Notes on Noncommutative Geometry

accompanying “Dirac Operators and Spectral Geometry” by Joseph C. Várilly

MRI Masterclass 2009–2010: *Arithmetic Geometry and Noncommutative Geometry*

Scedule:

- Lectures on February 9, 16, March 2, 16, 30, April 13, 27, May 18 (2010)
- Exercise classes on February 23, March 9, 23, April 6, 20, May 11, 25 (2010)

Time: Tuesdays, 14-17 h
Place: Minnaert 207, Utrecht University till 20 April, Minnaert 016, April 27–May 25

N.P. Landsman

Radboud Universiteit Nijmegen
Faculty of Science
Institute for Mathematics, Astrophysics, and Particle Physics
Heyendaalseweg 135
6525 AJ NIJMEGEN
THE NETHERLANDS

email `landsman at math.ru.nl`
website: `http://www.math.ru.nl/~landsman/ncg2010.html`

[0]Draft: October 5, 2011

1 Introduction

Noncommutative geometry has roots in and is a synthesis of a number of diverse areas of mathematics, including:

- Hilbert space and single operator theory;
- Operator algebras (C*-algebras and von Neumann algebras);
- Spin geometry – Dirac operators – index theory;
- Algebraic topology – homological algebra.

It has certainly also been inspired by quantum mechanics, and, besides feedback to the above areas that it comes from, noncommutative geometry has applications to (at least):

1. Foliation theory;
2. Number theory – arithmetic algebraic geometry;
3. Deformation theory – quantization theory;
4. Quantum field theory – renormalization;
5. Elementary particle physics – Standard Model;
6. Solid state physics – Quantum Hall effect;

In this sense, as a general mathematical formalism with such a wide range of deep applications to both mathematics and physics, noncommutative geometry may be compared with Newton's calculus. The interaction between the above areas plays an important role in noncommutative geometry, especially the unexpected use of tools from algebraic topology (like *K*-theory) and homological algebra (like Hochschild (co)homology) in the context of operator algebras and more general complex associative algebras. But the reverse direction, where operator techniques are e.g. used to redevelop and generalize spin geometry and index theory, is at least as fruitful and is arguably even more unexpected.

The history of noncommutative geometry goes back to John von Neumann's work on the mathematical structure of quantum physics, as presented in his book *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932), and his subsequent invention of the theory of operator algebras (written down is a series of papers published between 1936 and 1949, partly with his assistant F.J. Murray). Other events of great importance to noncommutative geometry were the definition of C*-algebras and the first results in this area by Gelfand and Naimark in 1943, and the development of index theory by Atiyah and Singer from 1968 onwards.

Connes himself brought the "introverted" period in the history of operator algebras to a close with his magnificent classification of injective factors in 1976, and subsequently opened up the field by relating it to foliated manifolds and index theory. This led to a series of papers by Connes in the period 1979-1985 that launched noncommutative geometry as a new area of mathematics. An important feature of this area was and is the interplay between abstract theory and examples; what makes it difficult to enter the field is that both theory and examples are technically quite complicated.

From about 1995, the notion of a spectral triple began to play a dominant role in noncommutative geometry. This made the field more accessible, since unlike techniques like K-theory and cyclic cohomology, spectral triples are quite concrete objects, which can be studied using standard techniques in Hilbert space. Perhaps it is worth saying that the alternation of periods in which noncommutative geometry undergoes a more abstract development with periods where Hilbert space techniques are predominant reflects the dual nature of C*-algebras, which may be seen as either abstract Banach algebras, or concrete operator algebras on Hilbert space.

A basic library in noncommutative geometry should at least include:

- Alain Connes, *Noncommutative Geometry* (Academic Press, 1994), the *Old Testament*;
- Alain Connes and Matilde Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives* (Hindawi/AMS, 2008), the *New Testament*;
- J.M. Gracia-Bondia, J.C. Varilly, H. Figueroa, *Elements of Noncommutative Geometry* (Birkhäuser, 2001), the *Exegesis*.

All of these are easily found online. Recent introductions include J. Varilly, *An Introduction to Noncommutative Geometry* (EMS, 2006) and M. Khalkhali, *Basic Noncommutative Geometry* (EMS, 2009), which are best read in the opposite order (in that Khalkhali is more introductory than Varilly). On specialized topics within noncommutative geometry or at its interface with neighbouring areas of mathematics, the following technical monographs are useful: B. Blackadar, *K-Theory for Operator Algebras* (CUP, 1998); N. Higson and J. Roe, *Analytic K-Homology* (OUP, 2000), and J.-L. Loday. *Cyclic Homology* (Springer, 1998). Online lecture notes include R. Wulkenhaar, *Spektrale Tripel* (German),[1] `wwwmath.uni-muenster.de/u/raimar/lehre/WS08/SpektraleTripel/index.html` and *Lecture Notes on C*-algebras and K-theory* (2003-2004) by the present author, at `remote.science.uva.nl/~npl/CK.html`.

The basic strategy of noncommutative geometry is the following procedure:

1. Reformulate some object/notion from topology or geometry in terms of a *commutative* (and associative) algebra (over $\mathbb{C}$) and possible other operator-theoretic data;[2]
2. Check the validity of this reformulation through a *reconstruction theorem*, stating that the original object can be recovered from these data up to isomorphism (preferably as a categorical duality);
3. Try to define the corresponding "noncommutative" object by simply dropping the commutativity of the algebra in step 1, and see where you get!

The third step is a cardinal change compared with algebraic geometry, which also involves steps 1 and 2, even in quite perfect form: Grothendieck's correspondence between affine schemes and commutative rings is indeed a categorical duality.

[1] No typo! Its is really `wwwmath.uni-muenster.de` rather than `www.math.uni-muenster.de`.

[2] So far, the commutative algebras in question tend to be defined over $\mathbb{C}$, but this may change!

The motivating example for this strategy is, as well as its most perfect realization is, of course, the Gelfand Duality Theorem for commutative C*-algebras.[3]

Here the starting object from topology is a compact Hausdorff space X, which defines a commutative algebra $A = C(X) \equiv C(X, \mathbb{C})$. In order to reconstruct X from A, the latter needs to be seen as a unital commutative C*-algebra. Note that the map $X \mapsto C(X)$ extends to a contravariant functor C from the category CH of compact Hausdorff spaces (as objects) and continuous maps (as arrows) to the category UCCA of unital commutative C*-algebras (as objects) and unital *-homomorphisms (as arrows), since a map $\varphi : X \to Y$ induces the pullback $\varphi^* : C(Y) \to C(X)$ (i.e. $\varphi^*(f) = f \circ \varphi$), with the right behaviour for functoriality. In a contravariant situation like this, it is often convenient to say that C is a functor from CH to the *opposite* category $\mathsf{UCCA}^{\mathrm{op}}$, in which the direction of each arrow is reversed.

The key to the reconstruction of X from $C(X)$ is the *Gelfand spectrum* $\Sigma(A)$ of a (unital) commutative C*-algebra A. As a set, $\Sigma(A)$ consists of all nonzero multiplicative functionals $\varphi : A \to \mathbb{C}$, equipped with the topology of pointwise convergence (i.e. $\varphi_n \to \varphi$ iff $\varphi_n(a) \to \varphi(a)$ for all $a \in A$).[4] The map $A \mapsto \Sigma(A)$ extends to a functor $\Sigma : \mathsf{UCCA} \to \mathsf{CH}^{\mathrm{op}}$ by pullback, too, and the pair (C, Σ) defines a *duality* between the categories CH and UCCA (in other words, an *equivalence* between CH and $\mathsf{UCCA}^{\mathrm{op}}$). This means, in particular, that

$$\Sigma(C(X)) \cong X; \tag{1}$$
$$C(\Sigma(A)) \cong A, \tag{2}$$

for all compact Hausdorff spaces X and all unital commutative C*-algebras A. The first isomorphism is $x \mapsto \mathrm{ev}_x$, with $x \in X$ and the evaluation map $\mathrm{ev}_x \in \Sigma(C(X))$ given by $\mathrm{ev}_x(f) = f(x)$. The second isomorphism is implemented by the famous *Gelfand transform* $a \mapsto \hat{a}$, where $a \in A$ and $\hat{a} \in C(\Sigma(A))$ is defined by $\hat{a}(\varphi) = \varphi(x)$.

Finally, the grand move, i.e. step 3, is to declare that a "noncommutative space" is an arbitrary C*-algebra. This is a well-defined notion, which forms the basis of noncommutative geometry. The (second) Gelfand–Naimark Theorem, stating that any C*-algebra has

[3]Despite its perfection, one should realize that this is by no means the only way to proceed. If one likes order theory instead of operator algebras, an alternative way to capture the notion of a topological space X algebraically is to regard the topology $\mathcal{O}(X)$ as a *frame*; this is a complete distributive lattice such that $x \wedge \bigvee_\lambda y_\lambda = \bigvee_\lambda x \wedge y_\lambda$ for arbitrary families $\{y_\lambda\}$ (and not just for finite ones, in which case the said property follows from the definition of a distributive lattice). Indeed, the topology $\mathcal{O}(X)$ of any space X is a frame with $U \leqslant V$ if $U \subseteq V$. However, just like the fact that not any commutative unital algebra A per se is of the form $A \cong C(X)$ for some space X – motivating the idea of regarding $C(X)$ as a commutative unital C*-algebra in order to have a good reconstruction theorem – there are frames that are not isomorphic to $\mathcal{O}(X)$ for some space X. To put this right, one defines a *point* of a frame $\mathcal{F}$ as a frame map $\varphi : \mathcal{F} \to \{0, 1\} \equiv \mathcal{O}(*)$, where a *frame map* preserves finite meets and arbitrary joins. The set $\mathrm{Pt}(\mathcal{F})$ of points of a frame $\mathcal{F}$ may be topologized in a natural way by declaring its opens to be the subsets of $\mathrm{Pt}(\mathcal{F})$ that are of the form $\mathrm{Pt}(U) = \{\varphi \in \mathrm{Pt}(\mathcal{F}) \mid \varphi^{-1}(U) = 1\}$, where $U \in \mathcal{F}$. We say that a frame $\mathcal{F}$ is *spatial* if it is isomorphic (in the category of frames) to $\mathcal{O}(\mathrm{Pt}(\mathcal{F}))$. A topological space X is called *sober* if it is homeomorphic to $\mathrm{Pt}(\mathcal{O}(X))$. This leads, almost by definition, to a categorical duality between spatial frames and sober spaces. The move of defining 'generalised spaces' in this context, analogous to regarding a C*-algebra as a 'noncommutative space', would be to say that a generalised space is an arbitrary frame. See, for example, S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (Springer, New York, 1992). The connection between this lattice-theoretic notion of a generalized space and a noncommutative space is explained in C. Heunen, N.P. Landsman, and B. Spitters, A topos for algebraic quantum theory, *Commun. Math.* Phys. 291, 63–110 (2009).

[4]Strictly speaking, this only defines the correct topology on $\Sigma(A)$ if it is metrisable, which is the case iff A is norm-separable. In general, the topology on $\Sigma(A)$ is defined by specifying the following base of open sets: for each $\varepsilon > 0, n \in \mathbb{N}, a_1, \ldots, a_n \in A, \varphi_0 \in \Sigma(A)$, take $\{\varphi \in \Sigma(A) \mid |\varphi(a_i) - \varphi_0(a_i)| < \varepsilon \;\forall\, i = 1, \ldots, n\}$.

a faithful representation on some Hilbert space (and that conversely any norm-closed involutive subalgebra of $B(H)$ is a C*-algebra) then explains the importance of Hilbert space theory in the noncommutative geometry; this role has no counterpart in algebraic geometry and is responsible for the intimate connection between noncommutative geometry and quantum physics (whose mathematical formalism is based on Hilbert spaces).

The next example of the "strategy" comes from complex *vector bundles* $E \stackrel{\pi}{\to} X$. Step 1 is implemented by passing to

$$\Gamma(X, E) \equiv \Gamma(E) = \{\sigma : X \to E \mid \pi \circ \sigma = \mathrm{id}_X\}, \tag{3}$$

where we assume that σ, like π, is continuous (this is implicit in our definition of arrows). Observation 1 is that $\Gamma(E)$ is a (right) module over $C(X)$, with $\sigma \cdot f$, $f \in C(X)$, simply defined by $\sigma \cdot f(x) = \sigma(x) f(x)$. But just looking at right $C(X)$ modules is not enough to reconstruct vector bundels over X. The key observation is that $\Gamma(E)$ is *finitely generated projective* over $C(X)$. We say that a right module $\mathcal{E}$ over some algebra A is finitely generated projective or f.g.p. if $\mathcal{E}$ is isomorphic (as a right A-module) to $p(\oplus^n A)$, for some n, with $p \in M_n(A)$ (in which case we may assume that $p^2 = p$). Here $\oplus^n A \equiv A^n$ is the direct sum of n copies of A, seen as a right-module over itself through right-multiplication (n times extended to the direct sum), and $M_n(A)$ is the algebra of $n \times n$ matrices with entries in A. The map $p : A^n \to A^n$ is A-linear and hence an element of $M_n(A)$, acting on A^n from the left in the obvious way.

The technical step in proving this is the property of *complementation* of vector bundles: if $E \stackrel{\pi}{\to} X$ is a vector bundle over a (locally) compact Hausdorff space X, there exists another vector bundle $F \stackrel{\pi'}{\to} X$ such that $E \oplus F$ is trivial, i.e. isomorphic to $\mathbb{C}^n \times X$ for some n. From this, the property that $\Gamma(E)$ is f.g.p. over $C(X)$ easily follows: construct p as in

$$p : C(X)^n \to C(X, \mathbb{C}^n) \to \Gamma(X, \mathbb{C}^n \times X) \to \Gamma(E \oplus F) \to \Gamma(E) \oplus \Gamma(F) \to \Gamma(E),$$

where the first four arrows are the obvious isomorphisms and the last one is projection onto the first component.

Subsequently, one can prove that, for given unital commutative C*-algebra A, the category of f.g.p. (right) A-modules is dual to the category of vector bundles over $\Sigma(A)$, for fixed A. We will not do this in detail, but just mention the key steps, drawing attention to the analogy with Gelfand duality.

The passage from vector bundles over X to f.g.p. (right) $C(X)$-modules has just been dealt with. The first step in the opposite direction, of course, is to pass from A to $\Sigma(A)$, which we write as X, so that henceforth we may assume that a given f.g.p. A-module $\mathcal{E}$ is in fact a $C(X)$-module. Subsequently, the fibers E_x of $E = \bigcup_{x \in X} E_x$ are defined as

$$E_x = \mathcal{E} / \mathcal{E} \cdot C(X; x), \tag{4}$$

where $C(X; x) = \{f \in C(X) \mid f(x) = 0\}$. This also yields a canonical projection $\pi_x : \mathcal{E} \to E_x$ and hence a "Gelfand transform" $\sigma \mapsto \hat{\sigma}$, where $\sigma \in \mathcal{E}$ and $\hat{\sigma} : X \to E$ is given by $\hat{\sigma}(x) = \pi_x(\sigma)$. Clearly, $\hat{\sigma}(x) \in E_x$, i.e., $\pi \circ \hat{\sigma} = \mathrm{id}_X$ in terms of the obvious projection $\pi : E \to X$. We now define the topology on E as the weakest one such that $\Gamma(E) = \{\hat{\sigma} \mid \sigma \in \mathcal{E}\}$; a base for this topology is simply given by the collection of all $U \subset E$ for which $\hat{\sigma}^{-1}(U) \in \mathcal{O}(X)$ for all σ.

Finally, by step 3, a "noncommutative vector bundle" over a "noncommutative space" A (in the guise of a C*-algebra) is then simply defined as a f.g.p. right A-module. This is somewhat unsatisfactory, because the notion of f.g.p. is purely algebraic and lacks contact with Hilbert spaces and operator algebras. To put this right, we introduce the following:

Definition 1.1 *A* Hilbert module *over a C*-algebra A is a complete right A-module $\mathcal{E}$ with a $\mathbb{C}$-sequilinear map $\langle\, , \,\rangle : \mathcal{E} \times \mathcal{E} \to A$, called an A-*valued inner product*, satisfying*

$$\begin{aligned} \langle\psi, \varphi\rangle^* &= \langle\varphi, \psi\rangle; && (5)\\ \langle\psi, \varphi\rangle a &= \langle\psi, \varphi a\rangle; && (6)\\ \langle\psi, \psi\rangle &\geq 0; && (7)\\ \langle\psi, \psi\rangle = 0 &\Leftrightarrow \psi = 0, && (8) \end{aligned}$$

for all $\psi, \varphi \in \mathcal{E}$, $a \in A$. The requirement that $\mathcal{E}$ be complete refers to the norm

$$\|\psi\|^2 = \|\langle\psi, \psi\rangle\|_A. \quad (9)$$

Note that (9) indeed defines a norm on $\mathcal{E}$ because of (8).

The three main classes of examples are:

1. *Hermitian vector bundles*. If a complex vector bundle $E \xrightarrow{\pi} X$ is equipped with a hermitian structure,[5] then $\mathcal{E} = \Gamma(E)$ is a Hilbert $C(X)$-module in the obvious way, i.e. $\langle\psi, \varphi\rangle(x) = (\psi(x), \varphi(x))_x$. Conversely, each f.g.p. Hilbert $C(X)$-module is of this form.

2. *Hilbert spaces*. Each Hilbert space is obviously a Hilbert $\mathbb{C}$-module.

3. *C*-algebras*. Each C*-algebra A is a Hilbert A-module with right-module structure given by right-multiplication and $\langle a, b\rangle = a^* b$. Note that the norm (9) coincides with the original norm of A because of the C*-axiom $\|a^* a\| = \|a\|^2$.

Thus Hilbert modules generalize Hermitian vector bundles, Hilbert spaces, and C*-algebras, forming an ideal concept in noncommutative geometry. However, it is precisely this diversity that makes a reconstruction theorem unfeasible.

What's the next step? Ome might try to make the notion of a manifold "noncommutative", and indeed this has been done, leading to all sorts of "noncommutative differential calculi". Or perhaps one should focus on Riemannian manifolds? It was one of Connes's most penetrating insights, however, that it is the class of *spin manifolds* that leads to the most attractive noncommutative generalization in the form of (real) *spectral triples*. To understand the complete definition of a real spectral triple, it is mandatory to start with the commutative case.[6]

Hence we are now going to enter the world of Clifford algebras and Dirac operators, with Varilly's notes *Dirac Operators and Spectral Geometry* as our guide. Whenever we expand on his notes, this will usually have the purpose of clarifying the underlying ideas from classical (i.e. 'commutative') geometry a bit more.

[5]This is a family of inner products $(\, , \,)_x$ in E_x that is continuous in x in the sense that for all $\sigma, \tau \in \Gamma(E)$, the function $x \mapsto (\sigma(x), \tau(x))_x$ is in $C(X)$.

[6]Warning: the reconstruction theorem for real spectral triples is extremely difficult; see Alain Connes, On the spectral characterization of manifolds, arXiv:0810.2088. Before the appearance of this paper, even some of the best experts in noncommutative geometry had published incorrect versions and/or proofs of this theorem. The lecture notes by Wulkenhaar cited earlier give a nice introduction to Connes's paper.

2 Clifford algebras and spinor representations

2.1 Definition of $\mathrm{Cl}(V, g)$

Let $V \cong \mathbb{R}^n$ be a real finite-dimensional vector space. The *tensor algebra*

$$(T(V), \iota : V \hookrightarrow T(V))$$

is (by definition) the solution to the following universal problem: given a linear map $\varphi : V \to A$ from V into a real associative algebra A, there is a unique algebra homomorphism $\alpha : T(V) \to A$ such that $\varphi = \alpha \circ \iota$. The solution of any universal problem is only defined up to isomorphism (in this case, in the category of real associative algebras); the simplest realization of $T(V)$ is

$$T(V) = \bigoplus_{p=0}^{\infty} V^{\otimes p}, \tag{10}$$

where $V^{\otimes p} \equiv \otimes^p V$ is the p-times tensor product of V with itself, $V^{\otimes 0} \equiv \mathbb{R}$, and the direct sum is meant to be algebraic, i.e. there are only finitely many terms in each element of $T(V)$. These elements, then, are linear combinations of terms of the form $v_1 \otimes \cdots \otimes v_p$, with $v_i \in V$. The algebra structure of $T(V)$ is given by *concatenation*, i.e.

$$v_1 \otimes \cdots \otimes v_p \cdot w_1 \otimes \cdots \otimes w_q = v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q, \tag{11}$$

so that, symbolically, $V^{\otimes p} \cdot V^{\otimes q} \to V^{\otimes(p+q)}$. The injection ι is given by $\iota(v) = v$, seen as an element of $V^{\otimes 1} = V$, and the map α in the definition is just given by linear extension of

$$\alpha(v_1 \otimes \cdots \otimes v_p) = \varphi(v_1) \cdots \varphi(v_p). \tag{12}$$

Now suppose the map φ is, in addition, supposed to satisfy[7]

$$\varphi(v)\varphi(w) = -\varphi(w)\varphi(w). \tag{13}$$

The corresponding universal problem for such maps is solved by the *exterior algebra*

$$\Lambda^\bullet(V) = \bigoplus_{p=0}^{n} V^{\otimes_A p}, \tag{14}$$

where $n = \dim(V)$ and $V^{\otimes_A p}$ is the antisymmetrized tensor product of p copies of V. This is the subspace of $V^{\otimes p}$ spanned by totally antisymmetric expressions of the form

$$v_1 \wedge \cdots \wedge v_p = \frac{1}{p!} \sum_{\sigma \in \Pi_p} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}, \tag{15}$$

where Π_p is the symmetric group (aka permutation group) on p symbols and $|\sigma| = 0, 1$ is the signature of $\sigma \in \Pi_p$. One may also realize $\Lambda^\bullet(V)$ as

$$\Lambda^\bullet(V) \cong T(V)/\mathcal{I}_A, \tag{16}$$

[7]Similarly, the universal problem for maps satisfying $\varphi(v)\varphi(w) = \varphi(w)\varphi(w)$ is solved by the symmetric algebra $S(V) = \bigoplus_{p=0}^{n} V^{\otimes_S p}$, where $V^{\otimes_S p}$ is the totally symmetrized tensor product of p copies of V.

where $\mathcal{I}_A$ is the two-sided ideal in $T(V)$ generated by $v \otimes v$ for all $v \in V$, or, equivalently, by $v \otimes w + w \otimes v$ for all $v, w \in V$. The connection between (14) and (16) is as follows: if $\pi : T(V) \to T(V)/\mathcal{I}_A$ is the canonical projection, then $\pi(v_1 \otimes \cdots \otimes v_p) \mapsto v_1 \wedge \cdots \wedge v_p$ defines an isomorphism $\pi(V^{\otimes p}) \to V^{\otimes_A p}$ for each p, and hence an isomorphism $T(V)/\mathcal{I}_A \to \Lambda^\bullet(V)$, the right-hand side being defined as in (14).

Now suppose V is equipped with a *bilinear form* $g : V \times V \to \mathbb{R}$. We may then look for solutions of the universal problem for maps $\varphi : V \to A$ satisfying

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = 2g(v, w). \tag{17}$$

This solution exists and is called the *Clifford algebra* $\mathrm{Cl}(V, g)$. It may be realized in the spirit of (16) as

$$\Lambda^\bullet(V) = T(V)/\mathcal{I}_g, \tag{18}$$

where $\mathcal{I}_g$ is the two-sided ideal in $T(V)$ generated by $v \otimes v - g(v, v)$ for all $v \in V$, or, equivalently, by $v \otimes w + w \otimes v - 2g(v, w)$ for all $v, w \in V$. In terms of the canonical projection $\pi_g : T(V) \to T(V)/\mathcal{I}_g$, we often write

$$v_1 \cdots v_p = \pi_g(v_1 \otimes \cdots \otimes v_p), \tag{19}$$

and, with slight abuse of notation,

$$v_1 \wedge \cdots \wedge v_p = \pi_g(v_1 \wedge \cdots \wedge v_p). \tag{20}$$

In the latter case, we should carefully note whether we regard $v_1 \wedge \cdots \wedge v_p$ as an element of $\Lambda^\bullet(V)$ or of $\mathrm{Cl}(V, g)$. Using the notation (19), we have

$$vw + wv = 2g(v, w), \tag{21}$$

also if each side is sandwiched between other terms. Hence expressions of the type $v_1 \cdots v_p$ can often be simplified, whereas in $v_1 \wedge \cdots \wedge v_p$ – seen as an element of $\mathrm{Cl}(V, g)$ – no rewriting is possible, since no symmetric terms of the kind $vw + wv$ occur. Removing all symmetric occurrences in favour of factors g and terms of lower degree (in p), one can reduce any term of the form (19) to terms of the form (20). It follows that

$$\mathrm{Cl}(V, g) \cong \Lambda^\bullet(V) \tag{22}$$

as vector spaces, and that each side has the following basis, defined in terms of a basis $(e_i)_{i=1}^n$ of V:

$$1, e_1, \ldots, e_n, e_1 \wedge e_2, \ldots, e_1 \wedge \cdots \wedge e_n. \tag{23}$$

Hence, up to a ± 1 sign, each subset S of $\underline{n} = \{1, \ldots, n\}$ defines a basis vector, consisting of the antisymmetrized product of all basis vectors e_i corresponding to elements of S. Therefore, with $n = \dim(V)$,

$$\dim(\mathrm{Cl}(V, g)) = \dim(\Lambda^\bullet(V)) = 2^n. \tag{24}$$

Algebraically, $\mathrm{Cl}(V, g)$ and $\Lambda^\bullet(V)$ are quite different (except when $g = 0$): for example, in $\mathrm{Cl}(V, g)$ one has $v \cdot v = g(v, v)$, whereas $v \cdot v = 0$ in $\Lambda^\bullet(V)$.

2.2 Structure of $\mathrm{Cl}(V, g)$

The structure of $\mathrm{Cl}(V, g)$ is completely determined by the dimension of V and the signature of g. Indeed, by Sylvester's Theorem V has a basis in which g is diagonal, with possible entries $1, -1, 0$. Let us assume, from now on, that g is nondegenerate. In that case, the *signature* (p, q) of g, with $p + q = n$, expresses that $g \equiv g_{p,q}$ has p positive eigenvalues and q negative ones (which may therefore be taken to be ± 1 in a suitable basis). We write

$$\begin{aligned} \mathrm{Cl}_{p,q} &= \mathrm{Cl}(\mathbb{R}^{p+q}, g_{p,q}); && (25)\\ \mathrm{Cl}_p^+ &= \mathrm{Cl}_{p,0}; && (26)\\ \mathrm{Cl}_p^- &= \mathrm{Cl}_{0,p}. && (27) \end{aligned}$$

It turns out that the structure of $\mathrm{Cl}(V, g)$ can be found by computing a few low-dimensional cases and using periodicity results. This will be clarified in the following exercises (some of whose answers are easily distilled from Varilly's notes!). The simplest cases are:

$$\begin{aligned} \mathrm{Cl}_1^- &\cong \mathbb{C}; && (28)\\ \mathrm{Cl}_1^+ &\cong \mathbb{R} \oplus \mathbb{R}. && (29) \end{aligned}$$

We use the basis (u_1, u_2) of $\mathrm{Cl}_1^{\pm}$, where u_1 is $1 \in \mathbb{R}$ seen as $V^{\otimes 0}$ and u_2 is $1 \in \mathbb{R}$ seen as $V^{\otimes 1}$, with $V = \mathbb{R}$. By definition, u_1 is the unit of the algebra. For (28), the map $\varphi : V \to \mathbb{C}$ given by $\varphi(u_2) = i$ satisfies $\varphi(u_2)^2 = g(u_2, u_2) = -1$ and hence extends to $\varphi : \mathrm{Cl}_1^- \to \mathbb{C}$; on $\mathbb{R} = V^{\otimes 0}$ this extension is given by $\varphi(u_1) = 1$ and taking linear combinations. A dimension count shows that φ is bijective and hence is an isomorphism.

To prove (29), define $\varphi : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ by $\varphi(u_2) = (1, -1)$, so that $\varphi(u_2)^2 = g(u_2, u_2) = 1$. This extends to an isomorphism $\varphi : \mathrm{Cl}_1^+ \to \mathbb{R} \oplus \mathbb{R}$; specifically, $\varphi(u_1) = (1, 1)$.

Exercise 2.1 *Show that:*

$$\begin{aligned} \mathrm{Cl}_2^- &\cong \mathbb{H}; && (30)\\ \mathrm{Cl}_2^+ &\cong M_2(\mathbb{R}); && (31)\\ \mathrm{Cl}_{1,1} &\cong M_2(\mathbb{R}). && (32) \end{aligned}$$

Here the *quaternions* $\mathbb{H}$ are the real algebra with unit 1 and other generators I, J, K with relations $I^2 = J^2 = K^2 = -1$, and hence $IJK = -1$. It is convenient to realize $\mathbb{H}$ as the real subalgebra of $M_2(\mathbb{C})$ that consists of matrices of the form

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, z, w \in \mathbb{C} \right\}, \tag{33}$$

with generators realized as (for example)

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; && (34)\\ I &\mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; && (35)\\ J &\mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; && (36)\\ K &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. && (37) \end{aligned}$$

In the case of interest to us, the relevant periodicity results are

$$\mathrm{Cl}_{p+2}^- \cong \mathrm{Cl}_p^+ \otimes \mathrm{Cl}_2^-; \tag{38}$$

$$\mathrm{Cl}_{p+2}^+ \cong \mathrm{Cl}_p^- \otimes \mathrm{Cl}_2^+. \tag{39}$$

It follows that

$$\mathrm{Cl}_{p+8}^\pm \cong \mathrm{Cl}_p^\pm \otimes M_{16}(\mathbb{R}). \tag{40}$$

Exercise 2.2 *Varilly computes all* $\mathrm{Cl}_p^+ \equiv \mathrm{Cl}_{p0}$ *on p. 9. Similarly, compute all* Cl_p^- *for* $p = 1, \ldots, 8$.

You may be disturbed by the sign changes in (38) and (39). These are absent if one uses the language of superalgebras, aka $\mathbb{Z}_2$-graded algebras. We say that a (real or complex) associative algebra A is a *superalgebra*, if $A = A^0 \oplus A^1$ *as a vector space* and multiplication is: $A^0 \times A^0 \to A^0$, $A^1 \times A^1 \to A^0$, $A^0 \times A^1 \to A^1$, $A^1 \times A^0 \to A^1$. In order to avoid confusion with direct sum decompositions *as algebras*, we will henceforth write $A = A^0 \hat{\oplus} A^1$ in this case.

This is the case for $\mathrm{Cl}(V, g)$, since we may write

$$\mathrm{Cl}(V, g) = \mathrm{Cl}(V, g)^0 \hat{\oplus} \mathrm{Cl}(V, g)^1, \tag{41}$$

where $\mathrm{Cl}(V, g)^0/\mathrm{Cl}(V, g)^1$ is the linear span of all $v_1 \cdots v_p$ for p even/odd; this is well defined, since the relations (21) can only change the degree p by an even amount.

Next, we define the *super tensor product* $A \hat{\otimes} B$ of two superalgebras A and B. As a vector space, this coincides with the usual tensor product $A \otimes B$, but as an algebra one defines multiplication by linear extension of

$$a \otimes b_i \cdot a_j \otimes b = (-1)^{|b_i| \cdot |a_j|} a a_j \otimes b_i b. \tag{42}$$

Here we assume that $b_i \in B^i$ and $a_j \in A^j$, with $|b_i| = i$ and $|a_j| = j$; in other words, elements $a \in A^0$ ("even") have grading $|a| = 0$, whereas $|a| = 1$ for $a \in A^1$ ("odd"). The factor $(-1)^{|b_i| \cdot |a_j|}$ is therefore nontrivial iff both b^1 and a_j are odd.

Exercise 2.3 *Show that* $\mathrm{Cl}_{p+q}^\pm \cong \mathrm{Cl}_p^\pm \hat{\otimes} \mathrm{Cl}_q^\pm$.

2.3 Complexification

We start from a complex vector space $V_{\mathbb{C}} \cong \mathbb{C}^n$, still equipped with nondegenerate bilinear form $g_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. This time, there is always a basis of $V_{\mathbb{C}}$ in which g is the identity, so that (up to isomorphism) we may simply write

$$\mathbb{C}\mathrm{l}_n = \mathrm{Cl}(\mathbb{C}^n, 1_n), \tag{43}$$

defined as the solution to a universal problem as in the real case. This also means that

$$\mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n^+ \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cl}_n^- \otimes_{\mathbb{R}} \mathbb{C}, \tag{44}$$

and in fact any $\mathrm{Cl}_{p,q}$ with $p + q = n$ could have been used here: complexification washes out the signature of g.

The structure of such complex Clifford algebras may be inferred either directly, or from the real case: either way, we obtain

$$\mathbb{C}\mathrm{l}_1 \cong \mathbb{C} \oplus \mathbb{C}; \tag{45}$$

$$\mathbb{C}\mathrm{l}_2 \cong M_2(\mathbb{C}), \tag{46}$$

with the periodicity theorem

$$\mathbb{C}\mathrm{l}_{n+2} \cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} M_2(\mathbb{C}). \tag{47}$$

Exercise 2.4 1. *Prove* (45) *both from* (29) *and* (28).

2. *Prove* (47) *directly (e.g., by adapting the argument in Lemma 1.5 of Varilly).*

The last three results combined immediately give

$$\mathbb{C}\mathrm{l}_{2k} \cong M_{2^k}(\mathbb{C}); \tag{48}$$

$$\mathbb{C}\mathrm{l}_{2k+1} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}). \tag{49}$$

It is important not to confuse (49) with the decomposition

$$\mathbb{C}\mathrm{l}_{2k+1} = \mathbb{C}\mathrm{l}^0_{2k+1} \hat{\oplus}\, \mathbb{C}\mathrm{l}^1_{2k+1} \cong M_{2^k}(\mathbb{C}) \hat{\oplus} M_{2^k}(\mathbb{C}), \tag{50}$$

where we used

$$\mathbb{C}\mathrm{l}^0_{2k+1} \cong \mathbb{C}\mathrm{l}_{2k} \tag{51}$$

as algebras. Similarly,

$$\mathbb{C}\mathrm{l}^0_{2k} \cong \mathbb{C}\mathrm{l}_{2k-1}. \tag{52}$$

Exercise 2.5 *Prove* (51) *and* (52).

Namely, in (49) the $\oplus$ is meant in the sense of algebras, whereas in (50) the symbol $\hat{\oplus}$ just refers to *vector spaces*, as already indicated by our notation. The relationship between the two copies of $M_{2^k}(\mathbb{C})$ in (50) is clarified by the introduction of

$$\gamma = (-i)^k e_1 \cdots e_n, \tag{53}$$

with either $n = 2k$ or $n = 2k+1$ and (e_i) a basis of $\mathbb{C}^n$ (the sign of γ depends on the orientation of this basis); see Varilly, §1.5. Note that

$$\gamma^2 = 1. \tag{54}$$

- For $n = 2k+1$, γ obviously lies in the odd part $\mathbb{C}\mathrm{l}^1_{2k+1}$, and is in the center of $\mathbb{C}\mathrm{l}_{2k+1}$ (whose center is actually spanned by 1 and γ). Any element of the odd part may be written uniquely as $y = \gamma x$, where x is even.
- For $n = 2k$, γ is even and generates the center of $\mathbb{C}\mathrm{l}^0_{2k}$; the center of $\mathbb{C}\mathrm{l}_{2k}$ is trivial.

2.4 $\mathbb{C}\mathrm{l}_n$ as a C*-algebra

As it stands, $\mathbb{C}\mathrm{l}_n$ is a complex associative algebra. It is easy to define an involution on $\mathbb{C}\mathrm{l}_n$, namely $z^* = \overline{z}$ for $z \in \mathbb{C}$ and

$$(v_1 \cdots v_p)^* = \overline{v}_p \cdots \overline{v}_1; \tag{55}$$

this is well-defined, since it respects the relations $vw + vw = 2g(v,w)$ that may be used to rewrite either side. To find an appropriate norm, we consider the canonical representation of $\mathbb{C}\mathrm{l}_n$ on itself by left multiplication, and turn $\mathbb{C}\mathrm{l}_n \cong \Lambda^\bullet(\mathbb{C}^n)$ into a complex Hilbert space of dimension 2^n by declaring the basis (23) to be an orthonormal one. More poshly, one may put

$$(v_1 \cdots v_p, w_1 \cdots w_q) = \delta_{pq} \det\left(g(\overline{v}_i, w_j)\right), \tag{56}$$

and an even more sophisticated description is

$$(x, y) = \tau(x^* y), \tag{57}$$

where τ is the trace on $\mathbb{C}\mathrm{l}_n$ defined in Varilly, §1.3. With respect to any of these (equivalent) inner products, the left action of $\mathbb{C}\mathrm{l}_n$ on itself defines a faithful *-representation π, so that $\mathbb{C}\mathrm{l}_n$ is a C*-algebra in the associated norm $\|x\| = \|\pi(x)\|$. One may then see (48) and (49) as isomorphisms of C*-algebras.

It is clear from these isomorphisms that the above representation π cannot be irreducible: the irreducible representation (irrep) of $\mathbb{C}\mathrm{l}_n$ for $n = 2k$ has dimension 2^k instead of 2^{2k}, and for $n = 2k + 1$ one has two inequivalent irreducible representations of dimension 2^k. It is possible to construct the irreps of $\mathbb{C}\mathrm{l}_n$ in a more or less natural way, which also gives insight into the relationship between real and complex Clifford algebras. Our treatment below is simpler but perhaps less profound than Varilly's.[8]

We start with $n = 2k$, with associated real Clifford algebra Cl^+_{2k}, with complexification $\mathbb{C}\mathrm{l}_n$. The basic trick is to turn the real vector space $\mathbb{R}^{2k}$ into $\mathbb{C}^k$, instead of complexifying it to $\mathbb{C}^{2k}$. This can be done by introducing the $2k \times 2k$ real matrix

$$J = \begin{pmatrix} 0 & 1_k \\ -1_k & 0 \end{pmatrix}, \tag{58}$$

which satisfies $J^2 = -1$. We then turn $\mathbb{R}^{2k}$ into a complex vector space by defining $iv = Jv$. This gives an isomorphism $\mathbb{R}^{2k} = \mathbb{R}^k \oplus \mathbb{R}^k \cong \mathbb{C}^k$, explicitly: $(x, y) \mapsto x + iy$. Subsequently, we make $\mathbb{R}^{2k}$ a complex Hilbert space of dimension k through the inner product (in terms of the positive definite form g on $\mathbb{R}^{2k}$)

$$(v, w)_J = g(v, w) + ig(Jv, w). \tag{59}$$

Exercise 2.6 *Verify that this defines a sesquilinear form on $\mathbb{R}^{2k}$ with respect to the complex structure $iv = Jv$ and show that this form is positive definite.*

We then represent $\mathbb{C}\mathrm{l}_{2k}$ on

$$H^{(k)}_F = \Lambda^\bullet(\mathbb{C}^k) \cong \mathbb{C}^{2^k} \tag{60}$$

(the "fermionic Fock space") in two steps.

1. For $v \in \mathbb{R}^{2k}$ we put

$$\pi_F(v) = a(v) + a^\dagger(v), \tag{61}$$

   (also written by Varilly as $a = \epsilon$ and $a^\dagger = \iota$), with $a^\dagger(v)z = zv$ and $a(v)z = 0$ for $z \in \mathbb{C}$, and, with $u_i \in \mathbb{R}^{2k} \cong \mathbb{C}^k$ for all i,

$$a(v)u_1 \wedge \cdots \wedge u_p = \sum_{j=1}^{p} (-1)^{j-1} (v, u_j)_J \, u_1 \wedge \cdots \hat{u}_j \cdots \wedge u_p; \tag{62}$$

$$a^\dagger(v)u_1 \wedge \cdots \wedge u_p = v \wedge u_1 \wedge \cdots \wedge u_p, \tag{63}$$

[8] As a compromise between his approach and the one we follow, consider $\Lambda^\bullet(\mathbb{C}^{2k})$, on which $\mathbb{C}\mathrm{l}_{2k}$ is represented by left multiplication. This representation is reducible, and one may find an irreducible subspace isomorphic to $H^{(k)}_F$ by projecting onto the antisymmetrized combinations of vectors of the type (v, iv), $v \in \mathbb{C}^k$. For one thing, this projection removes all subspaces $\Lambda^p(\mathbb{C}^{2k})$ for $p > k$.

where $\hat{u}_j$ means that u_j is omitted. This satisfies

$$\pi_F(v)\pi_F(w) + \pi_F(w)\pi_F(v) = 2g(v,w) \cdot 1,$$

and hence by universality the map $\pi_F : \mathbb{R}^{2k} \to \mathrm{End}(H_F^{(k)})$ thus defined extends to a representation π_F of Cl_{2k}^+ on $H_F^{(k)}$. This representation is *real*, in the sense that

$$\pi_F(v)^* = \pi_F(v) \tag{64}$$

for all $v \in \mathbb{R}^{2k}$.

2. This representation $\pi_F(\mathrm{Cl}_n^+)$ is then extended to $\mathbb{C}\mathrm{l}_n$ by complex linearity: whenever a factor i acts on a vector v, one uses $iv = Jv$. It follows from the previous item that this is a *-representation in that $\pi_F(x)^* = \pi_F(x^*)$ for all $x \in \mathbb{C}\mathrm{l}_n$.

The irreducibility of π_F is proved by Varilly, §1.9, but it is useful to prove it for yourself.

Exercise 2.7 *Prove that $\pi_F(\mathrm{Cl}_{2k}^+)$ and $\pi_F(\mathbb{C}\mathrm{l}_{2k})$ are faithful and irreducible. Hint: for faithfulness it suffices to look at the action on $\mathbb{C}$. For irreducibility, by Schur's lemma and the structure of Clifford algebras it is enough to prove that $[T, \pi_F(v)] = 0$ for all $v \in V$ implies $T = c \cdot 1$, and for this it is enough to prove that $[T, a(v)] = [T, a^\dagger(v)] = 0$ for all $v \in V$ implies $T = c \cdot 1$.*

The odd-dimensional case is now handled as follows: for $n = 2k + 1$ we use $H_F^{(k)}$ as defined for $n = 2k$, i.e. (60), and represent the even part $\mathbb{C}\mathrm{l}_n^0$ of $\mathbb{C}\mathrm{l}_n$ through (51). For the odd part $\mathbb{C}\mathrm{l}_n^1$, by (54) there are two possibilities: either $\pi_F(\gamma) = 1$, or $\pi_F(\gamma) = -1$. We write these as $\pi_F^\pm$, so that

$$\pi_F^\pm(\gamma x) = \pm\pi_F(x), \tag{65}$$

where $x \in \mathbb{C}\mathrm{l}_n^0$ and hence $\gamma x \in \mathbb{C}\mathrm{l}_n^1$. This gives two inequivalent irreps $\pi_F^\pm$ of $\mathbb{C}\mathrm{l}_n$ for $n = 2k + 1$, both realized on $H_F^{(k)} \cong \mathbb{C}^{2^k}$, in terms of which

$$\pi_F(\mathbb{C}\mathrm{l}_{2k+1}) = \pi_F^+(\mathbb{C}\mathrm{l}_{2k+1}) \oplus \pi_F^-(\mathbb{C}\mathrm{l}_{2k+1}). \tag{66}$$

In view of (52), an analogous phenomenon occurs for $\pi_F(\mathbb{C}\mathrm{l}_n^0)$ for $n = 2k$. The operator $\pi_F(\gamma)$, still with γ as in (53), which lies in the even part, commutes with $\pi_F(\mathbb{C}\mathrm{l}_n^0)$ and reduces it, so that

$$\pi_F(\mathbb{C}\mathrm{l}_{2k}^0) = \pi_F^+(\mathbb{C}\mathrm{l}_{2k}^0) \oplus \pi_F^-(\mathbb{C}\mathrm{l}_{2k}^0), \tag{67}$$

where π_F^+ and π_F^- are both realized on $H_F^{(k-1)} \cong \mathbb{C}^{2^{k-1}}$. Each is irreducible and the pair is inequivalent as representations of $\mathbb{C}\mathrm{l}_{2k}^0$. This phenomenon will recur the next subsection.

We will use the following notation: For $n = 2k$, we write $\mathbb{S}_{2k} = \mathbb{C}^{2^k}$ for the irreducible representation space on which $\pi_F(\mathbb{C}\mathrm{l}_{2k})$ is defined. Its decomposition under $\mathbb{C}\mathrm{l}_n^0$ is written

$$\mathbb{S}_n = \mathbb{S}_n^+ \oplus \mathbb{S}_n^-, \tag{68}$$

with $\mathbb{S}_{2k}^\pm = \mathbb{C}^{2^{k-1}}$. This decomposition induces a *grading* operator $\gamma_n : \mathbb{S}_n \to \mathbb{S}_n$, given by $\gamma_n\psi_\pm = \pm\psi_\pm$ for $\psi_\pm \in \mathbb{S}_\pm$; in fact, in terms of (53) one simply has $\gamma_n = \pi_F(\gamma)$. For $n = 2k+1$, we write $\mathbb{S}_{2k+1} = \mathbb{C}^{2^k}$ for the irreducible representation space carrying either $\pi_F^\pm$ (of which we typically use π_F^+).

2.5 Spin(n) and Spin$^{\mathbb{C}}$(n)

In order to understand "spin geometry", we need to know about the so-called Spin(n) and Spin$^{\mathbb{C}}$(n) groups, where the label n refers to the Clifford algebras Cl_n and $\mathbb{C}\mathrm{l}_n$ containing these groups. Our treatment will be brief, since a full understanding requires more knowledge about Lie groups and Lie algebras than we may reasonably assume.[9] It is the lift from SO(n) to Spin(n) that will later enable us to pass from Riemannian geometry to spin geometry.

We start with Cl_n^+ and write $V \equiv \mathbb{R}^n$ (although an analogous story can be told for all the $\mathrm{Cl}_{p,q}$, with corresponding groups Spin(p,q) covering $SO(p,q)$). Following Varilly's notation, we define an anti-automorphism ! of Cl_n^+ by linear extension of

$$v_1 \cdots v_p! = v_p \cdots v_1; \tag{69}$$

this is easily seen to be well-defined, because it preserves the relations (21) (officially, it is best to define ! by putting $A = (\mathrm{Cl}_n^+)^{\mathrm{op}}$ in the universal property of Cl_n^+ and putting $\varphi(v) = v$)).

Exercise 2.8 *Show that* $xvx! \in V$ *for any* $x = v_1 \cdots v_p \in \mathrm{Cl}_n^+$ *and* $v, v_i \in V \subset \mathrm{Cl}_n^+$.

We now define

$$\mathrm{Spin}(n) = \{tv_1 \cdots v_p \mid p \text{ even}, t = \pm 1, v_i \in V, g(v_i, v_i) = 1\}, \tag{70}$$

where $p = 0$ is included, so $\mathbb{Z}_2 = \{-1, 1\}$, seen as a subgroup of $\mathbb{R} \subset \mathrm{Cl}_n^+$, lies in Spin($n$). This is a group contained in Cl_n^+ under the product given by multiplication in Cl_n^+: the unit element is simply $1 \in \mathbb{R}$, and the inverse is $(v_1 \cdots v_p)^{-1} = v_1 \cdots v_p!$.[10]

A slightly more complicated argument than the one in the exercise (based on the fact that SO(n) is generated by reflections) shows that the map $v \mapsto xvx! = xvx^{-1} \equiv \lambda(x)v$ from V to V is in SO(n). This gives a homomorphism $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$, which is part of the SES (Short Exact Sequence)

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1. \tag{71}$$

Indeed, the kernel of λ is just $\mathbb{Z}_2$. It can be shown that for $n \geq 3$, Spin(n) is connected and simply connected, so in that case it is the universal cover of SO(n). The lowest-dimensional examples are

$$\mathrm{Spin}(3) \cong SU(2); \tag{72}$$

$$\mathrm{Spin}(4) \cong SU(2) \times SU(2). \tag{73}$$

The definition of Spin$^{\mathbb{C}}$(n) is a bit less natural.[11] With the torus $\mathbb{T} = \{z \in \mathbb{C} \mid \bar{z}z = 1\}$, we define

$$\mathrm{Spin}^{\mathbb{C}}(n) = \{zv_1 \cdots v_p \mid p \text{ even}, z \in \mathbb{T}, v_i \in V, g(v_i, v_i) = 1\}, \tag{74}$$

[9] For a very good treatment of the Spin(n) and Spin$^{\mathbb{C}}$(n) groups see R. Goodman and N.R. Wallach, *Representations and Invariants of the Classical Groups* (CUP, 1998).

[10] For arbitrary p one obtains the group Pin(n), which covers $O(n)$.

[11] With Varilly's approach to the definition of Spin$^{\mathbb{C}}$(n), it needs to be shown that any $u_1 \cdots u_p$ for $u_i \in \mathbb{C}^n$ with $\|u_i\| = 1$ can be rewritten as $zv_1 \cdots v_p$ with $z \in \mathbb{C}$, $\|z\| = 1$ and $v_i \in \mathbb{R}^n$ with $\|v_i\| = 1$. This is far from obvious, and his text is ambiguous whether one should impose $u_i = \lambda_i v_i$, $\lambda_i \in \mathbb{T}$ from the start.

again with $p = 0$ included so that $\mathbb{T} \subset \mathrm{Spin}^{\mathbb{C}}(n)$; this defines $\mathrm{Spin}^{\mathbb{C}}(n)$ as a group contained in the algebra $\mathbb{C}\mathrm{l}_n$, borrowing its multiplication. This immediately yields

$$\mathrm{Spin}^{\mathbb{C}}(n) \cong \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathbb{T}, \tag{75}$$

which stands for the quotient of $\mathrm{Spin}(n) \times \mathbb{T}$ by the equivalence relation $(x, z) \sim (-x, -z)$. Indeed, the map $\mathrm{Spin}(n) \times \mathbb{T} \to \mathrm{Spin}^{\mathbb{C}}(n)$ given by $(x, z) \mapsto zx$ assigns the same image to (x, z) and (x', z') iff $(x', z') = (\pm x, \pm z)$. The homomorphism $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ extends to $\mathrm{Spin}^{\mathbb{C}}(n)$, using the same formula $\lambda(x)v = xvx^{-1}$ for $x \in \mathrm{Spin}^{\mathbb{C}}(n)$. This gives our second SES

$$1 \to \mathbb{T} \to \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \to 1; \tag{76}$$

in terms of the realization (75), the second arrow is $z \mapsto [1, z]_{\mathbb{Z}_2}$. And the picture would be incomplete without the third SES

$$1 \to \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n) \to \mathbb{T} \to 1, \tag{77}$$

where the second arrow is $x \mapsto [x, 1]_{\mathbb{Z}_2}$ and the third is $[x, z]_{\mathbb{Z}_2} \mapsto z^2$.

Since $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^+ \subset \mathbb{C}\mathrm{l}_n$ and $\mathrm{Spin}^{\mathbb{C}}(n) \subset \mathbb{C}\mathrm{l}_n$, representations of $\mathbb{C}\mathrm{l}_n$ yield representations of $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ by restriction. Note that $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ are contained in the even part $\mathbb{C}\mathrm{l}_n^0$. We concentrate on the irreducible representations π_F. In general, irreps may become reducible when restricted to subalgebras or subgroups, and inequivalent representations might become equivalent upon restriction. However, none of this occurs here.

- $n = 2k$. Although $\pi_F(\mathbb{C}\mathrm{l}_{2k})$ on $\mathbb{S}_{2k}$ is irreducible, its restriction to $\mathbb{C}\mathrm{l}_{2k}^0$ is reducible, decomposing as a direct sum of two inequivalent irreducible subrepresentations $\pi_F^\pm$, realized on $\mathbb{S}_{2k}^\pm$; see (67). Further restricting $\pi_F^\pm(\mathbb{C}\mathrm{l}_{2k}^0)$ to $\mathrm{Spin}(n)$ or $\mathrm{Spin}^{\mathbb{C}}(n)$, these representations remain irreducible and inequivalent.[12] Hence both spin groups have a pair of inequivalent irreps $\pi_F^\pm$ of (complex) dimension 2^{k-1}. These are called *spinor* representations and are often denoted by $\Delta_\pm$ or $c_\pm$.
- $n = 2k + 1$. Here $\mathbb{C}\mathrm{l}_{2k+1}^0$ has a unique irrep on $\mathbb{S}_{2k+1}$, see (60), which remains irreducible if restricted to either $\mathrm{Spin}(n)$ or $\mathrm{Spin}^{\mathbb{C}}(n)$. Hence both spin groups have just one irreducible spinor representation π_F of dimension 2^k, often called Δ or c, carried by $\mathbb{S}_{2k+1}$

Furthermore, it can be shown that these representations are unitary with respect to the inner product on $H_F^{(k)}$ that has already been defined.

For example, it follows that Spin(3) has a unique spinor representation of dimension 2; this is just de defining (or "spin 1/2") representation $\mathcal{D}_{1/2}$ of $SU(2)$, cf. (72). For $n = 4$ we obtain two inequivalent 2-dimensional spinor representations of Spin(4); these are the irreps of $SU(2) \times SU(2)$ given by $(x, y) \mapsto \mathcal{D}_{1/2}(x)$ and $(x, y) \mapsto \mathcal{D}_{1/2}(y)$.

Finally, it follows from the definition of the spin groups that any representation π obtained by restriction of some representation $\pi(\mathbb{C}\mathrm{l}_n)$ is $\mathrm{Spin}(n)$-*covariant* in the following sense:

$$\pi(x)\pi(v)\pi(x^{-1}) = \pi(\lambda(x)v), \tag{78}$$

for each $x \in \mathrm{Spin}(n)$ and $v \in V$.

[12] Varilly's proof of irreducibility in §1.10 is not correct.

3 Spin manifolds

The following material cannot be found in Varilly's notes, but it seems necessary to me in order to really understand what is going on in his chapter 3. Our aim is to give three equivalent definitions of a Riemannian manifold: the first is intuitively clear, the second goes towards noncommutative geometry, and the third prepares for the definition of a spin manifold as a refinement of a Riemannian manifold.

3.1 What is a Riemannian manifold?

The traditional definition of a Riemannian manifold (M, g) is that M is a smooth manifold and g is a collection of symmetric bilinear forms $g_x : T_xM \times T_xM \to \mathbb{R}$, defined for each $x \in M$, such that each g_x is positive definite, and the collection $\{g_x\}$ is smooth in x. The latter requirement can be formulated either locally or, equivalently, globally.

Locally, we cover $M = \cup_\alpha U_\alpha$ by open sets $U_\alpha \cong \mathbb{R}^n$, with associated (bijective) charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, writing $(x_1, \ldots, x_n)_\alpha = \varphi_\alpha(x)$ for the coordinates of $x \in U_\alpha$ in the chart φ_α. The definition of a smooth manifold then entails that the transition functions $(x_1, \ldots, x_n)_\alpha \mapsto (x_1, \ldots, x_n)_\beta$, defined whenever $x \in U_\alpha \cap U_\beta$, are smooth. The components of the metric at $x \in U_\alpha$ w.r.t. chart φ_α are defined by

$$g_{ij}(x_1, \ldots, x_n)_\alpha = g_x\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right). \tag{79}$$

These, then, are required to be smooth functions of the coordinates $(x_1, \ldots, x_n)_\alpha$, for each α.

Globally, we require the function $x \mapsto g_x(X(x), Y(x))$ from M to $\mathbb{R}$ to be smooth for all smooth vector fields X and Y on M. This is equivalent to the local smoothness condition, for locally we may write $X(x) = \sum_i X_i(x_1, \ldots, x_n)_\alpha \partial/\partial x_i$, $Y(x) = \sum_i Y_i(x_1, \ldots, x_n)_\alpha \partial/\partial x_i$, so that by bilinearity of g one has

$$g_x(X(x), Y(x)) = \sum_{i,j} g_{ij}(x_1, \ldots, x_n)_\alpha X_i(x_1, \ldots, x_n)_\alpha Y_j(x_1, \ldots, x_n)_\alpha. \tag{80}$$

This global definition of smoothness is the main step towards Varilly's Definition 2.1 of a Riemannian metric on p. 18, which, however, works with continuity instead of smoothness. (This is very unusual in Riemannian geometry, as the curvature and other tensors cannot be defined if the metric is merely continuous.). In effect, his definition states that $\mathcal{X}(M)$ is something like a real Hilbert module over $C(M)$, or, in the smooth case, a pre Hilbert module over $C^\infty(M)$ in the sense that $\mathcal{E} = \mathcal{X}(M)$ is not complete in the norm $\|X\|^2 = \sup_{x \in M} g_x(X(x), X(x))$.

Exercise 3.1 *Show that the traditional definition of a Riemannian manifold is equivalent to Varilly's Definition 2.1 (adapted to the smooth setting). In particular, explain how the maps* $g_x : T_xM \times T_xM \to \mathbb{R}$ *are obtained from the single map* $g : \mathcal{X}(M) \times \mathcal{X}(M) \to C(M)$ *in Varilly's Definition 2.1.*

Our main goal, however, is a third approach to the definition of a Riemannian manifold, available if M is *oriented*.[13] This necessitates a crash course in principal fibre bundles and their associated vector bundles.

[13] Let $n = \dim(M)$ as usual, and take the top power $\Omega^n(M) = \Gamma(M, \Lambda^n(M))$, where $\Lambda^p(M)$ is the bundle of

3.2 Principal fibre bundles

Principal fibre bundles can be defined in the continuous setting or in the smooth setting. We choose the latter setting, so that in what follows P and M are manifolds and G is a Lie group (think of $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$).

Definition 3.2 *Let P be a manifold and G a Lie group that acts on P from the right, written $(p,g) \mapsto pg$. We also write $M = P/G$, with canonical projection $\pi : P \to M$. The pair (P,G) is a* principal G-bundle *if:*

1. *The G-action is free (i.e. $pg \neq p$ for all p and all $g \neq e$);*
2. *Each $x \in M$ has an open neighbourhood U such that there is a bundle morphism $P_{|U} \equiv \pi^{-1}(U) \to U \times G$ intertwining the given G-action on P with the natural G-action on $U \times G$, i.e. $(x,h)g = (x,hg)$.*

It follows that:

- the G-action on P is proper;[14]
- the quotient space M is a manifold;
- the projection $\pi : P \to M$ is G-invariant in that $\pi(pg) = \pi(p)$;
- the projection π is a surjective submersion.[15]

The simplest example is, of course, $P = M \times G$ with obvious right G action; this bundle is called *trivial*. To give our first nontrivial example, we recall that if $E \to M$ and $F \to M$ are (real or complex) vector bundels over M, one can form the vector bundle

$$\mathrm{Hom}(E,F) = \cup_{x \in M} L(E_x, F_x), \tag{81}$$

over M, whose fiber at x consists of all linear maps $E_x \to F_x$. We denote the projection $\mathrm{Hom}(E,F) \to M$ by $\tilde{\pi}$, i.e. any $\varphi \in \mathrm{Hom}(E,F)$ is by definition an element of some $L(E_x, F_x)$, so that $\varphi : E_x \to F_x$, in which case $\tilde{\pi}(\varphi) = x$. The smooth structure on $\mathrm{Hom}(E,F)$ is defined in terms of E and F. This may either be done locally, or globally, as follows. Let $\tilde{\phi} : M \to \mathrm{Hom}(E,F)$ satisfy $\tilde{\pi} \circ \tilde{\phi} = \mathrm{id}_M$, i.e., $\tilde{\phi}(x) \in L(E_x, F_x)$. For each $\sigma \in \Gamma(E)$, we obtain a map $\phi(\sigma) : M \to F$ in the obvious way by

$$\phi(\sigma)(x) = \tilde{\phi}(x)(\sigma(x)); \tag{82}$$

---

p-forms over M and $\Gamma(M, \Lambda^p(M))$ is the $C^\infty(M)$-module of smooth sections thereof. The fibers of $\Lambda^n(M)$ are 1-dimensional. A *volume form* is an element Vol of $\Omega^n(M)$ that nowhere vanishes, i.e. $\mathrm{Vol}(x) \neq 0$ for all $x \in M$. A manifold M is *orientable* if it has a volume form. If so, the volume forms fall into two equivalence classes, where $\mathrm{Vol}_1 \sim \mathrm{Vol}_2$ iff $\mathrm{Vol}_1(x) = c(x)\mathrm{Vol}_2(x)$, where $c \in C^\infty(M)$ is strictly positive. An *orientation* of an orientable manifold M is a choice of one of these equivalence classes. Such a choice makes an orientable manifold *oriented*.

A local basis $(e_1(x), \dots, e_n(x))$ of T_xM, is said to be oriented if the associated n-form $\theta_1(x) \wedge \cdots \theta_n(x)$ lies in the equivalence class defined by the given orientation, i.e. if $\theta_1(x) \wedge \cdots \theta_n(x) = c(x)\mathrm{Vol}(x)$ with $c(x) > 0$, where Vol is a representative of the equivalence class of nonvanishing volume forms defining the orientation. Of course, this depends on the order in which the $e_i(x)$ are listed!

[14]This means that the map $(p,g) \mapsto (pg,p)$ from $P \times G$ to $P \times P$ is proper, i.e. the inverse image of a compact set is compact.

[15]A smooth surjective map $f : M \to N$ between two manifolds is a *submersion* if $f_*(x) : T_xM \to T_{f(x)}N$ is a surjection for all $x \in M$. This is not automatic from surjectivity.

clearly, $\phi(\sigma)$ is a section of F. But this section is not necessarily smooth. We now define

$$\Gamma(\mathrm{Hom}(E,F)) = \{\tilde{\phi} : M \to \mathrm{Hom}(E,F) \mid \tilde{\pi} \circ \tilde{\phi} = \mathrm{id}_M, \phi(\sigma) \in \Gamma(F)\,\forall\,\sigma \in \Gamma(E)\}. \tag{83}$$

In other words, we require by definition (of the left-hand side) that

$$\Gamma(\mathrm{Hom}(E,F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E),\Gamma(F)), \tag{84}$$

where the bijection $\tilde{\phi} \leftrightarrow \phi$ is given by (82). Finally, the smooth structure on any vector bundle may be defined by stating what its smooth sections are (see Chapter 1 above for the topological case), so that we have now also (albeit implicitly) defined the smooth structure on $\mathrm{Hom}(E,F)$.

We now apply this construction to the real vector bundles $E = M \times \mathbb{R}^n$ and $F = TM$, with $n = \dim(M)$ as usual; this yields the vector bundle $\mathrm{Hom}(M\times\mathbb{R}^n, TM)$ over M, whose fiber $\mathrm{Hom}_x(M \times \mathbb{R}^n, TM)$ at x consists of all linear maps $p_x : \mathbb{R}^n \to T_xM$. For a given such map p_x, we may consider the set of n vectors in T_xM given by $(p_x(e_1), \dots, p_x(e_n))$, where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$.

Definition 3.3 *The* (oriented) frame bundle $F_{(+)}(M)$ *on an (oriented) manifold M is the sub-bundle of* $\mathrm{Hom}(M \times \mathbb{R}^n, TM)$ *consisting of all linear maps $p_x : \mathbb{R}^n \to T_xM$, $x \in M$, for which $(p_x(e_1), \dots, p_x(e_n))$ is an (oriented) basis of T_xM. The bundle structure and the smooth structure of this bundel are inherited from* $\mathrm{Hom}(M\times\mathbb{R}^n, TM)$. *In particular, the projection $\pi_F : F_{(+)}(M) \to M$ is given by $\pi_F(p_x) = x$, where $p_x \in \mathrm{Hom}_x(M \times \mathbb{R}^n, TM)$.*

So the idea is that each $p_x \in F_x(M)$ defines a basis for $T_x(M)$, in other words, a *frame*. A local section $p : U \to F(M)$, $x \mapsto p_x$, then defines a "moving frame" $(e_1(x), \dots, e_n(x))$, with $e_i(x) = p_x(e_i)$, defined for $x \in U \subset M$. It is important to realize that unlike the vector bundle $\mathrm{Hom}(M\times\mathbb{R}^n, TM)$ of which it is a sub-bundle, $F(M)$ typically has no global smooth sections.[16]

We now show that $F(M)$ may be seen as a principal $GL(n,\mathbb{R})$ bundle. Indeed, this group acts on $F(M)$ by

$$(pg)(v) = p(gv), \tag{85}$$

where $p \in F(M)$, $g \in GL(n,\mathbb{R})$, and $v \in \mathbb{R}^n$. If g is given by the matrix g_{ij}, and $(e_1(x), \dots, e_n(x))$ with $e_i(x) = p(e_i)$, $p \equiv p_x$, is a basis of T_xM, then pg induces the (ordered) set of vectors $(e'_1(x), \dots, e'_n(x))$ with $e'_i(x) = (pg)(e_i) = \sum_j g_{ji}e_j(x)$. The condition $\det(g) \neq 0$ then precisely guarantees that $(e'_1(x), \dots, e'_n(x))$ is again a basis of T_xM. Furthermore, this action is free, because the defining $GL(n,\mathbb{R})$ action on $\mathbb{R}^n$ is free. A local trivialization $TM_{|U} \cong U \times \mathbb{R}^n$ of TM for $U \subset M$ (with $TM_{|U} = \pi^{-1}(U)$) induces a local trivialization of $\mathrm{Hom}(M \times \mathbb{R}^n, TM)$ as

$$\mathrm{Hom}(M \times \mathbb{R}^n, TM)_{|U} \cong U \times L(\mathbb{R}^n, \mathbb{R}^n) \equiv U \times M_n(\mathbb{R}),$$

which in turn induces a local trivialization of $F(M)$ as

$$F(M)_{|U} \cong U \times GL(n,\mathbb{R}). \tag{86}$$

This diffeomorphism is explicitly given as follows. If $\xi \in T_xM$ corresponds to $(x, \tilde{\xi}) \in U \times \mathbb{R}^n$, with $\tilde{\xi} = \sum_i \tilde{\xi}_i e_i$, then $p \in F_x(M)$, $x \in U$, corresponds to $(x, \tilde{p}) \in U \times GL(n,\mathbb{R})$, where $\tilde{p}_{ij} = (e_i, \widetilde{p(e_j)})$ in terms of the standard inner product on $\mathbb{R}^n$. It follows that (86) is equivariant, since pg corresponds to the matrix product $\tilde{p}g$. This shows that $F(M)$ is

[16] By Brouwer's Theorem, the two-sphere $M = S^2$ already illustrates this phenomenon.

a principal $GL(n,\mathbb{R})$ bundle under the action (85), and analogously $F_+(M)$ is a principal $GL_+(n,\mathbb{R})$ bundle (where $GL_+(n,\mathbb{R})$ consists of all $g \in M_n(\mathbb{R})$ with $\det(g) > 0$).

Now suppose that M is oriented and Riemannian. This leads to the following refinement of the frame bundle.

Definition 3.4 *The* orthonormal frame bundle $\mathrm{SO}(M)$ *on an oriented Riemannian manifold* (M, g) *is the sub-bundle of* $F_+(M)$ *consisting of all linear maps* $p_x : \mathbb{R}^n \to T_xM$, $x \in M$, *for which*

$$g_x(p_x(e_i), p_x(e_j)) = \delta_{ij}. \tag{87}$$

In fact, $(p_x(e_1), \ldots, p_x(e_n))$ is automatically a basis of T_xM by (87), so we could equivalently have defined $\mathrm{SO}(M)$ as the sub-bundle of $\mathrm{Hom}(M\times\mathbb{R}^n, TM)$ consisting of all $p_x : \mathbb{R}^n \to T_xM$ for which (87) holds and $(p_x(e_1), \ldots, p_x(e_n))$ is oriented. Clearly, $\mathrm{SO}(M)$ is no longer a principal $GL_+(n,\mathbb{R})$ bundle, since (87) may be violated by the action of g. However, it is a principal $\mathrm{SO}(n)$ bundle over M by the same action (85).

Exercise 3.5 *Prove this in detail.*

So a Riemannian structure on M defines a specific principal $\mathrm{SO}(n)$-bundle over M. Could we, perhaps, *define* Riemannian manifolds in such a way? Yes, we can, but to do so, a certain construction on bundles is needed.

3.3 Associated vector bundles

Let P be a principal G bundle and let G act linearly on some vector space V (i.e., a representation $G \to \mathrm{Hom}(V)$ is given). From these data we construct a vector bundle Eover M with typical fibre V, as follows. The total space of the bundle is

$$E = P \times_G V \equiv (P \times V)/G, \tag{88}$$

where the quotient is defined by the obvious right G action on $P\times V$, i.e. $(p, v)g = (pg, g^{-1}v)$. We denote elements of E by equivalence classes $[p, v]$, so that $[pg, v] = [p, gv]$. In terms of these, the bundle projection $\tilde{\pi} : E \to M$ is given by

$$\tilde{\pi}([p, v]) = \pi(p), \tag{89}$$

where $\pi : P \to M$ is the bundle projection of P.

Exercise 3.6 1. *Show that a local trivialization* $P_{|U} \cong U\times G$ *corresponds bijectively to a choice of a local section* $s : U \to P$ *and explain how the latter also induces a local trivialization* $E_{|U} \cong U \times V$.

2. *Prove that the space of smooth sections of* $P \times_G V$ *can be realized as*

$$\Gamma(P \times_G V) \cong C^\infty(P, V)^G \equiv \{f \in C^\infty(P, V) \mid f(pg) = g^{-1}f(p)\,\forall\, p \in P, g \in G\}. \tag{90}$$

For us, the first example of this construction is $F(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n$, defined with respect to the usual action of $GL(n, R)$ on $\mathbb{R}^n$. It then turns out that

$$F(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \cong TM; \tag{91}$$

the isomorphism in question is given by $[p, v] \mapsto p(v)$. Similarly,

$$\mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathbb{R}^n \cong TM. \tag{92}$$

Exercise 3.7 *Show that* $\Gamma(TM) \cong \Gamma(\mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathbb{R}^n)$*, using* (90).

Definition 3.8 *An oriented Riemannian manifold is an oriented manifold* M*,* $\dim(M) = n$*, together with a principal* $\mathrm{SO}(n)$ *bundle* $\mathrm{SO}(M)$ *over* M *and an explicit isomorphism* (92).

Exercise 3.9 *Show that this definition of a Riemannian manifold is equivalent to the traditional one.*

3.4 Spin manifolds

We now recall the groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ from Chapter 2, as well as their action λ on $\mathbb{R}^n$. With respect to the latter, we arrive at the following.

Definition 3.10

1. *A* Spin manifold *is an oriented manifold* M*,* $\dim(M) = n$*, together with a principal* $\mathrm{Spin}(n)$ *bundle* $\mathrm{Spin}(M)$ *over* M *and an explicit isomorphism*

$$\mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{R}^n \cong TM. \tag{93}$$

2. *Similarly, A* $\mathrm{Spin}^{\mathbb{C}}$ manifold *is an oriented manifold* M*,* $\dim(M) = n$*, together with a principal* $\mathrm{Spin}^{\mathbb{C}}(n)$ *bundle* $\mathrm{Spin}^{\mathbb{C}}(M)$ *over* M *and an explicit isomorphism*

$$\mathrm{Spin}^{\mathbb{C}}(M) \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{R}^n \cong TM. \tag{94}$$

This is actually a refinement of a Riemannian manifold:

Exercise 3.11 *Show that a Spin manifold or a* $\mathrm{Spin}^{\mathbb{C}}$ *manifold automatically carries a Riemannian structure.*

Conversely, a spin structure on M is *compatible* with a given Riemannian structure if the latter coincides with the one of this exercise. Let us note, however, that this can always be achieved by modifying (93) if necessary.

The richness of Spin geometry compared with Riemannian geometry is a consequence of the existence of certain vector bundles associated with $\mathrm{Spin}(M)$ that are different from the tangent bundle or tensor powers thereof. Namely, as shown in the previous chapter, the groups $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ have *spinor representations*.

Recall that for any n, we have a spinor representation of $\mathrm{Spin}(n)$ and $\mathrm{Spin}^{\mathbb{C}}(n)$ on $\mathbb{S}_n$, reducible for n even, and irreducible for n odd. With respect to these, a Spin manifold M defines a complex vector bundle

$$\mathcal{S}_n = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{S}_n, \tag{95}$$

called the *spinor bundle* over M. Similarly, a $\mathrm{Spin}^{\mathbb{C}}$ manifold defines a complex vector bundle of the same dimension by

$$\mathcal{S}_n^{\mathbb{C}} = \mathrm{Spin}(M)^{\mathbb{C}} \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{S}_n. \tag{96}$$

The relationship between Spin and $\mathrm{Spin}^{\mathbb{C}}$ manifolds, and hence between (95) and (96), is not straightforward and will be analysed below.

3.5 Charge conjugation

The notion of *charge conjugation*, originally developed in the context of particle physics, is the key to the relationship between Spin and $\mathrm{Spin}^{\mathbb{C}}$ manifolds. As just described, let $\mathbb{S}_n$ simultaneously carry spinor representations of $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$ as well as of $\mathrm{Spin}^{\mathbb{C}}(n) \subset \mathbb{C}\mathrm{l}_n$; for the moment we will be interested in the inclusion

$$\mathrm{Cl}_n^{\pm} \subset \mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n^{\pm} \otimes_{\mathbb{R}} \mathbb{C}, \tag{97}$$

in so far as $\mathrm{Cl}_n^{\pm}$ (for a fixed choice of + or −, to which the analysis below is quite sensitive) is included in the complex algebra $\mathbb{C}\mathrm{l}_n$ as a *real* subalgebra and both act irreducibly on the *complex* vector space $\mathbb{S}_n$. Suppose, more generally, that one has a real subalgebra A of its complexification $\mathbb{A} = A \otimes_{\mathbb{R}} \mathbb{C}$, and that $\mathbb{A}$ (and hence A) acts irreducibly on a complex vector space $\mathbb{S}$ through a representation $\pi : \mathbb{A} \to \mathrm{End}(\mathbb{S})$. Note that $\mathbb{A}$ typically has many different real subalgebras of which it is the complexification; for example, for $\mathbb{A} = \mathbb{C}\mathrm{l}_n$ one might take $A = \mathrm{Cl}^+$, or $A = \mathrm{Cl}^-$, or any other $\mathrm{Cl}_{p,q}$ with $p + q = n$.

In such a situation, two questions may arise:

1. Does $\mathbb{S}$ have a *real* subspace S that is stable under $\pi(A)$? In that case, $\pi(\mathbb{A})$ would be the complexification of $\pi(A)$ restricted to S, in that $\mathbb{S} = S \otimes_{\mathbb{R}} \mathbb{C}$ and $\pi(a + bi) = \pi(a) + i\pi(b)$ for $a, b \in A$ and $a + bi \in \mathbb{A}$.

2. Does the complex representation $\pi(A)$ on $\mathbb{S}$ extend to a *quaternionic* representation, i.e., does the given $\mathbb{C}$-action on $\mathbb{S}$ (which commutes with $\pi(A)$ by its complex linearity) extend to an $\mathbb{H}$-action on $\mathbb{S}$ that still commutes with $\pi(A)$?

Let us give an easy *sufficient* condition for a positive answer to either question; using Schur's lemma, this condition is also *necessary* (given the assumed irreducibility of π).

1. Suppose there is an antilinear map $J : \mathbb{S} \to \mathbb{S}$ with $J^2 = 1$, such that $[J, \pi(a)] = 0$ for all $a \in A$. In that case, we may put $S = \{v \in \mathbb{S} \mid Jv = v\}$, and the answer to question 1 is positive. Such a J is called a *real* structure for $(\mathbb{A}, A, \pi, \mathbb{S})$.

2. Suppose there is an antilinear map $J : \mathbb{S} \to \mathbb{S}$ with $J^2 = -1$, such that $[J, \pi(a)] = 0$ for all $a \in A$. In that case, the generators (I, J, K) of $\mathbb{H}$ act on $\mathbb{S}$ by $Iv = iv$, J as given, and $K = IJ$. Thus the answer to question 2 is positive. Such a J is called a *quaternionic* structure for $(\mathbb{A}, A, \pi, \mathbb{S})$.

Either way, if π is faithful and irreducible, we may *reconstruct* A from $\mathbb{A}$ and J by[17]

$$A = \{a \in \mathbb{A} \mid [J, \pi(a)] = 0\}. \tag{98}$$

More generally, J identifies a real subalgebra

$$\mathrm{End}_{\mathbb{R}}(\mathbb{S}) = \{x \in \mathrm{End}(\mathbb{S}) \mid JxJ^* = x\} \tag{99}$$

of $\mathrm{End}(\mathbb{S})$; if $\pi(\mathbb{A})$ is faithful and irreducible, and $\mathbb{S}$ is finite-dimensional, we have $\mathbb{A} \cong \pi(\mathbb{A}) \cong \mathrm{End}(\mathbb{S})$ and $A \cong \pi(A) \cong \mathrm{End}_{\mathbb{R}}(\mathbb{S})$.

[17]This provides some justification for Connes' denoting J as a *real*structure even if $J^2 = -1$ (i.e. in the quaternionic case); (98) holds with either sign of J^2.

As an application of (98), for n *even* we always have either a real or a quaternionic structure $J_n^{\pm}$ for $\mathrm{Cl}_n^{\pm} \subset \mathbb{C}\mathrm{l}_n$ for both signs with respect to the spinor or Fock representation π of $\mathbb{C}\mathrm{l}_n$ (which is faithful and irreducible), so that

$$\mathrm{Cl}_n^{\pm} = \{x \in \mathbb{C}\mathrm{l}_n \mid [J_n^{\pm}, \pi(x)] = 0\}. \tag{100}$$

Finally, we may altogether forget $\mathbb{A}$ and start from an irreducible representation $\pi(A)$ of the real algebra A on a real vector space S; this includes the complex case $\mathbb{S}$, since we may regard $\mathbb{S}$ as a real vector space (of twice its complex dimension). The *commutant* $\pi(A)'$ consists of all (bounded) real-linear maps $T : S \to S$ such that $[T, \pi(a)] = 0$ for all $a \in A$. We then have the following possibilities.

- $\pi(A)' \cong \mathbb{R}$ iff the $\mathbb{C}$-linear extension $\pi_{\mathbb{C}}(A)$ of $\pi(A)$ to the complexification $S \otimes_{\mathbb{R}} \mathbb{C}$ of S (which, in this case, is the complex vector space $\mathbb{S}$ in the discussion above) admits a real structure; regarded as a *real* representation, $\pi_{\mathbb{C}}(A)$ on $\mathbb{S} = S \otimes_{\mathbb{R}} \mathbb{C}$ is *reducible*.
- $\pi(A)' \cong \mathbb{C}$ iff S is itself complex (coinciding with $\mathbb{S}$ above), $\pi(A)$ is $\mathbb{C}$-linear, irreducible over $\mathbb{C}$ *and* $\mathbb{R}$, and admits neither a real not a quaternionic structure.
- $\pi(A)' \cong \mathbb{H}$ iff S is complex (coinciding with $\mathbb{S}$ above) and $\pi(A)$ admits a quaternionic structure, so that it is $\mathbb{H}$-linear (hence *a fortiori* $\mathbb{C}$-linear); it is irreducible over all of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

In case that no real or quaternionic structure exists, one may ask for either of these for a smaller real subalgebra of $\mathbb{A}$ than A, or, say, for a group contained in A. In odd dimensions n this will be the case for $A = \mathrm{Cl}_n$, the smaller subalgebra being Cl_n^0 (i.e. the even part of Cl_n and hence also for the group $\mathrm{Spin}(n)$. Thus for all n we will find an operator $J_n : \mathbb{S}_n \to \mathbb{S}_n$ with $J_n^{\pm} = 1$ and $[J, \pi(x)] = 0$ for all $x \in \mathrm{Cl}_n^0$ and hence also for all $x \in \mathrm{Spin}(n)$. Appropriate analogues of (98) then hold. For example, for *odd* n one has

$$(\mathrm{Cl}_n^{\pm})^0 = \{x \in \mathbb{C}\mathrm{l}_n^0 \mid [J_n^{\pm}, \pi(x)] = 0\}; \tag{101}$$

$$\mathrm{Spin}(n) = \{x \in \mathrm{Spin}^{\mathbb{C}}(n) \mid [J_n^{\pm}, \pi(x)] = 0\}. \tag{102}$$

These are not valid for *even* n, since $\pi((\mathrm{Cl}_n^{\pm})^0)$ and $\pi(\mathrm{Spin}(n))$ are not irreducible, but this can be remedied by a simple trick: if (68) is the decomposition of $\mathbb{S}_n$ into irreducibles (for both $\pi((\mathrm{Cl}_n^{\pm})^0)$ and $\pi(\mathrm{Spin}(n))$), and γ_n is the grading on $\mathbb{S}_n$, then for *even* n one has

$$(\mathrm{Cl}_n^{\pm})^0 = \{x \in \mathbb{C}\mathrm{l}_n^0 \mid [J_n^{\pm}, \pi(x)] = 0, [\gamma_n, \pi(x)] = 0\}; \tag{103}$$

$$\mathrm{Spin}(n) = \{x \in \mathrm{Spin}^{\mathbb{C}}(n) \mid [J_n^{\pm}, \pi(x)] = 0, [\gamma_n, \pi(x)] = 0\}. \tag{104}$$

Finally, let $\mathbb{S}$ be a Hilbert space. In that case one requires J to be anti-unitary, i.e., $J^*J = JJ^* = 1$. This guarantees that, for both signs of J^2, the representations $\pi(A)$ on $\mathbb{S}$ and $\overline{\pi}(A)$ on $\overline{\mathbb{S}}$ are unitarily equivalent, namely through $J : \mathbb{S} \to \overline{\mathbb{S}}$ (which is *linear*!). Here $\overline{\mathbb{S}}$ is the same set and additive group as $\mathbb{S}$, but as a complex vestor space it carries the conjugate $\mathbb{C}$-action; the representation $\overline{\pi}(A)$ is the same as $\pi(A)$, but now seen as acting on $\overline{\mathbb{S}}$. Conversely, if one has a unitary equivalence between $\pi : A \to \mathrm{End}(\mathbb{S})$ and $\overline{\pi} : A \to \mathrm{End}(\overline{\mathbb{S}})$, then this may arise from either a real or a quaternionic structure.[18]

[18]Connes and his followers call such a J *real* in both cases, i.e., even if $J^2 = -1$. Physicists call both real and quaternionic structures *charge conjugations*, often denoted by C instead of J.

3.6 Charge conjugation for Clifford algebras

Let us work out the details for the case of $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^{\pm} \subset \mathbb{C}\mathrm{l}_n$. Both $\mathrm{Cl}_n^{\pm}$ and $\mathbb{C}\mathrm{l}_n$ act irreducibly on $\mathbb{S} = \mathbb{S}_n$ by the spinor representation $\pi = \pi_F$ (see section 2.4). As shown in section 2.5, the spin group $\mathrm{Spin}(n)$ (as well as $\mathrm{Spin}^{\mathbb{C}}(n)$) then acts on the same space $\mathbb{S}_n$ by restricting $\pi(\mathbb{C}\mathrm{l}_n)$ to $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n$; the representation $\pi(\mathrm{Spin}(n))$ is irreducible for odd n, whilst it is the direct sum of two irreps for even n. We recall the structure of $\mathrm{Cl}_n^{\pm}$ and $\mathbb{C}\mathrm{l}_n$ for $n = 1, \ldots, 8$ (extended to arbitrary n by periodicity mod 8), and give existence of $J_n^{\pm}$ and sign of its square (here x in the tabel means that $J_n^{\pm}$ does not exist for that sign and n).

n	Cl_n^+	Cl_n^-	$\mathbb{C}\mathrm{l}_n$	$\mathbb{S}_n$	$(J_n^+)^2$	$(J_n^-)^2$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}$	1	x
2	$M_2(\mathbb{R})$	$\mathbb{H}$	$M_2(\mathbb{C})$	$\mathbb{C}^2$	1	−1
3	$M_2(\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$	$\mathbb{C}^2$	x	−1
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$\mathbb{C}^4$	−1	−1
5	$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$	$\mathbb{C}^4$	−1	x
6	$M_4(\mathbb{H})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$\mathbb{C}^8$	−1	1
7	$M_8(\mathbb{C})$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$	$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$	$\mathbb{C}^8$	x	1
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$\mathbb{C}^{16}$	1	1

For example, take $n = 1$. The spinor representation $\pi(\mathrm{Cl}_1^+)$ on $\mathbb{S}_1 = \mathbb{C}$ is given by $\pi(1) = 1$ and $\pi(e_1) = 1$; accordingly, there is a real structure given by $Jz = \overline{z}$. This corresponds to fact that $\pi(\mathrm{Cl}_1^+)$ on $\mathbb{C}$ is the complexification of the real representation $\pi_{\mathbb{R}}$ of Cl_1^+ on $S_1 = \mathbb{R}$, given by the same expressions $\pi_{\mathbb{R}}(1) = \pi_{\mathbb{R}}(e_1) = 1$, with commutant $\pi_{\mathbb{R}}(\mathrm{Cl}_1^+) = \mathbb{R}$. In contrast, regarding $\mathbb{C} \cong \mathbb{R}^2$ as a real irreducible representation space for Cl_1^-, one has $\pi(\mathrm{Cl}_1^-)$ on $\mathbb{S}_1 = \mathbb{C}$ given by $\pi(1) = 1 \cong 1_2 = \mathrm{diag}(1, 1)$ and $\pi(e_1) = i \cong J$, with J given by (58) (for $k = 1$). This admits neither a real nor a quaternionic structure, whilst $\mathbb{R}^2$ has no invariant subspace under $\pi(\mathrm{Cl}_1^-)$, despite the presence of $J \in \pi(\mathrm{Cl}_1^-)'$.

For $n = 2$, the case Cl_2^+ is obvious; the real structure is again given by $Jz = \overline{z}$, $z \in \mathbb{C}^2$. For Cl_2^-, we use the spinor representation $\pi(e_1) = I$, $\pi(e_2) = J$, and hence $\pi(e_1 e_2) = K$, with I, J, K given in (35) – (37). We relabel these as I', J', K' and now introduce a second $\mathbb{H}$-action on $\mathbb{C}^2$, which commutes with the one just defined, by $Iz = iz$, $J(z_1, z_2) = (\overline{z}_2, -\overline{z}_1)$, and hence $K(z_1, z_2) = (i\overline{z}_2, -i\overline{z}_1)$.

Exercise 3.12 *Check the table for $n = 3$ and $n = 4$.*

The entries for $n = 5, 6, 7, 8$ follow from the previous ones and the periodicity results (38) and (39). For example, for $n = 5$ we have (with $\otimes \equiv \otimes_{\mathbb{R}}$)

$$\mathrm{Cl}_5^+ \cong \mathrm{Cl}_3^- \otimes \mathrm{Cl}_2^+; \tag{105}$$
$$\mathrm{Cl}_5^- \cong \mathrm{Cl}_3^+ \otimes \mathrm{Cl}_2^-. \tag{106}$$

The tensor product of a quaternionic and a real structure, as in (105), is a quaternionic one, explaining the sign in $(J_5^+)^2 = -1$. On the other hand, since Cl_3^+ has neither, so does Cl_5^-. In general, real tensor real is real, quaternionic tensor quaternionic is real, and, as already used, real tensor quaternionic is quaternionic (since $1 \times 1 = 1$, $-1 \times -1 = 1$, and $1 \times -1 = 1$, respectively).

Let us now look at the peculiar cases Cl_1^- and Cl_5^-, for which neither a real nor a quaternionic structure exists. A way out of this situation, favoured by Connes and his followers, is to restrict $\mathbb{C}l_{2k+1}$, $k = 0,2$ to its even part $(\mathbb{C}l_{2k+1})^0$, which by (51) is isomorphic to $\mathbb{C}l_{2k}$. This does have a J for all k, reducing the situation at $n = 2k+1$ to the one at $n = 2k$ (mod 8, i.e., $n = 1$ is reduced to $n = 8$). In particular, J_{2k}^- provides a real or quaternionic structure for $(\mathrm{Cl}_{2k+1}^-)^0$ and hence commutes with all *even* elements of Cl_{2k+1}^-. The price one pays is that for $k = 0$ or $k = 2$, the operator J_{2k+1} fails to commute with the *odd* elements of Cl_{2k+1}^- (for otherwise all of Cl_{2k+1}^- would have a real or quaternionic structure). However, this failure is under control, since explicit computations show that for $k = 0,2$ one does have a J_{2k+1}^- $(= J_{2k}^-)$ that *commutes* with all *even* elements of Cl_{2k+1}^- and *anticommutes* with all *odd* elements.[19] With this proviso, one may replace the x under $(J_1^-)^2$ by 1 (i.e. the sign of $(J_8^-)^2$) and the x under $(J_5^-)^2$ by -1 (i.e. the sign of $(J_4^-)^2$). In what follows, we shall use J_n^- for any n, understood in this way, i.e., as an antiunitary operator on $\mathbb{S}_n$ that:

1. commutes with the $(\mathrm{Cl}_n^-)^0$-action on $\mathbb{S}_n$;
2. for $n = 2,3,4,6,7,8$ mod 8 also commutes with the $(\mathrm{Cl}_n^-)^1$- action on $\mathbb{S}_n$, and hence commutes with the entire Cl_n^--action;
3. for $n = 1,5$ mod 8 anticommutes with the $(\mathrm{Cl}_n^-)^1$- action on $\mathbb{S}_n$.

The final matter to be settled, for $n = 2k$, is the behaviour of J_n^- under the grading γ_{2k} of $\mathbb{S}_{2k}$ with respect to the decomposition $\mathbb{S}_{2k} = \mathbb{S}_{2k}^+ \oplus \mathbb{S}_{2k}^-$ relative to $(\mathbb{C}l_{2k})^0$ or Spin$(2k)$. Also this can simply be computed for given J_{2k}^-, so that one finally obtains the famous sign table of Connes, which (by convention) refers to Cl_n^-:[20]

n	ε	ε'	ε''
1	1	-1	
2	-1	1	-1
3	-1	1	
4	-1	1	1
5	-1	-1	
6	1	1	-1
7	1	1	
8	1	1	1

Here $(J_n^-)^2 = \varepsilon$, $J_n^- x = \varepsilon' x J_n^-$ for all $x \in (\mathrm{Cl}_n^-)^1$ (i.e., $\varepsilon' = 1$ if J_n^- is real or quaternionic for all of Cl_n^- and $\varepsilon' = -1$ if J_n^- is real or quaternionic just for $(\mathrm{Cl}_n^-)^0$ whereas it anticommutes with the odd part $(\mathrm{Cl}_n^-)^1$), and finally $J_n^-\gamma_n = \varepsilon''\gamma_n J_n^-$, where γ_n is the grading on $\mathbb{S}_n$ (defined only for n even).

This table has deep implications for the nature of Reality.

[19]This is immediately clear for $n = 1$, where one may put $J_1^- z = \overline{z}$, so that $[J_1^- z, \pi(1)] = 0$ but $J_1^-\pi(e_1)z = \overline{iz} = -\pi(e_1)J_1^- z$. To understand $n = 5$, we need to deal with $\mathrm{Cl}_3^+ \cong \mathrm{Cl}_1^- \otimes \mathrm{Cl}_2^+$. With J_1^- as just defined, which is real, for Cl_3^+ we have real tensor real, which is real. For Cl_5^-, the isomorphism (106) implies real tensor quaternionic, which is quaternionic. Hence we have an operator J_5^- that commutes with $(\mathrm{Cl}_5^-)^0$ and anticommutes with $(\mathrm{Cl}_5^-)^1$, and satisfies $(J_5^-)^2 = -1$.

[20]To really appreciate this table, one needs *KO*-theory.

3.7 Bundles of Clifford algebras over a Riemannian manifold

Let (M, g) be an oriented Riemannian manifold, defined according to Definition 3.8. Define the following real and complex associated vector bundles over M:

$$\mathrm{Cl}^{\pm}(M, g) = \mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathrm{Cl}_n^{\pm}; \tag{107}$$
$$\mathbb{C}\mathrm{l}(M, g) = \mathrm{SO}(M) \times_{\mathrm{SO}(n)} \mathbb{C}\mathrm{l}_n, \tag{108}$$

with respect to the following action of $\mathrm{SO}(n)$ on $\mathrm{Cl}_n^{\pm}$ and hence on $\mathbb{C}\mathrm{l}_n$:

$$R \cdot (v_1 \cdots v_p) = Rv_1 \cdots Rv_n. \tag{109}$$

Here $v_i \in \mathbb{R}^n$, and $v \mapsto Rv$ is the defining action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$; to define (108), the action (109) is extended to $v_i \in \mathbb{C}^n$ by complexification. For $p = 0$ we put $Rz = z$, $z \in \mathbb{C}$. This action is well defined, as by definition of $\mathrm{SO}(n)$ it respects the relations $v^2 - (v, v) = 0$, where the right-hand side is the usual inner product in $\mathbb{R}^n$: after all, one has

$$R(v^2 - (v, v)) = (Rv)^2 - (v, v) = (Rv)^2 - (Rv, Rv) = 0.$$

By definition, $\mathbb{C}\mathrm{l}(M, g)$ is locally given by $\mathbb{C}\mathrm{l}(M, g)_U \cong U \times \mathbb{C}\mathrm{l}_n$, i.e. its typical fiber is $\mathbb{C}\mathrm{l}_n$, and similarly for $\mathrm{Cl}^{\pm}(M, g)$. The trace τ turns $\mathbb{C}\mathrm{l}_n$ into a Hilbert space (Varilly, §1.3) and hence $\mathbb{C}\mathrm{l}(M, g)$ is a hermitian vector bundle. As we know, the *continuous* cross-sections of $\mathrm{Cl}^{\pm}(M, g)$ and $\mathbb{C}\mathrm{l}(M, g)$ are given by

$$B_g^{\pm} = C(\mathrm{Spin}(M), \mathrm{Cl}_n^{\pm})^{\mathrm{Spin}(n)}; \tag{110}$$
$$\mathbb{B}_g = \Gamma(M, \mathbb{C}\mathrm{l}(M, g)) = C(\mathrm{SO}(M), \mathbb{C}\mathrm{l}_n)^{\mathrm{SO}(n)}, \tag{111}$$

respectively. The latter is a (f.g.p.) Hilbert $C(M)$ module, because $\mathbb{C}\mathrm{l}(M, g)$ is a hermitian vector bundle. Explicitly, the $C(M)$-valued inner product on $\mathbb{B}_g$ is given by

$$\langle \beta, \gamma \rangle_{C(M)}(x) = \tau(\beta(\varphi)^* \gamma(\varphi)), \tag{112}$$

where $x \in M$, and φ is any element of $\mathrm{SO}(M)$ projecting to x; this is well defined (i.e. independent of the choice of φ) by $\mathrm{SO}(n)$-equivariance of β and γ and $\mathrm{SO}(n)$-invariance of the trace τ, in that $\tau(Rx) = \tau(x)$ for any $x \in \mathbb{C}\mathrm{l}_n$ and $R \in \mathrm{SO}(n)$.

Exercise 3.13 *Check this.*

However, $\mathbb{B}_g$ has interesting additional structure, because, as we have seen, $\mathbb{C}\mathrm{l}_n$ is a C^*-algebra, with norm written as $\| \cdot \|_{\mathbb{C}\mathrm{l}_n}$.

Proposition 3.14 *The Hilbert $C(M)$ module $\mathbb{B}_g$ is a C^*-algebra with respect to pointwise operations and norm*

$$\|\beta\| = \sup_{\varphi \in \mathrm{SO}(M)} \{\|\beta(\varphi)\|_{\mathbb{C}\mathrm{l}_n}\}. \tag{113}$$

Exercise 3.15 *Show that $\mathbb{B}_g$ is isomorphic (as a Hilbert $C(M)$ module and as a C^*-algebra) to Varilly's B_g (see his Lemma 2.3 on p. 19).*

We now prepare for the definition of the important notion of a Clifford module. In any case, the following pre-definition is interesting in its own right.[21]

[21] See my paper on `http://xxx.lanl.gov/pdf/math-ph/0008004` for a survey of Hilbert bimodules, Morita equivalence, etc.

Definition 3.16 1. *Let A be a C^*-algebra and $\mathcal{E}$ a Hilbert A module. The operator algebra* $\mathrm{End}_A(\mathcal{E})$ *consists of all* adjointable *linear maps $b : \mathcal{E} \to \mathcal{E}$, i.e., there exists a map $b^* : \mathcal{E} \to \mathcal{E}$ with the property that $\langle \eta, b\eta' \rangle_A = \langle b^*\eta, \eta' \rangle_A$ for all $\eta, \eta' \in \mathcal{E}$.*

2. *Let A and B be C^*-algebras. A* Hilbert B-A bimodule *is a Hilbert A module $\mathcal{E}$ along with a nondegenerate *-homomorphism $B \to \mathrm{End}_A(\mathcal{E})$.*

In part 1, it turns out that $b \in \mathrm{End}_A(\mathcal{E})$ is automatically bounded, that b^* is unique, and that $\mathrm{End}_A(\mathcal{E})$ is a C^*-algebra in the usual operator norm on the Banach space $\mathcal{E}$ and the adjoint $b \mapsto b^*$. Thus the notion of a *-homomorphism $B \to \mathrm{End}_A(\mathcal{E})$ in part 2 is well defined. Note that if $\mathcal{E}$ is f.g.p. over A, one has

$$\mathrm{End}_A(\mathcal{E}) \cong \mathcal{E} \otimes_A \overline{\mathcal{E}}, \tag{114}$$

where $\overline{\mathcal{E}}$ is the conjugate space to $\mathcal{E}$, seen as a *left* A module through the action $a \cdot \eta = \eta a^*$ (which is complex *linear* in a), and $\otimes_A$ is the algebraic tensor product over A (i.e. one identifies $\eta a \otimes_A \eta'$ with $\eta \otimes_A a\eta'$, for all $a \in A$).

There are many examples of Hilbert bimodules. The easiest is $\mathcal{E} = A$, initially as a Hilbert A module through $\langle \eta, \eta' \rangle_A = \eta^* \eta'$ and right mutliplication, and subsequently with $b(\eta) = b\eta$. as a Hilbert A-A bimodule; the definition is easily checked.

Definition 3.17 *A complex* Clifford module *on an oriented Riemannian manifold (M, g) is a f.g.p. Hilbert $\mathbb{B}_g$ - $C(M)$ bimodule $\mathcal{E} = \Gamma(E)$, for which the map $\mathbb{B}_g \to \Gamma(\mathrm{End}(E))$ defining the action of $\mathbb{B}_g$ on $\mathcal{E}$ is $C(M)$-linear.*

A real Clifford module on (M, g) is a real vector bundle E equipped with a $C(M, \mathbb{R})$-linear action of $\mathrm{Cl}^{\pm}(M, g)$ on $\Gamma(E)$; if a fiber metric g on E is given, this action is supposed to be symmetric with respect to g, making $\Gamma(E)$ a "real" Hilbert $B_g^{\pm}$ - $C(M, \mathbb{R})$ bimodule.

The simplest example is probably $\Omega^\bullet(M) = \Gamma(M, \Lambda^\bullet(T^*M))$ in the real case and its complexification $\Omega^\bullet_{\mathbb{C}}(M)$ in the complex case (where one simple replaces the cotangent bundle T^*M by the complexified cotangent bundle $T^*_{\mathbb{C}}M$, with fibers $T^*_xM \otimes_{\mathbb{R}} \mathbb{C}$).

Exercise 3.18 *Show that for an oriented Riemannian manifold one has*

$$\Lambda^\bullet(T^*M)) \cong \mathrm{SO}(M) \times_{\mathrm{SO}(n)} \Lambda^\bullet(\mathbb{R}^n), \tag{115}$$

as vector bundles, defined with respect to the natural action of $\mathrm{SO}(n)$ *on* $\Lambda^\bullet(\mathbb{R}^n)$ *induced from the defining* $\mathrm{SO}(n)$ *action on $\mathbb{R}^n$ (i.e., $R \cdot (v_1 \wedge \cdots \wedge v_p) = Rv_1 \wedge \cdots \wedge Rv_p$).*

Consequently, for the continuous cross-sections one has

$$\Omega^\bullet(M) \cong C(\mathrm{SO}(M), \Lambda^\bullet(\mathbb{R}^n))^{\mathrm{SO}(n)}, \tag{116}$$

and analogously for the complexification $\Omega^\bullet_{\mathbb{C}}(M)$. Now recall the natural action of $\mathbb{C}\mathrm{l}_n$ on $\Lambda^\bullet(\mathbb{C}^n)$, given by (62) - (63), or, equivalenty, by left-multiplication of $\mathbb{C}\mathrm{l}_n$ on itself, combined with the vector space isomorphism $\mathbb{C}\mathrm{l}_n \cong \Lambda^\bullet(\mathbb{C}^n)$. The $\mathbb{B}_g$-action on $\Omega^\bullet_{\mathbb{C}}(M)$ – in its realization (116) – is then given by

$$(\beta\omega)(p) = \beta(p) \cdot \omega(p), \tag{117}$$

where $\beta \in \mathbb{B}_g$, $\omega \in \Omega(M)$, and $p \in \mathrm{SO}(M)$. Elementary manipulations show that this action is well defined in the sense that $\beta\omega$ is $\mathrm{SO}(n)$-equivariant, provided ω itself is. By restriction, the same procedure turns $\Omega^\bullet(M)$ into a real Clifford module.

This is a special case of a general construction: if some (real) vector space V carries both a representation c of $\mathrm{Cl}_n^{\pm}$ and a representation ρ of $\mathrm{SO}(n)$, such that (cf. (78))

$$\rho(R)c(v)\rho(R^{-1}) = c(R \cdot v) \tag{118}$$

for all $R \in \mathrm{SO}(n)$ and $v \in \mathbb{R}^n$, then $B_g^{\pm}$ acts on the sections $\Gamma(E_V)$ of the vector bundle

$$E_V = \mathrm{SO}(M) \times_{\mathrm{SO}(n)} V \tag{119}$$

through the formula (117), *mutatis mutandis*. Similarly in the complex case: if $\mathbb{V}$ is a finite-dimensional Hilbert space carrying an $\mathrm{SO}(n)$-covariant representation of $\mathbb{C}\mathrm{l}_n$ in the sense of (118), then $\mathbb{B}_g$ acts on $\Gamma(E_{\mathbb{V}})$ by (117).

Now assume that M is a Spin or a $\mathrm{Spin}^{\mathbb{C}}$ manifold; the following formulae are written down for the Spin case, but one may add suffixes $\mathbb{C}$ as appropriate. The group $\mathrm{Spin}(n)$ acts on $\mathrm{Cl}_n^{\pm} \subset \mathbb{C}\mathrm{l}_n$ through $xz = z$ for $p = 0$ and

$$x \cdot (v_1 \cdots v_p) = xv_1 \cdots v_p x^{-1} = \lambda(x)v_1 \cdots \lambda(x)v_p, \tag{120}$$

in terms of the familiar map $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ and the $\mathrm{SO}(n)$ action (109). The formulae to be given now are defined with respect to this action.

Exercise 3.19 *Show that*

$$\mathrm{Cl}^{\pm}(M, G) \cong \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathrm{Cl}_n^{\pm}; \tag{121}$$
$$\mathbb{C}\mathrm{l}(M, G) \cong \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} \mathbb{C}\mathrm{l}_n; \tag{122}$$
$$B_g^{\pm} \cong C(\mathrm{Spin}(M), \mathrm{Cl}_n^{\pm})^{\mathrm{Spin}(n)}; \tag{123}$$
$$\mathbb{B}_g \cong C(\mathrm{Spin}(M).\mathbb{C}\mathrm{l}_n)^{\mathrm{Spin}(n)}, \tag{124}$$

Any representation c of $\mathrm{Cl}_n^{\pm}$ on some vector space W brings along a representation of $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^{\pm}$ on W by restriction of the $\mathrm{Cl}_n^{\pm}$-action. This makes c automatically $\mathrm{Spin}(n)$-covariant, as in (78); similarly for a representation of $\mathbb{C}\mathrm{l}$ on a complex vector space $\mathbb{W}$. As in (119), we then define

$$E_W = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W, \tag{125}$$

with associated actions of $B_g^{\pm}$ (in the real or complex case) and of $\mathbb{B}_g$ (in the complex case) defined in the (by now) obvious way. This extends our previous construction (117) of Clifford modules from $\mathrm{SO}(n)$-representations to $\mathrm{Spin}(n)$-representations. The most important example of this construction is given by the spinor representation (95), with continuous cross-sections of $\mathcal{S}_n$ given by

$$\Gamma(\mathcal{S}_n) = C(\mathrm{Spin}(M), \mathbb{S}_n)^{\mathrm{Spin}(n)}. \tag{126}$$

This immediately turns $\Gamma(\mathcal{S}_n)$ into a Clifford module by the action (119), or, explicitly,

$$\beta\psi(p) = \beta(p)\psi(p), \tag{127}$$

where $\beta \in \mathbb{B}_g$ or $\beta \in B_g^{\pm}$, realized as (124), etc., $\psi \in \Gamma(\mathcal{S}_n)$, and $p \in \mathrm{Spin}(M)$.

Exercise 3.20 *Show that this action is well defined.*

Hence $\Gamma(\mathcal{S}_n)$ is a Hilbert $\mathbb{B}_g$ - $C(M)$ Hilbert bimodule. It has a stronger property, however.

Definition 3.21 *Let A and B be unital C^*-algebras.*[22] *A Hilbert B - A bimodule $\mathcal{M}$ is a B - A* Morita equivalence *if:*

1. *M is f.g.p. over A;*
2. *M is full in A (i.e., the linear span of $\{\langle\mu,\nu\rangle_A \mid \mu,\nu \in \mathcal{M}\}$ is dense in A);*
3. *$B \cong \mathrm{End}_A(\mathcal{M})$.*

We say that A and B are Morita equivalent *if there exists a B - A Morita equivalence.*

Clearly, by (114) we then also have $B \cong \mathcal{M} \otimes_A \overline{\mathcal{M}}$. Note that if A and B are Morita equivalent, the bimodule implementing this equivalence is by now means unique, not even up to unitary equivalence! In fact, one may study things like the unitary equivalence class of all A - A Morita equivalences, see Varilly.

Our earlier example of $\mathcal{E} = A$ as an A - A Hilbert bimodule is actually a Morita equivalence, showing that A is Morita equivalent to itself (in fact, Morita equivalence is indeed an equivalence relation!). Perhaps more surprisingly, for any k, the matrix algebra $M_k(\mathbb{C})$ is Morita equivalent to $\mathbb{C}$ through $\mathcal{M} = \mathbb{C}^k$ with the obvious actions. This suggests the significance of Morita equivalence: A and B are Morita equivalent if they have equivalent representation categories.

For n odd, instead of (111) define[23]

$$\mathbb{B}_g^0 = \Gamma(M, \mathbb{C}\mathrm{l}(M,g)) = C(\mathrm{SO}(M), \mathbb{C}\mathrm{l}_n^0)^{\mathrm{SO}(n)}. \tag{128}$$

Theorem 3.22 *Let M be Spin or* $\mathrm{Spin}^{\mathbb{C}}$. *If n is even, then $\Gamma(\mathcal{S}_n)$ is a $\mathbb{B}_g$ - $C(M)$ Morita equivalence. If n is odd, $\Gamma(\mathcal{S}_n)$ is a $\mathbb{B}_g^0$ - $C(M)$ Morita equivalence.*

Locally, with $C(M) \equiv C(M,\mathbb{C}) \cong \Gamma(M \times \mathbb{C}$, we have

$$(\mathcal{S}_{2k})_{|U} \cong U \times \mathbb{S}_{2k} \cong U \times \mathbb{C}^{2^k}; \tag{129}$$

$$(\mathbb{B}_g)_{|U} \cong U \times \mathbb{C}\mathrm{l}_{2k} \cong U \times M_{2^k}(\mathbb{C}); \tag{130}$$

$$(M \times \mathbb{C})_{|U} \cong U \times \mathbb{C}. \tag{131}$$

Hence locally, Theorem 3.22 just restates the matrix example just given, for $\mathbb{C}\mathrm{l}_{2k}$ is Morita equivalent to $\mathbb{C}$. Globally, Theorem 3.22 states that these "local" Morita equivalence can be glued together so as to form a global Morita equivalence.

Exercise 3.23 *Prove Theorem 3.22.*

Theorem 3.22 holds for both Spin and $\mathrm{Spin}^{\mathbb{C}}$ manifolds. In the Spin case, however, there is additional structure. The following result is just stated for the minus sign in Cl_n^-, but an analogous result also holds for Cl_n^+.

Proposition 3.24 *Let M be a Spin manifold, with associated spinor bundle $\mathcal{S}_n$, and let $J_n^- : \mathbb{S}_n \to \mathbb{S}_n$ be a an antiunitary real or quaternionic structure with respect to Cl_n^- (for even n) or $(\mathrm{Cl}_n^-)^0$ (for odd n). Then the operator $C_n^- : \Gamma(\mathcal{S}_n) \to \Gamma(\mathcal{S}_n)$, given in terms of* (126) *by*

$$C_n^- \psi(p) = J_n^- \psi(p), \tag{132}$$

is well defined, antiunitary, and:

[22] See Landsman, `math-ph/0008004`, loc.cit., for the nonunital case.

[23] Alternatively, for odd n we may leave $\mathcal{S}_n$ as it is and replace $C(M)$ in the statement of Theorem (3.22) by $C(M, \mathbb{C} \oplus \mathbb{C})$, reflecting the fact (prove!) that $\mathbb{C}\mathrm{l}_{2k+1}$ is Morita equivalent to $\mathbb{C} \oplus \mathbb{C}$.

1. *commutes with the* $C(M,\mathbb{R})$ *action on* $\Gamma(\mathcal{S}_n)$*;*

2. *commutes with the* $(B_g^-)^0$*-action on* $\Gamma(\mathcal{S}_n)$*;*

3. *for* $n = 2,3,4,6,7,8$ *mod 8 commutes with the* $(B_g^-)^1$*- action on* $\Gamma(\mathcal{S}_n)$*, and hence commutes with the entire* B_g^-*-action;*

4. *for* $n = 1,5$ *mod 8 anticommutes with the* $(B_g^-)^1$*- action on* $\Gamma(\mathcal{S}_n)$*.*

Here B_g^- *is given by* (110)*, and for* $i = 0,1$*,*

$$(B_g^-)^i = C(\mathrm{Spin}(M), (\mathrm{Cl}_n^-)^i)^{\mathrm{Spin}(n)}. \tag{133}$$

This is almost immediate from the definitions. For C_n^- to be well defined, is is already sufficient that J_n^- commutes with the $\mathrm{Spin}(n)$ action on $\mathbb{S}_n$ (as it indeed does).

We are finally in a position to explain Connes's noncommutative geometry approach to Spin and $\mathrm{Spin}^{\mathbb{C}}$ manifolds. From the point of view of differential geometry, Connes's definitions are theorems. Roughly speaking,

$$\text{Spin manifold } = \mathrm{Spin}^{\mathbb{C}} \text{ manifold} + \text{charge conjugation}. \tag{134}$$

We state the following definition and exercise for n even; for the odd case, replace $\mathbb{B}_g$ by $\mathbb{B}_g^0$, etc.

Definition 3.25 1. *A* $\mathrm{Spin}^{\mathbb{C}}$ *manifold is a (unitary equivalence class of)* $\mathbb{B}_g$ *-* $C(M)$ *Morita equivalence(s)* $\mathcal{M}$*.*

2. *A Spin manifold is a* $\mathrm{Spin}^{\mathbb{C}}$ *manifold (as just defined) with a charge conjugation, i.e. an antiunitary operator* $C : \mathcal{M} \to \mathcal{M}$ *that satisfies the four properties listed in Proposition 3.24 (with* C_n^- *replaced by* C *and* $\Gamma(\mathcal{S}_n)$ *replaced by* $\mathcal{M}$*).*

When the going gets tough, the tough get going:

Exercise 3.26 *Show how a* $\mathrm{Spin}^{(\mathbb{C})}$ *structure as we defined it can be reconstructed from the data in Definition 3.25.* Hint*: We know that* $\mathcal{M} = \Gamma(M,F)$ *for some vector bundle* $F \to M$*. Construct the principal* $\mathrm{Spin}^{(\mathbb{C})}$*-bundle* $\Sigma^{(\mathbb{C})}(M)$ *as a suitable sub-bundle of* $\mathrm{Hom}(M\times\mathbb{S}_n, F)$*. Cf. Definition 3.3.*

For later use, we mention that the idea of (132) may also be used to define a grading Γ_n on $\Gamma(\mathcal{S}_n)$ whenever n is even: starting with the spinor grading operator $\gamma_n : \mathbb{S}_n \to \mathbb{S}_n$, introduced just below (68), we define an operator $\Gamma_n : \Gamma(\mathcal{S}_n) \to \Gamma(\mathcal{S}_n)$ by

$$\Gamma_n\psi(p) = \gamma_n\psi(p). \tag{135}$$

By the corresponding properties of γ_n, this operator satisfies $\Gamma_n^2 = 1$ and $\Gamma_n^* = \Gamma_n$ (with respect to the natural inner product on $\Gamma(\mathcal{S}_n)$), and hence Γ_n has eigenvalues ±1. The corresponding eigenspaces $\Gamma^{\pm}(\mathcal{S}_n)$ are evidently given by

$$\Gamma^{\pm}(\mathcal{S}_n) = C(\mathrm{Spin}(M), \mathbb{S}_n^{\pm})^{\mathrm{Spin}(n)}, \tag{136}$$

inducing a decomposition $\Gamma(\mathcal{S}_n) = \Gamma^+(\mathcal{S}_n) \oplus \Gamma^+(\mathcal{S}_n)$.

4 Dirac operators

Historically, Dirac operators first arose in physics, notably in a paper in 1928 by Dirac, who, in one of the most famous papers in 20th century physics, tried to write down a relativistic theory of electrons. As it turned out, the equation he gave not only described electrons, but also positrons (i.e. their antiparticles), which was a sensational new notion at the time, discovered through the mathematical structure of Dirac's equation.[24]

In 1962, the same class of operators was rediscovered in the context of mathematics by Atiyah and Singer, who used Dirac operators both as examples of their famous index theorem and as an ingredient in one of their proofs of the theorem in general. The physics and the mathematics lines met in the 1980s, where so-called anomalies in quantum field theory were mathematically understood (albeit with a lot of handwaving involving path intergals) on the basis of the mathematical Atiyah-Singer index theorem applied to the Dirac equation of physics. In noncommutative geometry, the Dirac operators of mathematics and physics meet once again. In particular, Dirac operators on spin manifolds provide the motivating example of a (commutative) spectral triple.

4.1 Connections and covariant derivatives on vector bundles

In what follows, $\pi : E \to M$ is a real or complex vector bundle, "linear" means $\mathbb{R}$- or $\mathbb{C}$-linear as appropriate, and $\Gamma(E)$ stands for the *smooth* sections of E (as opposed to the *continuous* sections), unless explicitly stated otherwise. Furthermore,

$$\Omega^\bullet(M) \equiv \mathcal{A}^\bullet(M) = \Gamma(\Lambda^\bullet(T^*M))$$

stands for the smooth differential forms on M. One may subsequently form the tensor product bundle $E \otimes \Lambda^\bullet(T^*M)$ of E-valued differential forms on M. For any two bundles E, F over M one has an isomorhpism of f.g.p. A-modules[25]

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F), \tag{137}$$

so that, in particular,

$$\Gamma(E \otimes \Lambda^\bullet(T^*M)) \cong \Gamma(E) \otimes_{C^\infty(M)} \Omega^\bullet(M). \tag{138}$$

The sub-bundle of one-forms $\Omega^1(M) = \Gamma(T^*M)$ plays a special role, with

$$\Gamma(E \otimes T^*M) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(T^*M).$$

Definition 4.1 *A* connection *on a vector bundle $\pi : E \to M$ is a linear map*

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{C^\infty(M)} \Gamma(T^*M) \tag{139}$$

that, for all $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$, satisfies the Leibniz rule

$$\nabla(\sigma \cdot f) = (\nabla\sigma) \cdot f + \sigma \otimes_{C^\infty(M)} df. \tag{140}$$

[24]With hindsight, one could argue that, in an equally famous paper, Schrödinger already wrote down the first example of what we now call a Dirac operator in 1926, namely the momentum operator $p = -id/dx$ on $L^2(\mathbb{R})$ for a quantum particle moving on the line.

[25]To see this, use the Serre–Swan Theorem to write $\Gamma(E) = p_E A^m$ for some $p_E \in M_m(A)$, $m \in \mathbb{N}$, and $A = C^\infty(M)$, and similarly $\Gamma(F) = p_F A^l$. Then $E \otimes F$ is (isomorphic to) the sub-bundle of the trivial bundle $M \times (\mathbb{C}^m \otimes \mathbb{C}^l) \cong M \times \mathbb{C}^{ml}$ whose fiber at $x \in M$ is the image of $p_E(x) \otimes p_F(x)$ in $\mathbb{C}^m \otimes \mathbb{C}^l$, so that $\Gamma(E \otimes F) \cong p_E \otimes p_F(A^{ml})$. On the other hand, since $A^m \otimes_A A^l \cong A^{ml}$, we have $\Gamma(E) \otimes_A \Gamma(F) \cong p_E \otimes p_F A^m \otimes_A A^l \cong p_E \otimes p_F A^{ml}$.

Here $\Gamma(E)$ is a right $C^\infty(M)$ module, whereas $\Gamma(T^*M)$, and more generally $\Omega^\bullet(M)$, is to be seen as a $C^\infty(M)$-$C^\infty(M)$ bimodule, where in this case the left- and right actions coincide (this is possible because $C^\infty(M)$ is commutative). Hence $\Gamma(E)\otimes_{C^\infty(M)}\Gamma(T^*M)$ is naturally a right $C^\infty(M)$ module, and (140) states that ∇ fails to be $C^\infty(M)$-linear in a controlled way. For example, take $E = M\times\mathbb{C}$, so that $\Gamma(E) = C^\infty(M)$ and hence $\Gamma(E)\otimes_{C^\infty(M)}\Gamma(T^*M) \cong \Gamma(T^*M)$. The exterior derivative $d : C^\infty(M) \to \Gamma(T^*M)$, which is the same as $d : \Omega^0(M) \to \Omega^1(M)$, then provides a connection on the trivial bundle E, in which case the Leibniz rule assumes the familiar form $d(gf) = gdf + (dg)f$. This illustrates the idea that the general case (139) is an attempt to generalize the exterior derivative from trivial vector bundles, whose sections are just functions on M, to arbitrary vector bundles.

To explore the multitude of connections a bundle may admit, consider two such, ∇ and ∇'. One immediately sees from (140) that

$$(\nabla - \nabla')(\sigma \cdot f) = (\nabla - \nabla')(\sigma)\cdot f, \tag{141}$$

so that, unlike a single connection, the *difference* between two connections on E is $C^\infty(M)$-linear. Hence (by the functorial form of the Serre–Swan Theorem),

$$\nabla - \nabla' : \Gamma(E) \to \Gamma(E)\otimes_{C^\infty(M)}\Gamma(T^*M)$$

is induced by a vector bundle map $E \to E\otimes T^*M$, or, equivalently,

$$\nabla - \nabla' \in \Gamma(\mathrm{End}(E)\otimes T^*M). \tag{142}$$

This is often written as $\nabla-\nabla' = A$, where $A \in \Gamma(\mathrm{End}(E)\otimes T^*M)$ is a so-called $\mathrm{End}(E)$-valued one-form on M. Technically, this implies that the space of all connections on E is an affine space modeled on $\Gamma(\mathrm{End}(E)\otimes T^*M)$, which means what we have just said, namely that any connection ∇ on E can be written as the sum of a fixed connection ∇' and an element of $\Gamma(\mathrm{End}(E)\otimes T^*M)$. Calling the latter element A, we may write $\nabla_A \equiv \nabla' + A$. Locally, we may take $\nabla' = d$, obtaining the physicists' formula $\nabla_A = d + A$.

In the following exercise, $\mathbb{R}$ may also be replaced by $\mathbb{C}$.

Exercise 4.2 *Take local trivializations of both $E_{|U} \cong U\times\mathbb{R}^p$ and $T^*M_{|U} \cong U\times\mathbb{R}^n$, for some open $U \subset M$.*

1. *What are the local expressions for $\sigma \in \Gamma(E_{|U})$ and $\omega \in \Gamma(T^*M_{|U})$?*
2. *What is the local expression of $A \in \Gamma(\mathrm{End}(E_{|U})\otimes T^*M_{|U})$?*
3. *Show directly that $\nabla_A = d + A$, with A of the local form found in the previous questions, defines a connection on the trivial bundle $E_{|U}$ (i.e. write down the local form of the Leibniz rule and verify that $d + A$ satisfies it).*
4. *Conversely, show directly that any connection is locally of this form.*

Later on, for general spectral triples (A, H, D), the space of sections $\mathcal{E} = \Gamma(E)$ will be generalized to an f.g.p. module over A, and a connection on $\mathcal{E}$ will be defined as a linear map $\nabla : \mathcal{E} \to \mathcal{E}\otimes_A\Omega^1_D(A)$, where $\Omega^1_D(A)$ consists of all finite linear combinations of operators of the form $a[D, b]$. This is an A-A bimodule under left- and right operator multiplication by elements of A, so that $\mathcal{E}\otimes_A\Omega^1_D(A)$ is a right A module, like $\mathcal{E}$. The Leibniz rule then reads $\nabla(\sigma\cdot f) = \sigma\otimes_A df + (\nabla\sigma)\cdot f$, with $df = [D, f]$. As in the commutative case, such connections always exist in multitude.

If you don't like differential forms, the following equivalent definition of a connection may please you.

Definition 4.3 *A* covariant derivative *on a vector bundle $\pi : E \to M$ assigns a linear map $\nabla_X : \Gamma(E) \to \Gamma(E)$ to each smooth vector field X on M, such that*

$$\nabla_X(\sigma \cdot f) = (\nabla_X \sigma) \cdot f + \sigma \cdot (Xf). \tag{143}$$

The assignment $X \mapsto \nabla_X$ is linear and $C^\infty(M)$-linear, i.e., $\nabla_{sX+tY} = s\nabla_X + t\nabla_Y$ and $\nabla_{gX} = g\nabla_X$ for all $g \in C^\infty(M)$.

Exercise 4.4 *Explain and prove the equivalence between Definitions 4.1 and 4.3. In other words, show that a connection on E defines a covariant derivative on E and* vice versa.

A third way to look at connections or covariant derivatives arises if we realize vector bundels as associated bundles in terms of principal fibre bundles. We work in the setting of Definition 3.2.

Definition 4.5 *Let P be a principal G-bundle over M. An* Ehresmann connection *on P, simply called a* connection *in what follows, is a family of linear maps $h_p : T_{\pi(p)}M \to T_pP$, $p \in P$ such that:*[26]

1. *$\pi' \circ h_p = \mathrm{id}_{T_{\pi(p)}M}$, i.e., $\pi'(h_p(\xi)) = \xi$ for all $\xi \in T_{\pi(p)}M$;*
2. *$h_{pg} = R'_g \circ h_p$ (where $R_g(p) = pg$), for all $p \in P$ and $g \in G$;*
3. *the vector field $p \mapsto h_p(X)$ on P is smooth for any smooth vector field X on M.*

This family (h_p), $p \in P$, is called a *horizontal lift*; indeed, the idea is that TM is lifted into TP. Equivalently, one may define a connection on P as a family of subspaces $H_p \subset T_pP$, smoothly depending on $p \in P$, such that

$$\pi' : H_p \xrightarrow{\cong} T_{\pi(p)}M \tag{144}$$

is an isomorphism for each p, and

$$H_{pg} = R'_g H_p. \tag{145}$$

This structure arises from Definition 4.5 by declaring H_p to be the image $h_p(T_{\pi(p)}M)$; *vice versa*, a choice of H_p determines a decomposition

$$T_pP = H_p \oplus V_p, \tag{146}$$

where the *vertical tangent space* V_p at p consists of all tangent vectors v of the form $(vf)(p) = df(pg(t))/dt_{t=0}$ for smooth curves $t \mapsto g(t)$ in g with $g(0) = e$. Hence V_p is tangent to the G-orbit through p, whereas H_p is tangent to the 'horizontal' direction given by M. The crucial difference is that V_p is given by the bundle structure, whereas H_p is not canonically given but must be *chosen*. This is what an Ehresmann connection does.

A connection on P immediately leads to a covariant derivative on any bundle (88) associated to P through a representation of G on a vector space V. Namely, for $\sigma \in C^\infty(P, V)^G$ (see (90)) we put

$$\nabla_X \sigma(p) = h_p(X_{\pi(p)})\sigma(p). \tag{147}$$

[26] We write $\varphi' : TX \to TY$ for the derivative of a smooth map $\varphi : X \to Y$ between manifolds X, Y. In particular, $\varphi'_x : T_xX \to T_{\varphi(x)}Y$ is linear. We sometimes write φ' for φ'_x.

Exercise 4.6 *1. Show that* $\nabla_X\sigma \in C^\infty(P,V)^G$.

2. Show that ∇_X *satisfies the Leibniz rule* (143).

For no. 2, use

$$C^\infty(M) \cong C^\infty(P)^G = \{f \in C^\infty(P) \mid f(pg) = f(p)\, \forall p \in P, g \in G\},$$

so that the action of $C^\infty(M)$ on $\Gamma(E)$ is simply given by $(\sigma \cdot f)(p) = \sigma(p)f(p)$, $f \in C^\infty(P)^G$.

This is an extremely powerful construction, because any vector bundle can be realized as an associated bundle, and any connection is of the above form; cf. the next subsection.

To perform local computations with (147), we assume $P_{|U} \cong U \times G$ and $(P \times_G V)_{|U} \cong U \times V$, so that locally $\sigma : U \to E$ is represented by a V-valued function $\tilde{\sigma} : U \to V$ defined by

$$\tilde{\sigma}(x) = \sigma(x,e). \tag{148}$$

To proceed, you need to know that $T_eG = \mathfrak{g}$ is (by definition) the *Lie algebra* of G, and that $R'_g : T_eG \to T_gG$ is an isomorphism, so that we may also say that $T_gG \cong \mathfrak{g}$. With $p = (x,g)$, $x \in U \subset M$, $g \in G$, we then have $T_{(x,g)}(U \times G) \cong T_xM \oplus \mathfrak{g}$. Hence the lift h_p may locally be written as

$$h_{(x,g)}(\xi) = \xi - < A(x), \xi >, \tag{149}$$

where A is a $\mathfrak{g}$-valued 1-form on U (this is the "gauge field" of physics) and $< -, - >$ is the pairing between 1-forms and vectors on M. Using (148) and the G-equivariance of $\sigma \in C^\infty(P,V)^G$, the local expression for the covariant derivative turns out to be

$$\nabla_X\tilde{\sigma}(x) = X\tilde{\sigma}(x) + < A(x), X(x) > \tilde{\sigma}(x), \tag{150}$$

where $< A(x), X(x) > \in \mathfrak{g}$ acts on $\tilde{\sigma}(x) \in V$ through (the derivative of) the given representation of G on V defining the associated bundle $E = P \times_G V$. Hence locally $\nabla = d + A$, where the $\mathfrak{g}$-valued one-form A on U is defined in terms of the lifting maps h_p by (149).

In fact, this local construction has a global counterpart. Given the liftings h_p, define a $\mathfrak{g}$-valued 1-form $\mathbf{A}$ on P by

$$< \mathbf{A}, \xi >_p = 0 \text{ for all } \xi \in H_p; \tag{151}$$

$$< \mathbf{A}, \eta_Y > = Y \text{ for all } Y \in \mathfrak{g}, \tag{152}$$

where the vector field η_Y on P associated to $Y \in \mathfrak{g}$ is defined by

$$\eta_Y f(p) = \frac{d}{dt} f(pe^{tY})_{|t=0}. \tag{153}$$

This completely defines $\mathbf{A}$, since the vectors η_Y, $Y \in \mathfrak{g}$, span the vertical tangent space V_p, and (151) fixes the action of $\mathbf{A}$ on H_p. Property 2 in Definition 4.5, or rather its consequence (145), then leads to

$$R_g^*\mathbf{A} = \mathrm{Ad}(g)\mathbf{A}, \tag{154}$$

where Ad is the adjoint representation of G on its Lie algebra $\mathfrak{g}$ (informally, $\mathrm{Ad}(g)Y = gYg^{-1}$). If the above trivializations of P and E over U then correspond to a local cross-section $s : U \to P$, as in Exercise 3.6, the local $\mathfrak{g}$-valued one-form A on U is simply given in terms of $\mathbf{A}$ by $A = s^*\mathbf{A}$. Furthermore, the global version of (149) is

$$h_p(\xi) = \tilde{\xi} - \eta_{<\mathbf{A},\tilde{\xi}>}, \tag{155}$$

where $\tilde{\xi}$ is *any* lift of ξ, i.e., if $\xi \in T_{\pi(p)}M$, then $\tilde{\xi} \in T_pP$ such that $\pi'(\tilde{\xi}) = \xi$.

4.2 Levi-Civita, Clifford, and spin connections

Since there are many connections on a given vector bundle, extra conditions are sought to single out particularly nice ones, if at all possible unique (given the conditions). The oldest example is the *Levi-Civita connection* $\nabla^g : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$ on $E = TM$, where (M, g) is Riemannian. Equivalently, in the style of Definition 4.3, one has a family $\nabla^g_X : \Gamma(X) \to \Gamma(X)$, where $X \in \Gamma(TM)$. The "fundamental theorem of Riemannian geometry" states that (for a given metric g) there is a *unique* connection ∇^g on TM that is both *torsion-free*, in that

$$\nabla^g_X Y - \nabla^g_Y X = [X, Y], \tag{156}$$

and *metric*, in the sense that

$$g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z) = Xg(Y, Z), \tag{157}$$

for all smooth vector-fields $X, Y, Z \in \Gamma(TM)$.

Though initially defined on the tangent bundle TM, by the construction in the previous subsection the Levi-Civita connection ∇^g induces a connection on any vector bundle (119) associated to $SO(M)$ through an appropriate representation V of $SO(n)$. Indeed, through the isomorphism (92), ∇^g arises from a particular connection $h^g = (h^g_p)_{p \in P}$ on the principal $SO(n)$-bundle $SO(M)$ of orthonormal frames.[27] As a case in point, (115) shows that for $V = \Lambda^\bullet(\mathbb{R}^n)$, the Levi-Civita connection on TM induces a connection on $E_V = \Lambda^\bullet(T^*M)$. As such, the Levi-Civita connection ∇^g has a remarkable property, abstracted as follows.

Suppose (M, g) is Riemannian and suppose that $\mathcal{E} = \Gamma(E)$ is a Clifford module (see Definition 3.17). Using the embedding $\Gamma(TM) \subset B^\pm_g \subset \mathbb{B}_g$, given by Exercise 3.7, the canonical map $\mathbb{R}^n \to \mathbb{C}l_n$, and (111), we obtain an action of $\Gamma(TM)$ on $\mathcal{E}$, given by restriction of the $\mathbb{B}_g$ or $B^\pm_g$ action. We write c for this $\Gamma(TM)$-action, so that we have a map $c : \Gamma(TM) \to \mathrm{End}_{C(M)}\Gamma(E)$.

Definition 4.7 *A* Clifford connection *on a Clifford module* $\Gamma(E)$ *(with respect to a Riemannian manifold* (M, g)*) is a connection* ∇ *on* E *that satisfies*

$$\nabla_X(c(Y)\sigma) = c(Y)\nabla_X\sigma + c(\nabla^g_X Y)\sigma, \tag{158}$$

for all $X, Y \in \Gamma(TM)$ *and* $\sigma \in \mathcal{E}$.

A simple way of constructing Clifford connections on Clifford modules is as follows.

Exercise 4.8
1. *Show that the induced Levi-Civita connection on* (119) *is a Clifford connection, for any vector space* V *with representation* $\rho : SO(n) \to \mathrm{End}(V)$.
2. *As a special case, explain how the Levi-Civita connection on* $\Lambda^\bullet(T^*M)$ *defines a Clifford connection on the Clifford module* $\Omega^\bullet(M) = \Gamma(M, \Lambda^\bullet(T^*M))$.

The second major example of a Clifford connection on a Clifford module is the spin connection on the spinor bundle on a spin manifold. Recall that a spin manifold M is defined in terms of a principal $\mathrm{Spin}(n)$ bundle $\mathrm{Spin}(M)$ on M with an isomorphism (93), and that the associated Riemannian structure on M is given by the principal $SO(n)$ bundle $SO(M) = \mathrm{Spin}(M)/\mathbb{Z}_2$. Let $\pi : \mathrm{Spin}(M) \to SO(M)$ be the ensuing projection, with kernel $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is discrete, each linear map $\pi'_p : T_p(\mathrm{Spin}(M)) \to T_{\pi(p)}(SO(M))$ is an isomorphism. As we have seen, the Levi-Civita connection on TM induces an (Ehresmann) connection h^g on $SO(M)$.

[27] Those with a good background in differential geometry may try to find the explicit maps h_p that define the Levi-Civita connection; see Kobayashi and Nomizu, *Foundations of Differential Geometry, Vol. 1*.

Definition 4.9 *Let M be a spin manifold.*

1. *The* spin connection *on the principal* $\mathrm{Spin}(n)$ *bundle* $\mathrm{Spin}(M)$ *is the (Ehresmann) connection h^S defined by*
$$h^S_p = (\pi'_p)^{-1} \circ h^g_{\pi(p)}, \tag{159}$$
where h^g is the Levi-Civita connection on $\mathrm{SO}(M) = \mathrm{Spin}(M)/\mathbb{Z}_2$.

2. *For any representation of* $\mathrm{Spin}(n)$ *on a vector space W, the* spin connection *on the associated vector bundle* $E_W = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} W$ *is the connection induced by the spin connection* (159) *on* $\mathrm{Spin}(M)$ *through* (147).

3. *In particular, the* spin connection on the spinor bundle $\mathcal{S}_n$ *associated to* $\mathrm{Spin}(M)$ *by* (95) *is the connection defined in the previous item.*

The same method as in Exercise 4.8 now proves the following.

Proposition 4.10 *If a vector space W carries a* $\mathrm{Spin}(n)$ *representation defined as the restriction of a* $\mathrm{Cl}^{\pm}_n$*-action or of a* $\mathbb{C}\mathrm{l}_n$*-action on W, then the spin connection on any vector bundle E_W associated to* $\mathrm{Spin}(M)$ *by* (125) *is a Clifford connection.*

To make this a bit more explicit, we write the Levi-Civita connection on TM locally as
$$\nabla^g_X Y^a = X^b(e_b Y^a + \Gamma^a_{bc} Y^c), \tag{160}$$
where $Y = Y^a e_a$ in terms of a local *vielbein* (e_a) (i.e., an orthonormal frame), and all repeated indices are summed from 1 to n (Einstein summation convention). Equivalently,
$$\nabla^g = d + \Gamma, \tag{161}$$
where Γ is a local matrix-valued 1-form on U given by $\Gamma = \theta^b \Gamma_b$ in terms of the basis (θ^a) dual to (e_a), and each Γ_b is an $n \times n$ matrix with matrix elements $(\Gamma_b)^a_c$. Conceptually, the matrix Γ_b is an element of the Lie algebra $\mathfrak{so}(\mathfrak{n})$ of $\mathrm{SO}(n)$ acting in its defining representation on $\mathbb{R}^n$. The associated spin connection turns out to be given locally by
$$\nabla^S = d + \omega, \tag{162}$$
where ω is a one-form on U taking values in the Lie algebra of $\mathrm{Spin}(n)$ in its spinor representation on $\mathbb{S}_n$. As a matrix on $\mathbb{S}_n$, we have $\omega = \theta^b \omega_b$, with $\omega_b = \Gamma^a_{bc}[\gamma_a, \gamma_c]/4$, where $\gamma_a = c(e_a)$ is the representative of $e_a \in \mathbb{R}^n \subset \mathbb{C}\mathrm{l}_n$ acting on $\mathbb{S}_n$.[28] Hence
$$\nabla^S_X \psi = X^a(e_a + \frac{1}{4}\Gamma^b_{ac}[\gamma_b, \gamma_c])\psi. \tag{163}$$

To state the main example with an appropriate refinement, note that condition (157) has an analogue for complex vector bundles, called *hermiticity*: suppose E is a *hermitian* complex vector bundle, i.e. an f.g.p. Hilbert $C(M)$-module (M compact), with hermitian structure $\langle -, - \rangle_{C(M)} : \Gamma(E) \times \Gamma(E) \to C(M)$. We then say that a connection ∇ on E is *hermitian* if
$$\langle \nabla_X \sigma, \tau \rangle_{C(M)} + \langle \sigma, \nabla_X \tau \rangle_{C(M)} = X \langle \sigma, \tau \rangle_{C(M)} \tag{164}$$
for all $X \in \Gamma(TM)$ and all $\sigma, \tau \in \Gamma(E)$ (smooth sections).

[28] This formula arises from a computation of the inverse of the Lie algebra isomorphism $\lambda' : \mathfrak{spin}(\mathfrak{n}) \to \mathfrak{so}(\mathfrak{n})$ induced by the Lie group covering homomorphism $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ we have already encountered. If $\mathfrak{spin}(\mathfrak{n})$ is realized as the set of commutators of elements of $\mathbb{R}^n$ in $\mathrm{Cl}^{\pm}_n$, and $\mathfrak{so}(\mathfrak{n})$ is the set of antisymmetric $n \times n$ matrices with elements $(w \wedge v)_{ij} = \frac{1}{2}(w_i v_j - v_i w_j)$, one obtains $(\lambda')^{-1}(v \wedge w) = \mp[v, w]/4$. See Varilly, §1.7.

Proposition 4.11 *The spin connection ∇^S on the spinor bundle $\mathcal{S}_n$:*

1. *is a Clifford connection;*
2. *commutes with the charge conjugation C_n^-;*
3. *is hermitian with respect to the inner product on $\mathbb{S}_n$ that makes $\pi_F(\mathbb{C}\mathrm{l}_n)$ a *-representation (and hence makes the associated* Spin(n) *representation unitary).*

Moreover, ∇^S is the unique connection on $\mathcal{S}_n$ with these properties.

The last claim is the way Varilly introduces the spin connection, and he proves this proposition on pp. 29–30, using the local expressions for ∇^g and ∇^S.

Exercise 4.12 *Prove Proposition 4.11 globally, i.e., from Definition 4.9.*

4.3 Dirac operators: definition

Let $\Gamma(E)$ be a Clifford module, i.e., a $\mathbb{B}_g$ - $C^\infty(M)$ (pre-Hilbert) bimodule, over a Riemannian manifold (M, g), and let ∇ be *any* connection on E (not necessarily Clifford!). These data define a first-order partial differential operator D on $\Gamma(E)$ by

$$iD : \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M) \xrightarrow{\mathrm{flip}\circ\sharp} \Gamma(TM \otimes E) \cong \Gamma(TM) \otimes_{C^\infty(M)} \Gamma(E) \xrightarrow{c} \Gamma(E), \tag{165}$$

where

$$\mathrm{flip} \circ \sharp(\psi \otimes \omega) = \omega^\sharp \otimes \psi, \tag{166}$$

for $\psi \in \Gamma(E)$ and $\omega \in \Omega^1(M)$, in terms of the isomorphism $T^*M \to TM$, $\omega \mapsto \omega^\sharp$, given by the metric g (i.e., $g(\omega^\sharp, X) =< \omega, X >$ for all vector fields X).

Locally, D may be written as

$$D = -i \sum_a c(e_a)\nabla_{e_a}, \tag{167}$$

where (e_a) is an arbitrary *vielbein*. Indeed, we have $\nabla = \sum_a \theta^a \nabla_{e_a}$ and $(\theta^a)^\sharp = e_a$.

Exercise 4.13 *Show that $[D, f] = -ic(\nabla f)$ for all $f \in C^\infty(M)$, where ∇f is the vector field $\nabla f = (df)^\sharp$ and f acts on $\Gamma(E)$ as a multiplication operator.*

For example, the Dirac operator on $\Lambda^\bullet(T^*M)$ associated to the Levi-Civita connection ∇^g (see Exercise 4.8) turns out to be $D = d + d^*$, where d^* is the (formal) adjoint of d in terms of the natural inner product on $\Omega^\bullet(M)$. The golden example is:

Definition 4.14 The *Dirac operator $\not{D}$ is the Dirac operator on $\Gamma(\mathcal{S}_n)$ with respect to the spin connection ∇^S.*

To find a local expression for $\not{D}$, we initially write the $\Gamma(TM)$-action on $\Gamma(\mathcal{S}_n)$, given by (127) with $\beta = X$ a vector field, as

$$X\psi(p) = X(p)\psi(p) = \sum_a X^a(p)\gamma_a\psi(p), \tag{168}$$

where $p \in \mathrm{Spin}(M)$, $X(p) \in \mathbb{R}^n$ has components $(X^1(p), \ldots, X^n(p))$, and

$$\gamma_a = \pi_F(\hat{e}_a) \in \mathrm{End}(\mathbb{S}_n) \tag{169}$$

is the Fock representation (or Clifford action) of the a-th basis vector $\hat{e}_a$ of $\mathbb{R}^n$ on $\mathbb{S}_n$. A local section $s : U \to \mathrm{Spin}(M)$, $U \subset M$, trivializes both $\mathrm{Spin}(M)$ itself and all vector bundles associated to $\mathrm{Spin}(M)$, see Exercise 3.6. Explicitly, the bundle isomorphism

$$\begin{aligned} \mathrm{Spin}(M)_{|U} &\cong U \times \mathrm{Spin}(n); && (170)\\ p &\mapsto (x, \rho), && (171) \end{aligned}$$

is given by $x = \pi(p) \in U$, whilst $\rho \in \mathrm{Spin}(n)$ is defined by

$$p = s(x)\rho. \tag{172}$$

For any associated vector bundle $E = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n)} V$, using (172) we then have

$$\begin{aligned} E_{|U} &\cong U \times V; && (173)\\ [p, v] &\mapsto (x, \rho v). && (174) \end{aligned}$$

This induces a trivialisation, which, still using (172), is given by

$$\begin{aligned} \Gamma(E_{|U}) &\cong C^\infty(U, V); && (175)\\ \sigma &\mapsto \tilde{\sigma}, && (176)\\ \tilde{\sigma}(x) &= \sigma(s(x)); && (177)\\ \sigma(p) &= \rho^{-1}\tilde{\sigma}(x), && (178) \end{aligned}$$

where $\sigma \in C^\infty(\mathrm{Spin}(M), V)^{\mathrm{Spin}(n)}$. In particular, take $V = \mathbb{R}^n$ and $E = TM$, see (93). Given some section $s : U \to \mathrm{Spin}(M)$, under the isomorphism (93) the constant maps $x \mapsto (x, \hat{e}_a)$ from U to $U \times \mathbb{R}^n$ correspond to a specific local *vielbein* $TM_{|U}$. If a vector field $X \in \Gamma(TM)$ is given as an equivariant map $X \in C^\infty(\mathrm{Spin}(M), \mathbb{R}^n)^{\mathrm{Spin}(n)}$, the corresponding trivialisation $\tilde{X} : U \to \mathbb{R}^n$ then has components $(\tilde{X}^1(x), \ldots, \tilde{X}^n(x))$, with $X(s(x)) = \sum_a \tilde{X}^a(x)\hat{e}_a$. Therefore, if one also localises $\psi \in \Gamma(\mathcal{S}_n)$ to $\tilde{\psi} : U \to \mathbb{S}_n$, with $\tilde{\psi}(x) = \psi(s(x))$, setting $p = s(x)$ one finds that (168) localises to

$$\tilde{X}\tilde{\psi}(x) = \sum_a \tilde{X}^a(x)\gamma_a\tilde{\psi}(x), \tag{179}$$

so that the local Dirac operator (167) becomes

$$\not{D}\tilde{\psi}(x) = -i\sum_a \gamma_a \nabla^S_a \tilde{\psi}(x) \equiv -i\sum_a \gamma_a(e_a(x) + \frac{1}{4}\Gamma^b_{ac}(x)\,[\gamma_b, \gamma_c])\tilde{\psi}(x); \tag{180}$$

even more explicitly, one may write $e_a(x) = \sum_\mu e^\mu_a(x)\partial/\partial x^\mu$, as physicists do.

Dirac operators can be "coupled to a vector bundle" (in physics, this coupling is to internal degrees of freedom like electric charge, in the case of electrons and positrons, or color and flavour, in the case of quarks). We illustrate this for $\not{D}$. Suppose F is a vector bundle with connection called ∇^F. The spin connection ∇^S on $\mathcal{S}_n$ and ∇^F_A on F jointly determine a connection ∇ of $\mathcal{S}_n \otimes F$ by (check the Leibniz rule!)

$$\nabla_X(\sigma \otimes_{C^\infty(M)} \tau) = (\nabla^S_X\sigma) \otimes_{C^\infty(M)} \tau + \sigma \otimes_{C^\infty(M)} \nabla^F_X\tau, \tag{181}$$

and hence a Dirac operator $\not{D}^F$ on $\mathcal{S}_n \otimes F$ by the general procedure (165) or (167). If ∇^F is locally given by $\nabla^F = d + A$, we may write $\not{D}^F = \not{D}_A$, as physicists tend to do. We then have the local expression

$$\not{D}_A = -i\sum_a \gamma_a(\nabla^S_a + A_a), \tag{182}$$

with $A_a = < A, e_a >$, acting on a spinor field ψ taking values in $\mathbb{S}_n \otimes \mathbb{F}$, where $\mathbb{F}$ is the typical fiber of F (note that γ_a is a matrix acting on $\mathbb{S}_n$ and A_a is a matrix acting on $\mathbb{F}$).

4.4 Dirac operators: analysis

This material is well covered by Varilly, Chapter 3. We just have some minor additions.

Let (M, g) be a Riemannian manifold, which for the moment we allow to be non-compact. A good way to define the usual Laplacian $\Delta : C_c^\infty(M) \to C_c^\infty(M)$ is to embed $C_c^\infty(M) \subset L^2(M) \equiv L^2(M, \nu_g)$. In terms of $\nabla f = (df)^\sharp \in \Gamma(TM)$, where $f \in C_c^\infty(M)$, using the metric $g : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ we define a quadratic Q form on $C_c^\infty(M)$ by

$$Q(f_1, f_2) = \int_M d\nu_g \, g(\nabla \overline{f_1}, \nabla f_2). \tag{183}$$

By the theory of symmetric quadratic forms on Hilbert space, there is a unique operator $\Delta_g : C_c^\infty(M) \to C_c^\infty(M)$ implementing Q, in that

$$Q(f_1, f_2) = (f_1, \Delta_g f_2), \tag{184}$$

where the brackets on the right-hand side denote the inner product in $L^2(M)$. This operator is symmetric on $C_c^\infty(M)$, i.e, $\Delta_g \subset \Delta_g^*$, but not self-adjoint, i.e., $\Delta_g^* \neq \Delta_g$.

- If M is closed, i.e., compact without boundary, Δ_g is essentially self-adjoint, in the sense that its adjoint Δ_g^* is self-adjoint, or, in other words, that $\Delta_g^{**} = \Delta_g^*$.
- If M is an open set in $\mathbb{R}^n$, like $M = (0, 1)$, then Δ_g is not essentially self-adjoint on $C_c^\infty(M)$: it has a family of self-adjoint extensions, parametrised by boundary conditions on ∂M (such as Dirichlet, Neumann, ...). In spectral geometry, one usually takes Dirichlet boundary conditions, because in that case the corresponding self-adjoint extension of Δ_g remains *positive*, in the sense that $(f, \Delta_g f) \geq 0$ for all $f \in \mathrm{Dom}(\Delta_g)$. Hence an alternative way of arriving at this particular self-adjoint extension of Δ_g is to define it as the so-called *Friedrichs extension*.
- If M is a compact Riemannian manifold with boundary, the situation is the same, if one interprets $C_c^\infty(M)$ as $C_c^\infty(\overset{o}{M})$, where $\overset{o}{M}$ is the interior of M; think of $M = [0, 1]$ with $\overset{o}{M} = (0, 1)$.
- If M is noncompact without boundary, Δ_g is essentially self-adjoint on $C_c^\infty(M)$ when M is *geodesically complete*, in the sense that geodesics can be extended to arbitrary parameter values.[29]

The Laplacian $\Delta^E : \Gamma(E) \to \Gamma(E)$ of a (real or complex) hermitian connection ∇^E on a (real or complex) vector bundle E over a Riemannian manifold may be defined analogously.

1. Define a quadratic form Q^E on $\Gamma(E)$ by

$$Q^E(\psi_1, \psi_2) = \sum_a \int_M d\nu_g \, \langle \nabla_{e_a} \psi_1, \nabla_{e_a} \psi_2 \rangle_{C^\infty(M)} \tag{185}$$

where $\langle -, - \rangle_{C^\infty(M)} : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$ is the hermitian structure and (e_a) is a local *vielbein* (put together by a partition of unity).

[29] The geodesic equation $\Delta_{\dot\gamma}^g \dot\gamma = 0$ with initial data $\gamma(0) = x_0$ and $\dot\gamma(0) = u(x_0)$ has a local solution $t \mapsto \gamma(t)$, $\gamma : (-\varepsilon, \varepsilon) \to M$, for some $\varepsilon > 0$. The manifold is called *geodesically complete* if this solution exists for all $t \in \mathbb{R}$, for arbitrary initial data. Clearly, $M = (0, 1)$ with flat metric is not geodesically complete, like all bounded open sets in $\mathbb{R}^n$.

2. Define $\Delta^E = (\nabla^E)^*\nabla^E$ in the same way as in (184), namely as the unique operator satisfying
$$Q^E(\psi_1, \psi_2) = (\psi_1, \Delta^E \psi_2), \tag{186}$$
where the brackets on the right-hand side denote the natural inner product on $\Gamma(E)$:
$$(\psi_1, \psi_2) = \int_M dv_g \, \langle \psi_1, \psi_2 \rangle_{C^\infty(M)}. \tag{187}$$

3. Extend Δ^E to a self-adjoint operator on $L^2(E)$, defined as the completion of $\Gamma(E)$ with respect to the inner product (187); the details are the same as for the scalar Laplacian.

Yet another way to define Δ^E is to start from the expression
$$\Delta^E_{X,Y} = \nabla^E_{\nabla^g_X Y} - \nabla^E_X \nabla^E_Y, \tag{188}$$
where $X, Y \in \Gamma(TM)$ and ∇^g is the Levi-Civita connection on TM. Next, define
$$\Delta^E = \sum_a \Delta^E_{e_a, e_a} \tag{189}$$

Exercise 4.15 1. *Show that* (189) *coincides with the operator defined by* (186).

2. *Show that* $\Delta^E_{X,Y} - \Delta^E_{Y,X} : \Gamma(E) \to \Gamma(E)$ *is* $C^\infty(M)$*-linear.*

For the following exercise, see Fact 3.12 on p. 39 of Varilly.

Exercise 4.16 *Show that if* $N_\Delta(\lambda) \sim \lambda^\alpha$ *for* $\lambda \to \infty$, *then* $\lambda_k \sim k^{1/\alpha}$ *for* $k \to \infty$, *where* (λ_k) *are the eigenvalues of* Δ_g *listed in increasing order, including multiplicities.*

Finally, for Weyl's Theorem, Varilly refers to lectures [Hig] by Nigel Higson for a proof (see also the homepage of this course for the pdf file). Unfortunately:

Exercise 4.17 *Look at Theorem 1.13 in Higson's lectures and find the gap in his proof!*

5 Noncommutative calculus

Spectral triples (A, H, D) are the noncommutative generalisations of compact spin manifolds M. The elementary part of the definition is given as Definition 4.1 in Varilly;

- H is a Hilbert space;
- $A \subset B(H)$ is a unital *-algebra of bounded operators acting on H;
- $D : \mathrm{Dom}(D) \to H$ is a selfadjoint operator on H (generally unbounded) with dense domain $\mathrm{Dom}(D) \subset H$.
- If $\psi \in \mathrm{Dom}(D)$, then $a\psi \in \mathrm{Dom}(D)$ for all $a \in A$ (or, symbolically, $A\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D)$), and $[D, a]$ (which is well-defined on $\mathrm{Dom}(D)$ by the previous condition) is bounded (i.e., initially on $\mathrm{Dom}(D)$, but subsequently $[D, a]$ extends to a bounded operator on H, still called $[D, a]$). Symbolically: $[D, a] \in B(H)$ for all $a \in A$.
- For each $\lambda \in \rho(D)$, the operator $(D - \lambda)^{-1}$ is compact (equivalently, $(D + i)^{-1}$ is compact).

Exercise 5.1 *Show that for the canonical commutative spectral triple* $(C^\infty(M), L^2(\mathcal{S}_n), \not{D})$ *the operator* $[\not{D}, a]$ *is (indeed) bounded for all* $a \in C^\infty(M)$*.* Hint:[30]

1. *The domain* $\mathrm{Dom}(\not{D})$ *of* $\not{D}$ *is the completion of* $\Gamma(\mathcal{S}_n)$ *in the norm* $\|\psi\|_1^2 = \|\psi\|^2 + \|\not{D}\psi\|^2$*, where* $\|\cdot\|$ *is the* L^2*-norm on* $\Gamma(\mathcal{S}_n)$*. (Note that* $\mathrm{Dom}(\not{D}) = H^1(\mathcal{S}_n)$*, the Sobolev space consisting of all distributional sections* ψ *of* $\mathcal{S}_n$ *for which* $\psi \in L^2(\mathcal{S}_n)$ *and* $\not{D}\psi \in L^2(\mathcal{S}_n)$*.)*
2. *Show that* $\| -ic(\nabla f)\| = \|\nabla f\|_\infty$*, where the left-hand side is the operator norm on* $L^2(\mathcal{S}_n)$*, and the norm on the right is* $\|X\|_\infty^2 = \sup_{x\in M}\{g_x(X(x), X(x))\}$ *for any vector field* X *on* M*.*

The aim of the noncommutative calculus is to adapt the notions of integration and differentiation to spectral triples.[31] We start with integration. The idea is to define $⨍ a$ for $a \in A$, generalizing $\int_M a$ for $a \in C^\infty(M)$ (which is defined through the measure ν_g on M induced by the Riemannian metric g). The definition will be applicable when (A, H, D) is *p-summable* for some $p \in [1, \infty)$, in which case

$$⨍ a = \mathrm{Tr}^+(a|D|^{-p}). \tag{190}$$

Here $a \in A$ and Tr^+ is the *Dixmier trace* on H. In fact, if some $p \in [1, \infty)$ exists such that the operator $|D|^{-p}$ lies in the domain of Tr^+, then p is unique under the additional property that (190) is nonzero for some $a \in A$. For the canonical commutative spectral triple one finds $p = n = \dim(M)$, and (190) is computed via *Connes' trace theorem*. This states that

[30] The first part is true because $\not{D}$ is elliptic. The Sobolev space $H^1(\mathcal{S}_n)$ may equivalently be defined as the space of all $\psi \in L^2(\mathcal{S}_n)$ for which all first derivatives in all coordinate patches of M (with respect to a "good" cover) are in $L^2(\mathcal{S}_n)$.

[31] It should be mentioned, however, that something like Newton's fundamental theorem of calculus, stating that integration (defined as the area of the surface enclosed by the graph of a function $f : I \to \mathbb{R}$ and $I \subseteq \mathbb{R}$) is the inverse of differentiation, modernized by Stokes' formula $\int_M d\omega = \int_{\partial M} \omega$, appears to be lacking in noncommutative geometry (though the local index theorem of Connes and Moscovici gives an even more remarkable connection between differentiation and integration in the noncommutative case).

for any positive elliptic pseudodifferential operator P of order $-n$ on a complex vector bundle $E \to M$, one has

$$\mathrm{Tr}^{+}(P) = \frac{1}{n(2\pi)^n}\mathrm{Wred}(P), \tag{191}$$

where Wred(P) is the so-called *Wodzicki residue* of P. This formula also applies to the spinor bundle $E = \mathcal{S}_n$ if one takes $P = f|\not{D}|^{-n}$, $f \in C^\infty(M)$, although this P may not be elliptic. This eventually yields the justification for calling (190) an integral, namely

$$⨍ f = C_n \int_M f, \tag{192}$$

where C_n is some constant.

It has to be stated that the details of the noncommutative calculus are extremely technical, relying on partly unusual constructions in (functional) analysis. In any case, let us explain at least the terminology and the basic ideas.

5.1 The Dixmier trace

Let $T \in K(H)$ be a compact operator on H. The *singular values* of T are $s_k(T) = \sqrt{\lambda_k(T^*T)}$, where $\lambda_k(T^*T)$ are the eigenvalues of T^*T (which is positive, so that $\lambda_k(T^*T) \geq 0$ and the square root poses no problems). Let us form the sum

$$\sigma_N(T) = \sum_{k=0}^{N} s_k(T), \tag{193}$$

where the $s_k(T)$ are listed in decreasing order $s_0(T) \geq s_1(T) \geq \ldots$.

You may be familiar with the algebra $B_1(H)$ of *trace-class operators*, consisting of all $T \in K(H)$ for which $\lim_{N\to\infty} \sigma_N(T)$ exists. The *trace* of $T \in B_1(H)$ is then defined as $\mathrm{Tr}\,(T) = \sum_i (e_i, Te_i)$, where (e_i) is some o.n.b. of H (the result turns out to be finite and independent of the the choice of this basis).

Definition 5.2 1. *We say that $T \in K(H)$ lies in the class $B_1^+(H)$ if* $\sup_N\{\sigma_N(T)/\log(N)\} < \infty$.

2. *A positive operator $T \in K(H)$ is called* measurable *if* $\lim_{N\to\infty} \sigma_N(T)/\log(N$ *exists (and is finite). In that case we write $T \in B_1^{(+)}(H)$ (note that trivially $B_1^{(+)}(H) \subset B_1^+(H)$).*

3. *A general operator $T \in K(H)$ is called measurable if each $T_i \geq 0$, $i = 1, 2, 3, 4$, in the decomposition of T into positive operators is measurable.*[32]

The Dixmier trace Tr^{+} *of $T \in B_1^{(+)}(H)$ is defined by*

$$\mathrm{Tr}^{+}(T) = \lim_{N\to\infty} \sigma_N(T)/\log(N). \tag{194}$$

It is a highly nontrivial fact that this expression actually defines a *trace*.[33] To motivate this definition, let us look at the eigenvalue asymptotics of the Dirac operator $\not{D}$. It follows from Lichnerowicz's formula and Weyl's asymptotics for the eigenvalues of the Laplacian that $\lambda_k(|\not{D}|) \sim k^{1/n}$ for $k \to \infty$, so that $s_k(|\not{D}|^{-n}) \sim 1/k$ and hence $\sigma_N(|\not{D}|^{-n}) \sim \sum_k^N 1/k \sim$

[32] That is, $T = T_a + iT_b$, $T_a^* = T_a$, $T_b^* = T_b$, $T_a = (T + T^*)/2$, $T_b = (T - T^*)/2i$, $T_a = T_1 - T_2$, $T_b = T_3 - T_4$, $T_i \geq 0$.

[33] There exists an uncountable (and uncomputable) family of traces Tr^{+}_{ω} on $B_1^+(H)$, all of which reduce to Tr^{+} on the class of measurable operators $B_1^{(+)}(H) \subset B_1^+(H)$. See Varilly, §4.4.

$\log(N)$. Hence $|\not{D}|^{-n} \in B_1^+(H)$, and a more detailed analysis shows that $|\not{D}|^{-n} \in B_1^{(+)}(H)$. Here the operators $|\not{D}|$ and $|\not{D}|^{-n}$ are most easily defined using the spectral theorem, i.e. if $\not{D} = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$ is the spectral resolution of $\not{D}$, then $|\not{D}| = \sum_k |\lambda_k| |\psi_k\rangle\langle\psi_k|$ and similarly $|\not{D}|^{-n} = \sum_k |\lambda_k|^{-n} |\psi_k\rangle\langle\psi_k|$. In case that $\not{D}$ has zero eigenvalue(s), the corresponding eigenvectors are simply omitted in the expression for $|\not{D}|^{-n}$ (in other words, $|\not{D}|^{-1}$ is defined as zero on the kernel of $|\not{D}|$ and is the usual inverse of $|\not{D}|$ on the orthogonal complement of its kernel, where $|\not{D}|$ is invertible).[34]

Exercise 5.3 *Read* Example 4.4 *and do* ***Exercise 4.5*** *in Varilly, p. 43.*

These considerations suggest the following alternative and closely related axioms for spectral triples, all of which are satisfied by the canonical commutative spectral triple $(C^\infty(M), L^2(\mathcal{S}_n), \not{D})$ for $p = n = \dim(M)$. Let $B_p^{(+)}(H)$ be the set of all $T \in K(H)$ for which $\lim_{N\to\infty} \sigma_N(T)/N^{\frac{p-1}{p}})$ exists (and is finite), and let $B_p^+(H)$ consist of all $T \in K(H)$ for which $\sup_N\{\sigma_N(T)/N^{(p-1)/p}\} < \infty$. We may regard $B_1^{(+)}(H)$ and $B_1^+(H)$ as limiting cases as $p \to 1$.

Definition 5.4 *1. A spectral triple (A, H, D) is said to have* spectral dimension $p \in \mathbb{N}$ *if $\lambda_k(|D|^{-1}) = O(k^{-1/p})$ as $k \to \infty$ (for the smallest such p).*

2. *A spectral triple is* strongly p-summable *if* $|D|^{-p} \in B_1^{(+)}(H)$.

3. *A spectral triple is* p-summable *if* $|D|^{-p} \in B_1^+(H)$.

4. *A spectral triple is* strongly p^+-summable *if* $|D|^{-1} \in B_p^{(+)}(H)$.

5. *A spectral triple is* p^+-summable *if* $|D|^{-1} \in B_p^+(H)$.

The terminology "strongly p-summable" is our own. The implications are $1 \to 5 \to 3$, $2 \to 3$, $4 \to 5$, and $4 \to 2$; it is not so clear what the best "summability" axiom should be. Connes's reconstruction theorem (in which A is commutative) requires strong p-summability, which is derived from the first property in Definition 5.4 and some further axioms (see Section 6). Clearly, if (A, H, D) is (strongly) $p^{(+)}$-summable, then it is also (strongly) $q^{(+)}$-summable for all $q > p$. The interesting value of p for a (strongly) $p^{(+)}$-summable spectral triple is therefore the smallest p for which it is (strongly) $p^{(+)}$-summable. In all nonpathological cases, this coincides with the spectral dimension of the spectral triple in question. If 1 and 4 apply, it follows that $\mathrm{Tr}^+(|D|^{-p} > 0$ for the spectral dimension p (whereas $\mathrm{Tr}^+(a|D|^{-q} = 0$ for all $a \in A$ and all $q > p$).

If (A, H, D) is a strongly[35] p-summable spectral triple, the noncommutative integral (190) actually defines a trace on A, in the sense that it is linear and has the tracial property

$$⨍ ab = ⨍ ba, \tag{195}$$

for all $a, b \in A$. This is far from obvious; see Varilly, Theorem 6.5 (from which you'll see that (195) even holds for $a \in A$ and $b \in B(H)$). If (A, H, D) is regular, (195) is true for all $b \in B(H)$ and all $a \in \Omega_D(A)$, i.e., the algebra generated by A and $dA \equiv \{[D, a], a \in A\}$.[36]

[34] In fact, expressions like this can be defined for any closed operator, whether or not its is selfadjoint and/or has discrete spectrum.

[35] If (A, H, D) is just p-summable, a similar statement holds for te integral defined by any Dixmier trace Tr_ω.

[36] See F. Cipriani, D. Guido, S. Scarlatti, A remark on trace properties of K-cycles, J. Operator Theory 35, 179–189 (1996).

Finally, for spectral dimension p consider the complex function

$$\zeta_D(z) = \mathrm{Tr}\,(|D|^{-z}), \tag{196}$$

where for large real part of z (i.e., $\mathrm{Re}(z) > p$) the right-hand side is defined as $\sum_k |\lambda_k|^{-z}$, and by analytic continuation otherwise.[37] For example, for the canonical commutative spectral triple it turns out that $\zeta_{\not D}$ is meromorphic with simple poles at $z = n, n-1, n-2, \ldots$ (this is a classical result of Minakshisundaram and Pleijel from 1949). In general, the complete set of singularities of ζ_D forms the *dimension spectrum* of the triple. These singularities are the same if, for any $a \in A$, ζ_D is replaced by (with some abuse of notation)

$$\zeta_a(z) = \mathrm{Tr}\,(a|D|^{-z}). \tag{197}$$

If ζ_D has simple poles only and the spectral triple is p-summable, the residue of this function at $z = 0$ is then related to the Dixmier trace by

$$\mathrm{Res}_{z=0}\zeta_a = \mathrm{Tr}^{+}(a|D|^{-p}). \tag{198}$$

5.2 Pseudodifferential operators

Pseudodifferential operators (ΨDO's) enter the discussion because operators of the type $a|\not D|^{-n}$ on $H = L^2(\mathcal{S}_n)$, where $a \in C^\infty(M)$, are in this class. For our purposes (in which M is compact), it is enough to consider compactly supported symbols. Starting with scalar symbols on $\mathbb{R}^n$, we therefore have operators of the type V(5.1) (= eq. (5.1) in Varilly), where p has compact support in x and satisfies the bound V(5.4) for $K = \mathrm{supp}_x(p)$. If the symbol p of P satisfies V(5.4), we say that $P \in \Psi\mathrm{DO}_m(\mathbb{R}^n)$. One may regard ΨDO's as maps between endless varieties of space of smooth functions, Hilbert spaces, or distributions; for our purposes it is convenient to see them as linear maps

$$P : C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),$$

or into $C_c^\infty(\mathbb{R}^n)$ if $K = \mathrm{supp}_x(p)$ is compact, and, for each open $U \subset \mathbb{R}^n$ (or $U \subset K$), as

$$P : C_c^\infty(U) \to C^\infty(U).$$

The definition of classical symbols in Varilly is not quite correct: in V(5.5), the p_j are supposed to satisfy the scaling relation

$$p_j(x, t\xi) = t^j p_j(x, \xi) \tag{199}$$

only for $\|\xi\| \geq c_j > 0$. If $P \in \Psi\mathrm{DO}_m(\mathbb{R}^n)$ for all $m \in \mathbb{Z}$, we write $P \in \Psi\mathrm{DO}_{-\infty}(\mathbb{R}^n)$ and say that P is a *smoothing operator*.

Our main interest will lie in the singularities of the kernel K_P of P, defined as

$$K_P(x,y) = \int_{\mathbb{R}^n} \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi(x-y)} p(x,\xi), \tag{200}$$

which is a distribution on $\mathbb{R}^{2n}$ defined as an *oscillatory intergal* (rather than as a Lebesgue integral, which unlike (200) for $m \geq 0$ ought to be absolutely convergent). This has the

[37] Higson's Trieste Lectures [Hig] give a very clear introduction to such matters.

following meaning: take a cutoff function $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(\xi) = 1$ if $0 \le \|\xi\| \le 1$, $0 \le \chi(\xi) \le 1$ if $1 \le \|\xi\| \le 2$, and $\chi(\xi) = 0$ whenever $\|\xi\| \ge 2$. Then by definition

$$K_P(x,y) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{d^n\xi}{(2\pi)^n}\, \chi(\varepsilon\xi) e^{i\xi(x-y)} p(x,\xi), \tag{201}$$

where the limit is meant in the sense that

$$\int_{\mathbb{R}^n} d^n y\, K_P(x,y) f(y) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} d^n y\, K_P^\varepsilon(x,y) f(y),$$

where $K_P^\varepsilon(x,y)$ is the expression on the right-hand side of (201), and $f \in C_c^\infty(\mathbb{R}^n)$, However, if P is a smoothing operator, K_P is C^∞ and the integral (200) is defined as a Lebesgue integral. In general, the location of the singularities of K_P is controlled by:[38]

Lemma 5.5 *If $P \in \Psi\mathrm{DO}_m(\mathbb{R}^n)$ and $\mathrm{supp}_x(p)$ is compact, then for each $\varepsilon > 0$ one can decompose $P = P_1 + P_2$, where P_1 is ε-local in the sense that for each $x \in \mathrm{supp}(Pf)$ there is $y \in \mathrm{supp}(f)$ with $|x - y| < \varepsilon$, and P_2 is a smoothing operator.*

Consequently, the singularities of K_P must lie on the diagonal $x = y$; it follows that the limit in (201) exists pointwise for $x \neq y$. More detailed analysis shows that for a classical $P \in \Psi\mathrm{DO}_m(\mathbb{R}^n)$, $m \ge 0$, the singularities of K_P near the diagonal take the form

$$K_P(x,y) = \sum_{k=-m-n}^{-1} h_k(x, x-y) - h_0(x)\log(|x-y|) + R(x,y), \tag{202}$$

where $h_k(x, tz) = t^k h(x,z)$ for small z, and R is smooth. Each function h_k can be computed in terms of symbols p_j for which $p \sim \sum_j p_j$. This should be clear from naive power counting, since $p_j(x,\xi)$ behaves as $\|\xi\|^j$ for $\xi \to \infty$. The case of interest turns out to be

$$h_0(x) = \int_{S^{n-1}} d^{n-1}\sigma(\xi)\, p_{-n}(x,\xi). \tag{203}$$

If P has compact support in x, the *Wodzicki residue*

$$\mathrm{Wred}(P) = \int_{\mathbb{R}^n} d^n x\, h_0(x) = \int_{S^*\mathbb{R}^n} p_{-n} \tag{204}$$

is well defined, where $S^*\mathbb{R}^n = \{(x,\xi) \in T^*\mathbb{R}^n \mid \|\xi\| = 1\}$ is the *cosphere bundle* on $\mathbb{R}^n$. One then has:

Theorem 5.6 *The classical pseudodifferential operators on $\mathbb{R}^n$ whose symbols are compactly supported in x form an algebra under operator multiplication (i.e., composition), and the Wodzicki residue* (204) *defines a trace on this algebra.*

All this may be generalized to compact Riemannian manifolds M and even to vector bundles E over M. To do so, we first generalize the theory on $\mathbb{R}^n$ to operators $P : C_c^\infty(\mathbb{R}^n, \mathbb{C}^p) \to C^\infty(\mathbb{R}^n, \mathbb{C}^p)$, defined by symbols p for which $p(x,\xi) \in M_p(\mathbb{C})$. Estimates like V(5.4) now relate to the matrix norm of $p(x,\xi)$. With this modification, (classical)

[38] Note that P_1 in the is lemma is *not* local in the sense that $\mathrm{supp}(P_1 f) \subseteq \mathrm{supp}(f)$ for all $f \in C_c^\infty(\mathbb{R}^n)$ (in which case P_1 would be a differential operator by Peetre's Theorem). The difference is that P_1 of the lemma may actually increase support, but only in a tiny way.

pseudodifferential operators of order m on $\mathbb{R}^n$ may then be defined in the same way as in the scalar case.

Second, cover $M = \cup_\alpha U_\alpha$ in such a way that the open sets U_α locally trivialise $E_{|U_\alpha} \cong U_\alpha \times \mathbb{C}^p$ *and* admit coordinatization $U_\alpha \cong B_\alpha \subset \mathbb{R}^n$ into disjoint open balls B_α in $\mathbb{R}^n$.

We then have:

Definition 5.7 *A linear map $P : \Gamma(E) \to \Gamma(E)$ is a pseudodifferential operator of order m if one (and hence all) of the following equivalent conditions is satisfied:*

1. *Each of its localizations $P : C_c^\infty(B_\alpha, \mathbb{C}^p) \to C^\infty(B_\alpha, \mathbb{C}^p)$ is a pseudodifferential operator of order m on $\mathbb{R}^n$.*

2. *For some partition of unity (φ_α) subordinate to the cover (U_α), each map $P_\alpha : f \mapsto \varphi_\alpha P(\varphi_\beta f)$ from $C_c^\infty(U_\alpha, \mathbb{C}^p)$ to $C_c^\infty(U_\beta, \mathbb{C}^p)$ is a pseudodifferential operator of order m on $\mathbb{R}^n$.*

3. *$P = \sum_\alpha P_\alpha + P_2$, where each $P_\alpha : C_c^\infty(B_\alpha, \mathbb{C}^p) \to C^\infty(B_\alpha, \mathbb{C}^p)$ is a pseudodifferential operator of order m on $\mathbb{R}^n$, and P_2 is of the form*

$$P_2\psi(x) = \int_M dv_g(y) K_2(x, y)\psi(y), \tag{205}$$

for some smooth (matrix-valued) kernel K_2 (so that $K_2(x, y) : E_y \to E_x$ is a linear map).

Note that with some abuse of notation (meant to avoid the unnecessary complications that strictly correct notation would involve), we have identified e.g. $P : \Gamma(E_{|U_\alpha}) \to \Gamma(E_{|U_\alpha})$ with $P : C_c^\infty(B_\alpha, \mathbb{C}^p) \to C^\infty(B_\alpha, \mathbb{C}^p)$, omitting both the trivializing map $E_{|U_\alpha} \to U_\alpha \times \mathbb{C}^p$ and the coordinatization $U_\alpha \to B_\alpha$.

The equivalence of these conditions follows from Lemma 5.5; operators of the form (205) with smooth kernel K_2 are precisely the smoothing operators on $\Gamma(E)$. The formula for the Wodzicki residue now reads

$$\mathrm{Wred}(P) = \int_{S^*M} \mathrm{tr}\, p_{-n}, \tag{206}$$

where the cosphere bundle

$$S^*M = \{X \in T^*M \mid g(X, X) = 1\} \tag{207}$$

is defined in terms of the Riemannian metric g, and tr is the usual trace on $M_p(\mathbb{C})$. Theorem 5.6 then adapts in the obvious way (Guillemin–Wodzicki):

Theorem 5.8 *The classical pseudodifferential operators on $\Gamma(E)$ form an algebra under composition, and the Wodzicki residue* (206) *defines a trace on this algebra.*

The proof of (192) is given by Varilly, Proposition 6.1 and Example 5.16.

Exercise 5.9 *Give a self-contained exposition of the computation leading to* (192).

6 Smooth structures

Differential calculus on spectral triples (A, H, D) emerges through the self-adjoint operator $D : \mathrm{Dom}(D) \to H$. This operator will define two different "smooth structures", the first one lying in H, the second in $B(H)$.

6.1 Smooth vectors

Definition 6.1 *The subspace $H^\infty \subset H$ of* smooth vectors *in H (relative to D) is defined as*

$$H^\infty = \cap_{m\in\mathbb{N}} H^m; \tag{208}$$
$$H^m = \mathrm{Dom}(D^m). \tag{209}$$

It can be shown that H^∞ is (norm) dense in H. Given the property that D has compact resolvent, we may write for $m \in \mathbb{N}$,

$$D = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|, \tag{210}$$
$$|D|^m = \sum_k |\lambda_k|^m |\psi_k\rangle\langle\psi_k|,; \tag{211}$$
$$H^m = \mathrm{Dom}(|D|^m) = \mathrm{Dom}(D^m) = \left\{\psi = \sum_k c_k\psi_k, \sum_k |c_k|^2 < \infty, \sum_k |c_k|^{2m} < \infty\right\}, \tag{212}$$

where $(\psi_k)_k$ is the o.n.b. of H consisting of eigenvectors of D with eigenvalues λ_k (repeated in the sum (210) in case of multiplicity), and $|\psi_k\rangle\langle\psi_k|$ is the orthogonal projection onto $\mathbb{C}\psi_k$. Each H^m may be seen in two quite different ways:

1. H^m is a dense subspace of H in the inner product (and hence norm) of H, so that $|D|^m : H^m \to H$ and $D^m : H^m \to H$ are *unbounded* operators;
2. H^m is a Hilbert space in its own right in the inner product

$$(\psi, \varphi)_m = (\psi, \varphi) + (|D|^m\psi, |D|^m\varphi), \tag{213}$$

so that $|D|^m : H^m \to H$ and $D^m : H^m \to H$ are *bounded* (hence continuous) operators.

Although H^∞ is no longer a Hilbert space in any sense, in slightly different form both aspects pertain to H^∞ as well:

1. H^∞ is a dense subspace of H in the norm of H, and all $|D|^m : H^\infty \to H$ and $D^m : H^\infty \to H$, $m \in \mathbb{N}$, are unbounded operators;
2. H^∞ is a *Fréchet space*,[39] and each $D^m : H^\infty \to H^\infty$ or $|D|^m : H^\infty \to H^\infty$ is a continuous linear map.

Indeed, the seminorms $\|\cdot\|$ on H^∞ are the Hilbert space norms

$$\|\psi\|_m^2 = \|\psi\|^2 + \||D|^m\psi\|^2, \tag{214}$$

[39] I.e. a locally convex vector space defined by a *countable* number of seminorms $\|\cdot\|_m$ that is sequentially complete, in the sense that a sequence that is Cauchy with respect to all $\|\cdot\|_m$ converges.

and H^∞ is sequentially complete because each Hilbert space H^m is. Furthermore, D^m maps H^l into H^{l-m} for $l \geq m$, so that H^∞ is stable under D^m and $|D|^m$. One may now introduce pseudodifferential operators of order l as those operators $P : H^\infty \to H^\infty$ that extend to bounded operators $H^m \to H^{m-l}$ for any $m \geq l$. In particular, the pseudodifferential operators of order zero map $H^\infty \to H^\infty$ and are bounded with respect to any norm $\|\cdot\|_m$.

For the canonical commutative spectral triple $(C^\infty(M), L^2(\mathcal{S}_n), \not{D})$, it follows from standard arguments involving elliptic regularity and Sobolev embedding that

$$H^\infty = \Gamma(\mathcal{S}_n), \tag{215}$$

where the right-hand side is the space of smooth sections of the spinor bundle $\mathcal{S}_n \to M$. Hence the smooth structure of the spinor bundle $\mathcal{S}_n \to M$, though lost in forming the completion $L^2(\mathcal{S}_n)$ of $\Gamma(\mathcal{S}_n)$, can be recovered from the domain of the Dirac operators and its higher powers. Another example is $D = -id/dx$ on $L^2(\mathbb{R})$, for which $L^2(\mathbb{R})^\infty = C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, as followd from Sobolev's theorems.

6.2 Derivations

A *derivation* on an algebra A (over $\mathbb{R}$ or $\mathbb{C}$) is a linear map $\delta : A \to A$ satisfying the *Leibniz rule* $\delta(ab) = \delta(a)b + a\delta(b)$. Two key examples are:

1. *Classical geometry*: $A = C^\infty(M)$, $\delta_X(f) = Xf$ for any smooth vector field X on M. Conversely, any derivation on $C^\infty(M)$ is of this form.

2. *Quantum geometry*: $A = B(H)$, $c \in B(H)$,

$$\delta(a) = i[c, a]. \tag{216}$$

   Conversely, any strongly continuous derivation $\delta : B(H) \to B(H)$ is of this form (Kadison).

It can be shown that nonzero derivations on $C(M)$ as a C^*-algebra do not exist (Sakai). To cover the case of classical geometry from an operator-algebraic point of view, one needs to introduce *unbounded derivations* on Banach algebras. Apart from $\delta_X : C^\infty(M) \to C^\infty(M)$, which is an unbounded derivation on $C(M)$ with domain $C^\infty(M)$, we note the examples of the type $\delta(a) = i[c, a]$ for unbounded $c : \mathrm{Dom}(c) \to H$. This is an unbounded derivation on $B(H)$ whose domain consists of all $a \in B(H)$ satisfying $a\mathrm{Dom}(c) \subseteq \mathrm{Dom}(c)$ and $[c, a] \in B(H)$.

Just as for unbounded operators on H, we say that $\delta : \mathrm{Dom}(\delta) \to B$, where $Dom(\delta) \subseteq B$ is a dense subset of a Banach algebra B, is *closed* if its graph its closed, i.e., if $\{(a, \delta(a)), a \in \mathrm{Dom}(\delta)\} \subset B \times B$ is closed. This is the case iff $a_n \to a$ and $\delta(a) \to b$ imply $a \in \mathrm{Dom}(\delta)$ and $\delta(a) = b$.

If A is a *-algebra, we say that : $A \to A$ is *symmetric* if

$$\delta(a^*) = \delta(a)^*. \tag{217}$$

If $A = B(H)$ and $c^* = c$, then (216) is symmetric; this explains the i. Similarly, if δ is an unbounded derivation, we say that it is symmetric if $a^* \in \mathrm{Dom}(\delta)$ whenever $a \in \mathrm{Dom}(\delta)$ and (217) holds.

Exercise 6.2 *Show that if D is selfadjoint (and hence closed), then the unbounded derivation*

$$\tilde{\delta}(a) = i[D, a], \tag{218}$$

is closed with respect to the strong operator topology on $B(H)$ *(in which* $a_n \to a$ *iff* $a_n\psi \to a\psi$ *in* H *for all* $\psi \in H$*).*

6.3 Smooth operators

To define the smooth operators, we introduce the unbounded derivation

$$\delta(a) = i[|D|, a], \tag{219}$$

defined for $a \in B(H)$ satisfying $a\text{Dom}(|D|) \subseteq \text{Dom}(|D|)$ and $[|D|, a] \in B(H)$. Hence δ is an unbounded linear map on the Banach space $B(H)$ whose domain $\text{Dom}(\delta)$ consists of all $a \in B(H)$ with the properties just stated. Note that $\text{Dom}(D) = \text{Dom}(|D|)$, but by no means $\text{Dom}(\delta) = \text{Dom}(\tilde{\delta})$, with $\tilde{\delta}(a) = i[D, a]$!

By induction, we define $\delta^k : \text{Dom}(\delta^k) \to B(H)$ by saying that $a \in \text{Dom}(\delta^k)$ if:

1. $a \in \text{Dom}(\delta^{k-1})$;
2. $\delta^{k-1}(a)\text{Dom}(|D|) \subseteq \text{Dom}(|D|)$;
3. $\delta^k(a) = i[|D|, \delta^{k-1}(a)] \in B(H)$.

Definition 6.3 *The subspace $B^\infty(H) \subset B(H)$ of* smooth operators *on H (relative to D) is defined as*

$$B^\infty(H) = \cap_{m \in \mathbb{N}} \text{Dom}(\delta^m). \tag{220}$$

Curiously, it does not seem to be known what this space is for the canonical commutative spectral triple $(C^\infty(M), L^2(\mathcal{S}_n), \not{D})$. We do know, however, that

$$B^\infty(H) \quad \supseteq \quad \Psi\text{DO}_0(\mathcal{S}_n), \tag{221}$$

where the right-hand side is the set of pseudodifferential operators of order zero on $\mathcal{S}_n$.[40]

We are now in a position to state the basic regularity axioms for spectral triples, using the notation $dA \equiv \{[D, a], a \in A\}$. We have already dealt with what we now call Axiom 0, but repeat it for convenience (see (218)):

Axiom 0: $A \subset \text{Dom}(\tilde{\delta})$, or $dA \subset B(H)$;

Axiom 1: $A \subset B^\infty(H)$ and $dA \subset B^\infty(H)$;

Axiom 2: H^∞ is finitely generated projective over A (with respect to the given A-action on H, restricted to H^∞).

Of course, in order to state Axiom 2, we need $AH^\infty \subseteq H^\infty$.

Exercise 6.4 *Prove from Axiom 1 that $a\psi \in H^\infty$ whenever $\psi \in H^\infty$ (Hint: see Varilly).*

It would be more natural to state Axiom 1 in terms of D rather than $|D|$ in (219), but in that form it is not true for the canonical commutative spectral triple!

Axiom 0 holds for the canonical commutative spectral triple because of Exercise 4.13. Axiom 2 follows from (215) and the Serre–Swan Theorem. Axiom 1 is not so easy to verify. The proof (due to Connes and Moscovici) has three steps.

1. $a \in B^\infty(H)$ iff $a \in \text{Dom}(L^k \circ R^l)$ for all $k, l \in \mathbb{N}$. See Varilly Prop. 6.13.
2. $L^k \circ R^l(a) \in \Psi\text{DO}_0(\mathcal{S}_n)$ for all $k, l \in \mathbb{N}$. See VarillyCor. 6.14.
3. $\Psi\text{DO}_0(\mathcal{S}_n) \subset B(L^2(\mathcal{S}_n))$; this is the Calderon–Vaillancourt Theorem.

[40] In the lecture I stated that one has equality in (221) and attributed this result to H.O. Cordes, but in fact, so far only the inclusion (221) seems to have been proved. See Lemma I.2 in Connes and Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. 5, 174-243 (1995).

6.4 Orientability

The *Orientability Axiom* is stated in terms of *Hochschild Homology*. This is the simplest homology theory for algebras. Let A be a complex algebra with unit, and let M be an A-A bimodule (later we will take $\mathsf{M} = A$). The chain complex leading to Hochschild Homology is defined for each $n \in \mathbb{N}$ by

$$C_n(A, \mathsf{M}) = \mathsf{M} \otimes A^{\otimes n}, \tag{222}$$

with $C_0(A, \mathsf{M}) = \mathsf{M}$. The boundary maps $b_n : C_n(A, \mathsf{M}) \to C_{n-1}(A, \mathsf{M})$ are given by

$$\begin{aligned} b_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= (ma_1) \otimes a_2 \otimes \cdots \otimes a_n + (-1)^n (a_n m) \otimes a_1 \otimes \cdots \otimes a_{n-1} \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n, \end{aligned} \tag{223}$$

and $b_0 = 0$. For example, $b_1(a \otimes m) = ma - am$.

Exercise 6.5 *Show that $b_{n-1} \circ b_n = 0$ (i.e., "$b^2 = 0$").*

The *Hochschild Homology of A with coefficients in* M is then defined as usual, i.e., by

$$HH_*(A, \mathsf{M}) = \bigoplus_{i=0}^{\infty} HH_i(A, \mathsf{M}); \tag{224}$$

$$HH_i(A, \mathsf{M}) = \frac{\ker(b_i)}{\operatorname{im}(b_{i+1})} \equiv \frac{Z_i(A, \mathsf{M})}{B_i(A, \mathsf{M})}. \tag{225}$$

For example,

$$HH_0(A, \mathsf{M}) = \frac{\ker(b_0)}{\operatorname{im}(b_1)} = \frac{\mathsf{M}}{[A, \mathsf{M}]}, \tag{226}$$

where $[A, \mathsf{M}]$ consists of all elements of m of the form $ma - am$ and linear combinations thereof. We write $HH_i(A, A) \equiv HH_i(A)$.

Let us take a look at $A = C^\infty(M)$, with $\mathsf{M} = A$, as always for a compact manifold M. Recall that Ω^p stands for the $C^\infty(M)$-module of smooth p-forms on M. We then have maps

$$\mu_i : C_i(A, A) \to \Omega^i(M); \tag{227}$$

$$f_0 \otimes f_1 \otimes \cdots \otimes f_i \mapsto f_0 df_1 \wedge \cdots df_i; \tag{228}$$

$$\varepsilon_i : \Omega^i(M) \to C_i(A, A); \tag{229}$$

$$f_0 df_1 \wedge \cdots df_i \mapsto \sum_{\sigma \in \Pi_i} (-1)^{|\sigma|} f_0 \otimes f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(i)}, \tag{230}$$

where Π_p is the permutation group on p symbols.

Exercise 6.6 *Show that $b_i \circ \varepsilon_i = 0$ and $\mu_i \circ b_{i+1} = 0$.*

Consequently, μ_i and ε_i induce maps

$$\tilde{\mu}_i : HH_i(C^\infty(M)) \to \Omega^i(M); \tag{231}$$

$$\tilde{\varepsilon}_i : \Omega^i(M) \to HH_i(C^\infty(M)), \tag{232}$$

respectively. If one replaces the algebraic tensor product by the projective one, these maps are inverse to each other, so that $HH_*(C^\infty(M)) \cong \Omega^*(M)$ (Connes–Hochschild–Kostant–Rosenberg Theorem). In order to obtain the de Rham cohomology of M, one needs to pass from Hochschil homology to *periodic cyclic homology*.

To state the Axiom of Orientability, we need the following map, assuming A is part of a spectral triple (A, H, D).

$$\pi_D : C_i(A, A) \rightarrow B(H); \quad (233)$$
$$\pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_i) \mapsto a_0 da_1 \cdots da_i, \quad (234)$$

with $da = [D, a]$; the analogy with (228) is hard to miss. The Axiom of Orientability relies on a two others, which we state first.

Axiom 3: (A, H, D) has spectral dimension n, i.e., $\lambda_k(|D|^{-1}) = O(k^{-1/n})$ as $k \rightarrow \infty$.

Axiom 4: If n is even, then (A, H, D) is *graded*, with grading operator Γ.

Axiom 4 means that there exists an operator $\Gamma : H \rightarrow H$ with $\Gamma^2 = 1$, $\Gamma^* = \Gamma$, and $\Gamma D + D\Gamma = 0$. Such an operator leads to a decomposition $H = H_+ \oplus H_-$, where $H_\pm$ are the eigenspaces of H with eigenvalues ± 1, such that $D : H_\pm \rightarrow H_\mp$. As we have seen, the canonical commutative spectral triple is indeed graded.

Axiom 5: There exists an antisymmetric Hochschild boundary $c \in Z_n(A, A)$ such that $\pi_D(c) = \Gamma$ if d is even and $\pi_D(c) = 1$ if n is odd.

This Axiom of Orientability is only understandable if one sees its verification in the canonical commutative spectral triple, for which we refer to Varilly, pp. 80–81.

Exercise 6.7 *Do **Exercise 7.2** in Varilly, p. 81.*

In combination with the preceding axioms 1–4, Axiom 5 also has an important (yet highly technical) consequence, namely the fact that (A, H, D) is strongly p-summable, so that the noncommutative integral (190) is defined (see Wulkenhaar, p. 17). In combination with Axiom 2, this enables us to state:

Axiom 6: If $(\, , \,)$ is the inner product on H (as usual) and $\langle \, , \, \rangle_A$ is the canonical A-valued inner product on H^∞ (seen as a finitely generated projective A-module $H^\infty \cong pA^m$, with $p \in M_m(A)$, so that $\langle \psi, \varphi \rangle_A = \sum_{i=1}^m \psi_i^* \varphi_i$ for $\psi = (\psi_1, \ldots, \psi_m)$, $\psi_i \in A$, etc.), then

$$(\psi, \varphi) = \int\!\!\!\!\!- \langle \psi, \varphi \rangle_A. \quad (235)$$

There are two remaining axioms.

Axiom 7: (A, H, D) is *real*, in that there exists an antilinear operator $J : H \rightarrow H$ satisfying $J^* J = 1$ (i.e., J is antiunitary), as well as $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\Gamma = \varepsilon'' \Gamma J$, where the signs ε, ε', ε'' are ± 1 according to the table on p. 24.

Axiom 8: The multiplicity of the von Neumann algebra A'' generated by A is equal to 2^k, where $n = 2k$ or $n = 2k + 1$.

Connes' reconstruction theorem then states the following:

Theorem 6.8 *A commutative spectral triple satisfying Axioms 1–8 is isomorphic to the canonical commutative spectral triple* $(C^\infty(M), L^2(\mathcal{S}_n), \not{D})$*, with* $\dim(M) = n$*, and* vice versa.

The proof is very difficult, but the first step is appealing, namely $A = A'' \cap B^\infty(H)$.

However, the axioms are not necessarily of this form to achieve this. In fact, in interesting noncommutative examples strange phenomena appear (e.g., the spectral dimension of Axiom 3 may not coincide with the KO-dimension appearing in Axiom 6, and there is no good reason why the number n in Axiom 7 should be equal to either of these). Hence enough remains to be done for newcomers!