Letters in Mathematical Physics **46:** 181–188, 1998.

Twisted Lie Group C^*-Algebras as Strict Quantizations

N. P. LANDSMAN*
Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands. e-mail:npl@wins.uva.nl

(Received: 26 June 1998)

Abstract. A nonzero 2-cocycle $\Gamma \in Z^2(\mathfrak{g}, \mathbb{R})$ on the Lie algebra $\mathfrak{g}$ of a compact Lie group G defines a twisted version of the Lie–Poisson structure on the dual Lie algebra $\mathfrak{g}^*$, leading to a Poisson algebra $C^\infty(\mathfrak{g}^*_{(\Gamma)})$. Similarly, a multiplier $c \in Z^2(G, \mathrm{U}(1))$ on G which is smooth near the identity defines a twist in the convolution product on G, encoded by the twisted group C^*-algebra $C^*(G, c)$.

Further to some superficial yet enlightening analogies between $C^\infty(\mathfrak{g}^*_{(\Gamma)})$ and $C^*(G, c)$, it is shown that the latter is a strict quantization of the former, where Planck's constant $\hbar$ assumes values in $(\mathbb{Z}\backslash\{0\})^{-1}$. This means that there exists a continuous field of C^*-algebras, indexed by $\hbar \in 0 \cup (\mathbb{Z}\backslash\{0\})^{-1}$, for which $\mathfrak{A}^0 = C_0(\mathfrak{g}^*)$ and $\mathfrak{A}^\hbar = C^*(G, c)$ for $\hbar \neq 0$, along with a cross-section of the field satisfying Dirac's condition asymptotically relating the commutator in $\mathfrak{A}^\hbar$ to the Poisson bracket on $C^\infty(\mathfrak{g}^*_{(\Gamma)})$. Note that the 'quantization' of $\hbar$ does not occur for $\Gamma = 0$.

Mathematics Subject Classifications (1991): 81S99, 46L60, 22D25.

Key words: quantization, C^*-algebras, Poisson manifolds.

1. Introduction

There now exists a satisfying C^*-algebraic definition of quantization, which enables one to link Poisson and symplectic geometry with operator algebras and noncommutative geometry. The main functional-analytic idea behind this goes back to Rieffel [1], who showed how the idea of 'formal' deformation quantization [2] may be adapted to an operator-algebraic context. Later modifications by Rieffel himself and by the author have culminated in the following definition (see [3] for references and comments). Recall that $C_0(P)$ is the commutative C^*-algebra of continuous functions on P which vanish at infinity, equipped with the supremum-norm.

DEFINITION 1. A continuous quantization of a Poisson manifold P consists of a subset $I \subseteq \mathbb{R}$ (containing 0 as an accumulation point), a continuous field of C^*-algebras $(\{\mathfrak{A}^\hbar\}_{\hbar \in I}, \mathfrak{C} \subset \prod_{\hbar \in I} \mathfrak{A}^\hbar)$ with $\mathfrak{A}^0 = C_0(P)$, a Poisson algebra $\tilde{\mathfrak{A}}^0$

* Supported by a fellowship from the Royal Netherlands Academy of Arts and Sciences (KNAW).

which lies densely in $C_0(P)$, and a distinguished collection $\{\mathcal{Q}(f)\}_{f\in\tilde{\mathfrak{A}}^0} \subset \mathfrak{C}$ of cross-sections, such that

$$\mathcal{Q}_0(f) = f; \tag{1}$$

$$\mathcal{Q}_\hbar(f^*) = \mathcal{Q}_\hbar(f)^*; \tag{2}$$

for all $\hbar \in I$ and $f \in \tilde{\mathfrak{A}}^0$. Finally, Dirac's condition

$$\lim_{\hbar\to 0} \| \frac{i}{\hbar}[\mathcal{Q}_\hbar(f), \mathcal{Q}_\hbar(g)] - \mathcal{Q}_\hbar(\{f, g\}) \|= 0 \tag{3}$$

should hold for all $f, g \in \tilde{\mathfrak{A}}^0$.

We refer to Dixmier [4] for the concept of a continuous field of C^*-algebras; the collection $\mathfrak{C}$ of cross-sections determines a continuity structure on $\{\mathfrak{A}^\hbar\}_{\hbar\in I}$, and has to satisfy a number of conditions which are listed in [4]. It should be noted that Definition 1 guarantees the property

$$\lim_{\hbar\to 0} \| \mathcal{Q}_\hbar(f)\mathcal{Q}_\hbar(g) - \mathcal{Q}_\hbar(fg) \|= 0 \tag{4}$$

for all $f, g \in \tilde{\mathfrak{A}}^0$. In addition, the function $\hbar \to \| \mathcal{Q}_\hbar(f) \|$ is continuous on I, so that, in particular, one has

$$\lim_{\hbar\to 0} \| \mathcal{Q}_\hbar(f) \|=\| f \|_\infty, \tag{5}$$

where the right-hand side is the supremum-norm of f. Conversely, one has

LEMMA 1. *Suppose one has a Poisson manifold P, a family $\{\mathfrak{A}^\hbar\}_{\hbar\in I}$ of C^*-algebras indexed by a discrete subset $I \subset \mathbb{R}$ containing 0 as an accumulation point, a Poisson algebra $\tilde{\mathfrak{A}}^0$ whose (sup-norm) closure is $\mathfrak{A}^0 = C_0(P)$, and a collection of linear maps $\{\mathcal{Q}_\hbar: \tilde{\mathfrak{A}}^0 \to \mathfrak{A}^\hbar\}_{\hbar\in I}$ satisfying* (1)–(5).

There exists a family $\mathfrak{C} \subset \prod_{\hbar\in I}$ making $(\{\mathfrak{A}^\hbar\}_{\hbar\in I}, \mathfrak{C})$ into a continuous field of C^*-algebras, such that $\mathfrak{C}$ contains all maps $\{\mathcal{Q}_\hbar(f)\}_{\hbar\in I}$, $f \in \tilde{\mathfrak{A}}^0$.

The continuous field in question is uniquely determined when the set $\{\mathcal{Q}_\hbar(f)\}_{f\in\tilde{\mathfrak{A}}^0}$ is dense in $\mathfrak{A}^\hbar$ for all $\hbar \in I$, but we shall not need this. A proof of this lemma may be found in [3].

Apart from its more stringent definition of convergence, strict quantization as defined by Definition 1 differs from deformation quantization in the sense of [2], as well as from the corresponding notion in [1], in that $\mathcal{Q}_\hbar(\tilde{\mathfrak{A}}^0)$ is not necessarily closed under multiplication (in $\mathfrak{A}^\hbar$). If it is, and if $\mathcal{Q}_\hbar$ is nondegenerate in that $\mathcal{Q}_\hbar(f) = 0$ iff $f = 0$ for each $\hbar$, one may define an associative 'deformed' product $\cdot_\hbar$ in $\tilde{\mathfrak{A}}^0$ with the property $\mathcal{Q}_\hbar(f)\mathcal{Q}_\hbar(g) = \mathcal{Q}_\hbar(f \cdot_\hbar g)$ (and, of course,

$f \cdot_0 g = fg$).The conditions on a strict quantization may then be rephrased in terms of this product in the obvious way, leading to the framework of [1]. However, there are many examples of strict quantization that are not deformation quantizations, including the ones in this paper that correspond to nontrivial group extensions.

A conceptually rather pleasing class of examples of strict quantization (in the original definition of [1]) was discovered by Rieffel [5]. Consider a Lie group G with Lie algebra $\mathfrak{g}$. The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Poisson manifold under the well-known Lie–Poisson bracket [6]

$$\{f, g\}(\theta) = -\theta([\mathrm{d}f_\theta, \mathrm{d}g_\theta]). \tag{6}$$

The symplectic leaves of a Poisson manifold P (along with their covering spaces) play the role of 'classical' irreducible representations of the corresponding Poisson algebra $C^\infty(P)$ [3]. As shown by Kirillov [7] (also cf. [6]), the symplectic leaves of $\mathfrak{g}^*$ with respect to this Poisson structure are (the connected components of) its coadjoint orbits. This result is reminiscent of the bijective correspondence between the (nondegenerate) irreducible representations of the group C^*-algebra $C^*(G)$ and the irreducible unitary representations of G [8]. Since the latter may be seen as the quantum counterparts of the coadjoint orbits of G, Kirillov's result already suggests that $C^*(G)$ should be the C^*-algebraic analogue of the Poisson algebra $C^\infty(\mathfrak{g}^*)$.

The correspondence between the coadjoint orbits in $\mathfrak{g}^*$ and the unitary irreducible representations of G is at its best (namely, bijective and functorial) when G is nilpotent, connected, and simply connected. In that case, Rieffel [5] showed that $C^*(G)$ is related to $C^\infty(\mathfrak{g}^*)$ by a strict quantization, with $I = \mathbb{R}$. Under the stated assumptions G is exponential, so that one may identify G with $\mathfrak{g}$. Translated into the setting of Definition 1, the quantization maps $\mathcal{Q}_\hbar$ are given by

$$\mathcal{Q}_\hbar(f)\colon X \to \int_{\mathfrak{g}^*} \frac{\mathrm{d}^n\theta}{(2\pi\hbar)^n}\, \mathrm{e}^{i\theta(X)/\hbar} f(\theta). \tag{7}$$

Here $f \in \tilde{\mathfrak{A}}^0 = \mathcal{S}(\mathfrak{g}^*)$, the Schwartz space of test functions on $\mathfrak{g}^*$.

In Section 2, we show that an analogous statement holds for arbitrary compact Lie groups; given the results on strict quantization on Riemannian manifolds in [9], this is a simple exercise. In Section 3 we modify the Lie–Poisson structure on $\mathfrak{g}^*$ by a nonzero 2-cocycle Γ on $\mathfrak{g}$, and show that $C^*(G)$ should then be replaced by the twisted group C^*-algebra $C^*(G, c)$, defined by a multiplier c on G which is smooth near the identity. Similar representation-theoretic analogies as in the untwisted case then hold. The main point of this Letter is made in Section 4, where we extend the strict quantization of the Lie–Poisson structure to the twisted case. That is, when Γ is the derivative of c in a suitable sense, we show that the C^*-algebra $C^*(G, c)$ and the twisted Poisson algebra $C^\infty(\mathfrak{g}^*_{(\Gamma)})$ are related by a strict quantization. Surprisingly, this only works if the interval $I = \mathbb{R}$ is replaced by the discrete set $I = 0 \cup (\mathbb{Z}\backslash\{0\})^{-1}$. The quantization maps, however, are still given by (7).

2. Strict Quantization of the Lie–Poisson Structure for Compact Lie Groups

In this section we modify (7) so as to make it applicable to compact Lie groups. Firstly, the Fourier transform of $f \in L^1(\mathfrak{g}^*)$ is defined by

$$\grave{f}(X) = \int_{\mathfrak{g}^*} \frac{\mathrm{d}^n\theta}{(2\pi)^n}\, \mathrm{e}^{i\theta(X)} f(\theta), \tag{8}$$

where $\mathrm{d}^n\theta$ is Lebesgue measure on $\mathfrak{g}^* \simeq \mathbb{R}^n$, whose normalization is fixed by that of the Haar measure $\mathrm{d}x$ on G, as follows. When f has support near e, we can write $\int_G \mathrm{d}x\, f(x) = \int_{\mathfrak{g}} \mathrm{d}^n X\, J(X) f(\mathrm{Exp}(X))$, where $\mathrm{d}^n X$ is a Lebesgue measure on $\mathfrak{g}$, and J is some Jacobian. The normalization is now fixed by the condition $J(0) = 1$. In turn, the normalization of the Lebesgue measure $\mathrm{d}^n\theta$ on $\mathfrak{g}^*$ is fixed by requiring the inversion formula $f(\theta) = \int_{\mathfrak{g}} \mathrm{d}^n X\, \mathrm{e}^{-i\theta(X)} \grave{f}(X)$. We define $C^\infty_{\mathrm{PW}}(\mathfrak{g}^*)$ as the class of functions on $\mathfrak{g}^*$ whose Fourier transform $\grave{f}$ is in $C^\infty_c(\mathfrak{g})$. This is a Poisson subalgebra of $C^\infty(\mathfrak{g}^*_\pm)$.

We choose a smooth cutoff function κ on $\mathfrak{g}$ which equals 1 in a neighbourhood $\tilde{\mathcal{N}}$ of 0, is invariant under inversion $X \to -X$, and has support in the neighbourhood $\mathcal{N}$ of 0 on which Exp is a diffeomorphism. When G is compact one may assume that κ is Ad-invariant, i.e., satisfies $\kappa(\mathrm{Ad}(y)X) = \kappa(X)$ for all $y \in G$. This may always be achieved by averaging.

We now modify (7) as follows: for $x \notin \mathrm{Exp}(\mathcal{N})$ we put $\mathcal{Q}_\hbar(f)(x) = 0$, whereas for $x \in \mathrm{Exp}(\mathcal{N})$ we put

$$\mathcal{Q}_\hbar(f)(x) = \hbar^{-n}\kappa(\mathrm{Exp}^{-1}(x))\, \grave{f}(\mathrm{Exp}^{-1}(x)/\hbar). \tag{9}$$

The restriction $f \in C^\infty_{\mathrm{PW}}(\mathfrak{g}^*)$ implies that for small enough $\hbar$ the operator $\mathcal{Q}_\hbar(f)$ is independent of κ.

THEOREM 1. *Suppose G is an n-dimensional compact Lie group. The collection of maps $\mathcal{Q}_\hbar\colon C^\infty_{PW}(\mathfrak{g}^*) \to C^*(G)$ defined by* (9) *and preceding text, where $\hbar \in \mathbb{R}\backslash\{0\}$, satisfies* (1)–(5). *Hence, there exists a strict quantization (cf. Definition 1) of $\mathfrak{g}^*$ on $I = \mathbb{R}$ for which $\tilde{\mathfrak{A}}^0 = C^\infty_{PW}(\mathfrak{g}^*)$, $\mathfrak{A}^0 = C_0(\mathfrak{g}^*)$, and $\mathfrak{A}^\hbar = C^*(G)$ for $\hbar \notin 0$, the maps $\mathcal{Q}_\hbar(f)$ being cross-sections of the associated continuous field of C^*-algebras.*

The conclusion of the theorem is immediate from Lemma 1. To prove that the assumptions of the lemma are satisfied, we identify $C^*(G)$ with $\pi_L(C^*(G))$, where π_L is the left-regular representation on $L^2(G)$ [8]; this representation is faithful because compact groups are amenable. Also, we identify $C^\infty(\mathfrak{g}^*)$ as a Poisson algebra with the subalgebra $C^\infty(T^*G)^R$ of right-invariant smooth functions on T^*G, equipped with the canonical cotangent bundle Poisson bracket [6]. This identification is inherited by $C^\infty_{\mathrm{PW}}(\mathfrak{g}^*) \simeq C^\infty_{\mathrm{PW}}(T^*G)^R$, where on the right-hand side the class C^∞_{PW} is defined relative to the Fourier transform in the fiber direction [9, 6]; recall that $T^*_x G \simeq \mathfrak{g}^*$.

A compact Lie group G admits a right-invariant Riemannian metric $\mathbf{g}$, such that the exponential map $\exp_e$ obtained from $\mathbf{g}$ coincides with the map Exp defined by the Lie group structure [10]. Using such a metric, the generalized Weyl quantization prescription on Riemannian manifolds of [9], restricted to $C^\infty_{\mathrm{PW}}(T^*G)^R$, coincides with $\mathcal{Q}_\hbar$ as defined by (9). All claims then follow from Theorem 1 in [9]. □

3. The Twisted Lie–Poisson Algebra vs the Twisted Group Algebra

Let $\Gamma \in Z^2(\mathfrak{g}, \mathbb{R})$ be a 2-cocycle on $\mathfrak{g}$ with values in $\mathbb{R}$ [6]. This leads to a modification of the Lie–Poisson structure on $\mathfrak{g}^*$, in which one adds a term $-\Gamma(\mathrm{d}f, \mathrm{d}g)$ to the right-hand side of (6). In canonical co-ordinates on $\mathfrak{g}^*$ (relative to a basis $\{T_a\}_{a=1,\dots,n}$ of $\mathfrak{g}$), the ensuing bracket reads

$$\{f, g\}^{(\Gamma)}_{\pm} = -(C^c_{ab}\theta_c + \Gamma_{ab})\frac{\partial f}{\partial\theta_a}\frac{\partial g}{\partial\theta_b}, \tag{10}$$

where the C^c_{ab} are the structure constants of $\mathfrak{g}$ in the given basis, and $\Gamma_{ab} = \Gamma(T_a, T_b)$. We denote the space $\mathfrak{g}^*$, seen as a Poisson manifold through (10), by $\mathfrak{g}^*_{(\Gamma)}$, with associated Poisson algebra $C^\infty(\mathfrak{g}^*_{(\Gamma)})$.

The 2-cocycle Γ defines a central extension $\mathfrak{g}_\Gamma$ of $\mathfrak{g}$ as well. As a vector space one has $\mathfrak{g}_\Gamma = \mathfrak{g} \oplus \mathbb{R}$; denoting the central element by T_0 (this is a basis vector in the extension $\mathbb{R}$), the new Lie bracket is $[X, Y]_\Gamma = [X, Y] + \Gamma(X, Y)T_0$. This also equips the dual $\mathfrak{g}^*_\Gamma$ with the Lie–Poisson structure. Let ω^0 be the basis element in $\mathfrak{g}^*_\Gamma$ dual to T_0. Then $J_1 : \mathfrak{g}^*_{(\Gamma)} \to \mathfrak{g}_\Gamma$ given by $J_1(\theta) = \theta + \omega^0$ (where $\mathfrak{g}^*$ is embedded in $\mathfrak{g}^*_\Gamma$ as the annihilator of the extension $\mathbb{R}$) is a Poisson map.

PROPOSITION 1. *The canonical identification of $C^\infty(\mathfrak{g}^*_\Gamma)/\ker(J_1^*)$ with $C^\infty(\mathfrak{g}^*_{(\Gamma)})$ is a Poisson isomorphism.*

Proof. This is immediate from the definitions and (10). □

The identification between the symplectic leaves in $\mathfrak{g}^*$ with respect to the Lie-Poison structure and the coadjoint orbits has the following generalization to the twisted case [6, 3]. Let γ be a symplectic cocycle on G with the property that $\Gamma(X, Y) = -(\mathrm{d}/\mathrm{d}t)\gamma(\mathrm{Exp}(tX))(Y)_{|t=0}$. The symplectic leaves of $\mathfrak{g}^*_{(\Gamma)}$ then coincide with the G-orbits in $\mathfrak{g}^*$ under the twisted coadjoint action $\mathrm{Co}^\gamma(x)\theta = \mathrm{Co}(x)\theta + \gamma(x)$, where Co stands for the usual coadjoint action.

We pass from Poisson algebras to C^*-algebras. The role of Γ is now played by a multiplier $c \in Z^2(G, \mathrm{U}(1))$ on G which is smooth near the identity [11, 3]. The quantum analogue of the twisted Poisson algebra $C^\infty(\mathfrak{g}^*_{(\Gamma)})$ is the twisted group algebra $C^*(G, c)$. This is defined as a suitable C^*-completion of $L^1(G)$, under the twisted convolution product

$$f * g(x) = \int_G \mathrm{d}y\, c(xy^{-1}, y) f(xy^{-1}) g(y), \tag{11}$$

and the twisted involution

$$f^*(x) = \overline{c(x, x^{-1}) f(x^{-1})}. \quad (12)$$

The bijective correspondence between the nondegenerate (irreducible) representations of $C^*(G)$ and the continuous unitary (irreducible) representations of G is generalized to a bijective correspondence between the nondegenerate (irreducible) representations of $C^*(G, c)$ and the continuous projective unitary (irreducible) representations of G with multiplier c; see [12, 3].

Furthermore, a multiplier c defines a central extension G_c of G by $\mathrm{U}(1)$ [11]. A quantum analogue of Proposition 1 is as follows.

PROPOSITION 2. *Let G be a compact Lie group with multiplier c, and write π^k for the representation of $C^*(G_c)$ corresponding to the representation $U^k(G_c)$ induced by $U_k(\mathrm{U}(1))$, where $k \in \mathbb{Z}$ and $U_k(z) = z^k$ for $z \in \mathbb{T} = \mathrm{U}(1)$. For each $k \in \mathbb{Z}$ there are isomorphisms*

$$C^*(G, c^k) \simeq \pi^k(C^*(G_c)) \simeq C^*(G_c)/\ker(\pi^k). \quad (13)$$

Explicitly, under the first isomorphism the function $\pi^k(f) \in C^*(G, c^k)$ is

$$\pi^k(f) : x \to \int_{\mathbb{T}} \mathrm{d}z\, z^k f(x, z). \quad (14)$$

Here $\mathrm{d}z$ is the normalized Haar measure on $\mathbb{T}$. Given a projective representation $\mathrm{U}(G)$ with multiplier c^k, one defines an associated representation U_{c^k} of G_c by $U_{c^k}(x, z) = z^k \mathrm{U}(x)$, and verifies that U_{c^k} is unitarily equivalent to the representation $U^k(G_c)$ induced by $U_k(\mathrm{U}(1))$. □

This proposition is closely related to the decomposition

$$C^*(G_c) \simeq \oplus_{k \in \mathbb{Z}} \pi^k(C^*(G_c)), \quad (15)$$

which follows from the isomorphism $C^*(G_c) = \pi_L(C^*(G_c))$ and the Peter-Weyl theorem applied to G_c.

4. Strict Quantization of the Twisted Lie–Poisson Structure for Compact Lie Groups

Comparing the comment after the proof of Proposition 1 with the one following (12), and also comparing Propositions 1 and 2, it is clear that the twisted group C^*-algebra $C^*(G, c)$ is indeed a quantum version of the twisted Poisson algebra $C^\infty(\mathfrak{g}^*_{(\Gamma)})$. Inspired by the analogies in question, we now generalize Theorem 1 to the twisted case.

We identify $\mathbb{R}$ in $\mathfrak{g}_\Gamma = \mathfrak{g} \oplus \mathbb{R}$ with the Lie algebra $\mathfrak{u}_c(1)$ of the central subgroup $\mathrm{U}(1) \subset G_c$ defining the extension, and write $\mathrm{Exp}\colon \mathfrak{u}(1) \to \mathrm{U}(1)$ for the exponential map, conventionally realized as $\mathrm{Exp}(X) = \exp(-iX)$. In a neighbourhood $\mathcal{N}_e \times \mathcal{N}_e$ of (e, e) we can write $c = \mathrm{Exp}(\chi)$, where $\chi: \mathcal{N}_e \times \mathcal{N}_e \to \mathfrak{u}_c(1)$. Then define $\Gamma: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$\Gamma(X, Y) = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}[\chi(\mathrm{Exp}(tX), \mathrm{Exp}(sY)) - \\ - \chi(\mathrm{Exp}(sY), \mathrm{Exp}(tX))]_{|s=t=0}. \tag{16}$$

It is easy to see that $\Gamma \in Z^2(\mathfrak{g}, \mathbb{R})$ when $c \in Z^2(G, \mathrm{U}(1))$.

THEOREM 2. *Suppose G is an n-dimensional compact Lie group, with multiplier $c \in Z^2(G, \mathrm{U}(1))$, and define a 2-cocycle $\Gamma \in Z^2(\mathfrak{g}, \mathbb{R})$ on $\mathfrak{g}$ by* (16). *Regard $C^\infty_{PW}(\mathfrak{g}^*_{(\Gamma)})$ as a Poisson subalgebra of $C^\infty(\mathfrak{g}^*_{(\Gamma)})$ with respect to the Poisson bracket* (10), *and regard* (9) *as a map from $C^\infty_{PW}(\mathfrak{g}^*_{(\Gamma)})$ to $\mathfrak{A}^\hbar = C^*(G, c^{1/\hbar})$, where $\hbar \in (\mathbb{Z}\backslash\{0\})^{-1}$.*

The collection of maps $\mathcal{Q}_\hbar$, thus construed, satisfies (1)–(5). *Hence, there exists a strict quantization (cf. Definition* 1*) of $\mathfrak{g}^*_{(\Gamma)}$ for which $\tilde{\mathfrak{A}}^0 = C^\infty_{PW}(\mathfrak{g}^*_{(\Gamma)})$, $\mathfrak{A}^0 = C_0(\mathfrak{g}^*_{(\Gamma)})$, and $\mathfrak{A}^\hbar = C^*(G, c^{1/\hbar})$ for $\hbar \in (\mathbb{Z}\backslash\{0\})^{-1}$, the maps $\mathcal{Q}_\hbar(f)$ being cross-sections of the associated continuous field of C^*-algebras.*

It is obvious that (2) holds. The proof of the other properties is based on the analogy between Propositions 1 and 2. Extend $f \in C^\infty_{\mathrm{PW}}(\mathfrak{g}^*)$ to a function $\tilde{f} \in C^\infty_{\mathrm{PW}}(\mathfrak{g}^*_\Gamma)$, such that $f(\theta) = \tilde{f}(1, \theta)$ and

$$\tilde{f}(\theta_0 \neq 1, \theta) < \tilde{f}(\theta_0 = 1, \theta) = f(\theta); \tag{17}$$

in particular, one has

$$\| f \|_\infty = \| \tilde{f} \|_\infty . \tag{18}$$

In view of (10) this automatically means that

$$\{\tilde{f}, \tilde{g}\}(1, \theta) = \{f, g\}^\Gamma(\theta), \tag{19}$$

since the left-hand side does not involve derivatives with respect to θ_0.

We denote (9) as defined on $C^\infty_{\mathrm{PW}}(\mathfrak{g}^*_{(\Gamma)})$, taking values in $C^*(G, c^{1/\hbar})$, by $\mathcal{Q}_\hbar$, whereas the map defined in the same way, but now on $C^\infty_{\mathrm{PW}}(\mathfrak{g}^*_\Gamma)$, taking values in $C^*(G_c)$, is written as $\tilde{\mathcal{Q}}_\hbar$. A short computation using (14) and an elementary oscillatory integral shows that

$$\pi^{1/\hbar}(\tilde{\mathcal{Q}}_\hbar(\tilde{f})) = \mathcal{Q}_\hbar(f) \tag{20}$$

for $\hbar \in (\mathbb{Z}\backslash\{0\})^{-1}$ small enough so that the right-hand side is independent of κ. In particular, the left-hand side only depends on the value of $\tilde{f}$ at $\theta_0 = 1$; this is a special case of the fact that, for $\hbar$ small enough, $\pi^k(\tilde{\mathcal{Q}}_\hbar(\tilde{f}))$ only depends on $\tilde{f}(\theta_0 = k\hbar)$. This follows by a similar calculation as the one leading to (20).

Theorem 1 applied to G_c implies that $\lim_{\hbar\to 0} \| \tilde{\mathcal{Q}}_\hbar(\tilde{f}) \| = \| \tilde{f} \|_\infty$. On the other hand, according to (15) one has $\| A \| = \sup_{k\in\mathbb{Z}} \| \pi^k(A) \|$ for all $A \in C^*(G_c)$. Combining the two of these equations with the last remark of the preceding paragraph and the property (17), we conclude that

$$\lim_{\hbar\to 0} \| \tilde{\mathcal{Q}}_\hbar(\tilde{f}) \| = \lim_{\hbar\to 0} \| \pi^{1/\hbar}(\tilde{\mathcal{Q}}_\hbar(\tilde{f})) \| = \| \tilde{f} \|_\infty . \tag{21}$$

Together with (18) and (20) this proves (5).

Equations (4) and (3) now follow from (20), Proposition 2, (19), Theorem 1 (once again applied to G_c), and the inequality $\| \pi^k(A) \| \leqslant \| A \|$ in $C^*(G_c)$. □

While proved for compact G, Theorem 2 may hold in other situations. For example, let $G \simeq \mathfrak{g}^* = \mathbb{R}^{2n}$, with Γ given by $\Gamma(P_i, P_j) = \Gamma(Q^i, Q^j) = 0$ and $\Gamma(P_i, Q^j) = -\delta_i^j$, and c defined by $c((u, v), (u', v')) = \mathrm{e}^{i(uv'-vu')/2}$. Then the statement of Theorem 2 holds as well.

References

1. Rieffel, M. A.: *Comm. Math. Phys.* **122** (1989), 531–562.
2. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D.: *Ann. Phys. (NY)* **110** (1978), 61–110, 111–151.
3. Landsman, N. P.: *Mathematical Topics between Classical and Quantum Mechanics*, Springer, New York, 1998.
4. Dixmier, J. *C^*-Algebras*, North-Holland, Amsterdam, 1977.
5. Rieffel, M. A.: *Amer. J. Math.* **112** (1990), 657–686.
6. Libermann, P. and Marle, C.-M.: *Symplectic Geometry and Analytical Mechanics*, D. Reidel, Dordrecht, 1987.
7. Kirillov, A. A.: *Russian Math. Surveys* **31** (1976), 55–75.
8. Pedersen, G. K.: *C^*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.
9. Landsman, N. P.: *J. Geom. Phys.* **12** (1993), 93–132.
10. Milnor, J.: *Adv. Math.* **21** (1976), 293–329.
11. Tuynman, G. M. and Wiegerinck, W. A. J. J.: *J. Geom. Phys.* **4** (1987), 207–258.
12. Busby, R. C. and Smith, H. A.: *Trans. Amer. Math. Soc.* **149** (1970), 503–537.