = (a \circ \psi)(\sigma). \tag{4.110}$$

for all $a \in z_A A^{**}$ and $\sigma \in P(A) \cup \{0\}$.

Proof: Let $\varphi\colon P(A) \to P_n(z_A A^{**})$ be the composition of the map from Corollary 3.32, which maps a state on $P(A)$ to a normal state on A^{**}, and the identification of $P_n(A^{**})$ with $P_n(z_A A^{**})$ given in Corollary 4.33. So φ is a bijection from $P(A)$ onto $P_n(z_A A^{**})$ given by

$$\varphi(\sigma) = \widetilde{\sigma}_{|_{z_A A^{**}}}, \tag{4.111}$$

where $\widetilde{\sigma}$ is the normal extension of $\sigma \in P(A)$. We can extend φ to a bijection

$$\varphi_A\colon P(A) \cup \{0\} \to P_n(z_A A^{**}) \cup \{0\} \tag{4.112}$$

by setting $\varphi(0) = 0$. Similarly, let

$$\varphi_B\colon P(B) \cup \{0\} \to P_n(z_B B^{**}) \cup \{0\} \tag{4.113}$$

be the bijection from $P(B) \cup \{0\}$ onto $P_n(z_B B^{**}) \cup \{0\}$. Because Ψ induces ψ in the sense of Remark 4.39, we have

$$\begin{aligned} &\Psi^* = \varphi_A \circ \psi \circ \varphi_B^{-1}\colon P_n(z_B B^{**}) \cup \{0\} \to P_n(z_A A^{**}) \cup \{0\}, \quad \text{and} \\ &\psi = \varphi_A^{-1} \circ \Psi^* \circ \varphi_B\colon P(B) \cup \{0\} \to P(A) \cup \{0\}. \end{aligned} \tag{4.114}$$

Now let $a \in z_A A^{**}$ and $\sigma \in P(A) \cup \{0\}$. Then

$$\begin{aligned} (a \circ \psi)(\sigma) &= a\big((\varphi_A^{-1} \circ \Psi^* \circ \varphi_B)\sigma\big) \\ &= a\big(\varphi^{-1}(\widetilde{\sigma}_{|_{z_A A^{**}}} \circ \Psi)\big) \\ &= (\widetilde{\sigma}_{|_{z_A A^{**}}} \circ \Psi)(a) \\ &= \widetilde{\sigma}_{|_{z_A A^{**}}}(\Psi(a)) \\ &= \Psi(a)(\sigma), \end{aligned} \tag{4.115}$$

which proves the claim. ■

By Theorem 4.41 the C*-algebras A and B are weakly perfect, i.e.,

$$\{a \in z_A A^{**} \mid a, a^*a \text{ and } aa^* \text{ uniformly continuous on } P(A) \cup \{0\}\} = z_A A$$
$$\{b \in z_B B^{**} \mid b, b^*b \text{ and } bb^* \text{ uniformly continuous on } P(B) \cup \{0\}\} = z_B B. \quad (4.116)$$

Let $a \in z_A A$. Because ψ is uniformly continuous, the maps

$$\begin{aligned} a \circ \psi &\colon P(B) \cup \{0\} \to \mathbb{C} \\ a^*a \circ \psi &\colon P(B) \cup \{0\} \to \mathbb{C} \\ aa^* \circ \psi &\colon P(B) \cup \{0\} \to \mathbb{C} \end{aligned} \quad (4.117)$$

are uniformly continuous, so $a \circ \psi \in z_B B$. Using the claim, we see that $\Psi(a) \in z_A A$, hence

$$\Psi(z_A A) \subseteq z_B B. \quad (4.118)$$

Similarly, because ψ^{-1} is uniformly continuous, we have

$$\Psi^{-1}(z_B B) \subseteq \Psi(z_A A). \quad (4.119)$$

We conclude that Ψ is a *-isomorphism that maps $z_A A$ onto $z_B B$ and induces ψ in the sense of Remark 4.39. Let

$$\begin{aligned} \varphi_A &\colon A \to z_A A; \\ \varphi_B &\colon B \to z_B B \end{aligned} \quad (4.120)$$

be the *-isomorphisms from Corollary 4.34. Then

$$\Phi \equiv \varphi_B^{-1} \circ \Psi \circ \varphi_A \colon A \to B \quad (4.121)$$

is a *-isomorphism from A onto B that induces ψ. □

4.6 Wigner symmetries that preserve q-closed sets

Instead of requiring a Wigner symmetry to be uniformly continuous and has uniformly continuous inverse, we can equivalently require that it preserves ***q-closed sets***. This result was stated as a corollary of Proposition 4.38 by Shultz (1982), citing results by Akemann (1969) and Giles, Kummer, and Sneddon (1971), but he did not give a proof.

Definition 4.43. *Let A be a unital C^*-algebra. A set $X \subseteq P(A)$ is said to be* ***q-closed*** *if X consists of all pure states of some w^*-closed face of $S(A)$. We denote the set of q-closed sets by $Q(A)$.*

Remark 4.44. Let $X \in Q(A)$. By Theorem 1.8, the w^*-closed face F of $S(A)$ such that $X = F \cap S(A)$ is uniquely determined by X. Hence by Theorem 3.38 there is a bijective correspondence between closed projections of A^{**} and q-closed sets of A.

Corollary 4.45. *Let A and B be unital C*-algebras and $\psi\colon P(B) \to P(A)$ a bijection. Then ψ is induced by a *-isomorphism of A onto B if and only if ψ preserves transition probabilities and orientation, and ψ and ψ^{-1} preserve q-closed sets.*

One direction of the proof (assuming that ψ is induced by a *-isomorphism) is almost trivial, as we will see at the end of this section. The other direction however, requires some work. Shultz here invokes Theorem 5.13 of Giles, Kummer, and Sneddon (1971), but we will take a different route to avoid a lot of new notation. We will prove Corollary 4.45, by proving that if ψ is induced by $\Psi\colon z_A A^{**} \to z_B B^{**}$ and ψ^{-1} preserve q-closed sets, then Ψ^* and $(\Psi^*)^{-1}$ preserve ***q-continuous*** elements of A^{**}. Then, using a result from Akemann, Pedersen, and Tomiyama (1973), we can conclude that Ψ bijectively maps $z_A A$ onto $z_B B$, which proves the corollary.

Definition 4.46. *Let A be a C*-algebra. We say that an element $a \in A^{**}_{sa}$ is **q-continuous** if each open set in its spectrum corresponds to an open spectral projection. We denote the set of q-continuous elements by $C_q(A)$.*

It was proven by Akemann (1970) that for unital C*-algebras, the q-continuous elements of A^{**} are self-adjoint elements of A. We also want to know if the reverse is true, i.e., if all self-adjoints elements of A are q-continuous. This was done by Akemann, Pedersen, and Tomiyama (1973), but in an even more general case. To state the theorem, we need the ***multiplier algebra***.

Definition 4.47. *Let A be a C*-algebra. The **multiplier algebra** of A is given by*

$$M(A) = \{a \in A^{**} \mid aA \subseteq A \text{ and } Aa \subseteq A\}. \tag{4.122}$$

Lemma 4.48. *Let A be a C*-algebra, which we embed in A^{**} using the canonical embedding. Then the self-adjoint part of $M(A)$ is equal to the set of q-continuous elements of A^{**}.*

Proof. See Akemann, Pedersen, and Tomiyama (1973), Theorem 2.2. □

Note that for a unital C*-algebra A, its multiplier algebra is just equal to A. In that case, Lemma 4.48 implies that

$$\{a \in A^{**} \mid a \text{ is q-continuous}\} = A_{\mathrm{sa}}. \tag{4.123}$$

We now have to relate q-continuous elements of A^{**} to elements of $z_A A^{**}$. The following lemma does just that, by proving that every closed projection is uniquely determined by its atomic part. Denote the set of closed projections of a C*-algebra A by $\mathcal{P}_c(A)$ and denote the set of open projections by $\mathcal{P}_o(A)$.

Lemma 4.49. *Let A be a unital C*-algebra and let z be the central projection such that zA^{**} is the atomic part of A^{**}. If $p, q \in A^{**}$ are either open or closed projections and $zp \geq zq$, then $p \geq q$.*

Proof. See Akemann (1969), Theorem II.17. □

It is clear that the converse also holds, i.e., if p and q are open or closed projections and $p \geq q$, then $zp \geq zq$. Hence the open and closed projections are uniquely determined by their atomic parts, i.e., for $p, q \in \mathcal{P}_{c/o}(A)$ we have $z_A p = z_A q$ if and only if $p = q$. Now define

$$\mathcal{P}_c(zA) = ((zA)^+)_m = (zA^+)_m; \tag{4.124}$$
$$\mathcal{P}_o(zA) = ((zA)^+)^m = (zA^+)^m. \tag{4.125}$$

We call elements of $\mathcal{P}_o(zA)$ ***q-open projections*** and elements of $\mathcal{P}_c(zB)$ ***q-closed projections***. Note that by the previous discussion,

$$\mathcal{P}_c(zA) = z\mathcal{P}_c(A); \tag{4.126}$$
$$\mathcal{P}_o(zA) = z\mathcal{P}_o(A). \tag{4.127}$$

Also, define $C_q(zA)$ to be the elements $a \in (zA^{**})_{\text{sa}} = z(A^{**}_{\text{sa}})$ such that each open set in its spectrum corresponds to a q-open spectral projection. By the previous discussions we have $C_q(zA) = zC_q(A)$.

Corollary 4.50. *Let A and B be unital C*-algebras, and let $z_A A^{**}$, respectively $z_B B^{**}$ be the atomic parts of A and B, respectively. Let $\Psi\colon z_A A^{**} \to z_B B^{**}$ be a *-isomorphism. Then Ψ bijectively maps $\mathcal{P}_c(z_A A)$ onto $\mathcal{P}_c(z_B B)$.*

Proof. We prove that

$$\Psi(\mathcal{P}_c(z_A A)) \subseteq \Psi(z_B \mathcal{P}_c(z_B B)). \tag{4.128}$$

The proof that

$$\Psi^{-1}(\mathcal{P}_c(z_B B)) \subseteq \Psi(\mathcal{P}_c(z_A A)) \tag{4.129}$$

is analogous.

By Lemma 4.49 there is a bijective correspondence between closed projections of A^{**} and their atomic parts. According to Theorem 3.38 and Remark 4.44 we also have a bijective correspondence between closed projections and q-closed sets. This gives us a bijective correspondence between $\mathcal{P}_c(z_A A)$ and $Q(A)$. So let $p \in \mathcal{P}_c(A)$ and let X_p be the unique corresponding q-closed set, i.e.,

$$X_p = \{\sigma \in P_n(z_A A^{**}) \mid \sigma(p) = 1\}. \tag{4.130}$$

We want to show that $\Psi(p) \in \mathcal{P}_c(B)$. Note that

$$\begin{aligned} X_{\Psi(p)} &= \{\rho \in P_n(z_B B^{**}) \mid \rho(\Psi(p)) = 1\} \\ &= \{\sigma \circ \Psi^{-1} \mid \sigma \in P_n(z_A A^{**}) \text{ and } \sigma(p) = 1\} \\ &= \{\sigma \circ \Psi^{-1} \mid \sigma \in X_p\} \\ &= (\Psi^{-1})^*(X_p) = (\Psi^*)^{-1}(X_p). \end{aligned} \tag{4.131}$$

Claim: Ψ^* and $(\Psi^*)^{-1}$ preserve q-closed sets.

Proof: Let $X \subseteq P(A)$ be q-closed, i.e., X consists of all pure states of a w^*-closed face of $S(A)$, say F. Because Ψ^* is an affine homeomorphism (Theorem 4.1), $\Psi^*(F)$ is a w^*-closed face of $S(A)$. This shows that $\Psi^*(X)$ consists of all pure states of a w^*-closed face of $S(A)$, namely $\Psi^*(F)$. Analogously, $(\Psi^*)^{-1}$ preserves q-closed sets because

$$(\Psi^*)^{-1} \colon S(A) \to S(B) \tag{4.132}$$

is an affine homeomorphism. ■

The claim shows that $X_{\Psi(p)} = (\Psi^*)^{-1}(X_p) \in Q(B)$, and hence by the remark following Theorem 3.38, $\Psi(p) \in \mathcal{P}_c(z_B B)$. □

So if ψ is induced by a *-isomorphism $\Psi \colon z_A A^{**} \to z_B B^{**}$, then Ψ and Ψ^{-1} map q-closed projections onto q-closed projections. We now want to prove that Ψ maps q-continuous elements of A onto q-continuous elements of B.

Corollary 4.51. *Let A and B be unital C^*-algebras and let $z_A A^{**}$, respectively $z_B B^{**}$ be their atomic parts. Let $\Psi \colon z_A A^{**} \to z_B B^{**}$ be a *-isomorphism. Then Ψ bijectively maps $C_q(z_A A)$ onto $C_q(z_B B)$.*

Proof. We prove that

$$\Psi(z_A C_q(A)) \subseteq \Psi(z_B C_q(B)). \tag{4.133}$$

The proof that

$$\Psi^{-1}(z_B C_q(B)) \subseteq \Psi(z_A C_q(A)) \tag{4.134}$$

is analogous.

Let $a \in C_q(z_A A)$ and let $b = \Psi(a)$. We want to show that $b \in C_q(z_B B)$. So let E_a be the spectral measure of a, E_b be the spectral measure of b, and let $I \subset \mathbb{R}$ be an open interval. Then, using the fact that Ψ is a *-isomorphism, we find

$$\begin{aligned} b_I \equiv \int_I \lambda dE_b(\lambda) &= \int_I \lambda d(\Psi \circ E_a)(\lambda) \\ &= \Psi\Big(\int_I \lambda dE_a(\lambda)\Big) \equiv \Psi(a_I). \end{aligned} \tag{4.135}$$

Because $a \in C_q(z_A A)$, we have $a_I \in \mathcal{P}_o(z_A A)$. Corollary 4.50 proved that Ψ preserves q-closed projections and hence also q-open projections. This shows that $b_I = \Psi(a_I) \in \mathcal{P}_o(z_B B)$, from which we conclude that $b \in C_q(z_B B)$. □

Proving Corollary 4.45 is now only a matter of assembling the parts.

Proof of Corollary 4.45. First suppose that ψ is induced by a *-isomorphism $\Psi\colon A \to B$. Then by Proposition 4.1 and Theorem 4.21,

$$\Psi^*\colon S(B) \to S(A) \tag{4.136}$$

is an affine homeomorphism that preserves orientation, so ψ preserves orientation. It follows from a proof similar to that of Proposition 4.38, that ψ also preserves transition probabilities. So it only remains to be shown that ψ and ψ^{-1} preserve q-closed sets. This proof is practically the same as the proof of Corollary 4.50.

Now suppose that ψ preserves transition probabilities and orientation, and ψ and ψ^{-1} preserve q-closed sets. Let $z_A A^{**}$ be the atomic part of A^{**} and $z_B B^{**}$ be the atomic part of B^{**}. From Proposition 4.38 it follows that ψ is induced by a *-isomorphism

$$\Psi\colon z_A A^{**} \to z_B B^{**}. \tag{4.137}$$

Just as in the proof of 4.42, it remains to be shown that Ψ maps $z_A A$ onto $z_B B$. Fortunately, this follows directly from Lemma 4.48 and Corollary 4.51. □

4.7 Equivalence between Wigner symmetries and Jordan symmetries

If a C*-algebra A is not a factor, then not all Jordan symmetries are given by *-isomorphisms, or *-anti-isomorphisms. Hence the condition that a Wigner symmetry is orientation-preserving or reversing is too restrictive to find an equivalence between *all* Wigner symmetries and *all* Jordan symmetries. In this section we slightly alter the proof of Proposition 4.38 and Theorem 4.42 to find the correct equivalence.

Lemma 4.52. *Let A and B be C^*-algebras. A bijective map $\psi\colon P(B) \to P(A)$ is induced by a Jordan symmetry of the atomic part of A^{**} onto the atomic part of B^{**} if and only if ψ preserves transition probabilities.*

Proof. First suppose that ψ is induced by a Jordan symmetry Ψ from the atomic part of A^{**} onto the atomic part of B^{**}, in the sense of Remark 4.39. We need to show that ψ preserves transition probabilities.

Claim: Let $\sigma \in P_n(z_B B^{**})$. Then $\Psi^{-1}\colon z_B B^{**} \to z_A A^{**}$ maps the carrier projection of σ onto the carrier projection of $\psi(\sigma) = \Psi^*(\sigma)$.

Proof: Let p_σ be the carrier projection of σ and $p_{\psi(\sigma)}$ the carrier projection of $\psi(\sigma)$. Because Ψ is Jordan symmetry, $\Psi^{-1}(p_\sigma)$ is a projection. Then

$$\psi(\sigma)(\Psi^{-1}(p_\sigma)) = \sigma\big((\Psi \circ \Psi^{-1})p_\sigma\big) = \sigma(p_\sigma) = 1, \tag{4.138}$$

which shows that $p_{\psi(\sigma)} \leq \Psi^{-1}(p_\sigma)$. On the other hand, $\Psi(p_{\psi(\sigma)})$ is a projection and,

$$\sigma(\Psi(p_{\psi(\sigma)})) = \big(\sigma \circ \Psi \circ \Psi^{-1}\big)\big(\Psi(p_{\psi(\sigma)})\big) = (\Psi^* \circ \sigma)(p_{\psi(\sigma)}) = \psi(\sigma)(p_{\psi(\sigma)}) = 1, \tag{4.139}$$

which shows that $p_\sigma \leq \Psi(p_{\psi(\sigma)})$ and hence $\Psi^{-1}(p_\sigma) \leq \Psi^{-1}\big(\Psi(p_{\psi(\sigma)})\big) = p_{\psi(\sigma)}$. We conclude that $p_{\psi(\sigma)} = \Psi^{-1}(p_\sigma)$. ■

Let $\sigma, \rho \in P_n(z_B B^{**})$. Then by Theorem 3.79 and the above claim we have

$$\tau(\psi(\sigma), \psi(\rho)) = \psi(\sigma)(p_{\psi(\rho)}) = (\sigma \circ \Psi)(\Psi^{-1}(p_\rho)) = \sigma(p_\rho) = \tau(\sigma, \rho), \tag{4.140}$$

i.e., ψ preserves transition probabilities.

Conversely suppose that $\psi\colon P(B) \to P(A)$ preserves transition probabilities. Corollary 4.28 implies that it preserves equivalence of pure states. Using Corollary 3.32, identify the state space of A (respectively B), with the normal state space of A^{**} (respectively B^{**}). Hence ψ is a bijection from $\{\rho \in P_n(B^{**}) \mid \rho \sim \sigma\}$ onto $\{\rho \in P_n(A^{**}) \mid \rho \sim \psi(\sigma)\}$. By Lemma 4.35 and equation (4.73) the atomic part of B^{**} is the direct sum of type I factors $c([\sigma])B^{**} \cong B(H_\sigma)$, where $[\sigma]$ denotes the equivalence class of $\sigma \in P(B)$. Using the Corollary 4.33, identify the states in the equivalence class of $\sigma \in P(A)$ with the pure normal states of $c([\sigma]A^{**})$. Because ψ

preserves the equivalence pure states, ψ bijectively maps the pure normal states of $c([\sigma])B^{**} \cong B(H_\sigma)$ onto those of $c([\psi(\sigma)])A^{**} \cong B(H_{\psi(\sigma)})$. So we have a bijection

$$W\colon P_n(B(H_\sigma)) \to P_n(B(H_{\psi(\sigma)})), \tag{4.141}$$

which preserves transition probabilities. Then by Theorem 4.37 this bijection extends to a unique affine isomorphism

$$K\colon S_n(B(H_\sigma)) \to S_n(B(H_{\psi(\sigma)})). \tag{4.142}$$

Furthermore, K is induced by a unique Jordan symmetry

$$J\colon B(H_{\psi(\sigma)}) \to B(H_\sigma), \tag{4.143}$$

which is either a *-isomorphism or a *-anti-isomorphism by Proposition 2.26. Let

$$\Psi_{[\sigma]}\colon c([\psi(\sigma)])A^{**} \to c([\sigma])B^{**} \tag{4.144}$$

be the corresponding Jordan symmetry between $c([\psi(\sigma)])A^{**}$ and $c([\sigma])B^{**}$. Then $\Psi_{[\sigma]}$ is the unique Jordan symmetry that induces

$$\psi\colon P_n(c[\psi(\sigma)])A^{**}) \to P_n(c([\sigma])B^{**}). \tag{4.145}$$

Note that because ψ preserves the equivalence of pure states we have

$$\bigoplus_{[\sigma]\in P(B)/\sim} c([\psi(\sigma)])A^{**} = \bigoplus_{[\sigma]\in P(A)/\sim} c([\sigma])A^{**}. \tag{4.146}$$

Let Ψ be the direct sum of all Jordan symmetries $\Psi_{[\sigma]}$, i.e.,

$$\Psi = \bigoplus_{[\sigma]\in P(B)/\sim} \Psi_{[\sigma]}. \tag{4.147}$$

By equation (4.73) this is a Jordan symmetry that maps the atomic part of A^{**} onto the atomic part of B^{**}, and induces ψ. □

Note that the the proof is almost exactly the same as the proof of Proposition 4.38. The only difference is that because we do not assume that ψ preserves or reverses orientation, we cannot make a consistent choice of either only *-isomorphisms or only *-anti-isomorphisms that induce ψ on the summands of the atomic part. In general the map Ψ that induces ψ will be a mix of *-isomorphisms and *-anti-isomorphisms. We now come to the main result of this thesis.

Theorem 4.53. *Let A and B be C*-algebras. Then $\Psi \mapsto \Psi^*$ is a bijective correspondence between Jordan symmetries Ψ from A onto B, and Wigner symmetries from $P(B) \cup \{0\}$ onto $P(A) \cup \{0\}$.*

Proof. Let $\Psi\colon A \to B$ be a Jordan symmetry. By Proposition 4.4

$$\Psi^*\colon S(B) \cup \{0\} \to S(A) \cup \{0\} \tag{4.148}$$

is a Kadison symmetry, which implies that it preserves the extreme boundary, i.e., Ψ^* maps $P(B)\cup\{0\}$ onto $P(A)\cup\{0\}$. The proof that $\psi = \Psi^*\colon P(B)\cup\{0\} \to P(A)\cup\{0\}$ preserves transition probabilities is the same as in Lemma 4.52. So we only need to show that ψ and ψ^{-1} are uniformly continuous. This proof is analogous to that of Theorem 4.42, and hence will be omitted.

Now suppose that $\psi\colon P(B) \cup \{0\} \to P(A) \cup \{0\}$ is a Wigner symmetry, i.e., a bijection with $\psi(0) = 0$ that is uniformly continuous, has uniformly continuous inverse, and preserves transition probabilities. Let z_A be the central projection such that $z_A A^{**}$ is the atomic part of A^{**}, and let z_B be the central projection such that $z_B B^{**}$ is the atomic part of B^{**}. By Lemma 4.52, ψ is induced by a Jordan symmetry $\Psi\colon z_A A^{**} \to z_B B^{**}$. It remains to be shown that Ψ maps $z_A A$ onto $z_B B$.

Claim: Let $\Psi\colon z_A A^{**} \to z_B B^{**}$ be the Wigner symmetry that induces

$$\psi\colon P(A) \cup \{0\} \to P(B) \cup \{0\}. \tag{4.149}$$

Then

$$\Psi(a)(\sigma) = (a \circ \psi)(\sigma), \tag{4.150}$$

for all $a \in z_A A^{**}$ and $\sigma \in P(A) \cup \{0\}$.

Proof: See the proof of Theorem 4.42. ■

By Theorem 4.41, the C*-algebras A and B are weakly perfect, i.e.,

$$\begin{aligned} &\{a \in z_A A^{**} \mid a, a^*a \text{ and } aa^* \text{ uniformly continuous on } P(A) \cup \{0\}\} = z_A A \\ &\{b \in z_B B^{**} \mid b, b^*b \text{ and } bb^* \text{ uniformly continuous on } P(B) \cup \{0\}\} = z_B B. \end{aligned} \tag{4.151}$$

Let $a \in z_A A$. Because ψ is uniformly continuous, the maps

$$\begin{aligned} a \circ \psi\colon P(B) \cup \{0\} &\to \mathbb{C} \\ a^*a \circ \psi\colon P(B) \cup \{0\} &\to \mathbb{C} \\ aa^* \circ \psi\colon P(B) \cup \{0\} &\to \mathbb{C} \end{aligned} \tag{4.152}$$

are uniformly continuous, so $a \circ \psi \in z_B B$. Using the claim, we see that $\Psi(a) \in z_A A$, hence

$$\Psi(z_A A) \subseteq z_B B. \tag{4.153}$$

Similarly, because ψ^{-1} is uniformly continuous, we have

$$\Psi^{-1}(z_B B) \subseteq \Psi(z_A A). \tag{4.154}$$

We conclude that Ψ is a Wigner symmetry that maps $z_A A$ onto $z_B B$ and induces ψ in the sense of Remark 4.39. Let

$$\begin{aligned} &\varphi_A \colon A \to z_A A, \quad \text{and} \\ &\varphi_B \colon B \to z_B B \end{aligned} \tag{4.155}$$

be the *-isomorphisms from Corollary 4.34. Then

$$\Phi \equiv \varphi_B^{-1} \circ \Psi \circ \varphi_A \colon A \to B \tag{4.156}$$

is a Wigner symmetry from A onto B that induces ψ. $\square$

Appendix A

Functional analysis and operator algebras

Throughout the main text we use many results concerning functional analysis and operator algebras. In an effort to make this thesis almost self-contained, this chapter offers an overview of the relevant results. Although most results are highly non-trivial, we state them without proof because they are too general to be included in this thesis. The proofs can be found in standard text books. For functional analysis we recommend Rudin (1973) and MacCluer (2009). For operator algebras we recommend Dixmier (1977), Pedersen (1979), Kadison and Ringrose (1983), Kadison and Ringrose (1986), Blackadar (2006) and Murphy (1990). The appendix of Landsman (2017) can also be used as a complete introduction into functional analysis as well as operator algebras.

A.1 Hilbert spaces

We define a vector space to be a *complex* vector space.

Definition A.1. *Let V be a vector space and take $x, y \in V$.*

*(a) A **sesquilinear form** on V is a map $V \times V \to \mathbb{C}$, written $(x, y) \mapsto \langle x, y\rangle$ that is conjugate-linear in the first coordinate and linear in the second.*

*(b) A **hermitian form** on V is a sesquilinear form that satisfies $\langle y, x\rangle = \overline{\langle x, y\rangle}$.*

*(c) A **pre-inner product** on V is a **positive** hermitian form, i.e., $\langle x, x\rangle \geq 0$.*

(d) *An **inner product** on V is a pre-inner product that is positive definite, i.e., $\langle x, x\rangle = 0$ if and only if $x = 0$. A vector space V together with an inner product on V is called an **inner product space**.*

Theorem A.2 (Cauchy-Schwarz)**.** *Let V be a vector space and $\langle\, ,\, \rangle$ a (pre-)inner product on V. Then*

$$\big|\langle x, y\rangle\big| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle} \quad \forall x, y \in V. \tag{A.1}$$

Definition A.3. *Let V be a vector space. A **norm** on V is a function $\|\cdot\| : V \to \mathbb{R}^+$ that for all $x, y \in V$ and $\lambda \in \mathbb{C}$ satisfies:*

(a) $\|x + y\| \leq \|x\| + \|y\|$;

(b) $\|\lambda x\| = |\lambda|\|x\|$;

(c) $\|x\| = 0$ *if and only if* $x = 0$.

One can easily check that an inner product induces a norm by setting, for all $x, y \in V$,

$$\|x\| = \sqrt{\langle x, y\rangle}. \tag{A.2}$$

Definition A.4. *Let V be a set. A **metric** on V is a function $d\colon V \times V \to \mathbb{R}^+$ that for all $x, y, z \in V$ satisfies:*

(a) $d(x, z) \leq d(x, y) + d(y, z)$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) = 0$ *if and only if* $x = y$.

One can easily check that a norm induces a metric by setting, for all $x, y \in V$,

$$d(x, y) = \|x - y\|. \tag{A.3}$$

Hence, combining (A.2) and (A.3), an inner product also induces a metric by setting

$$d(x, y) = \sqrt{\langle x - y, x - y\rangle}, \quad \forall x, y \in V. \tag{A.4}$$

Definition A.5. *A* ***seminorm*** *on a vector space V is a function $p\colon V \to \mathbb{R}$ such that for all $x, y \in V$ and $\lambda \in V$*

(a) $p(x+y) \leq p(x) + p(y)$*, and*

(b) $p(\lambda x) = |\lambda|\, p(x)$*.*

Remark A.6. A seminorm is a weaker notion than a norm. A seminorm is a norm if it sasitfies $p(x) = 0$ if and only if $x = 0$.

Because a seminorm is in general not a norm it does not generate a metric and hence a topology on the vector space. However, an appropriate family of seminorms can generate a topology, as the following theorem shows.

Theorem A.7. *Let V be a vector space and Γ a family of seminorms on V that separates the points of V, i.e., if $x \in V$ and $x \neq 0$ then there is an element $p \in \Gamma$ such that $p(x) \neq 0$. Then there is a locally convex topology on V in which, for each $x_0 \in V$, the family of all sets*

$$V(x_0; p_1, \ldots, p_m; \epsilon) = \{x \in V \mid p_j(x - x_0) < \epsilon \,\forall j = 1, \ldots m\}, \qquad \text{(A.5)}$$

where $\epsilon > 0$ and $p_1, \ldots, p_m \in \Gamma$, is a base of neighbourhoods of x_0. With this topology, each of the seminorms in Γ is continuous.

A net $(x_\lambda)_{\lambda\in\Lambda}$ converges with regard to the topology stated in Theorem A.7 if and only if $\lim_{\lambda\in\Lambda} p(x_\lambda - x) = 0$ for all $p \in \Gamma$.

Definition A.8. *A* ***Banach space*** *is a normed vector space, that is complete in the associated metric given by* (A.3)*.*

A ***Hilbert space*** *is an inner product space, that is complete in the associated metric given by* (A.4)*.*

The difference between an inner product space and a Hilbert space is that an inner product space is not necessarily complete. However, we can complete the vector space and extend the inner product to turn an inner product space into a Hilbert space. Hence, an inner product space is also called a ***pre-Hilbert space***.

Proposition A.9. *Let H be an inner product space and let $\widehat{H}$ be the completion of H with respect to the metric given by equation* (A.4)*. Then there is a unique inner product on $\widehat{H}$ extending the inner product of H. We call $\widehat{H}$ endowed with this inner product the* ***Hilbert space completion*** *of H.*

Definition A.10. *Let V be a vector space with sesquilinear form. Two elements $x, y \in V$ are said to be* ***orthogonal****, denoted $x \perp y$, if $\langle x, y\rangle = 0$. Two subsets $E, F \subset V$ are said to be* ***orthogonal****, denoted $E \perp F$, if for every $x \in E$ and $y \in F$, $\langle x, y\rangle = 0$. We denote by $E^\perp$ the set of all $x \in V$ such that $x \perp y$ for all $y \in E$. This set is called the* ***orthogonal complement*** *of E. Similarly, we denote the* ***double orthocomplement*** *by $E^{\perp\perp} = (E^\perp)^\perp$, etc.*

Definition A.11. *An* ***orthonormal set*** *in a Hilbert space H is a set $V \subseteq H$ that satisfies*

(a) $\langle v, v\rangle = 1$ for all $v \in v$,

(b) $\langle v, w\rangle = 0$ for all distinct vectors $v, w \in V$.

Lemma A.12. *Let $\{v_i\}_{i\in I}$ be an orthonormal set in H. We have Bessel's Inequality*

$$\sum_{i\in I} \left|\langle v_i, x\rangle\right|^2 \leq \|x\|^2 \quad (x \in H). \tag{A.6}$$

Theorem A.13. *Let $B = \{v_i\}_{i\in I}$ be an orthonormal subset of a Hilbert space H. The following conditions are equivalent (and each defines B to be an* ***orthonormal basis*** *of H):*

(i) Any $x \in H$ can be written as $x = \sum_{i\in I} c_i v_i$.

(ii) For each $x \in H$, one has ***Parseval's inequality***

$$\sum_{i\in I} \left|\langle v_i, x\rangle\right|^2 = \|x\|^2 . \tag{A.7}$$

(iii) For any $x, y \in H$ one has

$$\langle x, y\rangle = \sum_{i\in I} \langle x, v_i\rangle\langle v_i, y\rangle. \tag{A.8}$$

(iv) B is maximal, i.e., B is not properly contained in any other orthonormal set.

(v) $B^\perp = \{0\}$.

(vi) $B^{\perp\perp} = H$.

(vii) The closure of the linear span of B is H.

Theorem A.14. *Let H be a Hilbert space and $Y \subset H$ a closed subspace of of H, then*

$$H = Y \oplus Y^{\perp},$$

i.e., Y and $Y^{\perp}$ are closed subspaces of H whose intersection is $\{0\}$ and whose sum is H.

Corollary A.15. *Let H be a Hilbert space and $Y \subseteq H$ a subspace.*

(i) *If Y is a closed subspace of a Hilbert space H, then $Y^{\perp\perp} = Y$.*

(ii) *$Y^{\perp\perp} = \overline{Y}$.*

The above theorem shows that we can decompose a Hilbert space into two orthogonal closed subspaces, each of which we can interpret as a Hilbert space in its own right. Conversely, given two (or more) Hilbert spaces, we can form a new Hilbert space by taking the *direct sum.*

Theorem A.16. *Let $H_1, \ldots, H_n$ be Hilbert spaces with respective inner products $\langle\, , \,\rangle_i$ $(i = 1, \ldots, n)$. Let H be the set of all n-tuples $(x_1, \ldots, x_n)$ with $x_i \in H_i$ $(i = 1, \ldots, n)$. Then there is a Hilbert space structure on H in which the algebraic operations and inner product are defined by*

$$\lambda(x_1, \ldots, x_n) + \mu(y_1, \ldots, y_n) = (\lambda x_1 + \mu y_1, \ldots, \lambda x_n + \mu y_n); \tag{A.9}$$

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \langle x_1, y_1 \rangle_1 + \cdots + \langle x_n, y_n \rangle_n. \tag{A.10}$$

The resulting Hilbert space H is called the ***direct sum*** *of $H_1, \ldots, H_n$, and is denoted by $H_1 \oplus \cdots \oplus H_n$ or $\oplus_{i=1}^{n} H_i$.*

The above theorem applies to *finite* sums of Hilbert spaces. We can generalise this notion to sums of Hilbert spaces over arbitrary index sets.

Theorem A.17. *Given Hilbert spaces H_i $(i \in I)$, let L be the set of all families $\{x_i\}$ such that $x_i \in H_i$ and $\sum_{i \in I} \|x_i\|^2 < \infty$. Then there is a Hilbert space structure on H in which the algebraic operations and inner product are defined by*

$$\lambda\{x_i\} + \mu\{y_i\} = \{\lambda x_i + \mu y_i\}; \tag{A.11}$$

$$\langle \{x_i\}, \{y_i\} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle_i, \tag{A.12}$$

where the sum over i on the right-hand side is defined as the supremum (in $\mathbb{R}$) of the set of all sums $\sum_{i \in F} \langle x_i, y_i \rangle_i$ over finite subsets $F \subset I$ of the index set I in which i takes values. The resulting Hilbert space H is called the ***direct sum*** *of H_i $(i \in I)$, and is denoted by $\oplus_{i \in I} H_i$.*

A.2 Bounded operators

Definition A.18. *Let V and W be normed vector spaces. A linear map $a\colon V \to W$ is called a* ***bounded linear operator*** *from V to W if there is a finite constant $C > 0$ such that $\|ax\|_W \leq C\|x\|_V$ for all $x \in V$.*

We denote the set of all bounded linear operators from V to W by $B(V, W)$. When $V = W$ we write $B(V)$ instead of $B(V, V)$.

Proposition A.19. *If $a\colon V \to W$ is a linear map from a normed vector space V to a normed vector space W, the following are equivalent:*

(i) a is bounded;

(ii) a is continuous;

(iii) a is continuous at 0.

Note that $B(V, W)$ is more than just a set. First, we can define scalar multiplication and addition pointwise by

$$(\lambda a)(x) = \lambda ax \tag{A.13}$$
$$(a + b)(x) = ax + bx, \tag{A.14}$$

which turns $B(V, W)$ into a vector space. In the special case that $V = W$ we can define multiplication by composition, i.e., $(ab)(x) = a(b(x))$, for all $a, b \in B(V)$ and $x \in V$. This gives $B(V)$ the structure of an algebra with unit $1_V\colon V \to V; x \mapsto x$. We can impose even more structure, like a norm.

Definition A.20. *Let V and W be normed vector spaces and let $a \in B(V, W)$. We define the* ***operator norm*** *on $B(V, W)$ by*

$$\|a\| = \sup\{ax \mid \|x\|_V \leq 1\}. \tag{A.15}$$

In the case that $V = W$, the operator norm is ***submultiplicative***, i.e., for all $a, b \in B(V)$

$$\|ab\| \leq \|a\|\|b\|\,. \tag{A.16}$$

Proposition A.21. *Let V and W be normed vector spaces. Then $B(V, W)$ is a normed vector space in the operator norm, where the vector operations are defined pointwise. If, in addition, W is a Banach space, then $B(V, W)$ is a Banach space.*

Theorem A.22. *Given Hilbert spaces H_1 and H_2 and $a \in B(H_1, H_2)$, there is a unique $a^* \in B(H_2, H_1)$ so that*

$$\langle ax, y\rangle_{H_2} = \langle x, a^*y\rangle_{H_1}, \tag{A.17}$$

for all $x \in H_1$ and $y \in H_2$.

The operator a^* in the previous theorem is called the ***adjoint*** of a. It has some nice properties, which we describe in the following proposition.

Proposition A.23. *For $a, b \in B(H)$, we have*

(i) $a^{**} = a$, *where* $a^{**} = (a^*)^*$;

(ii) $(a + b)^* = a^* + b^*$;

(iii) $(\lambda a)^* = \overline{\lambda} a^*$ *for* $\lambda \in \mathbb{C}$;

(iv) $(ab)^* = b^*a^*$.

The previous proposition can be easily proven by applying the definition of the adjoint. The following proposition is of great great importance, and will be one of the defining characteristics of C*-algebras, which we define in Section A.7.

Proposition A.24. *If $a \in B(H)$, then $\|a^*\| = \|a\|$ and $\|a^*a\| = \|a\|^2$.*

Theorem A.25. *If $a \in B(H)$, then*

$$\ker a^* = (\operatorname{Ran} a)^{\perp}; \tag{A.18}$$

$$\ker a = (\operatorname{Ran} a^*)^{\perp}. \tag{A.19}$$

The following two definitions describe special kinds of operators.

Definition A.26. *Let H_1 and H_2 be Hilbert spaces and $u \in B(H_1, H_2)$. We call u a* ***unitary operator*** *or a* ***unitary*** *if it satisfies one and hence all of the following equivalent conditions*

(a) $u^*u = 1_{H_1}$ *and* $uu^* = 1_{H_2}$;

(b) u *is surjective and* $\langle ux, uy\rangle_{H_2} = \langle x, y\rangle$ *for all* $x, y \in H_1$.

A related notion is that of the ***anti-unitary operator***.

Definition A.27. *Let H_1 and H_2 be Hilbert spaces.*

(a) A real-linear operator $u\colon H_1 \to H_2$ *is* ***anti-linear*** *if*

$$u(\lambda x) = \overline{\lambda} u(x) \quad (\lambda \in \mathbb{C}). \tag{A.20}$$

(b) An anti-linear operator $u\colon H_1 \to H_2$ *is* ***anti-unitary*** *if it is surjective, and*

$$\langle ux, uy\rangle_{H_2} = \overline{\langle x, y\rangle}_{H_1} \quad (x, y \in H_1). \tag{A.21}$$

The adjoint $u^*\colon H_2 \to H_1$ *of a bounded anti-linear operator* u *is defined by the property*

$$\langle u^* x, y\rangle_{H_1} = \overline{\langle x, uy\rangle}_{H_2} \quad (x \in H_2,\ y \in H_1), \tag{A.22}$$

in which case $u^*\colon H_2 \to H_1$ *is anti-linear too. Hence we may equally say that an anti-linear operator is anti-unitary if* $u^*u = 1_{H_1}$ *and* $uu^* = 1_{H_2}$.

Definition A.28. *An operator* $a \in B(H)$ *is called*

(a) ***self-adjoint*** *or* ***hermitian*** *if* $a^* = a$;

(b) ***normal*** *if* $aa^* = a^*a$;

(c) a ***projection*** *if* $a = a^* = a^2$.

Given a closed subspace Y of a Hilbert space H, Theorem A.14 states that each vector $x \in H$ can be uniquely decomposed as $x = y + z$ with $y \in Y$ and $z \in Y^{\perp}$. Hence we can define a linear operator $p_Y \in B(H)$ by

$$p_Y(y + z) = y, \tag{A.23}$$

where $y \in Y$ and $z \in Y^{\perp}$. It is easy to see that p_Y is a projection and we call it the ***(orthogonal) projection*** from H to Y. Conversely, given a projection $p \in B(H)$ we can define

$$M = \{px \in H \mid x \in H\} = \{x \in H \mid px = x\}, \tag{A.24}$$

which is a closed subspace of H. It is easy to see that $p_Y = p$. This leads to the following proposition.

Proposition A.29. *The relations* (A.23) *and* (A.24) *establish a one-to-one correspondence between closed subspaces* Y *of a Hilbert space* H *and projections* p *acting on* H.

Proposition A.30. *If* p *and* q *are the projections from a Hilbert space* H *onto closed subspaces* Y *and* Z*, respectively, the following conditions are equivalent:*

(i) $Y \subseteq Z$*;*

(ii) $qp = p$*;*

(iii) $pq = p$*;*

(iv) $\|px\| \leq \|qx\|$ *for all* $x \in H$*;*

(v) $p \leq q$*.*

Another special kind of operator is a ***compact operator***.

Definition A.31. *If* H *is a Hilbert space, a linear map* $a \colon H \to H$ *is called* ***compact*** *if the image* $a(H_{\leq 1})$ *of the closed unit ball in* H *is compact, or, equivalently, if the image* ax_n *of any bounded sequence in* H *has a convergent subsequence. We denote the set of compact operators on* H *by* $B_0(H)$*.*

Although we do not require that the linear map a is bounded, it follows immediately from the second characterisation of compact operators. So as the notation suggests $B_0(H)$ is a subset of $B(H)$. In fact, it forms a C*-subalgebra in $B(H)$, and it is a two-sided ideal.

Definition A.32. *Let* V *be a Banach space and let* $a \in B(V)$*. The* ***spectrum*** *of* a*, denoted* $\mathrm{sp}(a)$*, is the set of complex number* λ *such that* $\lambda 1_V - a$ *is not invertible.*

Lemma A.33. *Let* H *be a Hilbert space and let* $a \in B(H)$*. The following conditions are equivalent:*

(i) $\langle x, ax \rangle \geq 0$ *for all* $x \in H$*;*

(ii) $a^* = a$ *and* $\mathrm{sp}(a) \subseteq \mathbb{R}^+$*;*

(iii) $a = c^2$ *for some self-adjoint operator* c*;*

(iv) $a = b^* b$ *for some operator* b*.*

Definition A.34. *An operator* $a \in B(H)$ *satisfying one, and hence all, of the conditions in the previous lemma is called* ***positive*** *and we write* $a \geq 0$ *or* $0 \leq a$*. We denote the set of positive operators by* $B(H)^+$*.*

It is clear that a positive operator is in particular self-adjoint. For operators $a, b \in B(H)_{\mathrm{sa}}$ we write $a \leq b$ if and only if $0 \leq b - a$.Then "$\leq$" defines a partial ordening on the set of self-adjoint operators.

A.3 Symbolic calculus for normal operators

Definition A.35. *Let X be a set. A collection Σ of subsets of X is a* ***σ-algebra*** *on X if*

(a) $X \in \Sigma$;

(b) *for each set A that belongs to Σ, the set $A^c \equiv X \setminus A$ belongs to Σ;*

(c) *for each infinite sequence $\{A_i\}$ of sets that belong to Σ, the set $\bigcup_{i=1}^{\infty} A_i$ belongs to Σ, and;*

(d) *for each infinite sequence $\{A_i\}$ of sets that belong to Σ, the set $\bigcap_{i=1}^{\infty} A_i$ belongs to Σ.*

The smallest σ-algebra on X is the indiscrete σ topology given by $\{\emptyset, X\}$ and the largest on is the discrete σ-topology $\mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set of X. Every topological space X with open subsets $\mathscr{O}(X)$ gives rise to a σ-algebra called the **Borel σ-algebra**. This is the smallest σ-algebra of X that contains $\mathscr{O}(X)$. We denote it by $\mathscr{B}(X)$. Elements of $\mathscr{B}(X)$ are called **Borel sets**.

Definition A.36. *Let X be a set and Σ a σ-algebra on X. A* ***measure*** *is a map*

$$\mu \colon \Sigma \to [0, +\infty] \tag{A.25}$$

that satisfies $\mu(\emptyset) = 0$ and that is ***countably additive****, i.e.,*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i), \tag{A.26}$$

for each countable sequence $\{A_i\}$ of mutually disjoint sets that belong to Σ. A ***probability space*** *is a measure space (X, Σ, μ) for which $\mu(X) = 1$.*

Definition A.37. *A* ***measure space*** *is a triple (X, Σ, μ), where X is a set, Σ is a σ-algebra on X and μ is a measure.*

Definition A.38. *Let Ω be a compact Hausdorff space and H a Hilbert space. A* ***resolution of the identity*** *or* ***spectral measure*** *E relative to (Ω, H) is a map from the σ-algebra of all Borel sets of Ω to the set of projections in $B(H)$ such that*

(a) $E(\emptyset) = 0$, $E(\Omega) = 1$;

(b) *$E(S_1 \cap S_2) = E(S_1)E(S_2)$ for all Borel sets S_1, S_2 of Ω;*

(c) for all $x, y \in H$, the function

$$E_{x,y}\colon S \mapsto \langle y, E(S)x\rangle, \tag{A.27}$$

is a regular Borel complex measure on Ω.

Theorem A.39. *Let H be a Hilbert space and $a \in B(H)$ a normal operator. Then there exists a unique resolution of the identity E on the Borel subsets of $\mathrm{sp}(a)$ which satisfies*

$$a = \int_{\mathrm{sp}(a)} \lambda dE(\lambda), \tag{A.28}$$

by which we mean that for every $x, y \in H$:

$$\langle y, ax\rangle = \int_{\mathrm{sp}(a)} f dE_{x,y}. \tag{A.29}$$

Furthermore, every projection $E(\omega)$ commutes with every $b \in B(H)$ which commutes with a. We shall refer to this E as the ***spectral decomposition*** *of a.*

This gives us the ***symbolic calculus for normal operators***. Let $a \in B(H)$ be a normal operator and let f be a bounded Borel function on $\sigma(a)$. It is customary to denote the operator

$$\int_{\mathrm{sp}(a)} f dE \tag{A.30}$$

by $f(a)$. The mapping $f \mapsto f(a)$ is a homomorphism of the algebra of all bounded Borel functions on $\mathrm{sp}(a)$ into $B(H)$, which carries the function 1 to 1_H, which carries the identity function on $\mathrm{sp}(a)$ to a, and which satisfies

$$\overline{f}(a) = f(a)^*, \tag{A.31}$$

$$\big\|f(a)\big\| \leq \sup\{\big|f(\lambda)\big| \mid \lambda \in \mathrm{sp}(a)\}. \tag{A.32}$$

If $f \in C(\sigma(a))$, then equality holds in A.32, and therefore $f \mapsto f(a)$ is an isomorphism on $C(\mathrm{sp}(a))$ which satisfies

$$\big\|f(a)x\big\|^2 = \int_{\mathrm{sp}(a)} |f|^2\, dE_{x,x}. \tag{A.33}$$

If f_n converges uniformly to f, then $\big\|f_n(a) - f(a)\big\| \to 0$, as $n \to \infty$.
If $b \in B(H)$ and $ab = ba$, then $f(a)b = bf(a)$ for every bounded Borel function f.

Corollary A.40. *Every positive $a \in B(H)$ has a unique positive square root $b \in B(H)$, i.e., a positive operator b such that $b^2 = a$.*

We denote the square root of a positive operator a by $\sqrt{a}$. We define the ***absolute value*** of an operator a as the square root of a^*a and denote it by $|a|$.

Corollary A.41. *Any self-adjoint operator $a \in B(H)_{sa}$ has a decomposition*

$$a = a_+ - a_-, \tag{A.34}$$

where $a_\pm \geq 0$. These are unique if they also satisfy $a_+a_- = a_-a_+ = 0$. Furthermore, $a_\pm$ satisfy

$$|a| = a_+ + a_-. \tag{A.35}$$

A.4 The trace

For a complex $n \times n$ matrix $A \in M_n(\mathbb{C}) = B(\mathbb{C}^n)$ the ***trace*** is defined as the sum of its diagonal elements, i.e.,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}. \tag{A.36}$$

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^n$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \langle e_i, Ae_i \rangle, \tag{A.37}$$

and it is easy to see that this equality in fact holds for every orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$. We want to generalise the notion of trace to operators on (possibly infinite dimensional) Hilbert spaces. Let $\{v_i\}$ be an orthonormal basis for a Hilbert space H. We can naively define the trace of an operator $a \in B(H)$ to be

$$\mathrm{Tr}(a) = \sum_{i} \langle v_i, av_i \rangle, \tag{A.38}$$

but we run into problems if H is infinite dimensional. In that case the sum given in the previous equation is not independent of the choice of basis. However, it is easy to see that for all *positive* $a \in B(H)$ we have for any two bases $\{v_i\}$ and $\{v_i'\}$ that

$$\sum_{i} \langle v_i, av_i \rangle = \sum_{i} \langle v_i', av_i' \rangle, \tag{A.39}$$

where both sides may be infinite (as is the case for 1_H, if H is infinite dimensional). This gives us a well-defined map

$$\mathrm{Tr}\colon B(H)^+ \to [0, \infty]; \tag{A.40}$$

$$a \mapsto \sum_i \langle v_i, av_i \rangle, \tag{A.41}$$

where $\{v_i\}$ is an arbitrary orthonormal basis of H. Recall that the absolute value of an operator $a \in B(H)$ is defined by $|a| = \sqrt{a^*a}$. We then define the set of ***trace-class operators*** in $B(H)$ by

$$B_1(H) = \{a \in B(H) \mid \mathrm{Tr}(|a|) < \infty\}, \tag{A.42}$$

and the ***trace norm*** on $B_1(H)$ by

$$\|a\|_1 = \mathrm{Tr}(|a|). \tag{A.43}$$

Lemma A.42. *(i) For any $a \in B_1(H)$ we have*

$$\|a\| \leq \|a\|_1 = \|a^*\|_1 . \tag{A.44}$$

(ii) Any trace-class operator is compact, i.e., $B_1(H) \subseteq B_0(H)$.

(iii) For $b \in B(H)$ and $a \in B_1(H)$ one has

$$\big|\mathrm{Tr}(ab)\big| \leq \|a\|_1 \|b\| . \tag{A.45}$$

(iv) The trace-class operators $B_1(H)$ form a vector space with norm A.43.

We can now extend the trace to all trace-class operators.

Proposition A.43. *The map*

$$Tr\colon B_1(H) \to \mathbb{C}; \tag{A.46}$$

$$a \mapsto \sum_i \langle v_i, av_i \rangle, \tag{A.47}$$

where $\{v_i\}$ is some orthonormal basis of H, is well defined, linear, and independent of the choice of basis. Furthermore $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ *holds for all $a, b \in B_1(H)$.*

A.5 Topologies on the set of bounded operators

In this section H denotes a Hilbert space. We already defined a topology on $B(H)$, namely the topology induced by the operator norm (see Definition A.20). There are several other topologies on $B(H)$, which are induced by elements in H, trace-class operators, and compact operators. Figure A.1 gives a graphical representation of how these topologies relate to each other.

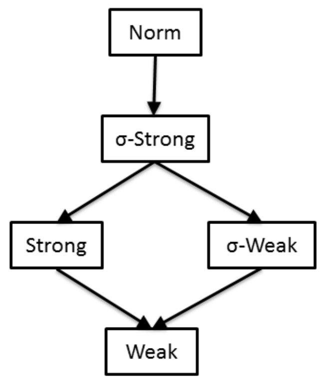


Figure A.1: This is a graphical representation of how the topologies on $B(H)$ relate. If an arrow goes from topology 1 to topology 2, this means that topology 1 is stronger than topology 2.

Definition A.44. *The* ***strong topology*** *on $B(H)$ is the topology generated by the family of seminorms of the form*

$$p_x\colon B(H) \to \mathbb{R}^+; \quad a \mapsto \|ax\|, \tag{A.48}$$

where $x \in H$. It follows that the strong topology is weaker than the norm topology on $B(H)$.

Definition A.45. *The* ***weak topology*** *on $B(H)$ is the topology generated by the family of seminorms of the form*

$$p_{x,y}\colon B(H) \to \mathbb{R}^+; \quad a \mapsto \left|\langle ax, y\rangle\right|, \tag{A.49}$$

where $x, y \in H$. It follows that the weak topology is weaker than the norm topology on $B(H)$, and weaker than the strong topology on $B(H)$.

Definition A.46. *The* ***σ-weak topology*** *or* ***ultraweak topology*** *on $B(H)$ is the topology on $B(H)$ generated by the family of seminorms of the form*

$$p_b\colon B(H) \to \mathbb{R}^+; \quad a \mapsto \left|\mathrm{Tr}(ab)\right|, \tag{A.50}$$

where $b \in B_1(H)$. Equivalently, it is the topology generated by the family of seminorms of the form

$$p_{\{x_i\},\{y_i\}}\colon B(H) \to \mathbb{R}^+; \quad a \mapsto |\sum_{i=1}^{\infty}\langle ax_i, y_i\rangle|, \tag{A.51}$$

where $\{x_i\}$ and $\{y_i\}$ are two sequences in H such that $\sum_{i=1}^{\infty}\|x_i\|^2 < \infty$ and $\sum_{i=1}^{\infty}\|y_i\|^2 < \infty$. It follows that the σ-weak topology is weaker than the norm topology on $B(H)$, but stronger than the weak topology.

Definition A.47. *The* ***σ-strong topology*** *or* ***ultrastrong topology*** *on $B(H)$ is the topology on $B(H)$ generated by the family of seminorms of the form*

$$p_b\colon B(H) \to \mathbb{R}^+; \quad a \mapsto \|ab\|, \tag{A.52}$$

where $b \in B_0(H)$. Equivalently, it is the topology generated by the family of seminorms of the form

$$p_{\{x_i\},\{y_i\}}\colon B(H) \to \mathbb{R}^+; \quad a \mapsto (\sum_{i=1}^{\infty}\|ax_i\|^2)^{1/2}, \tag{A.53}$$

where $\{x_i\}$ is a sequence in H such that $\sum_{i=1}^{\infty}\|x_i\|^2 < \infty$. It follows that the σ-strong topology is weaker than the norm topology on $B(H)$, but stronger than the strong topology and the σ-weak topology.

The following lemma summarises how these topologies relate to each other.

Lemma A.48. *Let A be a weakly closed *-subalgebra of $B(H)$. The σ-strong topology on A is stronger than the σ-weak topology, but weaker than the norm topology. The σ-strong topology on A is stronger than the strong topology, but these two topologies coincide on the unit ball A_1. The σ-weak toplogy on A is stronger than the weak topology, but these two topologies also coincide on the unit ball A_1.*

Lemma A.49. *The multiplication in a weakly closed *-subalgebra A of $B(H)$ is separately continuous in each variable with respect to both the σ-weak and the σ-strong topology. The multiplication is jointly continuous in both variables on the* *unit ball A_1 with respect to the σ-strong topology. The involution $a \mapsto a^*$ on A is continuous with respect to the σ-weak topology. The closures of a convex subset of $B(H)$ in the σ-weak and the σ-strong topology coincide. The unit ball A_1 is compact in the σ-weak topology.*

Proposition A.50. *If $\{a_\lambda\}$ is an increasing net bounded above in $B(H)_{sa}$, then $\{a_\lambda\}$ has a supremum $a \in B(H)_{sa}$, and a is also a strong (and weak) limit of $\{a_\lambda\}$. Similarly for a decreasing net and its greatest lower bound.*

A.6 Dual spaces

In the previous section we studied linear maps between normed vector spaces V and W. In this section we study the special case that $W = \mathbb{C}$. Since $\mathbb{C}$ is a Banach space, it follows from Proposition A.21 that $B(V, \mathbb{C})$ with the operator norm (A.15) is a Banach space. We call this Banach space the ***dual space*** of V and denote it by V^*. The elements of V^* are called ***functionals***. As in the previous section, we can impose the operator norm on V^*, which turns it into a Banach space by Proposition A.21. In the main text we often use the ***double dual***, that is $V^{**} = (V^*)^*$, as well as the fact that we can embed V into V^{**}, as stated in the following theorem.

Proposition A.51. *For any normed vector space V, the map $x \mapsto \widehat{x}$ from $V \to V^{**}$, given by*

$$\widehat{x}(\Lambda) = \Lambda(x), \quad \Lambda \in V^*, \tag{A.54}$$

is isometric (and hence injective), mapping V onto a closed subspace $\widehat{V} \subseteq V^{**}$.

Hence the map $V \to V^{**}$ is an isometric isomorphism onto its image $\widehat{V}$. We often use this fact to identify V with $\widehat{V}$ and view V as a closed subspace of V^{**}.

There is another norm on the set of functionals on V, which is weaker than the operator norm, we call it the **weak* topology** or **w*-topology**. It is the weakest topology that makes all functionals $\Lambda \in V^*$ continuous. It follows from Proposition A.19 that all functionals $\Lambda \in V^*$ are continuous, hence the w^*-topology is weaker than the operator topology. Equivalently, it is the **topology of pointwise convergence** because of the following proposition:

Proposition A.52. *Suppose that Λ_n is a sequence in V^*. We have $\Lambda_n \to \Lambda$ in the w^*-topology if and only if $\Lambda_n(x) \to \Lambda(x)$ in $\mathbb{C}$ for all $x \in V$.*

Theorem A.53 (Banach-Alaoglu). *Let V be a normed vector space. Then the unit ball of V^*, i.e., $V^*_{\leq 1} \equiv \{\Lambda \in V^* \mid \|V\| \leq 1\}$, is compact with respect to the w^*-topology.*

Theorem A.54. *For any Hilbert space H, we have dualities and double dualities*

$$B_0(H)^* \cong B_1(H); \tag{A.55}$$
$$B_1(H)^* \cong B(H); \tag{A.56}$$
$$B_0(H)^{**} \cong B(H); \tag{A.57}$$
$$B_1(H)^{**} \cong B(H)^*, \tag{A.58}$$

where the symbol $\cong$ stands for isometric isomophism. *Explicitly:*

- *Any norm-continuous linear map $\omega\colon B_0(H) \to \mathbb{C}$ takes the form*

  $$\omega(b) = \mathrm{Tr}(ab), \tag{A.59}$$

  for some $a \in B_1(H)$ uniquely determined by ω, and vice versa, giving a bijective correspondence between $\omega \in B_0(H)^*$ and $a \in B_1(H)$ satisfying

  $$\|\omega\| = \|a\|_1 . \tag{A.60}$$

  This equality remains valid if ω is regarded as an element of $B(H)^*$ via (A.58) *and the isometric embedding $B_1(H) \hookrightarrow B_1(H)^{**}$.*

- *Any norm-continuous linear map $\xi\colon B_1(H) \to \mathbb{C}$ takes the form*

  $$\xi(a) = \mathrm{Tr}(ab), \tag{A.61}$$

  for some $b \in B(H)$ uniquely determined by ξ, and vice versa, giving a bijective correspondence between $\xi \in B_1(H)^*$ and $b \in B(H)$ satisfying

  $$\|\xi\| = \|b\| . \tag{A.62}$$

A.7 C*-algebras

We saw in section A.2 that for each normed vector space V the set $B(V)$ has the structure of an algebra. Equation (A.16), Proposition A.23 and Proposition A.24 described some nice properties that the map $a \mapsto a^*$ has. We can generalise this to the notion of a C*-algebra. By an algebra, we always mean an associative algebra.

Definition A.55. *(a) A **normed algebra** is an algebra with submultiplicative norm, i.e.,*

$$\|ab\| \leq \|a\| \|b\| , \quad a, b \in A. \tag{A.63}$$

*We call a normed algebra algebra **unital** if it admits a unit, i.e., an element $1_A \in A$ such that $a1_A = 1_A a = a$ for all $a \in A$.*

(b) *A **Banach algebra** is a Banach space A that is simultaneously a normed algebra.*

(c) *An **involution** on an algebra A is a conjugate-linear map $a \mapsto a^*$ on A such that $a^{**} = a$ and $(ab)^* = b^*a^*$, for all $a, b \in A$. The pair $(A, *)$ is called a ***-algebra**.*

(d) *A **C*-algebra** is a Banach algebra A with involution in which*

$$\|aa^*\| = \|a\|^2, \quad a \in A. \tag{A.64}$$

Because a C*-algebra is in particular a normed algebra, it follows that $\|a\| = \|a^*\|$.

As in Definition A.28, we can define certain special kinds of elements.

Definition A.56. *Let A be a *-algebra. An element $a \in A$ is called:*

(a) ***self-adjoint** or **hermitian** if $a^* = a$;*

(b) ***normal** if $aa^* = a^*a$;*

(c) *a **projection** if $a = a^* = a^2$.*

*Furthermore, if A is unital, we call an element $a \in A$ **unitary** if $a^*a = aa^* = 1_A$.*

We denote the set of all self-adjoint elements by A_{sa}. Each element $a \in A$ has a unique decomposition $a = b + ic$ with b and c in A_{sa}, namely $b = \frac{1}{2}(a + a^*)$ and $c = -\frac{1}{2}i(a - a^*)$.

Not every algebra is unital, but it is much easier to work with unital algebras than with non-unital algebras. Therefore, we embed a non-unital algebra into an appropriate unital algebra, which we denote by $\widetilde{A}$.

Definition A.57. *Let A be an algebra. The **unitisation** $\widetilde{A}$ of A is $A \oplus \mathbb{C}$ as a vector space with multiplication given by*

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu). \tag{A.65}$$

The unit in $\widetilde{A}$ is $(0, 1)$. The map $a \mapsto (a, 0)$ gives an injective homomorphism from A to $\widetilde{A}$. If we identify A with its image under this homomorphism, we can view A as an ideal of $\widetilde{A}$. We then write $a + \lambda$ for (a, λ). If A is a normed algebra we can turn $\widetilde{A}$ into a normed algebra by setting $\|a + \lambda\| = \|a\| + |\lambda|$. In that case A is a closed subalgebra of $\widetilde{A}$. Observe furthermore that if A is a Banach space, then $\widetilde{A}$ is also a Banach space. If A is a C*-algebra, then $\widetilde{A}$ is also a C*-algebra by setting $(a + \lambda)^* = a^* + \overline{\lambda}$.

Definition A.58. *Let A be a C*-algebra and $a \in A$. The* ***spectrum*** *of a, denoted* $\mathrm{sp}(a)$*, is the set of complex numbers λ such that $\lambda 1_{\widetilde{A}} - a$ is not invertible in $\widetilde{A}$. The* ***point spectrum*** *of a, denoted* $\mathrm{sp}_{\mathrm{p}}(a)$ *is the set of its eigenvalues, so* $\mathrm{sp}_{\mathrm{p}}(a) \subseteq \mathrm{sp}(a)$*.*

Lemma A.59. *If $x \in A_{sa}$ then* $\mathrm{sp}(x) \subset \mathbb{R}$*. If $1_A \in A$ and u is unitary,* $\mathrm{sp}(u)$ *is contained in the unit circle.*

Definition A.60. *An element $x \in A$ is called* ***positive*** *if x is self-adjoint and* $\mathrm{sp}(x) \subset \mathbb{R}^+$*. We denote the set of all positive elements of A by A^+.*

Theorem A.61. *Let A be a C*-algebra and A^+ be the set of its positive elements. Then*

$$A^+ = \{a^2 \mid a \in A_{sa}\} \tag{A.66}$$
$$= \{a^* a \mid a \in A\}. \tag{A.67}$$

Lemma A.62. *Each self-adjoint element $a \in A$ has a decomposition*

$$a = a_+ - a_-, \tag{A.68}$$

where $a_+, a_- \in A^+$ and $a_+ a_- = 0$. These conditions determine $a_\pm$ uniquely, and $\|a\| = \max\{\|a_+\|, \|a_-\|\}$*.*

Using the positive elements in a C*-algebra, we can define something called an ***approximate unit***. This is another way to deal with non-unital C*-algebras.

Definition A.63. *Let A be a C*-algebra. An* ***approximate unit*** *is a increasing net $\{e_\lambda\}_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of A such that $a = \lim_\lambda a e_\lambda$ for all $a \in A$. Equivalently, $a = \lim_\lambda e_\lambda a$, for all $a \in A$.*

Theorem A.64. *Every C*-algebra A admits an approximate unit. Indeed, if Λ is the upwards-directed set of all $a \in A^+$ such that $\|a\| < 1$ and $e_\lambda = \lambda$, for all $\lambda \in \Lambda$, then $\{e_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for A, which we call the* ***canonical approximate unit****.*

Theorem A.65. *Let ω is a bounded linear functional on a C*-algebra A. The following are equivalent.*

(i) ω is positive.

(ii) For each approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ of A, we have $\|\omega\| = \lim_\lambda \omega(e_\lambda)$.

(iii) There exists an approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ of A such that $\|\omega\| = \lim_\lambda \omega(e_\lambda)$.

Definition A.66. *A* ****-homomorphism*** *between C*-algebras A and B is a linear map* $\varphi\colon A \to B$ *that for all* $a, b \in A$ *satisfies*

$$\varphi(ab) = \varphi(a)\varphi(b); \tag{A.69}$$
$$\varphi(a^*) = \varphi(a)^*. \tag{A.70}$$

If, further, φ *is bijective, it is called a* ****-isomorphism****. A* ****-anti-homomorphism*** *between C*-algebras A and B is a linear map* $\varphi\colon A \to B$ *that for all* $a, b \in A$ *satisfies*

$$\varphi(ab) = \varphi(b)\varphi(a); \tag{A.71}$$
$$\varphi(a^*) = \varphi(a)^*. \tag{A.72}$$

Similarly, a ****-anti-isomorphism*** *is a bijective *-anti-homomorphism.*

Theorem A.67. *Let* $\varphi\colon A \to B$ *be a nonzero *-homomorphism between C*-algebras.*

*(i) The *-homomorphism* φ *is continuous, with norm* $\|\varphi\| = 1$.

(ii) Its kernel $\ker\varphi$ *is an ideal in A.*

(iii) If φ *is injective, then it is isometric.*

*(iv) An *-isomorphism of C*-algebras is automatically isometric.*

(v) The range $\operatorname{Ran}\varphi$ *is a C*-subalgebra of B; in particular,* $\varphi(A)$ *is closed in B.*

A.8 Von Neumann Algebras

Definition A.68. *Let A be an algebra and* $B \subseteq A$*. We define its* ***commutant*** B' *to be the set of all elements of A that commute with all elements of B, i.e.,*

$$B' = \{a \in A \mid ab = ba \,\forall b \in B\}. \tag{A.73}$$

The ***double commutant*** B'' *of B is* $(B')'$*. Similarly,* $B''' = (B'')'$*, etc.*

The following lemma states some elementary properties of the commutant.

Lemma A.69. *Let A be an algebra and* $B \subseteq A$.

(i) B' *is a subalgebra of A.*

*(ii) If A is a *-algebra and B is self-adjoint, then* B' *is a *-subalgebra of A.*

(iii) $B \subseteq B''$ *and* $B' = B'''$.

(iv) If A *is a normed algebra, then* B' *is closed.*

Theorem A.70 (Von Neumann bicommutant theorem)**.** *Let* M *be a *-subalgebra of* $B(H)$ *containing the identity operator. Then the following conditions are equivalent.*

(i) $M = M''$;

(ii) M *is strongly closed;*

(iii) M *is weakly closed.*

Definition A.71. *A* ***von Neumann algebra*** *is a strongly closed C*-subalgebra of B(H) containing* 1_H.

Remark A.72. The von Neumann bicommutant theorem proves that every von Neumann algebra is also weakly closed. Combining Lemma A.69 and Theorem A.70 we see that for every *-subalgebra C of $B(H)$ the double commutant C'' is a von Neumann algebra.

The following corollary is an extension of von Neumann's bicommutant theorem, which shows that a von Neumann algebra is also σ-weakly and σ-strongly closed.

Corollary A.73. *Let* M *be a *-subalgebra of* $B(H)$ *containing the identity operator and denote the closed unit ball of* M *by* M_1. *The following are equivalent:*

(i) M *(or* M_1*) is weakly closed;*

(ii) M *(or* M_1*) is strongly closed;*

(iii) M *(or* M_1*) is* σ*-weakly closed;*

(iv) M *(or* M_1*) is* σ*-strongly closed;*

(v) $M = M''$.

By definition of a von Neumann algebra it is a strongly closed C*-algebra containing the unit element. So for all C*-algebras A acting on a Hilbert space H and containing 1_H, the strong closure $\overline{A}^s$ is a von Neumann algebra. But what can we say about the strong closure of a C*-algebra that does not contain 1_H? The following definition and proposition show that for certain non-unital C*-algebras their strong closures are von Neumann algebras.

Definition A.74. *We say that a C*-subalgebra A of $B(H)$ acts **non-degenerately** on H if for each non-zero vector $x \in H$ there is an element $a \in A$ with $a(x) \neq 0$.*

Proposition A.75. *Let A be a C*-subalgebra of $B(H)$ with strong closure M. Then M is a weakly closed C*-algebra with unit. Furthermore, if A acts non-degenerately on H, then $M = A''$.*

Definition A.76. *A **factor** is a von Neumann algebra whose center consists of only the scalar multiples of the identity.*

Remark A.77. If H is a Hilbert space, then $B(H)' = \mathbb{C}\,1_H$, i.e., $B(H)$ is a factor. For it is obvious that $(\mathbb{C}\cdot 1_H)' = B(H)$, and since $\mathbb{C}\cdot 1_H$ is a von Neumann algebra containing 1_H, the von Neumann double commutant theorem implies that $\mathbb{C}\cdot 1_H = (\mathbb{C}\cdot 1_H)''$, so $B(H)' = \mathbb{C}\cdot 1_H$.

Definition A.78. *If a is a self-adjoint element in a von Neumann algebra M, then the least projection $p \in M$ such that $pa = a$ is called the **range projection** of a and is denoted by $r(a)$.*

Theorem A.79. *If M is a von Neumann algebra, then it contains the range projection of all of its elements.*

Proposition A.80. *If M is a von Neumann algebra and $a, b \in M$ with $aMb = \{0\}$, then there is a projection z in the center $Z(M)$ such that $x \in zM$ and $y \in (1-z)M$.*

Corollary A.81. *Let M be a factor, and let a, b be a pair of elements in M. Assume that $aMb = 0$. Then it follows that $a = 0$ or $b = 0$.*

A.9 Representations

We view von Neumann algebras a weakly-closed *-subalgebras of $B(H)$ containing the unit 1_H, so all von Neumann algebras are *concretely represented*. However, we do not view C*-algebras as norm-closed *-subalgebras of $B(H)$ for a certain Hilbert space H. We use ***representations*** to map C*-algebras onto $B(H)$, which makes studying them a lot easier.

In this section we use the terminology "state" and "pure state". These concepts are defined in Chapter 3. In short, a state is a positive functional of norm 1, and a pure state is an extreme point of the convex set of states on a C*-algebra. We denote the set of states by $S(A)$ and the set of pure states by $P(A)$.

Definition A.82. *A* ***representation*** *of a C*-algebra A is a pair (H, π) where H is a Hilbert space and $\pi\colon A \to B(H)$ is a *-homomorphism. We say (H, π) is* ***faithful*** *if π is injective.*

Because $\pi(A)$ is a C*-subalgebra of $B(H)$, we can define non-degenerate representations, using Definition A.74.

Definition A.83. *A representation (H, π) is* ***non-degenerate*** *if $\pi(A)$ is non-degenerate on H.*

Definition A.84. *Let A be a C*-algebra and (H, π) a representation of A. A vector $x \in H$ is called* ***cyclic*** *if $\pi(A)x$ is dense in H. If (H, π) admits a cyclic vector it is called a* ***cyclic representation***.

To emphasise that a representation (H, π) is cyclic, with cyclic vector x, we sometimes write (H, π, x) instead of (H, π). It is clear from the definition that a cyclic representation is non-degenerate.

Definition A.85. *If $(H_i, \pi_i)_{i\in I}$ is a family of representations of A, their* ***direct sum*** *is the representation (H, π) obtained by setting $H = \oplus_{i\in I} H_i$ and $\pi(a)(\{x_i\}_i) = \{\pi_i(a)(x_i)\}_i$ for all $a \in A$ and $\{x_i\} \in H$.*

It follows from the definition of the direct sum of Hilbert spaces that (H, π) is indeed a representation of A. A consequence of this definition is stated in the following proposition.

Proposition A.86. *Let $(H_i, \pi_i)_{i\in I}$ be a family of representation and (H, π) its direct sum. If for each non-zero element $a \in A$ there is an index $i \in I$ such that $\pi_i(a) \neq 0$, then (H, π) is faithful.*

For a given C*-algebra A, we want to find a faithful representation. The ***Gelfand-Neumark-Segal construction*** or ***GNS-construction*** for short, will do just that. The ***GNS-representation***, is a representation that can be constructed from each positive functional, hence in particular from each state. We will explain the construction, but will leave out the details. Let σ be a positive functional on a C*-algebra A. Then

$$N_\sigma = \{a \in A \mid \sigma(a^*a) = 0\}, \tag{A.74}$$

is a closed left ideal of A and the map

$$\begin{aligned} (A/N_\sigma) \times (A/N_\sigma) &\to \mathbb{C} \\ (a + N_\sigma, b + N_\sigma) &\mapsto \sigma(a^*b) \end{aligned} \tag{A.75}$$

is a well defined inner product on A/N_σ. Hence A/N_σ is a pre-Hilbert space and Proposition A.9 extends it to a Hilbert space, which we call H_σ. For $a \in A$ define $\pi(a) \in B(A/N_\sigma)$ by setting

$$\pi(a)(b + N_\sigma) = ab + N_\sigma. \tag{A.76}$$

The inequality $\|\pi(a)\| \leq \|a\|$ holds and $\pi(a)$ has a unique extension to a bounded operator $\pi_\sigma(a)$ on H_σ. The map

$$\pi_\sigma \colon A \to B(H_\sigma); \tag{A.77}$$
$$a \mapsto \pi_\sigma(a) \tag{A.78}$$

is a *-homomorphism, hence (H_σ, π_σ) is a representation of A. It is called the GNS-representation associated to σ. It turns out that this representation is cyclic.

Theorem A.87. *Let A be a C^*-algebra and $\sigma \in S(A)$. Then there is a unique vector $x_\sigma \in H_\sigma$ such that for all $a \in A$,*

$$\sigma(a) = \langle a + N_\sigma, x_\sigma \rangle \tag{A.79}$$

Moreover, x_σ is a unit cyclic vector for (H_σ, π_σ), and

$$\pi_\sigma(a)x_\sigma = a + N_\sigma \tag{A.80}$$

for all $a \in A$.

Although the GNS-representation associated with a positive functional will in general not be faithful, we can take the direct sum of appropriate representations and use Proposition A.86 in order to obtain a faithful one.

Definition A.88. *Let A be a nonzero C^*-algebra. Then we define its* ***universal representation*** *to be the direct sum of all representation (H_σ, π_σ), where σ ranges over $S(A)$. We denote it by (H_u, π_u).*

Theorem A.89 (Gelfand-Neumark)**.** *If A is a C^*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

It is by Proposition A.86 enough to show that for every $a \in A$ there is a $\sigma \in S(A)$ such that $\pi_\sigma(a) \neq 0$. The following proposition states that we do not actually need *all* states to form a faithful representation. Just using the pure states turns out to be enough.

Proposition A.90. *If a is a non-zero element of a C^*-algebra A, there is a pure state $\sigma \in P(A)$ such that $\pi_\sigma(a) \neq 0$.*

This leads to another faithful representation called the ***atomic representation***.

Definition A.91. *Let A be a nonzero C^*-algebra. Then we define its **atomic representation** to be the direct sum of all representation (H_σ, π_σ), where σ ranges over $P(A)$. We denote it by (H_a, π_a).*

Definition A.92. *We say that two representations (H_1, π_1) and (H_2, π_2) of a C^*-algebra A are **unitarily equivalent** if there is a unitary $u\colon H_1 \to H_2$ such that $u\pi_1(a)u^* = \pi_2(a)$ for all $a \in A$.*

Proposition A.93. *Let (H_1, π_1, x_1) and (H_2, π_2, x_2) be cyclic representation of a C^*-algebra A. Then there is a unitary $u\colon H_1 \to H_2$ such that $x_2 = u(x_1)$ and $\pi_2(a) = u\pi_2(a)u^*$ for all $a \in A$ if and only if*

$$\langle x_1, \pi_1(a)x_1\rangle = \langle x_2, \pi_2(a)x_2\rangle, \tag{A.81}$$

for all $a \in A$.

Definition A.94. *A representation (H, π) of a C^*-algebra A is said to be **irreducible** if $\{0\}$ and H are the only closed subspaces invariant under $\pi(A)$.*

Theorem A.95. *Let (H, π) be a non-zero representation of a C^*-algebra A. The following conditions are equivalent:*

(i) *(H, π) is irreducible.*

(ii) *$\pi(A)' = \mathbb{C}\,1_H$.*

(iii) *$\pi(A)$ is strongly dense in $B(H)$.*

(iv) *For any two vectors $x, y \in H$ with $x \neq 0$ there is an $a \in A$ such that $\pi(a)x = y$.*

(v) *Each non-zero vector in H is cyclic for $\pi(A)$.*

(vi) *(H, π) is unitarily equivalent to the GNS-representation associated with a pure state of A.*

The equivalence between items (i) and (ii) is known as ***Schur's lemma***.

A.10 Lattice of projections

Definition A.96. *A **lattice** is a partially ordered set X in which for pair of elements $x, y \in X$ there exists:*

- *an element $x \vee y$, called the **supremum** of x and y, such that*

$$x \leq x \vee y; \tag{A.82}$$
$$y \leq x \vee y, \tag{A.83}$$

  and if $x \leq z$ and $y \leq z$ for some $z \in X$, then $x \vee y \leq z$;

- *an element $x \wedge y$, called the **infimum** or x and y, such that*

$$x \geq x \wedge y; \tag{A.84}$$
$$y \geq x \wedge y, \tag{A.85}$$

  and if $x \geq z$ and $y \geq z$ for some $z \in X$, then $x \wedge y \geq z$.

Definition A.97. *A lattice X is **complete** if every set $S \subseteq X$ has a supremum $\bigvee S$, as well as an infimum $\bigwedge S$. A complete lattice X has a smallest element $0 = \bigvee X$ and a largest element $1 = \bigwedge X$.*

Definition A.98. *A lattice X with 0 and 1 is **orthocomplemented** if there is a map $x \mapsto x^\perp$, called the **orthocomplementation**, that satisfies*

(a) $x^{\perp\perp} = x$*;*

(b) $x \leq y$ *implies* $y^\perp \leq x^\perp$*;*

(c) $x \vee x^\perp = 1$ *and* $x \wedge x^\perp = 0$.

*An orthocomplemented lattice X is called **orthomodular** if it additionally satisfies*

(d) if $x \leq y$, then $y = x \vee (x^\perp \wedge y)$.

Definition A.99. *Let X be a complete lattice. A non-zero element $x \in X$ is an **atom** if $y \leq x$ implies that $y = 0$ or $y = x$.*

Theorem A.100. *The set $\mathcal{P}$ of projections in a von Neumann algebra M is an orthomodular lattice under the ordening induced from M_{sa}, and $(pqp)^n \searrow p \wedge q$ strongly for each pair of elements $p, q \in M$. If M acts on a Hilbert space H, then the ordening in $\mathcal{P}$ corresponds to the natural ordening of the subspaces $p(H)$ associated* *with the projections $p \in M$, and for each pair $p, q \in \mathcal{P}$ the projection $p \wedge q$ is the projection onto $p(H) \cap q(H)$ and the projection $p \vee q$ is the projection onto the closed linear span of $p(H) \cup q(H)$. If p and q are two commuting projections in M, then*

$$p \wedge q = pq \quad \textit{and} \quad p \vee q = p + q - pq. \tag{A.86}$$

Corollary A.101. *The lattice of projections $\mathcal{P}$ in a von Neumann algebra M is complete. More specifically, an increasing net $(p_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{P}$ converges σ-weakly (and σ-strongly) to $p = \bigvee_{\lambda \in \Lambda} p_\lambda$, which is the supremum of $(p_\lambda)_{\lambda \in \Lambda}$ in M_{sa}, as well as in $B(H)_{sa}$ when M acts on a Hilbert space H. Similarly for decreasing nets and infima.*

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