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C^*-Algebraic Quantization and the Origin of Topological Quantum Effects

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Abstract. Concepts from the theory of abstract operator algebras are used to solve the problem of quantizing a particle moving on an arbitrary locally compact homogeneous space. Inequivalent quantizations are identified with inequivalent irreducible representations of the corresponding C^*-algebra. Topological terms in the action (or Hamiltonian) are found to be representation-dependent, and are automatically induced by the quantization procedure. Known charge quantization conditions turn out to be identically satisfied. Several examples are considered, among them the Dirac monopole and the Aharonov-Bohm effect.

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1. Introduction

The study of topological quantum effects, with such diverse areas of application as magnetic monopoles [1, 2], general relativity [3], anomalies in quantum field theory [4], the quantum Hall effect [5], and high-T_c superconductivity [6], is becoming one of the most important and fascinating branches of theoretical physics. In this Letter, we address two of the principal problems in this field: the quantization of a physical system formulated on a topologically nontrivial configuration space and, in direct relation to this, the origin and explicit form of so-called 'topological' terms in the Hamiltonian (or action) of such systems. The former problem will be solved for a special class of configuration spaces Q, namely the locally compact homogeneous ones, i.e., $Q = G/H$ for some locally compact group G with closed subgroup H. For technical reasons, our approach to the second issue at this stage is justified only for amenable type I groups; this class includes, among others, all compact groups, all locally compact Abelian groups, as well as all (semidirect) products of such groups.

It is well known that a number of systems of this type cannot be quantized by imposing canonical commutation relations, because these would conflict with the global structure of the system (cf., e.g., [3]), or lead to problems related to the (essential) self-adjointness of the observables involved [7]. Accordingly, several methods to deal with general systems possessing a topologically nontrivial configuration space

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have been proposed in the literature, like path-integral quantization [8, 9] and geometric quantization [10, 11], having been amended by certain cohomological techniques in [12] and [13], respectively, star (deformation) quantization [14], which has been reformulated as a C^*-algebra theory [15], Borel quantization [16], and various others. These techniques have led to important insights concerning the existence of various 'inequivalent' quantum theories corresponding to a given classical system, which appear to go hand in hand with certain 'topological' terms in the action (or Hamiltonian). Thus kinematical and dynamical aspects of the quantization procedure turn out to be inextricably linked to each other.

The main purpose of the quantization method (yet another one!) presented in this Letter is to explain this very linkage in a transparent algebraic language, providing a direct connection between the existence of inequivalent quantizations, which we identify with superselection sectors, and the emergence of topological terms in the Hamiltonian. To do so, we propose to rely on the insights of Segal [17] and Haag and Kastler [18], according to whom quantization of a given system amounts to the specification of a C^*-algebra $\mathscr{A}$ whose self-adjoint elements correspond to the physical observables of the system. (Note that a C^*-algebra is an object which is isomorphic to a norm-closed sub-algebra of the algebra of all bounded operators on some Hilbert space, assumed to be separable in the quantum-mechanical case.)

Before identifying $\mathscr{A}$, it should be remarked that the representation $Q = G/H$ is highly nonunique, so that the choice of G should be restricted by demanding that it respects certain additional structures, e.g., a metric. Even so, one may form arbitrary nontrivial extensions E of G by K ($G = E/K$), and let E act on Q via the canonical epimorphism $p : E \to G$. Such extensions can be classified by cohomological methods [19], and, as will rapidly become clear, inequivalent extensions will lead to different quantizations in the present method. (This is true for trivial extensions of G as well, but these just correspond to the incorporation of internal degrees of freedom.) In ordinary quantum mechanics, only central extensions are taken into account [20], whereas the proposed method is more general, so that one is faced with an *embarras du choix* which may be a blessing or a curse. In any case, it will turn out that *topological quantum effects are caused by a non-minimal choice of the group G.*

2. Algebraic Quantization

But let us agree on a particular representation $Q = G/H$. We then quantize the system at one stroke by stating* that the C^*-algebra $\mathscr{A}$ is given by the so-called crossed product (also called covariance algebra [24, 25])

$$\mathscr{A} = C_0(Q) \times_\alpha G. \tag{1}$$

* This proposal, which is just a C^*-algebraization of Mackey's quantization method for homogeneous spaces [21], was independently made in [22] and (later) in [23]. It should be stressed, that the embedding of $C_0(Q)$ in $\mathscr{A}$ is part of the identification of the quantum algebra, since G and H as such are not uniquely determined by $\mathscr{A}$.

Following [25], we define (1) by first introducing the semidirect product bundle $\mathscr{S}$, whose base space is G, and whose fibers are isomorphic to the C^*-algebra $C_0(Q)$, on which the group G acts via $*$-automorphisms α_x defined by $\alpha_x[\varphi](q) = \varphi(x^{-1}q)$ for $\varphi \in C_0(Q)$ (here $q \to xq$ denotes the action of G on Q). Elements of the bundle are then pairs $\langle A, x \rangle$, $A \in C_0(Q)$, $x \in G$, which are multiplied according to

$$\langle A, x \rangle \cdot \langle B, y \rangle = \langle A\alpha_x[B], xy \rangle. \tag{2}$$

The adjoint is given by

$$\langle A, x \rangle^* = \langle \alpha_{x^{-1}}[A^*], x^{-1} \rangle. \tag{3}$$

The crossed product (1) is the canonical C^*-completion [25] of the L^1-cross-sectional algebra of $\mathscr{S}$, equipped with the usual convolution product, and the involution $f^*(x) = \Delta(x^{-1})f(x^{-1})^*$, with Δ the modular function on G.

Once the appropriate C^*-algebra pertaining to a physical system has been specified, one needs to construct its representations by bounded operators on a concrete Hilbert space in order to investigate the quantum mechanics of the system. It may be shown [25] that the irreducible representations π of $\mathscr{A}$ are in one-to-one correspondence with the pairs (U, π'), where U is a unitary representation of G and π' is a representation of the C^*-algebra $C_0(Q)$ satisfying the 'covariance' condition

$$(U(x)\pi'(\varphi)U^*(x)) = \pi'(\alpha_x[\varphi]). \tag{4}$$

It then follows from the standard theory of induced group representations [21, 26] that *the irreducible representations* π *of (1) are in one-to-one correspondence with the irreducible unitary representations* $\hat{h}$ *of* H.

Given $\hat{h}$ and a (measurable) section $s : Q \to G$, the representation $\pi_{\hat{h}}(\mathscr{A})$ may be canonically realized on the Hilbert space $\mathscr{H} = L^2(G/H, \mu, \mathscr{H}_{\hat{h}})$ (i.e., the space of wave functions on Q with values in $\mathscr{H}_{\hat{h}}$), where μ is a certain (known) quasi-invariant measure on G/H, with associated Radon–Nikodym derivative ρ_x [26, 21], and $\mathscr{H}_{\hat{h}}$ is the carrier space of $\hat{h}$. For $\psi \in \mathscr{H}$ one then has

$$\begin{aligned}(\pi_{\hat{h}}(f)\psi)(q) &= \int_G \mathrm{d}x\, f(q, x)(U_{\hat{h}}(x)\psi)(q);\\ (U_{\hat{h}}(x)\psi)(q) &= \rho_x^{1/2}(q)\overline{D_{\hat{h}}(s(q)^{-1}xs(x^{-1}q))}\psi(x^{-1}q),\end{aligned} \tag{5}$$

where we have identified a section f of $\mathscr{S}$ with a function f on $Q \times G$. In case that canonical quantization applies, the canonical commutation relations are equivalent to (4) [21], which itself may be rederived from (5). The 'momenta' (divided by $\hbar$) are just the generators of the $U(x)$. Finally, note that these irreducible representations are not faithful, unless H is trivial.

3. Examples

Let me illustrate this with a few examples. The simplest case is $Q = \mathbb{R}$. Then H is trivial, and the algebra (1) has a unique irreducible representation. Thus one is immediately led to the Stone–von Neumann uniqueness theorem; in fact, the reasoning above here reduces to Mackey's proof of this theorem [21].

An interesting example is a particle moving on the half-line $\mathbb{R}^+$. Here $\mathbb{R}^+$ is to be regarded as a multiplicative group which acts on itself by left-multiplication. The method above then trivially leads to the correct quantization of this system, in which $\mathscr{H} = L^2(\mathbb{R}^+, dx/x)$, on which the position operator x acts in the usual way, but where the 'canonical' momentum operator is given by $p = -ix\, d/dx$, cf. [3].

Next, we consider the case $Q = \mathbb{R}^3$, interpreted as a flat Riemannian manifold. This allows the maximal symmetry group $E(3) = \mathrm{SO}(3) \circledS T^3$. Within the context of scalar quantum mechanics, one may take $G = \mathrm{SU}(2) \circledS T^3$. The canonical action of G on $\mathbb{R}^3$ shows that the little group H is SU(2), so that the irreducible representations of the quantum algebra (1) are labeled by the spin of the particle (compare this with Mackey's derivation of spin [21]).

Deleting the origin (or any other point) from the above configuration space gives $Q = \mathbb{R}^3 - \{0\} \equiv S^2 \times \mathbb{R}$, which may be written as a coset by choosing $G = \mathrm{SO}(3) \times \mathbb{R}$. Here the first factor acts on Q in the obvious way, whereas elements of the second factor act on the radial coordinate alone (cf. the next example). Hence $H = \mathrm{SO}(2)$, so that a quantum-mechanical particle moving on Q has inequivalent representations labeled by $\hat{h} \equiv n \in \mathbb{Z}$ (here $D_n(\alpha) = \exp(in\,\alpha)$ for $\alpha \in \mathrm{SO}(2)$). For later use, we note that, in terms of spherical coordinates on Q, a suitable section $s: Q \to G$ is given by

$$s(r, \phi, \theta) = (R(\phi, \theta, -\phi), \log r), \tag{6}$$

where $R \in \mathrm{SO}(3)$ is parametrized with the conventional Euler angles.

Finally, we examine the case $Q = \mathbb{R}^3 - \{z-\text{axis}\}$, which is equivalent to $Q = \mathbb{R} \times \mathbb{R} \times S^1$. Let $q = (r, z, \phi)$ in cylindrical coordinates, and choose $(a, b, c) \equiv x \in G = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to act on Q according to $xq = (e^a r, b + z, c + \phi \mid 2\pi)$. This shows that $H = Z$, whose irreducible unitary representations are labeled by an angle $\theta \in [0, 2\pi)$, so that $D_\theta(n) = \exp(in\,\theta)$. In agreement with other approaches [8, 27, 3] we thus find that, within 'scalar' quantum mechanics, the present system has inequivalent quantizations labeled by a θ-angle.

4. Dynamics and Topological Actions

In the algebraic description, time evolution in the algebra is not governed by a given Hamiltonian, but by a group of *-automorphisms (cf. [28]). In practically all infinite systems, and in the finite systems considered above as well, these automorphisms are *outer*. This means that one has $A_t = \alpha_t[A]$, whereas there is no unitary group in $\mathscr{A}$ implementing this by $A_t = U_t A U_t^*$. This reason that the time evolution operator cannot be in the algebra $\mathscr{A}$ is that this contains only localized objects, whereas the Hamiltonian is a global object. The latter may, however, be constructed in certain representations π_ω of $\mathscr{A}$ as the operator satisfying $e^{itH_\omega}\pi_\omega(A)\, e^{-itH_\omega} = \pi_\omega(A_t)$. Thus, the Hamiltonian H_ω derives its structure from both the morphism α_t, and the representation π_ω, which carries global, hence topological information; in short, it may depend explicitly on the parameter ω labeling the representation π_ω of the quantum algebra.

Thus, the first task is to actually construct the automorphism α_t, which is most easily done in a given representation π_c of $\mathscr{A}$ on $\mathscr{H}_c = L^2(G)$. This is defined by $\pi'_c(C_0(Q))$ given by $(\pi'_c(f)\psi)(x) = f(xq_0)\psi(x)$, in terms of an arbitrary point $q_0 \in Q$, and by the left-regular representation T^L of G. For G amenable π_c is faithful (and reducible for nontrivial H). Now in the case that G is type I, the unitary group corresponding to the Nelson operator [26] associated to T^L may be shown to define an outer *-automorphism on $\pi_c(\mathscr{A}) \equiv \mathscr{A}$.

One may then construct the Hamiltonian in arbitrary representations by the prescription in the preceding paragraph. It turns out, that *even though the time evolution on the abstract algebra is essentially free, the Hamiltonian (and thence the action) in a topologically nontrivial representation of $\mathscr{A}$ automatically contains topological terms*. Moreover, appropriate quantization conditions on coupling constants, well known from the path-integral formalism [9, 12, 29], are identically satisfied as a consequence of the 'quantized' representation theory of $\mathscr{A}$.

5. The Magnetic Monopole

We will illustrate these statements on the example of a magnetic monopole, which is generic to numerous topological quantum effects. The relevant feature of a charged particle moving in the field of a monopole is not that it moves in the field of a monopole, but that the location of the monopole (taken to be the origin) is excluded from its configuration space. To derive relevant aspects of the quantum mechanics of such a particle, we will therefore only assume that its configuration space is $Q = \mathbb{R}^3 - \{0\}$. According to Section 3, its quantum algebra $\mathscr{A}$ has inequivalent representations π_n, $n \in \mathbb{Z}$, defined by D_n of H and Equation (5). Let H_n denote the Hamiltonian in each of these representations. The construction in the preceding section implies that H_0 is the ordinary free Hamiltonian on $L^2(\mathbb{R}^3)$, which we write (in units such that $m = 1/2$) as $H_0 = p_r^2 + J^2/r^2$, where p_r is the radial momentum, and $J^2 = \Sigma\, J_i^2$ is the standard angular momentum operator of a spinless particle.

While the J_i generate the representation U_0 of SO(3) associated to π_0 according to (5) (with trivial $D = D_0$), explicit calculation (or some reflection) shows that in a general representation $\pi_n(\mathscr{A})$ one has $H_n = p_r^2 + (J^{(n)})^2/r^2$, where the $J_i^{(n)}$ generate the representation U_n of SO(3) associated to π_n according to (5), with $D_n(\alpha) = \exp(in\alpha)$ (actually, U_n is a representation of SO(3) $\times\, \mathbb{R}$, but the $\mathbb{R}$-dependence trivially factorizes). The generators $J_i^{(n)}$ may be explicitly computed from (5) and (6). The result is that one finds exactly the angular momentum operators of a charged particle moving in the field of a monopole of charge $-n$ as written down by Wu and Yang [2]! Here $n = eg$ is the monopole charge, and one sees that the Dirac quantization condition is automatically satisfied in the above description (note that half-integer charges may be obtained through the replacement of SO(3) by SU(2) in the above considerations). Indeed, the Hamiltonian can be written as $H_n = (\mathbf{p} - e\mathbf{A}_{-n})^2$, where $\mathbf{A}_n$ is the conventional monopole field with charge $g = n/e$ [1, 2].

There is, however, an essential technical difference between the above description of a quantum monopole, and the one by Wu and Yang [2]: we have not been talking about fiber bundles or co-ordinate patches at all (the $J_i^{(n)}$ mentioned above would correspond to the Wu–Yang operators defined on the Northern Hemisphere). This is because the section s employed in the construction of the representation (5) is not necessarily continuous; indeed, the choice (6) is discontinuous along the entire negative z-axis. On the other hand, since the $J_i^{(n)}$ are unbounded operators, one should specify their domain of definition. In the present case, a rather involved analysis shows that the Gårding domain [26] for the $J_i^{(n)}$ is given by the following restrictions on the wave function $\psi(\phi, \theta)$ (we suppress the dependence on the radial coordinate r, which 'goes for a free ride'): $\psi \in C^\infty([0, 2\pi] \times [0, \pi])$, with boundary conditions $\psi(0, \theta) = \psi(2\pi, \theta)$ (together with all derivatives); $\psi(\phi, 0) = \psi(0, 0)$; $(\partial^m\psi/\partial\phi^m)(\phi, 0) = 0$ for all m, and finally $\psi(\phi, \pi) = \exp(-2in\phi)\psi(0, \pi)$ (also cf. [30]). Finally, it goes without saying that the freedom to perform gauge transformations in the conventional approach corresponds to the arbitariness in the section s of which (6) is but one particular choice.

6. The Aharonov–Bohm effect

In similar vein, the existence of the Aharonov–Bohm effect follows from the single assumption that a charged particle moves in the configuration space $Q = \mathbb{R}^3 - \{z - \text{axis}\}$. According to the analysis in Section 3, the quantum algebra of such a particle has inequivalent representations π_θ, with $\theta \in [0, 2\pi)$, and the construction in Section 4 leads to a family of Hamiltonians H_θ. As in the monopole case, H_0 is just the free Hamiltonian on $L^2(\mathbb{R}^3)$, which we write in cylindrical coordinates as $H_0 = p_r^2 + p_z^2 + J^2/r^2$. Here $J = -i\partial/\partial\phi$ is the generator of the representation of the third factor $\mathbb{R}$ in G (cf. the previous section), which in the representation $\pi_0(\mathscr{A})$ is essentially self-adjoint on the domain consisting of wave functions ψ which are C^∞ in ϕ, and satisfy the boundary condition $\psi(r, z, 2\pi) = \psi(r, z, 0)$. It follows that in the topologically nontrivial θ-representations one has $H_\theta = p_r^2 + p_z^2 + J_\theta^2/r^2$, with $J_\theta = i\partial/\partial\phi + \theta/2\pi$, defined on the same (Gårding) domain as $J \equiv J_0$ (for related considerations cf. [31]).

One now may or may not be surprised to find that H_θ is precisely the Hamiltonian of a charged particle moving in the field generated by an infinitely thin solenoid coinciding with the z-axis, generating a flux equal to $-\theta/e$ (that is, the particle is minimally coupled to a static vector potential $e\mathbf{A} = eA_\phi = -\theta/2\pi r$). This is the configuration of the (idealized) Aharonov–Bohm experiment [32].

7. Some Conjectures

While the examples above have been constructed in a mathematically rigorous manner, it is natural to expect that a similar structure emerges in field-theoretic models, where the construction problem is still open. For example, the heuristic derivation of the quantization of the coupling constant in the (four- or two-dimensional) Wess–Zumino–Witten (WZW) action [29] is entirely analogous to that in the monopole case, so that we feel justified in assuming that it arises in the way sketched in Section 4. Note, however, that the configuration space of the SU(3) nonlinear σ-model is the space of appropriately defined smooth mappings from S^4 to SU(3), which has a natural group structure, but fails to be locally compact, so that the quantum algebra (1) should be constructed by an appropriate limiting procedure. If successful, our approach would immediately explain the connection between the projective representations of this group [33], the existence of Schwinger terms in its current algebra [34], and the emergence of the WZW-term in the action (also cf. [35, 36]).

Another well-known topological effect in field theory is the θ-angle in quantum chromodynamics (QCD). This parallels the θ-angle in the Aharonov–Bohm effect (or, equivalently, of the quantum particle moving on a circle [3]), so that we believe that the CP-violating topological action in QCD is also automatically induced in topologically nontrivial vacuum representations of this theory, by the mechanism discussed in this paper.

References

1. Dirac, P. A. M., *Proc. R. Soc. London* **A133**, 60 (1931).
2. Wu, T. T. and Yang, C. N., *Phys. Rev.* **D12**, 3845 (1975) and *Nucl. Phys.* **B107**, 365 (1976).
3. Isham, C. J., in DeWitt, B. S. and Stora, R. (eds.), *Relativity, Groups and Topology II* (Les Houches 1983), North-Holland, Amsterdam, 1984.
4. Bardeen, W. A. and White, A. R. (eds.), *Anomalies, Geometry, and Topology*, World Scientific, Singapore, 1985.
5. Pruisken, A. M. M., *Phys. Rev. Lett.* **61**, 1297 (1988).
6. Wiegman, P., *Phys. Rev. Lett.* **60**, 821 (1988).
7. Reed, M. and Simon, B., *Functional Analysis* and *Fourier Analysis, Self-adjointness*, Methods of Modern Mathematical Physics, Vols. I & II, Academic Press, New York, 1972 and 1975.
8. Laidlaw, M. G. G. and Morette DeWitt, C., *Phys. Rev.* **D3**, 1375 (1971).
9. Schulman, L. S., *Techniques and Applications of Path Integration*, Wiley, New York, 1981.
10. Souriau, J.-M., *Structure des Systèmes Dynamiques* Dunod, Paris, 1969.
11. Woodhouse, N., *Geometric Quantization*, Clarendon Press, Oxford, 1980.
12. Alvarez, O., *Commun. Math. Phys.* **100**, 279 (1985).
13. Gawedzki, K., in 't Hooft, G. *et al.* (eds.), *Nonperturbative Quantum Field Theory* (Cargèse 1987), Plenum, New York, 1988.
14. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., and Sternheimer, D , *Ann. Phys.* (*N.Y.*) **111**, 61, 111 (1978).
15. Rieffel, M. A., *Commun. Math. Phys.* **122**, 531 (1989).
16. Ali, S. T. and Doebner, H. B., in Kim, Y. S. and Zachary, W. W. (eds.), *The Physics of Phase Space*, SLNP 278, Springer, Berlin, 1987.
17. Segal, I. E., *Mathematical Problems of Relativistic Physics*, Am. Math. Soc., Providence, 1963, and in F. Lurçat (ed.), *Cargèse Lectures in Theoretical Physics 1965*, Gordon & Breach, New York, 1967.
18. Haag, R. and Kastler, D., *J. Math. Phys.* **5**, 848 (1964).
19. Robinson, D. J. S., *A Course in the Theory of Groups*, Springer, New York, 1982.
20. Weyl, H., *The Theory of Groups and Quantum Mechanics*, Dover, New York, 1950.
21. Mackey, G. W., *Induced Representations*, Benjamin, New York, 1968.
22. Majid, S., *Class. Quantum Grav.* **5**, 1587 (1988).

23. Landsman, N. P., *Concepts in Thermal Field Theory*, PhD thesis, University of Amsterdam, 1989.
24. Doplicher, S., Kastler, D., and Robinson, D. W., *Commun. Math. Phys.* **3**, 1 (1966).
25. Fell, J. M. G. and Doran, R. S., *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundels*, Vol. 2, Academic Press, San Diego, 1988.
26. Barut, A. O. and Raçka, R., *Theory of Group Representations and Applications*, PWN, Warsaw, 1977.
27. Horváthy, P. A., *Phys. Lett.* **A76**, 11 (1980).
28. Simon, B., in Lieb, E. H. *et al.* (eds.), *Studies in Mathematical Physics*, Princeton Univ. Press, Princeton, 1976.
29. Witten, E., *Nucl. Phys.* **B223**, 422 (1983).
30. Hurst, C. A., *Ann. Phys. (N.Y.)* **50**, 51 (1968).
31. Breitenecker, M. and Grümm, H. R., *Nuovo. Cimento* **55A**, 453 (1980).
32. Aharonov, Y. and Bohm, D., *Phys. Rev.* **115**, 485 (1959).
33. Faddeev, L. D. and Shatashvili, S., *Theor. Mat. Fiz.* **60**, 206 (1984).
34. Jackiw, R., *Comments Nucl. Part. Phys.* **15**, 99 (1985).
35. Rajeev, S. G., *Phys. Rev.* **D29**, 2944 (1984).
36. Alcalde, C. and Sternheimer, D., *Lett. Math. Phys.* **17**, 117 (1989).