CONSISTENT REAL-TIME PROPAGATORS FOR ANY SPIN, MASS, TEMPERATURE AND DENSITY

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The time-path method in finite temperature field theory is extended to arbitrary covariant fields. Explicit expressions for the free thermal propagators are obtained using the multi-mass Klein–Gordon divisor. The key formula which shows that the interacting theory is free of singularities is derived. Finally, a simple method for the determination of free massless propagators is given.

Over the last decade significant progress has been made in the real-time approach to relativistic quantum field theory at finite temperature and density [1–6]. It has been shown [1] that the algebraic operator method known as thermo field dynamics [2–4] leads to the same Feynman rules as the time-path method in its formulation by Niemi and Semenoff [5]. However, these considerations were restricted to some simple fields of spin $\leqslant 1$, and it is the purpose of this letter to extend the time-path method and the consistency proof it requires [5,6] to fields of arbitrary spin and mass.

Let $\varphi_\alpha(x)$ be a covariant complex field [‡1], transforming under some representation D of the Lorentz group and carrying an arbitrary number of charges q_A such that $[Q_A, \varphi_\alpha] = -q_A \varphi_\alpha$. Internal indices are suppressed here, and q_A could be a matrix in internal space. The free lagrangian is given by

$$\mathcal{L}_0 = \bar{\varphi}_\alpha \Lambda_{\alpha\beta}(\partial)\varphi_\beta \,, \tag{1}$$

where $\bar{\varphi}_\alpha = \varphi_\beta^\dagger A_{\beta\alpha}$ such that A intertwines the representations $D^\dagger$ and D^{-1}, and Λ is a differential operator of finite order satisfying $(A\Lambda(\partial))^\dagger = A\Lambda(-\partial)$. The mass spectrum m_l ($l = 1, ..., k$) and spin content of the field φ can be inferred from Λ [7].

In the time-path approach the first step in obtaining the thermal propagator consists in solving the equation

$$\Lambda_{\alpha\beta}(\partial_x) D_{0\beta\gamma}(x - x') = \delta_{\alpha\gamma}\delta_c(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x}') \,, \tag{2}$$

with the boundary conditions

$$D_{0\alpha\beta}(x - x') = \theta_c(t - t')D^+_{\alpha\beta}(x - x') + \theta_c(t' - t)D^-_{\alpha\beta}(x - x') \,, \tag{3}$$

$$D^+_{\alpha\beta}(t - t' - \mathrm{i}\beta, \boldsymbol{x} - \boldsymbol{x}') = \eta\, \mathrm{e}^{\alpha} D^-_{\alpha\beta}(x - x') \,. \tag{4}$$

Here the delta- and stepfunction are defined on a specified contour [5], and contourordering is covariant in the sense that the stepfunctions commute with the Klein–Gordon divisor. Eq. (4) is the KMS condition, with the linear combination of the independent chemical potentials $\alpha = -\beta \sum_A \mu_A q_A$, and the inverse temperature β. We now assume the existence of a multi-mass Klein–Gordon divisor $d(\partial)$ [7–9] satisfying

$$d_{\alpha\beta}(\partial)\Lambda_{\beta\gamma}(\partial) = (-1)^k \delta_{\alpha\gamma} \prod_{l=1}^{k} (\Box + m_l^2) \,, \tag{5}$$

in terms of which [‡2] eqs. (2)–(4) can be solved to yield

[‡1] For a real field the RHS of eqs. (1) and (5) must be multiplied by 1/2.

[‡2] A general formula for the multi-mass case is given in ref. [7]. Explicit forms of the KGD for the single mass > 0 case may be found in ref. [9].

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$$\Delta_{\alpha\beta}(p) = \int d^4x\, e^{ipx} D_{0\alpha\beta}(x)$$

$$= [\Delta(p, m_i)(1 + \eta N_p) - \eta\Delta^*(p, m_i) N_p]\, d_{\alpha\beta}(-ip)\,, \tag{6}$$

$$\Delta^{\pm}_{\alpha\beta}(p) = \int d^4x\, e^{ipx} D^{\pm}_{\alpha\beta}(x)$$

$$= [\Delta(p, m_i) - \Delta^*(p, m_i)]\, [\theta(\pm p_0) + \eta N_p]\, d_{\alpha\beta}(-ip)\,, \tag{7}$$

with

$$\Delta(p, m_i) = \prod_l (p^2 - m_l^2 + i\epsilon)^{-1}\,, \tag{8}$$

$$N_p = \theta(p_0)[\exp(\beta p_0 + \alpha) - \eta]^{-1} + \theta(-p_0)\{\exp[-(\beta p_0 + \alpha)] - \eta\}^{-1}\,. \tag{9}$$

In the single-mass case the terms in (6) proportional to N_p may be combined into the well-known mass-shell delta function [1–6]. In analogy with the scalar case [5] the free thermal 2 × 2-matrix propagator is now defined as [‡3]

$$iD^{11}_{0\alpha\beta}(p) = \int d^4x\, e^{ipx} iD_{0\alpha\beta}(x) = i\Delta_{\alpha\beta}(p)\,, \tag{10}$$

$$iD^{12}_{0\alpha\beta}(p) = \int d^4x\, e^{ipx} iD_{0\alpha\beta}(t + 1/2 i\beta, \boldsymbol{x})$$

$$= \exp(\beta p_0/2)\, i\Delta^{-}_{\alpha\beta}(p)\,, \tag{11}$$

$$iD^{21}_{0\alpha\beta}(p) = \int d^4x\, e^{ipx} iD_{0\alpha\beta}(t - 1/2 i\beta, \boldsymbol{x})$$

$$= \exp(-\beta p_0/2)\, i\Delta^{+}_{\alpha\beta}(p)\,, \tag{12}$$

$$iD^{22}_{0\alpha\beta}(p) = \int d^4x\, e^{ipx} (iD_{0\alpha\beta}(-x))^* = (i\Delta_{\alpha\beta}(p))^*\,. \tag{13}$$

In eq. (13) and in the following the complex conjugation refers to factors i and iϵ only, not to possible group matrices, etc. The explicit form (6), (7) of the Δ's allows us to write

$$D^{ij}_{0\alpha\beta}(p) = \mathbf{M}_\eta \begin{pmatrix} \Delta_{\alpha\beta}(p, m_i) & 0 \\ 0 & -\Delta^*_{\alpha\beta}(p, m_i) \end{pmatrix} \mathbf{M}_\eta\,, \tag{14}$$

in terms of the Bogoliubov transformation matrix

$$\mathbf{M}_\eta = \begin{pmatrix} \cos(\mathrm{h})\,\Theta & \eta\, e^{-\alpha/2} \sin(\mathrm{h})\,\Theta \\ e^{\alpha/2} \sin(\mathrm{h})\,\Theta & \cos(\mathrm{h})\,\Theta \end{pmatrix}\,, \tag{15}$$

with $\sin(\mathrm{h})^2\Theta = N_p$, cf. (9). The hyperbolic/goniometric functions arise in the bosonic/fermionic case.

Proceeding with the interacting case we define the full thermal propagator $D^{ij}_{\alpha\beta}(p)$ by

$$iD^{11}_{\alpha\beta}(p) = \int d^4x\, e^{ipx} \langle T\varphi_\alpha(x)\bar{\varphi}_\beta(0)\rangle\,, \tag{16}$$

$$iD^{12}_{\alpha\beta}(p) = \eta \int d^4x\, e^{ipx} \langle \bar{\varphi}_\beta(0)\varphi_\alpha(t + 1/2 i\beta, \boldsymbol{x})\rangle\,, \tag{17}$$

$$iD^{21}_{\alpha\beta}(p) = \int d^4x\, e^{ipx} \langle \varphi_\alpha(t - 1/2 i\beta, \boldsymbol{x})\bar{\varphi}_\beta(0)\rangle\,, \tag{18}$$

$$iD^{22}_{\alpha\beta}(p) = \int d^4x\, e^{ipx} \langle T^*\varphi_\alpha(x)\bar{\varphi}_\beta(0)\rangle\,. \tag{19}$$

The KMS condition then leads to

$$D^{21}_{\alpha\beta}(p) = \eta\, e^{\alpha} D^{12}_{\alpha\beta}(p)\,. \tag{20}$$

From the definitions (16)–(19) and eq. (20) we now immediately have

$$iD^{22}_{\alpha\beta}(p) = (iD^{11}_{\alpha\beta}(p))^*\,, \tag{21}$$

$$iD^{11}_{\alpha\beta}(p) + (iD^{11}_{\alpha\beta}(p))^* = [\eta \exp(\alpha + \beta p_0/2) + \exp(-\beta p_0/2)]\, iD^{12}_{\alpha\beta}(p)\,. \tag{22}$$

Now let us define a quantity $G_{\alpha\beta}(p)$ according to

$$iD^{11}_{\alpha\beta}(p) = (1 + \eta N_p) G_{\alpha\beta}(p) + \eta N_p G^*_{\alpha\beta}(p)\,, \tag{23}$$

then eqs. (21)–(23) imply that we may write the full matrix propagator in terms of the single quantity $G_{\alpha\beta}(p)$:

$$iD^{ij}_{\alpha\beta}(p) = \mathbf{M}_\eta \begin{pmatrix} G_{\alpha\beta}(p) & 0 \\ 0 & G^*_{\alpha\beta}(p) \end{pmatrix} \mathbf{M}_\eta\,, \tag{24}$$

in terms of the same Bogoliubov matrix as in eq. (14).

‡3 The equations in ref. [5] corresponding to our (11) and (12) have a sign error.

Of course, eqs. (10)–(14) are special cases of eqs. (16)–(24).

Eq. (24) is of the type discovered by Takahashi and Umezawa [2] and is by now more or less axiomatic in the construction of thermo field dynamics. For scalar fields eq. (24) was derived in ref. [5] on the assumption of the existence of a spectral representation, the use of which is not at all necessary, as we have demonstrated. The importance of the representation (24) for the full matrix propagator derives from its use in the proof [5,6] that the real-time method is consistent, i.e. free of singularities of the type $\delta^N(p^2 - m_i^2)$, $N > 1$, the occurrence of which would be naively expected from eqs. (10)–(13) for the free propagator. Starting from eq. (24), the consistency proof given for the scalar case in refs. [5,6] can be literally copied to conclude that the correctly applied real-time method leads to a consistent theory for any spin and mass (foregoing possible infrared divergences). Eq. (24) may also be used to define a thermal self-energy [5,6,10] which can be shown to be the standard analytic continuation of the imaginary-time self-energy [11].

We will finally elaborate on the derivation of the free propagator for massless theories. In order to obtain the correct finite temperature ϵ-prescription as in eq. (6) it is mandatory to avoid the use of nonlocal projection operators. For fields describing a single massless particle of helicity λ no problems will be encountered [12]. Gauge theories do not belong to this class, but for these our method involving the multimass Klein–Gordon divisor comes in handy. For the photon (or gluon) field A_μ we take the Stuckelberg lagrangian [13] with the differential operator

$$\Lambda_{\beta\gamma}(\partial) = g_{\beta\gamma}(\Box + \mu^2) + (\lambda - 1)\partial_\beta \partial_\gamma \,, \tag{25}$$

which possesses the mass spectrum $m_1^2 = \mu^2$, $m_2^2 = \mu^2/\lambda$. The two-mass Klein–Gordon divisor

$$d^{\alpha\beta}(\partial) = g^{\alpha\beta}(\Box + \mu^2/\lambda) - (1 - \lambda^{-1})\partial^\alpha\partial^\beta \tag{26}$$

satisfies eq. (5) for $k = 2$. We can at once set $\mu = 0$ to arrive with (6), (8) and (26) at the correct massless propagator, which had been previously obtained in refs. [14,15] by rather different methods. The method sketched above is of course applicable to zero temperature and density as well, and is somewhat simpler than the one described in textbooks [13].

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