A GAUGE-INDEPENDENT COUPLING CONSTANT IN THERMAL QCD

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Received 19 May 1989; revised manuscript received 30 August 1989

We employ the theory of the Vilkovisky–DeWitt effective action, trivially extendable to finite temperatures, to define a manifestly gauge- and vertex-independent coupling constant in the high-temperature phase of quantum chromodynamics. The renormalization-group flow of this effective interaction strength is determined on the basis of a recently constructed thermal generalization of the MOM subtraction scheme (which is briefly reviewed from a rigorous point of view). This scheme intends to minimize the effects of higher orders, and appears to be the most "physical" renormalization prescription at finite T. It is found that the one-loop β-function for the renormalized effective expansion parameter thus defined is *positive* at high T. It follows that, in contradiction to conventional wisdom, asymptotic freedom at high temperature (and/or density) cannot be derived from bare one-loop renormalization-group arguments.

1. Introduction

The study of perturbative quantum chromodynamics (QCD) in a many-body context was initiated by Collins and Perry [1], who argued that the strong interaction becomes weak not only at very high momentum transfer, as in deep inelastic scattering, but in ultra-dense nuclear matter as well. This seems evident from the usual heuristic picture of asymptotic freedom, and the argument was rapidly extended to the case of low densities but very high temperatures (~ 300 MeV) [2]. The idea of asymptotic freedom at high temperature and/or density led to the notion of the quark–gluon plasma [3,4]: the phase diagram of QCD should have at least two regions, a low-temperature phase in which quarks and gluons are confined, and a high-temperature phase (the quark–gluon plasma) in which these particles are "liberated", and interact only feebly.

The basic reasoning underlying the prediction of the existence of this deconfined phase of QCD was quite similar to the original perturbative argument leading to asymptotic freedom (see e.g., ref. [5]). To remove all divergences in a thermal field theory, it is sufficient to renormalize at $T=0$ [2,6], which produces a renormalized coupling constant $a_s(\mu)$ depending on the renormalization point μ (e.g., the scale in minimal subtraction, or the subtraction point in a momentum subtraction scheme. Let $G(k, T, \mu, a_s(\mu))$ be a thermal n-point Green function of dimension d, depending on an arbitrary set of momenta k. Dimensionality and renormalization group invariance give $G(k, T, \mu, a_s(\mu)) = T^d \zeta^{n/2} G(k/T, 1, 1, a_s(T))$, with $\zeta = Z_3(\mu=T)/Z_3(\mu=\mu)$, in obvious notation. If now the zero-momentum limit of G is under control, this relation seems to indicate that at high T the theory is governed by an effective coupling $a_s(T)$. Since the evolution of $a_s = a_s(\mu)$ is known from the (zero-temperature) renormalization group [5], it follows that $a_s(T) = [b \log(T/\Lambda)]^{-1}$ in terms of the RG-invariant scale Λ, and $b = (11N - 2N_f)/6$ ($N=3$ and $N_f=6$ in full QCD). Hence the coupling decreases at high T, and this fact, along with its high-density analogue, supposedly implies deconfinement.

While it has, by now, convincingly been shown [7,8] that the (alleged) smallness of the coupling constant of the quark–gluon plasma by no means leads to "perturbative" behaviour, until recently the above reasoning as such has been left unquestioned. It is the purpose of this letter to point out that, on top of the usual problems plagueing perturbative thermal QCD (for a recent overview please consult ref. [9]), the above argument neither correctly identifies the coupling constant of thermal QCD, nor provides its

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T-dependence in a meaningful way: in fact, a sensibly defined expansion parameter will turn out to grow with T, at least on the basis of a one-loop calculation. In our approach to the former issue we have been essentially inspired by the work of Rebhan [10,11], while our treatment of the latter is intended to refine the pioneering work of Nakkagawa et al. [12] (also cf. ref. [13]).

2. The thermal renormalization group

As reviewed, for example, in refs. [14,15] or ref. [16]{§5.3}, a physical quantity $f(g)$ is renormalization-scheme (RS) independent, yet its truncated perturbation series in a coupling constant g is not: if one truncates the series at order N, the scheme dependence enters at the next order. In an optimal RS, the next $(N+1)$ order contribution to f should be minimized. This poses the RS ambiguity problem: "which RS, among all the infinite number of possible RS's, should we choose in order to best compare the truncated, nth order QCD predictions with experimental results?" [15]. It is an unfortunate fact that at present no experimental data are available which allow a comparison with perturbative thermal QCD. Thus the relative virtues of several RS's in QCD at finite T ought to be examined on the basis of certain *internal* consistency requirements of the type discussed in the context of vacuum QCD [15,16].

The most straightforward criterion, and the one we shall adopt here, is the demand that a given scheme minimizes the coefficient of the next-to leading order term (but note that third- and higher-order coefficients may be expected to be large in any scheme, since the perturbation series is presumably an asymptotic one). As long as this requirement is met, one expects little numerical difference between various RS's [14]; at zero temperature, for many processes this applies to the $\overline{\mathrm{MS}}$ and the MOM (momentum subtraction) schemes. Nevertheless, despite its greater computational complexity, the latter scheme has been claimed, on the basis of explicit higher-order calculations, to be slightly superior as far as the above criterion is concerned (cf. refs. [16,17] and references therein). Moreover, it is a "physical" subtraction scheme, in that it satisfies the decoupling theorem of quantum field theory [18].

One may now ask which class of renormalization schemes is optimal at finite T. Although in principle this should be settled by explicit higher-order computations, a particular feature of thermal perturbation theory allows the immediate conclusion that the RS used in the argument in the Introduction, or any other scheme based on a zero-temperature renormalization prescription (possibly amended by the replacement of the momentum scale μ by T) cannot be satisfactory in the sense explained above. The reason is that, while such schemes do minimize the terms proportional to powers of log T, at least if the substitution $\mu\to T$ is made (they are summed by the renormalization group (RG) pertaining to this scheme for the same reason that leading logs are summed by the RG at $T=0$ [18], because vacuum terms $\sim\log(p^2/\mu^2)$ at finite T are replaced by terms $\sim\log(T^2/\mu^2)$), but higher-order terms proportional to powers of T itself are left unaffected. In the imaginary-time formalism [6], which we will exclusively use here, the former come from the non-static modes, while the latter derive from the zero modes. Thus an optimal (one might say, correct) RS should minimize the zero-mode contributions as well, as otherwise their contribution would grow indefinitely with T.

Although in absence of two-loop calculations the optimal thermal RS (if this exists) is obviously unknown, one may argue that such a scheme is at least approximated by a two-parameter RS originally introduced in the context of thermo field dynamics [19], but easily adaptable to the imaginary-time formalism [20–22]. Below we will introduce this scheme, and derive its associated RG equations [23], by extending the considerations of Zimmermann [24], in which no bare quantities ever enter, to finite T. In view of the ultimate goal of this paper, we will do so in the context of perturbative QCD, for concreteness' sake, in the usual covariant Landau gauge. In the following, $G^{(n)}(..., T)$ denotes the thermal (grand-canonical) average of a time-ordered product of n fields at temperature T, whereas $\Gamma^{(n)}(..., T)$ will denote the associated irreducible vertex function; also, we use a euclidean metric, and colour indices will mostly be omitted.

The idea is that the operator field $A_\rho(x)$ in a solution of the renormalized equations of motion, supplemented by initial conditions, which are stated in terms of normalization conditions on certain Green functions of A. Nothing forbids us to impose these conditions on the thermal Green functions $G^{(n)}(..., T_0)$ at a given temperature T_0. It is both convenient and physically meaningful (in view of a study of the infinite-temperature limit [20,21]) to normalize the *static* Green functions $G_0^{(2)}$ and $G_0^{(3)}$. An example of a suitable set of conditions is

$$\Pi_0^{\mathrm{L}}(\kappa=\mu, T=T_0)=0, \quad \Pi_0^{\mathrm{T}}(\kappa=\mu, T=T_0)=0,$$
$$G_0(\kappa=\mu, T=T_0)=g, \tag{1}$$

where $\kappa=|\boldsymbol{k}|$, and $\Pi_0^{\mathrm{L,T}}$ and G_0 are static structure functions defined through the two- and three-point Green function, respectively, as follows: firstly, in the Landau gauge the inverse full propagator $(G^{(2)})_{\rho\sigma}^{-1}$ satisfies the Ward identity [6] $k_\rho(G^{(2)}(k))_{\rho\sigma}^{-1}=0$, so that it allows the decomposition

$$(G^{(2)})_{\rho\sigma}^{-1}=L_{\rho\sigma}(1+\Pi^{\mathrm{L}})+T_{\rho\sigma}(1+\Pi^{\mathrm{T}}) \tag{2}$$

in terms of the kinematic tensors $T_{\rho\sigma}=\Delta_{\rho\sigma}\kappa^2-\kappa_\rho\kappa_\sigma$ and $L_{\rho\sigma}=g_{\rho\sigma}k^2-k_\rho k_\sigma-T_{\rho\sigma}$ (note that [6] $\Delta_{\rho\sigma}=g_{\rho\sigma}-U_\rho U_\sigma$, with $U_\rho=(1,\mathbf{0})$, and $\lambda_\rho=\Delta_{\rho\sigma}k_\sigma$; the functions $\Pi^{\mathrm{L,T}}$ defined above are not quite the same as $\tilde{\Pi}_{\mathrm{L,T}}$ of ref. [6]). Secondly, the spatial ("magnetic") part of the static irreducible three-point vertex function $\Gamma_0^{(3)}{}^{abc}_{ijk}(p,q,r)$ evaluated at the symmetric point $k^2=q^2=r^2\equiv\kappa^2$ may be decomposed as (cf. ref. [10])

$$\Gamma_0^{(3)}{}^{abc}_{ijk}(k,q,r)$$
$$=f^{abc}\{[g_{ij}(k-q)_k+\text{cycl.}]G_0(\kappa)+...\}, \tag{3}$$

where the dots stand for terms orthogonal to r^l.

Due to these normalization conditions, the field $A_\rho(x)$ effectively depends on g, μ, and T_0. We now call a triple $\{g,\mu,T_0\}$ equivalent to $\{g',\mu',T_0'\}$ if there exists a z^ρ such that $A_\rho(x;g',\mu',T_0')=(z^\rho)^{1/2}A_\rho(x;g,\mu,T_0)$ (no sum over ρ). Thus the RHS is independent of μ and T_0, and we are led to the RG equations for the vertex functions

$$\left(\tfrac{1}{2}\sum_{i=1}^{n}\alpha_T^{\rho_i}+\beta_T\frac{\partial}{\partial g}+T_0\frac{\partial}{\partial T_0}\right)\Gamma^{(n)}_{\rho_1\ldots\rho_n}(k,T,g,\mu,T_0)$$
$$=0, \tag{4}$$

$$\left(\tfrac{1}{2}\sum_{i=1}^{n}\alpha_\mu^{\rho_i}+\beta_\mu\frac{\partial}{\partial g}+\mu\frac{\partial}{\partial \mu}\right)\Gamma^{(n)}_{\rho_1\ldots\rho_n}(k,T,g,\mu,T_0)$$
$$=0, \tag{5}$$

(no sum over ρ), where k stands for the set of momenta on which Γ depends. The Callan–Symanzik type functions may be obtained from the normalization conditions (1) (cf. ref. [24]), which yields

$$\alpha_T^0=\left(T\frac{\partial\Pi_0^{\mathrm{L}}}{\partial T}\right)(\kappa=\mu, T=T_0),$$

$$\alpha_T^i=\left(T\frac{\partial\Pi_0^{\mathrm{T}}}{\partial T}\right)(\kappa=\mu, T=T_0), \tag{6}$$

$$\beta_T=-\tfrac{3}{2}g\alpha_T^i+\left(T\frac{\partial G_0}{\partial T}\right)(\kappa=\mu, T=T_0), \tag{7}$$

and similar expressions for α_μ^ρ and β_μ, with the replacement $T\partial/\partial T\to\kappa\partial/\partial\kappa$. Note that $\alpha^0\neq\alpha^i$ due to broken Lorentz invariance at finite T, and that the non-mixing of Π^{L} with Π^{T} in (6) is a result of our choice of *static* normalization conditions.

It should be mentioned that the RG equations (4) and (5) may also be derived from the conventional effective action formalism. Here the field is coupled *linearly* to the external source, so that one may show that $\partial\Gamma[\bar{A}]/\partial T_0=\partial\Gamma[\bar{A}]/\partial\mu=0$. On the other hand, the mean field $\bar{A}$ satisfies $\partial z^{1/2}\bar{A}(x;g,\mu,T_0)/\partial T_0=0$ (etc.), because the operator A, of which $\bar{A}$ is the expectation value in the presence of a T_0- and μ-independent source, does. Since the 1PI function $\Gamma^{(n)}$ is the coefficient of $\bar{A}^n$ in the Volterra expansion of Γ in powers of $\bar{A}$, consistency implies (4) and (5).

After these preparations, it should become clear why the present RS is an improvement over schemes based on vacuum subtractions: one chooses T_0 equal to the actual temperature T, and μ a relevant momentum scale of the process, so that the normalization conditions (1) imply that radiative corrections to the two- and four-point function *vanish* at the given temperature and momentum configuration. Assuming that the amplitudes are continuous functions of the (off-shell) momenta and of T, this implies that leading powers of T (which are definitely present in vacuum-based schemes) and log T (or κ and log κ) will indeed be absent in the renormalized Green functions at the given temperature and momentum scale. An additional feature of the thermal MOM RS is that it satisfies the dimensional reduction Ansatz at $T=\infty$ [22], in analogy to the fact that the vacuum MOM scheme obeys the decoupling theorem for heavy particles.

This, then, suggests that the appropriate effective coupling constant is the running one obtained from the β-functions in the above RS. The relevant computations have been performed, on the basis of slightly different normalization conditions, in refs. [12,13]. The results obtained in these publications indicate three serious drawbacks of the above procedure applied to QCD:

- One finds a strong vertex-dependence of the renormalized coupling constant. This means, that g heavily depends on the vertex chosen (i.e., the tri-gluon, the ghost–gluon, or the quark–gluon vertex) to satisfy the normalization condition of g. This problem is well known at $T=0$ as well, where it has been found that the heavy-quark contribution to the β-function (which introduces a large dimensionful parameter, similar to T) is very sensitive to the choice of the normalization vertex [25].
- Even if a vertex has been specified, the β-function is sensitive to the momentum configuration used to state the normalization condition. Indeed, refs. [12,13] employed slightly different kinematical configurations in the tri-gluon vertex, and obtained qualitatively different running coupling constants.
- Worst of all, the renormalized coupling is gauge-dependent: e.g., in covariant gauges the gauge parameter explicitly enters (in contrast to the state of affairs in the minimal subtraction scheme, which, unfortunately, is unphysical in other respects [18]).

3. The Vilkovisky–DeWitt effective action

As remarked by Rebhan [10], the drawbacks mentioned above are each caused by the fact that g has been defined in terms of the conventional effective action of QCD, which is gauge-variant (i.e., a gauge transformation does not leave Γ invariant) as well as gauge-dependent (that is, it depends on the gauge condition chosen to define the quantum gauge theory). Some years ago, however, the so-called Vilkovisky–DeWitt effective action $\Gamma_{\rm VD}$ has seen the light [26,27] (for reviews see refs. [11,28]). This action is explicitly gauge-invariant and gauge-independent, and it is natural to define the coupling constant of (thermal) QCD in terms of it [#1].

[#1] The Vilkovisky–DeWitt action has earlier been used in the study of the quark–gluon plasma in the context of the so-called plasmon damping problem [29–31].

For our purpose it is useful to first define an auxiliary quantity $\tilde{\Gamma}$, which was originally introduced by DeWitt [27] in the path-integral formalism as the solution of a certain functional equation. Here we directly define

$$\begin{aligned}\tilde{\Gamma}[\bar{A},A^*]:=&\sum_{n=2}^{\infty}\frac{1}{n!}\langle(\sigma^{i_1}[A^*,A]-\sigma^{i_1}[A^*,\bar{A}_0])\ldots\\&\times(\sigma^{i_n}[A^*,A]-\sigma^{i_n}[A^*,\bar{A}_0])\rangle_{\rm 1PI}\\&\times(\sigma_{i_1}[A^*,\bar{A}]-\sigma_{i_1}[A^*,\bar{A}_0])\ldots\\&\times(\sigma_{i_n}[A^*,\bar{A}]-\sigma_{i_n}[A^*,\bar{A}_0]).\end{aligned}\tag{8}$$

For brevity we employ the collective index notation, in which i stands for $\{x,\rho,\alpha\}$, and in which a double occurrence of i means that it is to be integrated and summed over. The average $\langle\ldots\rangle$ stands for the time-ordered grand-canonical one, and $\bar{A}_0$ denotes the value of $\bar{A}$ in the absence of a source. The crucial quantity appearing in (8) is the bi-tensor σ^i, which is defined by virtue of an essentially unique connection on the space of gauge fields [26]. For SU(N) Yang–Mills theories it has the form [32] (for clarity we now explicitly specify the indices)

$$\begin{aligned}\sigma^a_\rho(x)[A^*,A]=&A^{*a}_\rho(x)-A^a_\rho(x)\\&+\int d^4x'\,X^{ab}_\rho(x,x')[A^*,A]\\&\times[\delta^{bc}\partial_\sigma+gf^{bdc}A^{*d}_\sigma(x')][A^c_\sigma(x')-A^{*c}_\sigma(x')],\end{aligned}\tag{9}$$

where X is some involved functional.

The main point is that $\tilde{\Gamma}[\bar{A},A^*]$ is a gauge-invariant and gauge-independent functional of $\bar{A}$ for any choice of A^* [27,11,30]. This means, that any gauge condition may be used in the computation of the thermal averages in (8); the functional $\tilde{\Gamma}$ is insensitive to it. The dependence on the arbitrary reference point A^* in the field configuration space is usually removed by setting $A^*=\bar{A}$ [27], which defines the Vilkovisky–DeWitt action $\Gamma_{\rm VD}\equiv\tilde{\Gamma}[\bar{A},\bar{A}]$. This choice has the advantage that $\Gamma_{\rm VD}$ equals the conventional effective action if the field configuration space is flat (as in standard scalar field theories [26]) [#2], whereas the gauge invariance and independence is maintained. We are now going to use this object to

[#2] For footnote see next page.

define an effective coupling constant in thermal QCD.

4. A gauge-independent renormalized coupling

The functional $\Gamma_{\rm VD}$ has a conventional Volterra expansion

$$\Gamma_{\rm VD}[\bar{A}] = \sum \frac{1}{n!} \Gamma_{\rm VD}^{(n)i_1\ldots i_n}(\bar{A}_{i_1}-\bar{A}_{0i_1})\ldots(\bar{A}_{i_n}-\bar{A}_{0i_n}), \tag{10}$$

involving certain expansion coefficients $\Gamma^{(n)}_{\rm VD}$ which, as can be seen from (8), cannot be identified with 1PI correlation functions of A, or of any other operator (cf. ref. [28]). Nevertheless, the $\Gamma^{(n)}_{\rm VD}$ still satisfy the RG equations (4), (5). This is firstly because the Vilkovisky–DeWitt action is invariant under field reparametrizations [26,27], hence in particular under rescalings, so that it is an RG-invariant, like the conventional effective action (cf. section 2). Secondly, the mean field $\bar{A}$ is defined by [27,28] $\langle \sigma^i[\bar{A}, A]\rangle_j = 0$, so that according to (9) $\bar{A}$ scales in the same way as the operator A up to a gauge transformation, to which $\Gamma_{\rm VD}$ is insensitive. These observations imply (4) and (5) with $\Gamma^{(n)}$ replaced by $\Gamma^{(n)}_{\rm VD}$ as defined above.

By gauge invariance, one has the Ward identity, $k_\rho \Gamma^{(2)}_{\rm VD}{}^{ab}_{\rho\sigma}(k)=0$, so that we are allowed to make the kinematic decomposition (2), with $(G^{(2)})^{-1}$ replaced by $\Gamma^{(2)}_{\rm VD}$. This identifies two structure functions $\Pi^{\rm L,T}_{\rm VD}$ (which in general *cannot* be found by just computing self-energy diagrams), on which we impose normalization conditions similar to the first two members of (1), respecting gauge invariance. However, the crucial difference between $\Gamma_{\rm VD}$ and the conventional (Landau-gauge) Γ is that we now have a second Ward identity (which is, in fact, valid for any gauge-invariant effective action, cf. ref. [10]) $G_{\rm VD}(\kappa) = g(1+\Pi^{\rm T}_{\rm VD}(\kappa))$, where $G_{\rm VD}$ is defined in complete analogy to (3), with $\Gamma^{(3)}$ replaced by $\Gamma^{(3)}_{\rm VD}$.

[#2] This advantage is shared by the choice $A^* = \bar{A}_0$, which in fact seems more natural to us, since the action $\Gamma_{\rm OK}[\bar{A}] \equiv \bar{\Gamma}[\bar{A}, \bar{A}_0]$ has a covariant Volterra expansion $\Gamma_{\rm OK}[\bar{A}] = \sum(1/n!)\langle \sigma^{i_1}[\bar{A}_0, A]\ldots\sigma^{i_n}[\bar{A}_0, A]\rangle_{\rm 1PI}\sigma_{i_1}[\bar{A}_0, \bar{A}]\ldots\sigma_{i_n}[\bar{A}_0, \bar{A}]$, with 1PI correlation functions as coefficients. The Vilkovisky–DeWitt action does not admit any interpretation of this kind [28]. However, in this paper we will stick to the known object $\Gamma_{\rm VD}$.

This implies that the third member of (1) is now a consequence of the second one; in other words, the normalization of the coupling constant is now determined by gauge invariance (cf. ref. [10]). There is no free choice of a vertex or momentum configuration, and the present procedure is completely unambiguous (more precisely, a different choice of the normalization of g would spoil the gauge invariance of the renormalized action $\Gamma_{\rm VD}$). As an extra bonus of the use of the Vilkovisky–DeWitt action, the renormalized coupling is guaranteed to be independent of the gauge condition employed in computing its β-function.

One now still has the expressions (6), (7) for the thermal Callan–Symanzik functions, but due to the second Ward identity mentioned above, (7) and its μ-analogue in fact drastically simplify to

$$\beta_T = -\tfrac{1}{2} g\alpha_T, \quad \beta_\mu = -\tfrac{1}{2} g\alpha_\mu. \tag{11}$$

Hence in order to compute the β-functions one just has to calculate the Vilkovisky–DeWitt two-point function (compare this with an analogous situation in the usual background-field gauges [33]). The calculation may, of course, be performed in any gauge, but it drastically simplifies in the so-called Landau–DeWitt gauge [32], in which (9) simplifies to $\sigma^a_\rho(x)[A^*, A] = A^{*a}_\rho(x) - A^a_\rho(x)$. In that case the expansion coefficients $\Gamma^{(n)}_{\rm VD}$ can be evaluated from the usual irreducible n-point vertex functions, with Feynman rules given in ref. [10] (actually, the Landau–DeWitt gauge happens to be a special case of the covariant background-field gauges [33]).

The calculation itself is still fairly involved [#3], and here we just give the (relevant) high-temperature expansion (obtained using the techniques developed in ref. [21]). In terms of $a_{\rm s} \equiv g^2/4\pi^2$, $x \equiv T/\kappa$, and $\xi \equiv T_0/\mu$ we find

$$\Pi^{\rm T}_{0\rm VD}(x,\xi) = a_{\rm s}\left[b \log\left(\frac{T}{T_0}\right) + \tfrac{21}{16}\pi^2 N(\xi - x) + \frac{37\zeta(3)}{640\pi^2}(x^{-2}-\xi^{-2}) + {\rm O}(x^{-4}-\xi^{-4})\right]. \tag{12}$$

Here the logarithmic term is the "usual" one, coming

[#3] Related computations in the background field gauge have been performed in ref. [34], where the complementary non-static limit $\kappa \to 0$, $k_0 \neq 0$ is addressed.

from the nonstatic modes, whereas the term linear in x, which dominates the expression, derives from the static mode; this is precisely the one neglected in the conventional renormalization schemes mentioned in the beginning of this paper. The β-functions (11) may now be computed from (6) and (12), after which the running coupling is found by solving two coupled differential equations, entirely similar to the procedure followed in ref. [12]. It is convenient to pass to μ and ξ as independent variables, defining $\tilde{a}_s(\mu, \xi) = a_s(\mu, T_0)$. One then obtains

$$\overline{\beta_\mu} \equiv \mu \frac{\partial \tilde{a}_s}{\partial \mu} = -b\tilde{a}_s^2 + O(\tilde{a}_s^3), \qquad (13)$$

$$\beta_\xi \equiv \xi \frac{\partial \tilde{a}_s}{\partial \xi} = -\left(b - \tfrac{21}{16}\pi^2 N\xi - \frac{37\zeta(3)}{320\pi^2}\xi^{-2} + O(\xi^{-4})\right)\tilde{a}_s^2 + O(\tilde{a}_s^3). \qquad (14)$$

Thus for large ξ the thermal β-function $\overline{\beta_\xi}$ is *positive*. Eqs. (13), (14) may meaningfully be solved in the regime $1 \ll \xi \ll \tilde{a}_s^{-1}$, in which case one finds

$$\tilde{a}_s(\mu, \xi)^{-1} = b \log\left(\frac{\xi\mu}{\Lambda_{\mathrm{MOM}}}\right) - \tfrac{21}{16}\pi^2 N\xi + \frac{37\zeta(3)}{640\pi^2}\xi^{-2} + O(\xi^{-4}), \qquad (15)$$

where Λ_{MOM} is the usual QCD scale parameter at $T=0$.

To close this section, let us recall that the negative sign of the coefficient of x in (12), which ultimately is responsible for the positivity of β_T at high T, and which in the present formalism is gauge-independent, has also been found in gauge-dependent polarization scalars defined similar to Π^T in section 2, cf. ref. [35].

5. Discussion

The most conspicious feature of (15) is that the effective coupling *grows* with T at fixed μ. However, for $\xi \sim \tilde{a}_s^{-1}$ the one-loop approximation to the β-function looses its meaning, a fact which just reflects the justified folk wisdom that the static sector is not perturbatively under control in the stated regime. In addition, we should remark that the positivity of the β-function contradicts lattice data as well as physical intuition. This strongly suggests that the positivity found above is an artifact of the bare one-loop approximation; indeed, it has been suggested [36] that nonperturbative mechanisms like (approximate) mass generation or Bose condensation [8] would render the thermal β-function negative. What we can say with certainty is that the standard one-loop renormalization-group arguments supposedly leading to asymptotic freedom at high temperature (and/or density) [1,2] are rather misleading, because they deal with a coupling constant and a renormalization scheme which lack a proper physical meaning.

The nature of the renormalized effective interaction strength in the quark–gluon plasma has been addressed in several other publications [37–39, 12,13,40]. The results of this paper confirm part of the conclusions of Nakkagawa et al. [12] which, however, have been obtained by an explicitly gauge-dependent procedure. Yamada [40] follows a gauge-invariant procedure, investigating the interaction of a static q$\bar{\mathrm{q}}$ pair via a thermal Wilson loop. However, in contrast to an analogous procedure in QED, the effective coupling thus obtained (which is found to decrease at high T and fixed μ) has little to do with the perturbative expansion parameter in thermal QCD.

As a qualifying statement, we wish to stress that we obviously do not claim to have identified *the* unique coupling constant of thermal QCD. Firstly, as already hinted at in the second footnote to section 3, the notion of the Vilkovisky–DeWitt action is itself not free of ambiguities [27,28], so that it remains to be seen whether alternative definitions would produce the same running coupling as the one advertised in the present paper. Furthermore, other methods to extract gauge-independent and gauge-invariant results from the Green functions of a nonabelian gauge theory do exist [41,42], and do not necessarily all lead to the same result, as the controversy concerning the QCD plasma damping constant has shown [8,9,29–31,43]. Even so, the definition of the thermal coupling parameter suggested in this Letter appears to be a natural one, the idea simply being that a gauge-invariant effective action can depend on the gluon field A_ρ only via the field strength $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho + gA_\rho \times A_\sigma$, identifying the parameter g as the coupling, just as in the classical action.

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